1次元拡散作用素の固有関数の いくつかの具体例について

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確率解析とその周辺

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1. Introduction

Hermite polynomials

Hermite polynomials are defined by

$$H_n(x) = \frac{(-1)^n}{n!} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}, \quad n = 0, 1, \ldots$$

These are eigenfunctions of the Ornstein-Uhlenbeck operator

$$\frac{d^2}{dx^2} - x\frac{d}{dx}.$$

We have

$$\frac{d}{dx} H_n(x) = H_{n-1}(x).$$

$$
\begin{array}{cccc}
\text{eigenvalue} & & \xrightarrow{\;\frac{d}{dx}\;} & \mathbf{0} \\
\mathbf{0} & H_0(x) & \nearrow & H_0(x) \\
\mathbf{-1} & H_1(x) & \nearrow & H_1(x) \\
\mathbf{-2} & H_2(x) & \nearrow & H_2(x) \\
\mathbf{-3} & H_3(x) & \nearrow & H_3(x) \\
\vdots & \vdots & & \vdots
\end{array}
$$

In this talk, we give a general framework of this fact.

2. One dimensional diffusion processes

- $I = (l, r)$
- a, p: positive continuous functions on (l, r)

We consider the diffusion process generated by

(1) $$\mathfrak{A}u = \frac{1}{p}(apu')'.$$

This operator is regarded as a self-adjoint operator in $L^2(p)$. Here p denotes a measure $p(x)dx$ on (l, r).

By formal calculation, the associated Dirichlet form is

(2) $$\mathcal{E}(u, v) = \int_l^r u'v'ap\,dx.$$

Now we define an operator $V : L^2(p) \to L^2(ap)$ by

$$Vu = u' \tag{3}$$

Here $\mathbf{Dom}(V) = \mathbf{Dom}(\mathcal{E})$. The dual opetator $V^* : L^2(ap) \to L^2(p)$ of $V : L^2(p) \to L^2(ap)$ is given by

$$V^*\theta = -\frac{(ap\theta)'}{p}. \tag{4}$$

If we assume that a and p are C^2 functions, we have

$$\mathfrak{A}u = -V^*Vu = au'' + bu', \tag{5}$$

$$\hat{\mathfrak{A}}\theta = -VV^*\theta = a\theta'' + (b + a')\theta' + b'\theta. \tag{6}$$

Here $b = a' + a(\log p)'$ $\quad$ ($b + a' = a' + a(\log ap)'$).

Theorem 1. $\mathfrak{A}$ and $\hat{\mathfrak{A}}$ have the same spectrum except for $\mathbf{0}$. Here we impose the Neumann boundary condition on $\mathfrak{A}$ and the Dirichlet boundary condition on $\hat{\mathfrak{A}}$ if we need the boundary condition. Moreover the differentiation $u \mapsto u'$ gives rise to the correspondence between eigenfunctions.

3. Bessel functions

Squared Bessel process

We consider the squared Bessel process. We take $I = (0, \infty)$, $a = x$, $p = x^{\alpha}$. Then

$$\mathfrak{A}u = xu'' + (1+\alpha)u', \quad \hat{\mathfrak{A}}u = xu'' + (2+\alpha)u'.$$

Then, by Theorem 1, $\mathfrak{A}$ and $\hat{\mathfrak{A}}$ has the same spectrum. The differentiation gives rise to the correspondence between eigenfunctions.

Hypergeometric functions

We define

$$ {}_0F_1(c;x) = \sum_{n=0}^{\infty} \frac{1}{(c)_n n!} x^n. \tag{7} $$

For simplicity, we denote

$$ B(c;x) = {}_0F_1(c;x). \tag{8} $$

We use the notation $B(c\pm) = B(c \pm 1; x)$. Then we have

Proposition 2.

$$ B' = \frac{1}{c} B(c+), \quad xB' = (c-1)(B(c-) - B) $$

Eigenfunctoins are given as follows.

(a) $\alpha > -1$

The eigenfunction for the eigenvalue $-\xi$ ($\xi \geq 0$) is $B(1+\alpha; -\xi x)$ and

$$\frac{d}{dx}[B(1+\alpha; -\xi x)] = \frac{\lambda}{1+\alpha} B(2+\alpha; -\xi x).$$

Let us call these eigenfunctions as entrance family eigenfunctions.

(b) $\alpha < 0$

The eigenfunction for the eigenvalue $-\xi$ ($\xi \geq 0$) is $x^{-\alpha} B(1-\alpha; \xi x)$ and

$$\frac{d}{dx}[x^{-\alpha} B(1-\alpha; -\xi x)] = -\alpha x^{-\alpha-1} B(-\alpha; -\xi x)$$

Let us call these eigenfunctions as exit family eigenfunctions.

