

Spectra of 1-dimensional diffusion operators: some examples

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1. Introduction

Hermite polynomials

Hermite polynomials are defined by


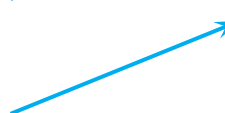

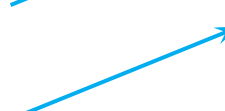
$$H_n(x) = \frac{(-1)^n}{n!} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}, \quad n = 0, 1, \dots$$

These are eigenfunctions of the Ornstein-Uhlenbeck operator

$$\frac{d^2}{dx^2} - x \frac{d}{dx}.$$

We have

$$\frac{d}{dx} H_n(x) = H_{n-1}(x).$$

eigenvalue		$\frac{d}{dx}$	$\mathbf{0}$
$\mathbf{0}$	$H_0(x)$		$H_0(x)$
-1	$H_1(x)$		$H_1(x)$
-2	$H_2(x)$		$H_2(x)$
-3	$H_3(x)$		$H_3(x)$
\vdots	\vdots		\vdots

In this talk, we give a general framework of this fact.

2. One dimensional diffusion processes

- $I = (l, r)$
- a, p : positive continuous functions on (l, r)

We consider the diffusion process generated by

$$(1) \quad \mathfrak{A}u = \frac{1}{p}(apu')'.$$

This operator is regarded as a self-adjoint operator in $L^2(p)$. Here p denotes a measure $p(x)dx$ on (l, r) .

By formal calculation, the associated Dirichlet form is

$$(2) \quad \mathcal{E}(u, v) = \int_l^r u'v'ap \, dx.$$

Now we define an operator $V : L^2(p) \rightarrow L^2(ap)$ by

$$(3) \quad Vu = u'$$

Here $\text{Dom}(V) = \text{Dom}(\mathcal{E})$. The dual operator $V^* : L^2(ap) \rightarrow L^2(p)$ of $V : L^2(p) \rightarrow L^2(ap)$ is given by

$$(4) \quad V^*\theta = -\frac{(ap\theta)'}{p}.$$

If we assume that a and p are C^2 functions, we have

$$(5) \quad \mathfrak{A}u = -V^*Vu = au'' + bu',$$

$$(6) \quad \hat{\mathfrak{A}}\theta = -VV^*\theta = a\theta'' + (b + a')\theta' + b'\theta.$$

Here $b = a' + a(\log p)'$ ($b + a' = a' + a(\log ap)'$).

Theorem 1. \mathfrak{A} and $\hat{\mathfrak{A}}$ have the same spectrum except for 0. Here we impose the Neumann boundary condition on \mathfrak{A} and the Dirichlet boundary condition on $\hat{\mathfrak{A}}$ if we need the boundary condition. Moreover the differentiation $u \mapsto u'$ gives rise to the correspondence between eigenfunctions.

3. Bessel functions

Squared Bessel process

We consider the squared Bessel process. We take $I = (0, \infty)$, $a = x$, $p = x^\alpha$. Then

$$\mathfrak{A}u = xu'' + (1 + \alpha)u', \quad \hat{\mathfrak{A}}u = xu'' + (2 + \alpha)u'.$$

Then, by Theorem 1, \mathfrak{A} and $\hat{\mathfrak{A}}$ has the same spectrum. The differentiation gives rise to the correspondence between eigenfunctions.

Hypergeometric functions

We define

$$(7) \quad {}_0F_1(c; x) = \sum_{n=0}^{\infty} \frac{1}{(c)_n n!} x^n.$$

For simplicity, we denote

$$(8) \quad B(c; x) = {}_0F_1(c; x).$$

We use the notation $B(c\pm) = B(c \pm 1; x)$. Then we have

Proposition 2.

$$B' = \frac{1}{c} B(c+), \quad xB' = (c-1)(B(c-) - B)$$

Eigenfunctoins are given as follows.

(a) $\alpha > -1$

The eigenfunction for the eigenvalue $-\xi$ ($\xi \geq 0$) is $B(1 + \alpha; -\xi x)$ and

$$\frac{d}{dx}[B(1 + \alpha; -\xi x)] = \frac{\lambda}{1 + \alpha} B(2 + \alpha; -\xi x).$$

Let us call these eigenfunctions as **entrance family** eigenfunctions.

(b) $\alpha < 0$

The eigenfunction for the eigenvalue $-\xi$ ($\xi \geq 0$) is $x^{-\alpha} B(1 - \alpha; \xi x)$ and

$$\frac{d}{dx}[x^{-\alpha} B(1 - \alpha; \xi x)] = -\alpha x^{-\alpha-1} B(-\alpha; \xi x)$$

Let us call these eigenfunctions as **exit family** eigenfunctions.

Remark 1. The function B is essentially the Bessel function.

$$(9) \quad B(\alpha + 1, -\xi x) = \Gamma(\alpha + 1)(\xi x)^{-\alpha/2} J_\alpha(\sqrt{4\xi x})$$

Remark 2. \mathfrak{A} has the spectral decomposition. Define the Hankel transform by

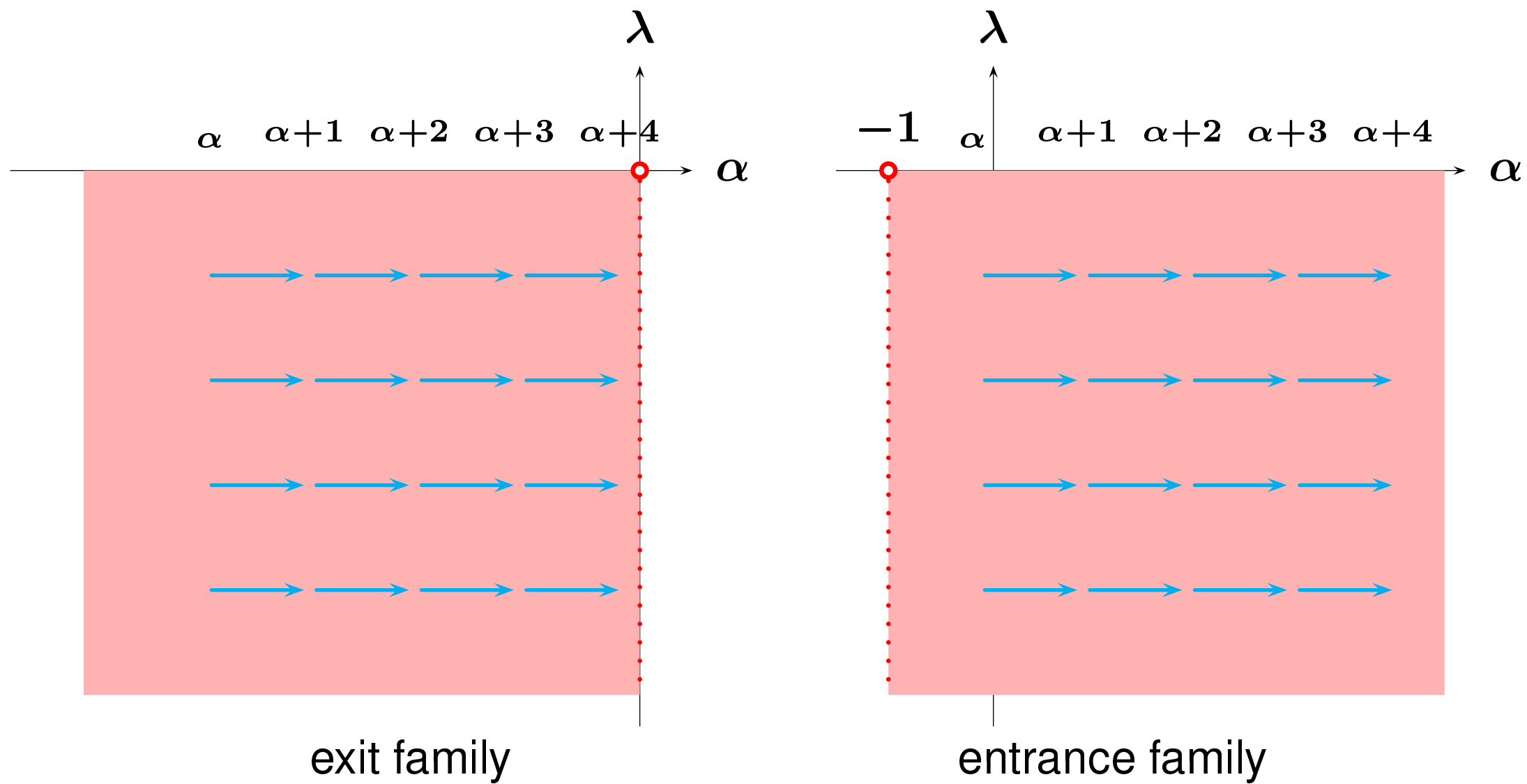
$$(10) \quad \hat{H}_\alpha[f](\xi) = \int_0^\infty f(x) \Gamma(\alpha + 1)^{-1} B(\alpha + 1; -\xi x) x^\alpha dx.$$

