1 次元拡散作用素の固有関数の
いくつかの具体例について

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確率解析とその周辺

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1. Introduction

Hermite polynomials

Hermite polynomials are defined by

\[ H_n(x) = \frac{(-1)^n}{n!} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}, \quad n = 0, 1, \ldots \]

These are eigenfunctions of the Ornstein-Uhlenbeck operator

\[ \frac{d^2}{dx^2} - x \frac{d}{dx}. \]

We have

\[ \frac{d}{dx} H_n(x) = H_{n-1}(x). \]
In this talk, we give a general framework of this fact.
2. One dimensional diffusion processes

- $I = (l, r)$
- $a, p$: positive continuous functions on $(l, r)$

We consider the diffusion process generated by

$$\mathcal{A}u = \frac{1}{p} (apu')'.$$

This operator is regarded as a self-adjoint operator in $L^2(p)$. Here $p$ denotes a measure $p(x)dx$ on $(l, r)$.

By formal calculation, the associated Dirichlet form is

$$\mathcal{E}(u, v) = \int_{l}^{r} u' v' ap \, dx.$$
Now we define an operator $V : L^2(p) \rightarrow L^2(ap)$ by

$$V u = u'$$

(3)

Here $\text{Dom}(V) = \text{Dom}(\mathcal{E})$. The dual operator $V^* : L^2(ap) \rightarrow L^2(p)$ of $V : L^2(p) \rightarrow L^2(ap)$ is given by

$$V^* \theta = -\frac{(ap\theta)'}{p}.$$ 

(4)

If we assume that $a$ and $p$ are $C^2$ functions, we have

$$\mathcal{A}u = -V^*Vu = au'' + bu',$$

(5)

$$\hat{\mathcal{A}} \theta = -VV^*\theta = a\theta'' + (b + a')\theta' + b'\theta.$$ 

(6)

Here $b = a' + a(\log p)'$ \quad ($b + a' = a' + a(\log ap)'$).
Theorem 1. \( \mathbf{A} \) and \( \hat{\mathbf{A}} \) have the same spectrum except for 0. Here we impose the Neumann boundary condition on \( \mathbf{A} \) and the Dirichlet boundary condition on \( \hat{\mathbf{A}} \) if we need the boundary condition. Moreover the differentiation \( u \mapsto u' \) gives rise to the correspondence between eigenfunctions.
3. **Bessel functions**

**Squared Bessel process**

We consider the squared Bessel process. We take $I = (0, \infty)$, $a = x$, $p = x^\alpha$. Then

$$Au = xu'' + (1 + \alpha)u', \quad \hat{A}u = xu'' + (2 + \alpha)u'. $$

Then, by Theorem 1, $A$ and $\hat{A}$ has the same spectrum. The differentiation gives rise to the correspondence between eigenfunctions.
Hypergeometric functions

We define

\[
_{0}F_{1}(c; x) = \sum_{n=0}^{\infty} \frac{1}{(c)_n n!} x^n.
\]

(7)

For simplicity, we denote

\[
B(c; x) = _{0}F_{1}(c; x).
\]

(8)

We use the notation \(B(c\pm) = B(c \pm 1; x)\). Then we have

**Proposition 2.**

\[
B' = \frac{1}{c} B(c+), \quad xB' = (c - 1)(B(c-) - B)
\]

Eigenfunctionstions are given as follows.
(a) \( \alpha > -1 \)

The eigenfunction for the eigenvalue \(-\xi\) \((\xi \geq 0)\) is \(B(1 + \alpha; -\xi x)\) and

\[
\frac{d}{dx} [B(1 + \alpha; -\xi x)] = \frac{\lambda}{1 + \alpha} B(2 + \alpha; -\xi x).
\]

Let us call these eigenfunctions as entrance family eigenfunctions.