Remark 1. The function B is essentially the Bessel function.

$$B(\alpha+1,-\xi x)=\Gamma(\alpha+1)(\xi x)^{-\alpha/2}J_\alpha(\sqrt{4\xi x}) \tag{9}$$

Remark 2. $\mathfrak{A}$ has the spectral decomposition. Define the Hankel transform by

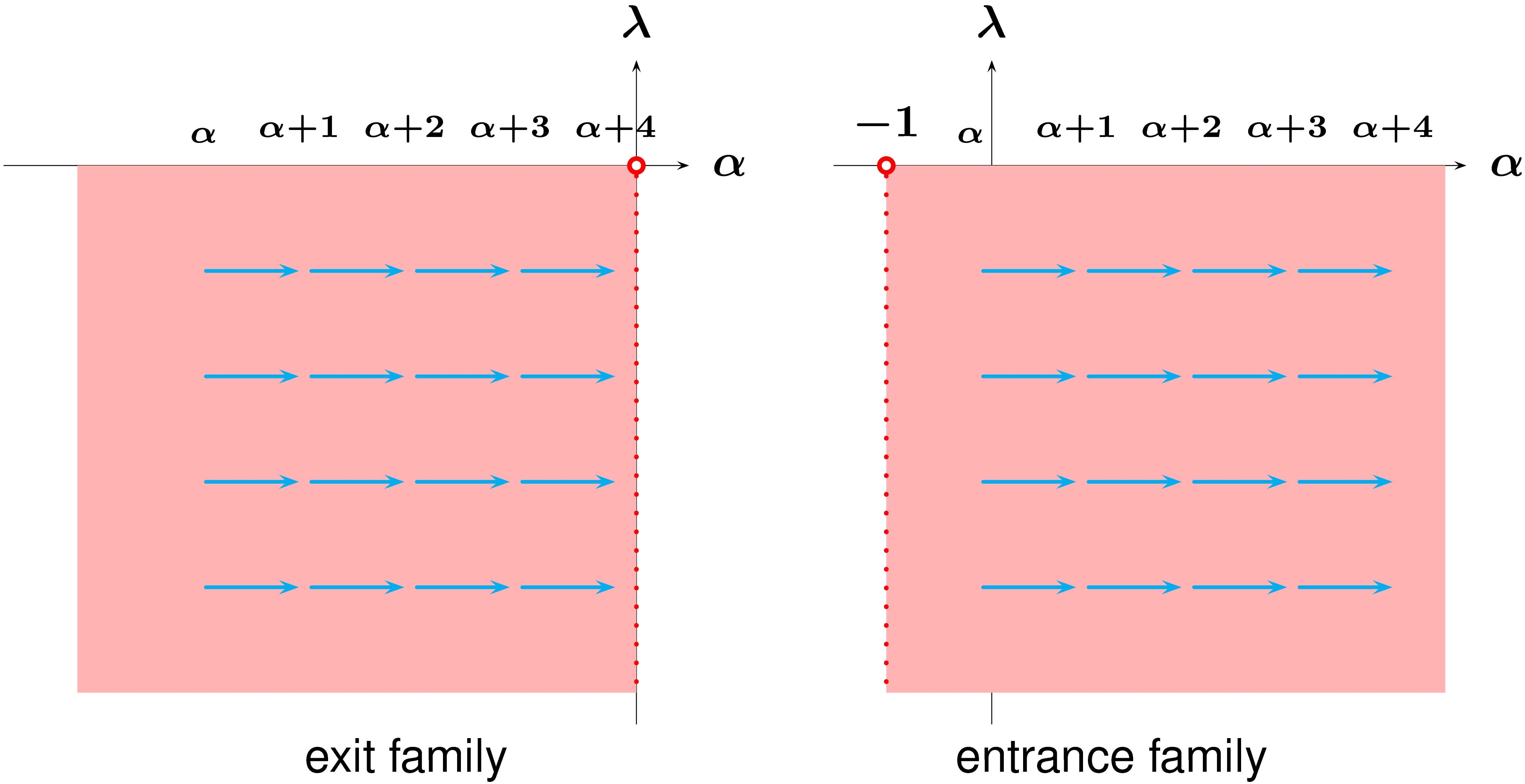
$$\hat{H}_\alpha[f](\xi)=\int_0^\infty f(x)\Gamma(\alpha+1)^{-1}B(\alpha+1;-\xi x)x^\alpha\,dx. \tag{10}$$