Then, we have

$$(11) \quad f(x) = \int_0^\infty \hat{H}_\alpha[f](\xi) \Gamma(\alpha + 1)^{-1} B(\alpha + 1; -\xi x) \xi^\alpha d\xi$$

and the following Plancherel identity

$$(12) \quad \int_0^\infty f(x)^2 x^\alpha dx = \int_0^\infty \hat{H}_\alpha[f](\xi)^2 \xi^\alpha d\xi.$$



Squared Bessel process on a finite interval

- $x \frac{d^2}{dx^2} + (1 + \alpha) \frac{d}{dx}$ on $I = (0, r)$.

On the boundary r , we consider the Dirichlet boundary condition and the Neumann boundary condition.

- $z(\alpha, n)$: n th zero point of the Bessel function J_α .

(a) $\alpha > -1$

[entrance, Neumann] family

The eigenfunction for the eigenvalue $-\frac{z(\alpha+1, n)^2}{4a}$ is

$$B\left(1 + \alpha; -\frac{z(\alpha+1, n)^2}{4a} x\right).$$

[entrance, Dirichlet] family

The eigenfunction for the eigenvalue $-\frac{z(\alpha, n)^2}{4a}$ is

$$B\left(1 + \alpha; -\frac{z(\alpha, n)^2}{4a} x\right).$$

The differentiation gives rise to the mapping from [entrance, Neumann] family to [entrance, Dirichlet] family.