(b) \( \alpha < 0 \)

The eigenfunction for the eigenvalue \(-\xi\) \((\xi \geq 0)\) is \(x^{-\alpha}B(1 - \alpha; \xi x)\) and

\[
\frac{d}{dx} [x^{-\alpha}B(1 - \alpha; -\xi x)] = -\alpha x^{-\alpha - 1}B(-\alpha; -\xi x)
\]

Let us call these eigenfunctions as exit family eigenfunctions.
Remark 1. The function $B$ is essentially the Bessel function.

$$B(\alpha + 1, -\xi x) = \Gamma(\alpha + 1)(\xi x)^{-\alpha/2}J_\alpha(\sqrt{4\xi x})$$

Remark 2. $A$ has the spectral decomposition. Define the Hankel transform by

$$\hat{H}_\alpha[f](\xi) = \int_0^\infty f(x)\Gamma(\alpha + 1)^{-1}B(\alpha + 1; -\xi x)x^\alpha \, dx.$$  \hspace{1cm} (10)

Then, we have

$$f(x) = \int_0^\infty \hat{H}_\alpha[f](\xi)\Gamma(\alpha + 1)^{-1}B(\alpha + 1; -\xi x)\xi^\alpha \, d\xi$$  \hspace{1cm} (11)

and the following Plancherel identity

$$\int_0^\infty f(x)^2 x^\alpha \, dx = \int_0^\infty \hat{H}_\alpha[f](\xi)^2 \xi^\alpha \, d\xi.$$  \hspace{1cm} (12)
Squared Bessel process on a finite interval

- \( x \frac{d^2}{dx^2} + (1 + \alpha) \frac{d}{dx} \) on \( I = (0, r) \).
  
  On the boundary \( r \), we consider the Dirichlet boundary condition and the Neumann boundary condition.

- \( z(\alpha, n) \): \( n \)th zero point of the Bessel function \( J_\alpha \).

(a) \( \alpha > -1 \)

[entrance, Neumann] family

The eigenfunction for the eigenvalue \( -\frac{z(\alpha+1,n)^2}{4a} \) is \( B(1 + \alpha; -\frac{z(\alpha+1,n)^2}{4a} x) \).

[entrance, Dirichlet] family

The eigenfunction for the eigenvalue \( -\frac{z(\alpha,n)^2}{4a} \) is \( B(1 + \alpha; -\frac{z(\alpha,n)^2}{4a} x) \).
The differentiation gives rise to the mapping from [entrance, Neumann] family to [entrance, Dirichlet] family.

(b) \( \alpha < 0 \)

[exit, Neumann] family

The eigenfunction for the eigenvalue \(- \frac{z(-\alpha,n)^2}{4a}\) is 
\[ x^{-\alpha} B(1 - \alpha; - \frac{z(-\alpha,n)^2}{4a} x) \].

[exit, Dirichlet] family

The eigenfunction for the eigenvalue \(- \frac{z(1-\alpha,n)^2}{4a}\) is 
\[ x^{-\alpha} B(1 - \alpha; - \frac{z(1-\alpha,n)^2}{4a} x) \].
[exit, Neumann] family

[exit, Dirichlet] family
4. Laguerre polynomials

We take \( I = (0, \infty) \), \( a = x, p = x^\alpha e^{-x} \). Then

\[
\mathcal{A}u = xu'' + (1 + \alpha - x)u', \quad \hat{\mathcal{A}}u = xu'' + (2 + \alpha - x)u' - u.
\]

These operators have the same spectrum except for 0.

To show the eigenfunction we need the hypergeometric functions defined by

\[
_{1}F_{1}(a; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n n!} x^n.
\]

We set

\[
M(a, c; x) = _{1}F_{1}(a; c; x)
\]
Theorem 3. We have

\[ M' = \frac{a}{c} M(a+, c+), \]

\[ xM' + (c - 1)M = (c - 1)M(c-), \]

\[ c(M' - M) = (a - c)M(c+), \]

\[ xM' + (1 - c - x)M = (c - 1)M(a-, c-). \]
(a) $\alpha > -1$ entrance family

The eigenfunction for the eigenvalue $-n$ ($n = 0, 1, \ldots$) is $M(-n, \alpha + 1; x)$ and

$$[M(-n, \alpha + 1; x)]' = -\frac{n}{\alpha + 1}M(-n + 1, \alpha + 2; x).$$

(b) $\alpha < 0$ exit family

The eigenfunction for the eigenvalue $-n + \alpha$ ($n = 0, 1, \ldots$) is $x^{-\alpha}M(-n, 1 - \alpha; x)$ and

$$[x^{-\alpha}M(-n, 1 - \alpha; x)]' = -\alpha x^{-\alpha-1}M(-n, -\alpha; x).$$

Remark 3. $M$ is essentially the Laguerre polynomial.