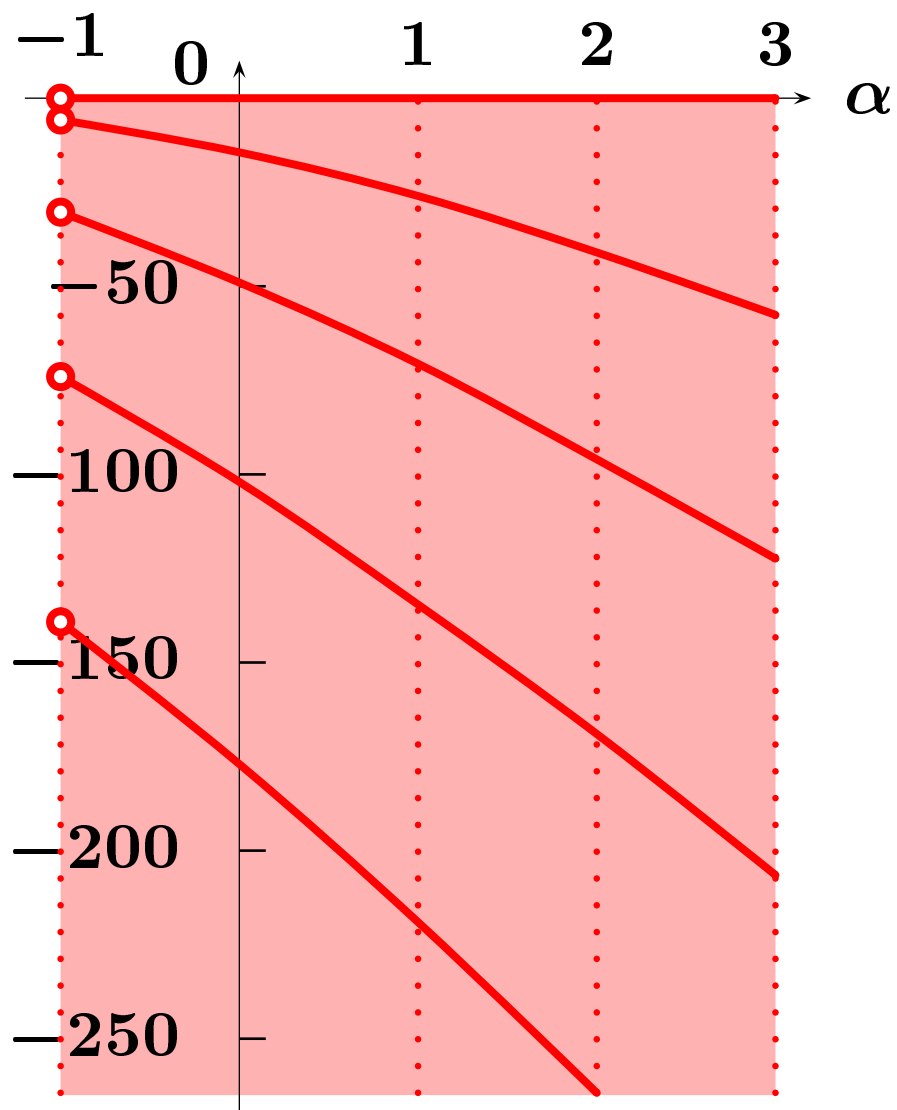
(b) $\alpha < 0$

[exit, Neumann] family

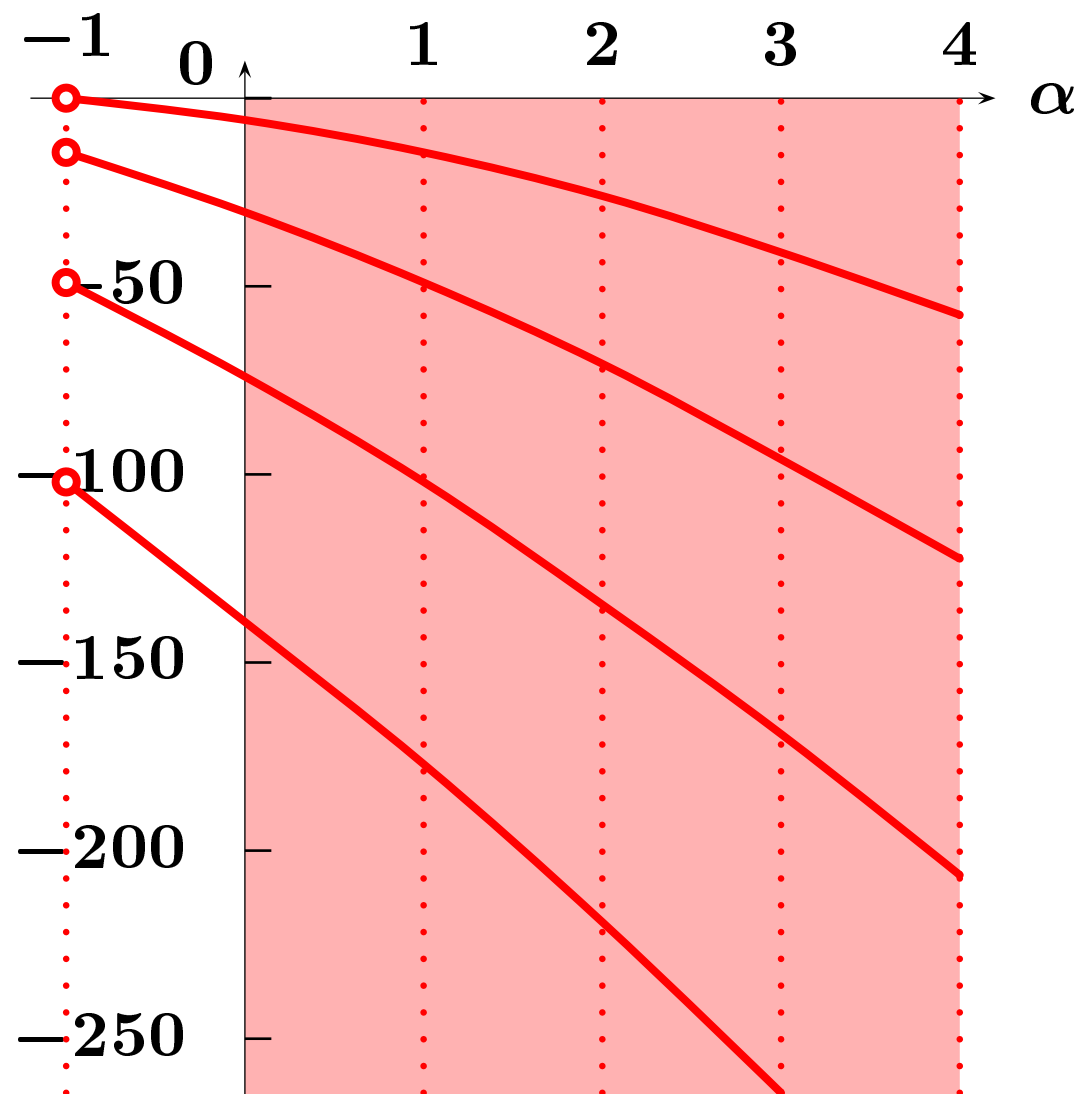
The eigenfunction for the eigenvalue $-\frac{z(-\alpha, n)^2}{4a}$ is $x^{-\alpha} B(1 - \alpha; -\frac{z(-\alpha, n)^2}{4a} x)$.

[exit, Dirichlet] family

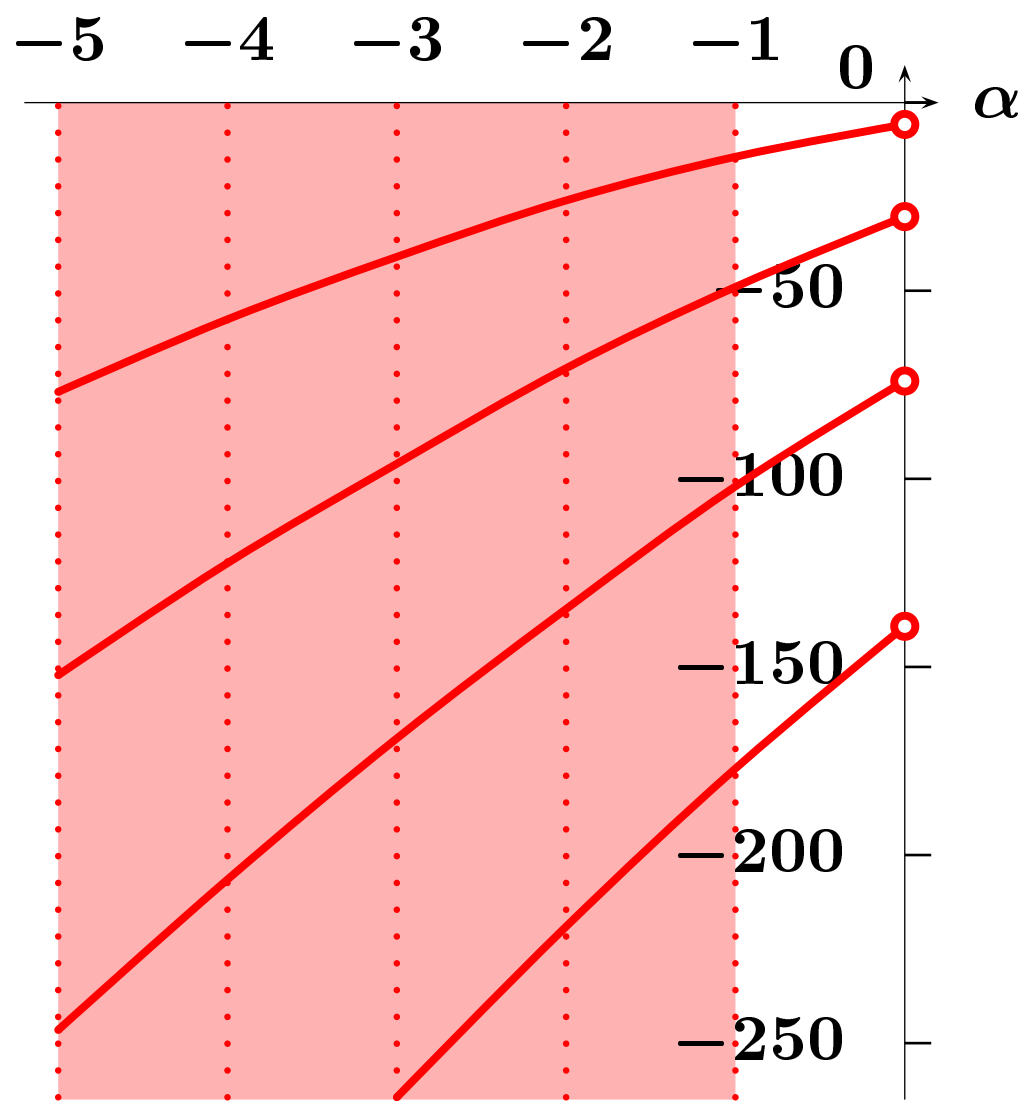
The eigenfunction for the eigenvalue $-\frac{z(1-\alpha, n)^2}{4a}$ is $x^{-\alpha} B(1 - \alpha; -\frac{z(1-\alpha, n)^2}{4a} x)$.



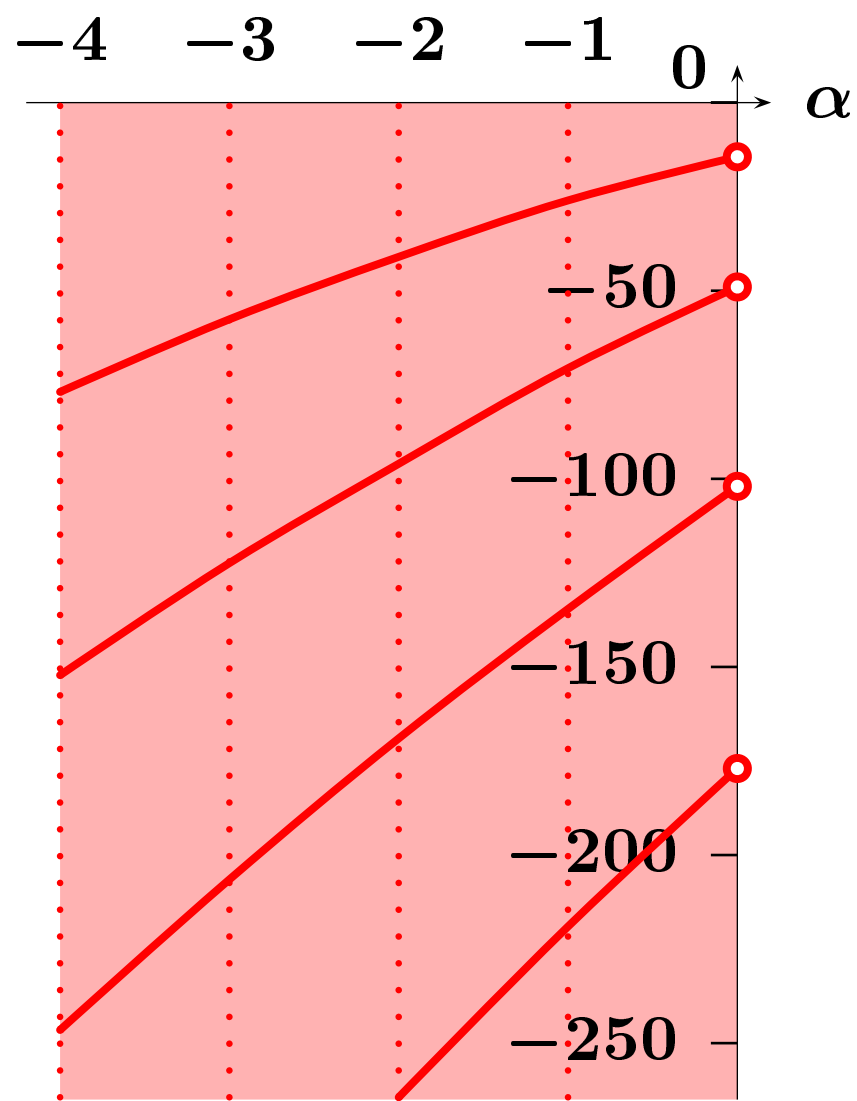
[entrance, Neumann] family



[entrance, Dirichlet] family



[exit, Neumann] family



[exit, Dirichlet] family

4. Laguerre polynomials

We take $I = (0, \infty)$, $a = x$, $p = x^\alpha e^{-x}$. Then

$$\mathfrak{A}u = xu'' + (1 + \alpha - x)u', \quad \hat{\mathfrak{A}}u = xu'' + (2 + \alpha - x)u' - u.$$

These operators have the same spectrum except for 0.

To show the eigenfunction we need the hypergeometric functions defined by

$$(13) \quad {}_1F_1(a; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n n!} x^n.$$

We set

$$(14) \quad M(a, c; x) = {}_1F_1(a; c; x)$$

Theorem 3. We have

$$(15) \quad M' = \frac{a}{c}M(a+, c+),$$

$$(16) \quad xM' + (c - 1)M = (c - 1)M(c-),$$

$$(17) \quad c(M' - M) = (a - c)M(c+),$$

$$(18) \quad xM' + (1 - c - x)M = (c - 1)M(a-, c-).$$

(a) $\alpha > -1$ entrance family

The eigenfunction for the eigenvalue $-n$ ($n = 0, 1, \dots$) is $M(-n, \alpha + 1; x)$ and