(19) \[ L_n^{(\alpha)}(x) = \frac{(\alpha + 1)_n}{n!}M(-n, \alpha + 1; x) \]
5. Jacobi polynomials

We take \( I = (0, 1) \), \( a = x(1 - x) \), \( p = x^\alpha (1 - x)^\beta \). Then

\[
\mathfrak{A}u = x(1 - x)u'' + ((\alpha + 1)(1 - x) - (\beta + 1)x)u',
\]

\[
\hat{\mathfrak{A}}u = x(1 - x)u'' + ((\alpha + 2)(1 - x) - (\beta + 2)x)u' - (\alpha + \beta + 2)u.
\]

To get eigenfunctions, we need hypergeometric function defined by

\[
(20) \quad _2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n.
\]

We set

\[
(21) \quad K(x) = K(\alpha, \beta, \gamma; x) = _2F_1(-\gamma, \alpha + \beta + \gamma + 1; \alpha + 1; x)
\]
Remark 4. \( K \) is essentially the Jacobi polynomial.

\[
P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha + n + 1)}{n! \Gamma(\alpha + 1)} K(\alpha, \beta, n; \frac{1 - x}{2}).
\]

Proposition 4. We have

\[
K' = -\frac{\gamma(\gamma + \alpha + \beta + 1)}{\alpha + 1} K(\alpha+, \beta+, \gamma-).
\]

\[
x K' + \alpha K = \alpha K(\alpha-, \beta+).
\]

\[
(1 - x) K' - \beta K = -\frac{(\alpha + \gamma + 1)(\beta + \gamma)}{\alpha + 1} K(\alpha+, \beta-).
\]

\[
x(1 - x) K' + (\alpha(1 - x) - \beta x) K = \alpha K(\alpha-, \beta-, \gamma+).
\]
(a) $\alpha > -1, \beta > -1$ [entrance,entrance] family

The eigenfunction for the eigenvalue $-n(n + \alpha + \beta + 1)$ $(n = 0, 1, \ldots)$ is $K(\alpha, \beta, n)$ and

$$K'(\alpha, \beta, n) = -\frac{\gamma(\alpha + \beta + \gamma + 1)}{\alpha + 1} K(\alpha + 1, \beta + 1, n - 1).$$

(b) $\alpha < 0, \beta > -1$ [entrance,exit] family

The eigenfunction for the eigenvalue $-(n - \alpha)(n + \beta + 1)$ $(n = 0, 1, \ldots)$ is $x^{-\alpha}K(-\alpha, \beta, n)$ and

$$[x^{-\alpha}K(-\alpha, \beta, n)]' = -\alpha x^{-\alpha - 1} K(-\alpha - 1, \beta + 1, n).$$
(c) $\alpha < 0$, $\beta < 0$  [exit,exit] family

The eigenfunction for the eigenvalue $-(n + 1)(n - \alpha - \beta)$ ($n = 0, 1, \ldots$) is $x^{-\alpha}(1 - x)^{-\beta}K(-\alpha, -\beta, \gamma)$ and

$$[x^{-\alpha}(1 - x)^{-\beta}K(-\alpha, -\beta, n)]'$$

$$= -\alpha x^{-\alpha - 1}(1 - x)^{-\beta - 1}K(-\alpha - 1, -\beta - 1, n + 1).$$
\[ \beta = \alpha + c \]
(a) $\alpha > -1, \beta > -1$
(b) $\alpha < 0, \beta > -1$

[exit, entrance] family
(c) $\alpha < 0, \beta < 0$

[exit, exit] family
# Summary

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Thanks !