$$[M(-n, \alpha + 1; x)]' = -\frac{n}{\alpha + 1} M(-n + 1, \alpha + 2; x).$$

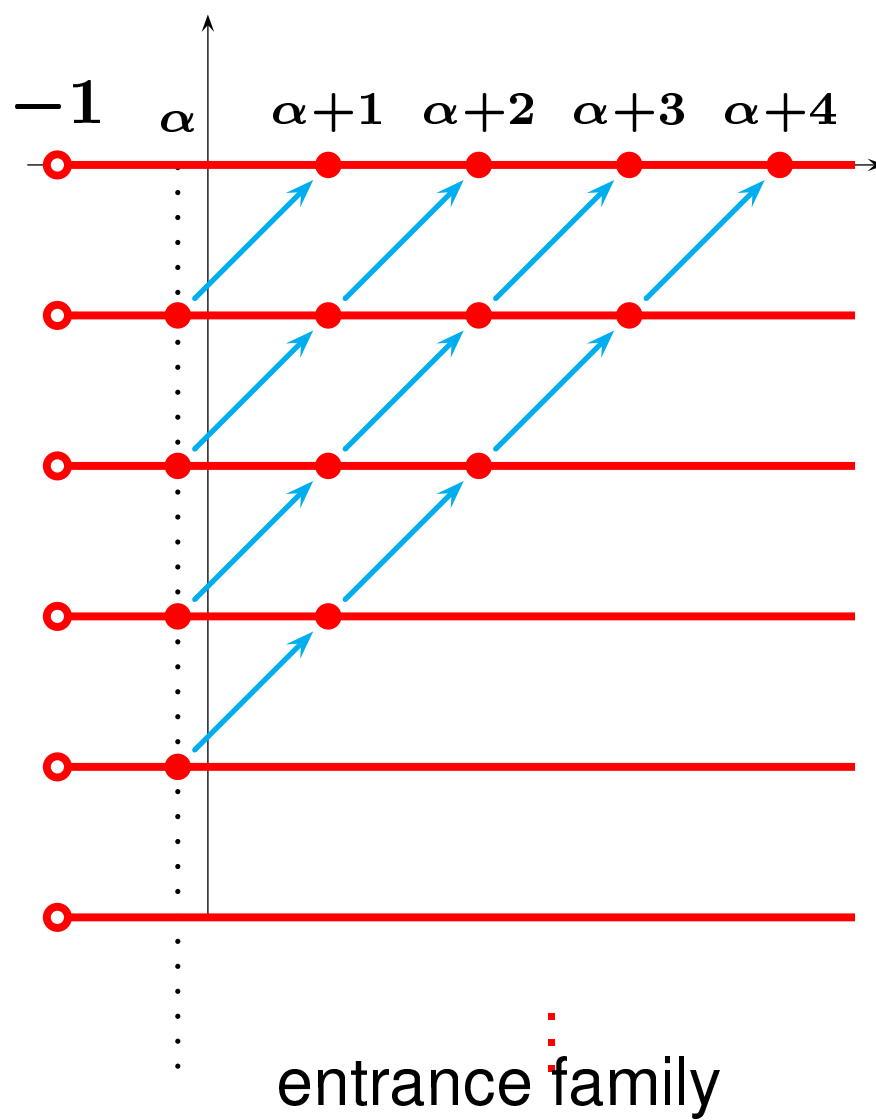
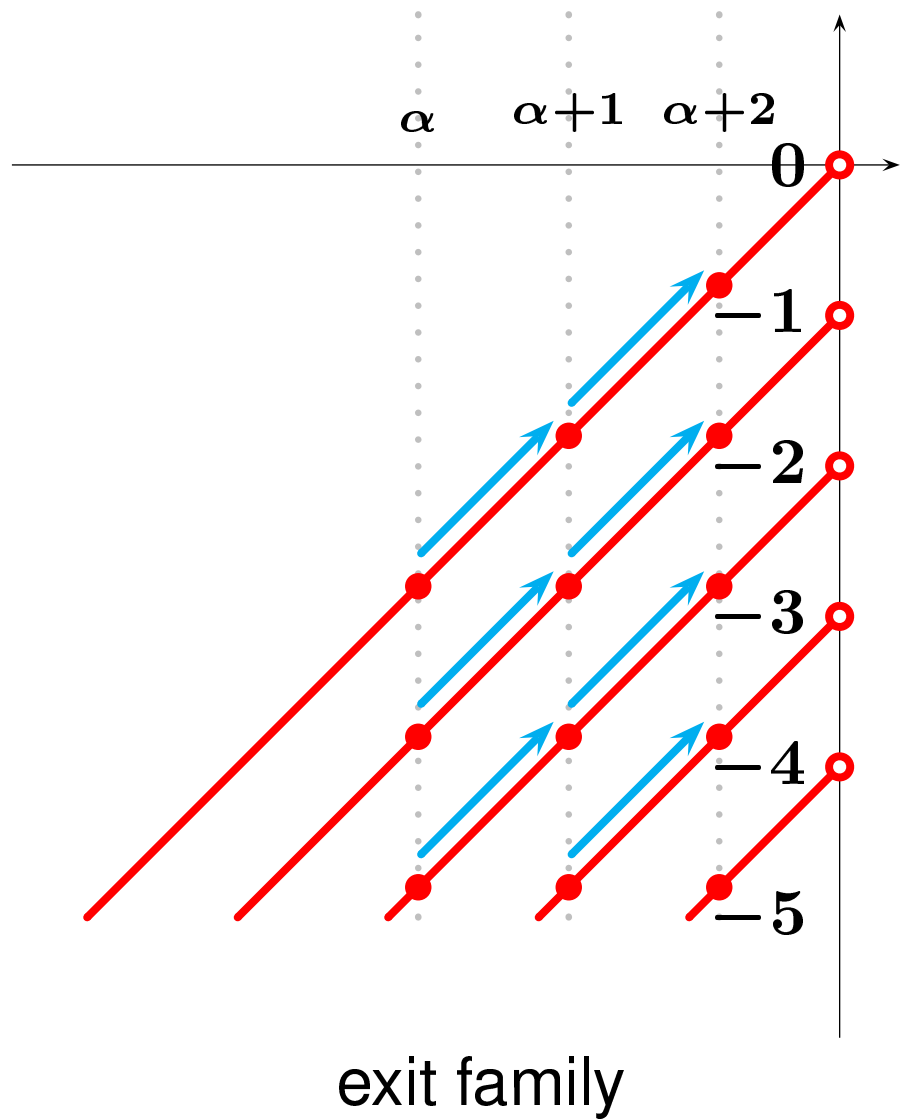
(b) $\alpha < 0$ exit family

The eigenfunction for the eigenvalue $-n + \alpha$ ($n = 0, 1, \dots$) is $x^{-\alpha} M(-n, 1 - \alpha; x)$ and

$$[x^{-\alpha} M(-n, 1 - \alpha; x)]' = -\alpha x^{-\alpha-1} M(-n, -\alpha; x).$$

Remark 3. M is essentially the Laguerre polynomial.

$$(19) \quad L_n^{(\alpha)}(x) = \frac{(\alpha + 1)_n}{n!} M(-n, \alpha + 1; x)$$



5. Jacobi polynomials

We take $I = (0, 1)$, $a = x(1 - x)$, $p = x^\alpha(1 - x)^\beta$. Then

$$\mathfrak{A}u = x(1 - x)u'' + ((\alpha + 1)(1 - x) - (\beta + 1)x)u',$$

$$\hat{\mathfrak{A}}u = x(1 - x)u'' + ((\alpha + 2)(1 - x) - (\beta + 2)x)u' - (\alpha + \beta + 2)u.$$

To get eigenfunctions, we need hypergeometric function defined by

$$(20) \quad {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n.$$

We set

$$(21) \quad K(x) = K(\alpha, \beta, \gamma; x) = {}_2F_1(-\gamma, \alpha + \beta + \gamma + 1; \alpha + 1; x)$$

Remark 4. K is essentially the Jacobi polynomial.

$$(22) \quad P_n^{(\alpha, \beta)}(x) = \frac{\Gamma(\alpha + n + 1)}{n! \Gamma(\alpha + 1)} K(\alpha, \beta, n; \frac{1-x}{2}).$$

Proposition 4. We have

$$(23) \quad K' = -\frac{\gamma(\gamma + \alpha + \beta + 1)}{\alpha + 1} K(\alpha+, \beta+, \gamma-).$$

$$(24) \quad xK' + \alpha K = \alpha K(\alpha-, \beta+).$$

$$(25) \quad (1-x)K' - \beta K = -\frac{(\alpha + \gamma + 1)(\beta + \gamma)}{\alpha + 1} K(\alpha+, \beta-).$$

$$(26) \quad x(1-x)K' + (\alpha(1-x) - \beta x)K = \alpha K(\alpha-, \beta-, \gamma+).$$

(a) $\alpha > -1, \beta > -1$ [entrance,entrance] family

The eigenfunction for the eigenvalue $-n(n + \alpha + \beta + 1)$
 ($n = 0, 1, \dots$) is $K(\alpha, \beta, n)$ and

$$K'(\alpha, \beta, n) = -\frac{\gamma(\alpha + \beta + \gamma + 1)}{\alpha + 1} K(\alpha + 1, \beta + 1, n - 1).$$

(b) $\alpha < 0, \beta > -1$ [entrance,exit] family

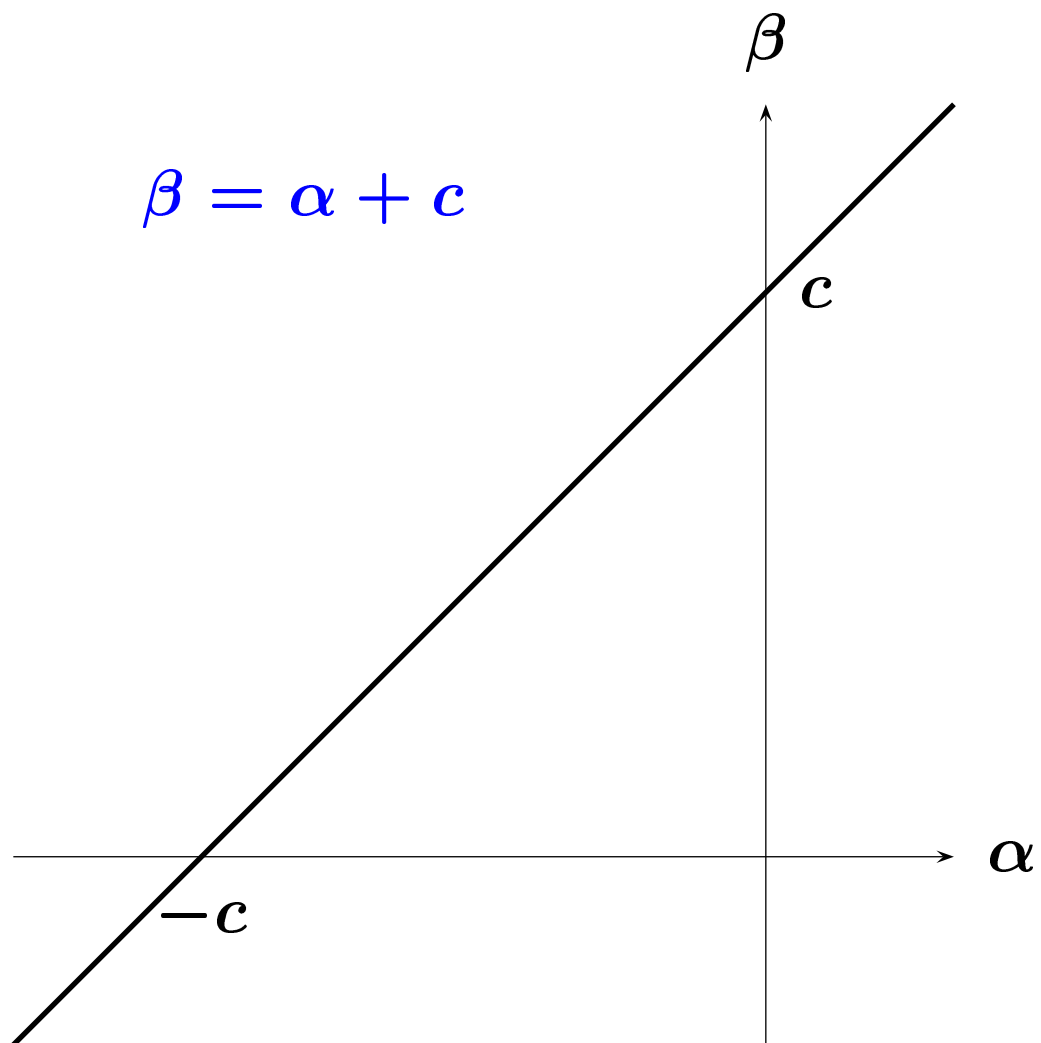
The eigenfunction for the eigenvalue $-(n - \alpha)(n + \beta + 1)$
 ($n = 0, 1, \dots$) is $x^{-\alpha} K(-\alpha, \beta, n)$ and

$$[x^{-\alpha} K(-\alpha, \beta, n)]' = -\alpha x^{-\alpha-1} K(-\alpha - 1, \beta + 1, n).$$

(c) $\alpha < 0, \beta < 0$ [exit,exit] family

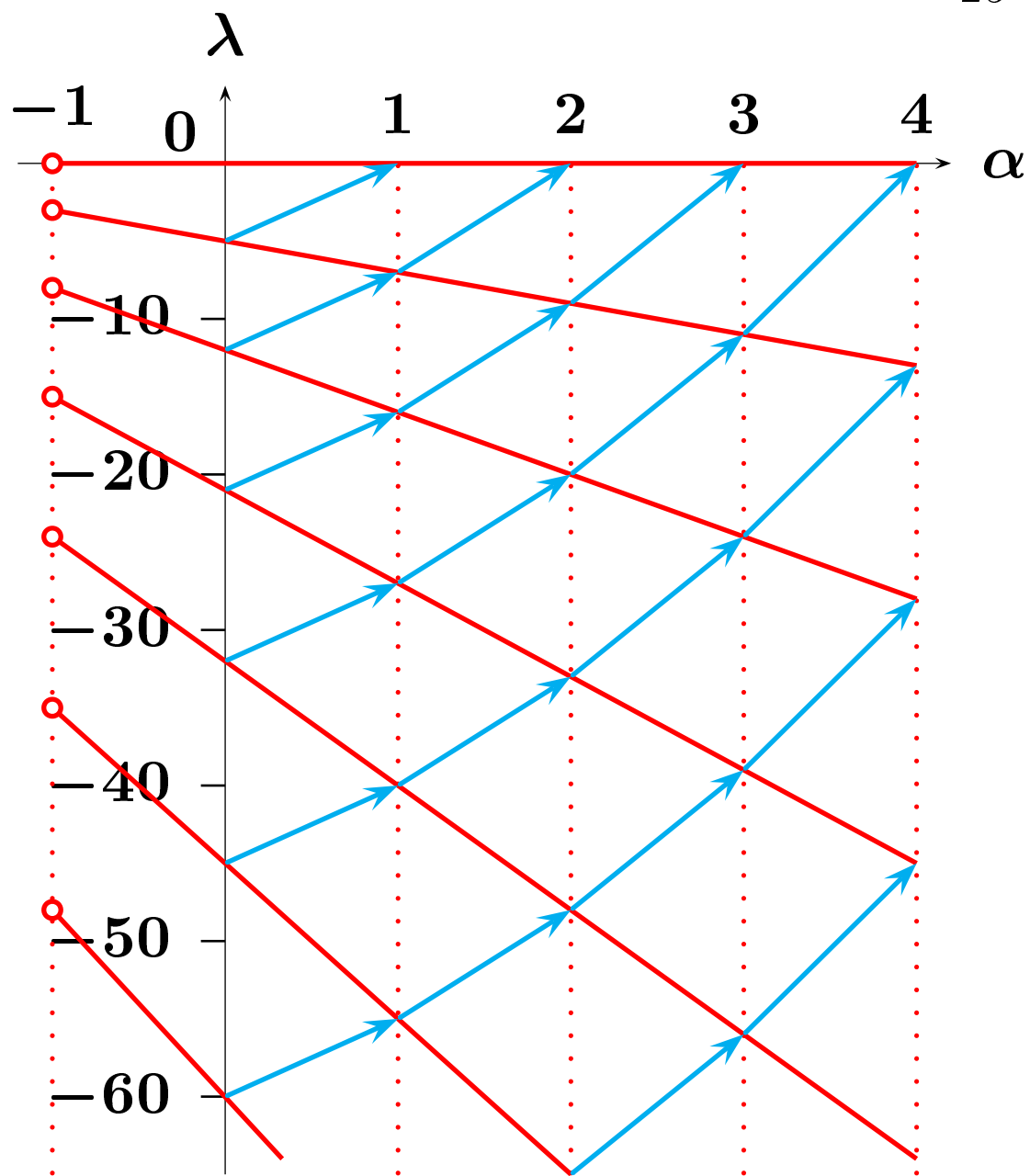
The eigenfunction for the eigenvalue $-(n+1)(n-\alpha-\beta)$ ($n = 0, 1, \dots$) is $x^{-\alpha}(1-x)^{-\beta}K(-\alpha, -\beta, \gamma)$ and

$$\begin{aligned} & [x^{-\alpha}(1-x)^{-\beta}K(-\alpha, -\beta, n)]' \\ & = -\alpha x^{-\alpha-1}(1-x)^{-\beta-1}K(-\alpha-1, -\beta-1, n+1). \end{aligned}$$

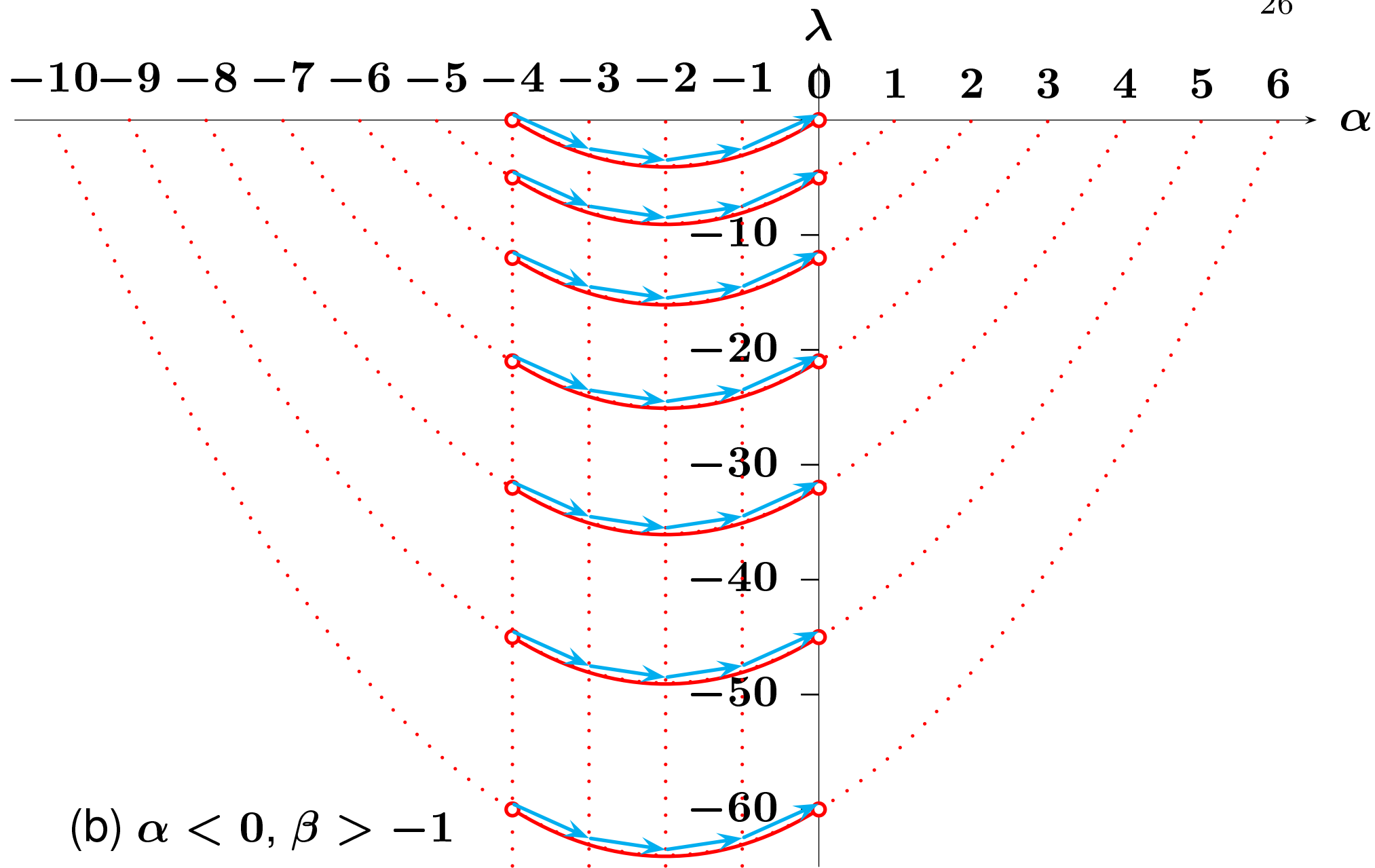


parameters

(a) $\alpha > -1, \beta > -1$



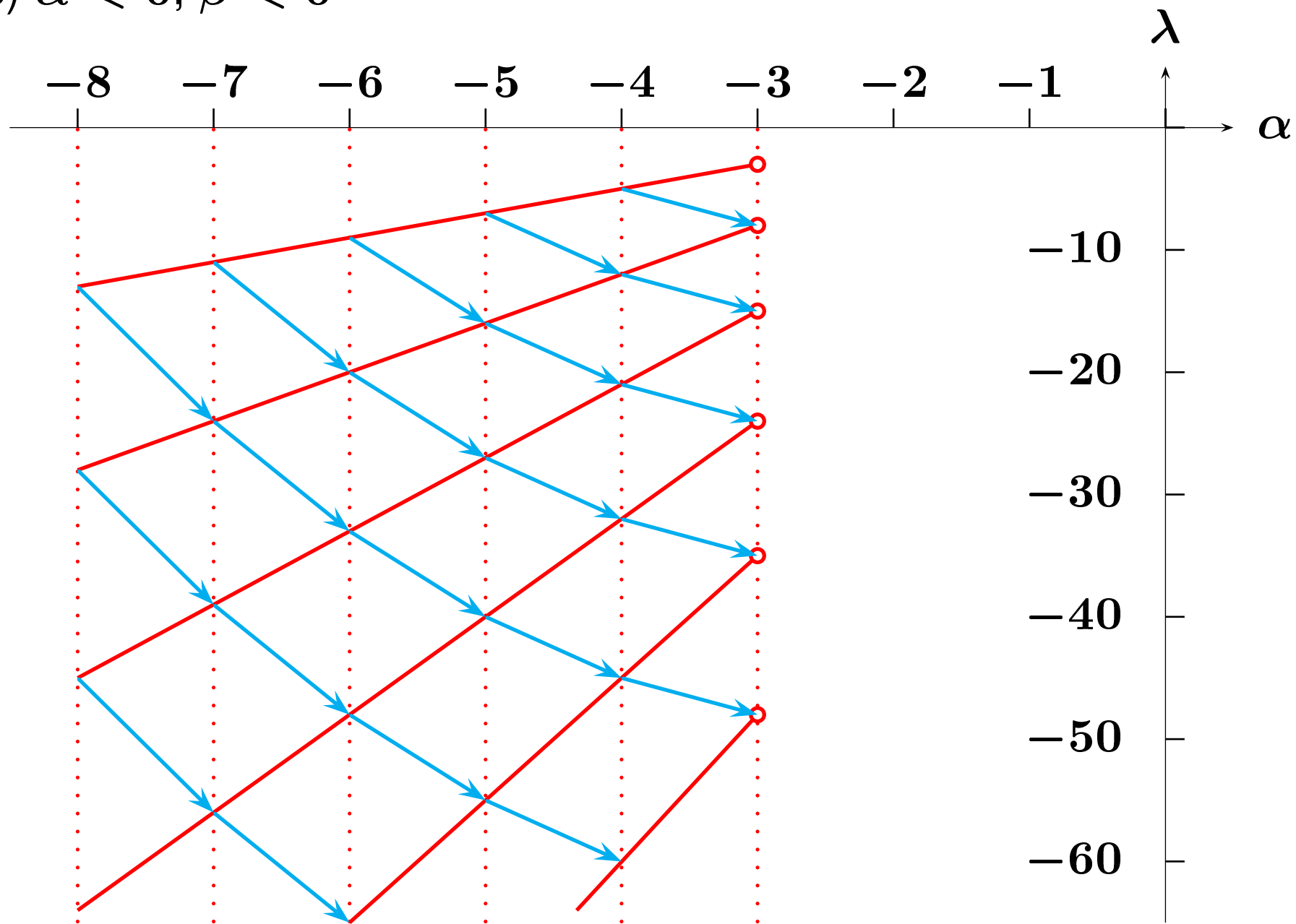
[entrance, entrance] family



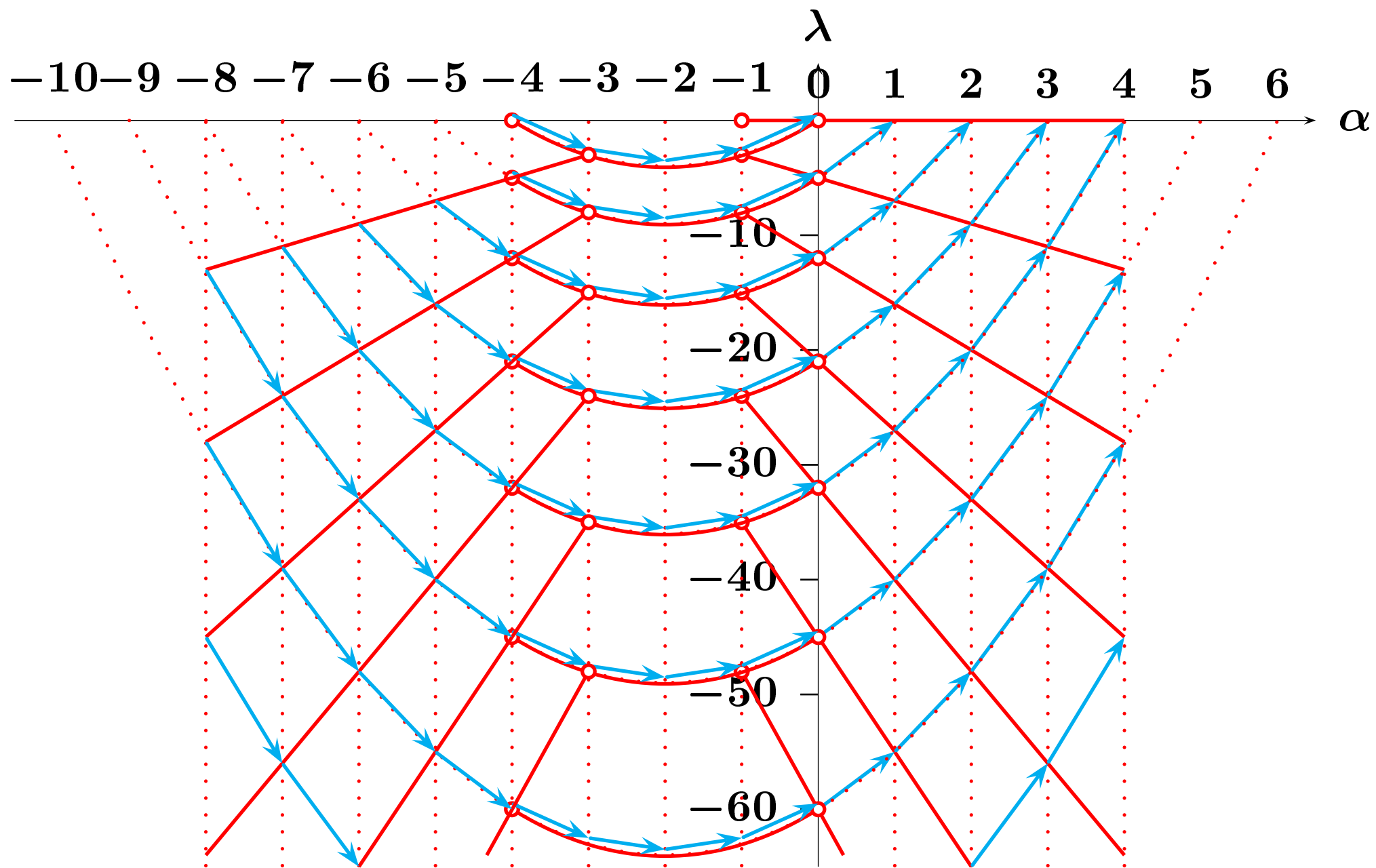
[exit, entrance] family

(c) $\alpha < 0, \beta < 0$

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[exit, exit] family



Summary

$x \frac{d^2}{dx^2} + (\alpha + 1) \frac{d}{dx}$	Bessel	${}_0F_1$
$x \frac{d^2}{dx^2} + (\alpha + 1 - x) \frac{d}{dx}$	Laguerre	${}_1F_1$
$x(1-x) \frac{d^2}{dx^2} + ((\alpha + 1)(1-x) - (\beta + 1)x) \frac{d}{dx}$	Jacobi	${}_2F_1$

Thanks !