Then, we have

$$f(x)=\int_0^\infty \hat{H}_\alpha[f](\xi)\Gamma(\alpha+1)^{-1}B(\alpha+1;-\xi x)\xi^\alpha\,d\xi \tag{11}$$

and the following Plancherel identity

$$\int_0^\infty f(x)^2x^\alpha\,dx=\int_0^\infty \hat{H}_\alpha[f](\xi)^2\xi^\alpha\,d\xi. \tag{12}$$



λ
α
α+1
α+2
α+3
α+4
α
exit family
λ
−1
α
α+1
α+2
α+3
α+4
α
entrance family

Squared Bessel process on a finite interval

- $x\frac{d^2}{dx^2} + (1+\alpha)\frac{d}{dx}$ on $I = (0, r)$.
  On the boundary r, we consider the Dirichlet boundary condition and the Neumann boundary condition.
- $z(\alpha, n)$: nth zero point of the Bessel function J_α.

(a) $\alpha > -1$

[entrance, Neumann] family

The eigenfunction for the eigenvalue $-\frac{z(\alpha+1,n)^2}{4a}$ is $B(1+\alpha; -\frac{z(\alpha+1,n)^2}{4a}x)$.

[entrance, Dirichlet] family

The eigenfunction for the eigenvalue $-\frac{z(\alpha,n)^2}{4a}$ is $B(1+\alpha; -\frac{z(\alpha,n)^2}{4a}x)$.

The differentiation gives rise to the mapping from [entrance, Neumann] family to [entrance, Dirichlet] family.

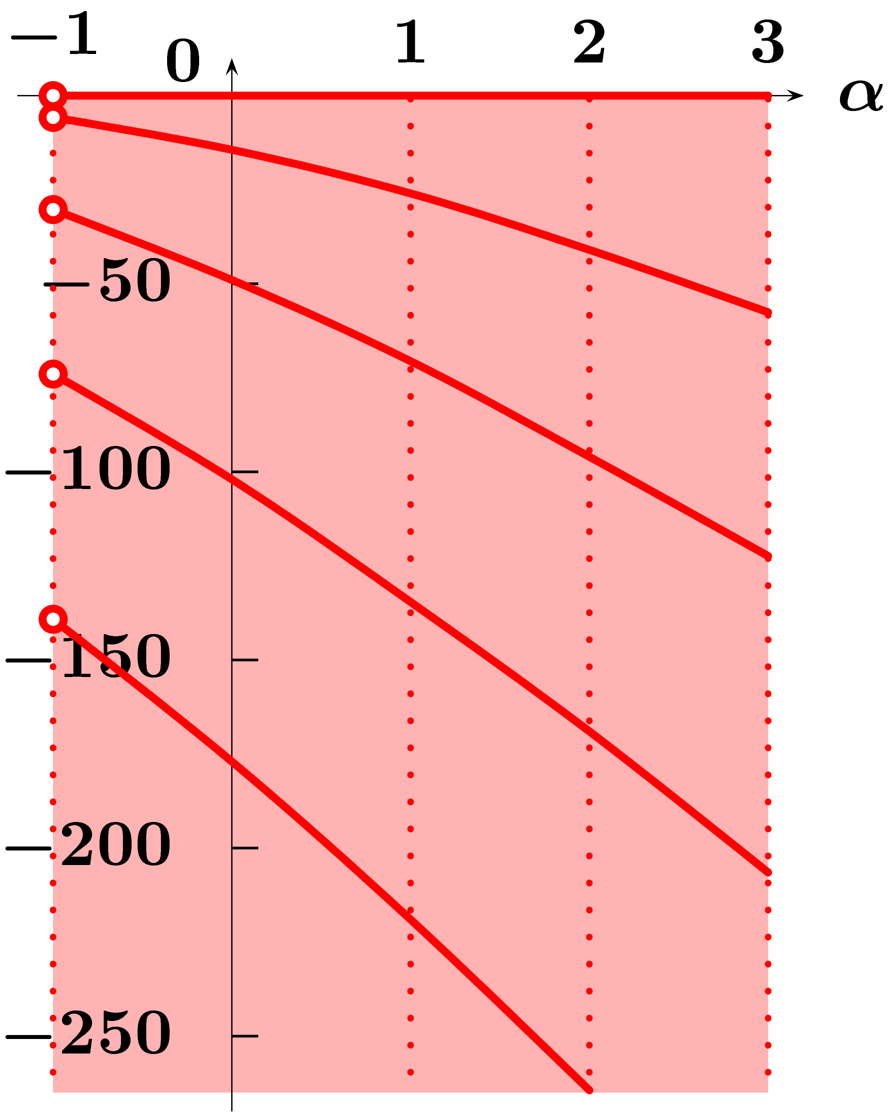
(b) $\alpha < 0$

[exit, Neumann] family

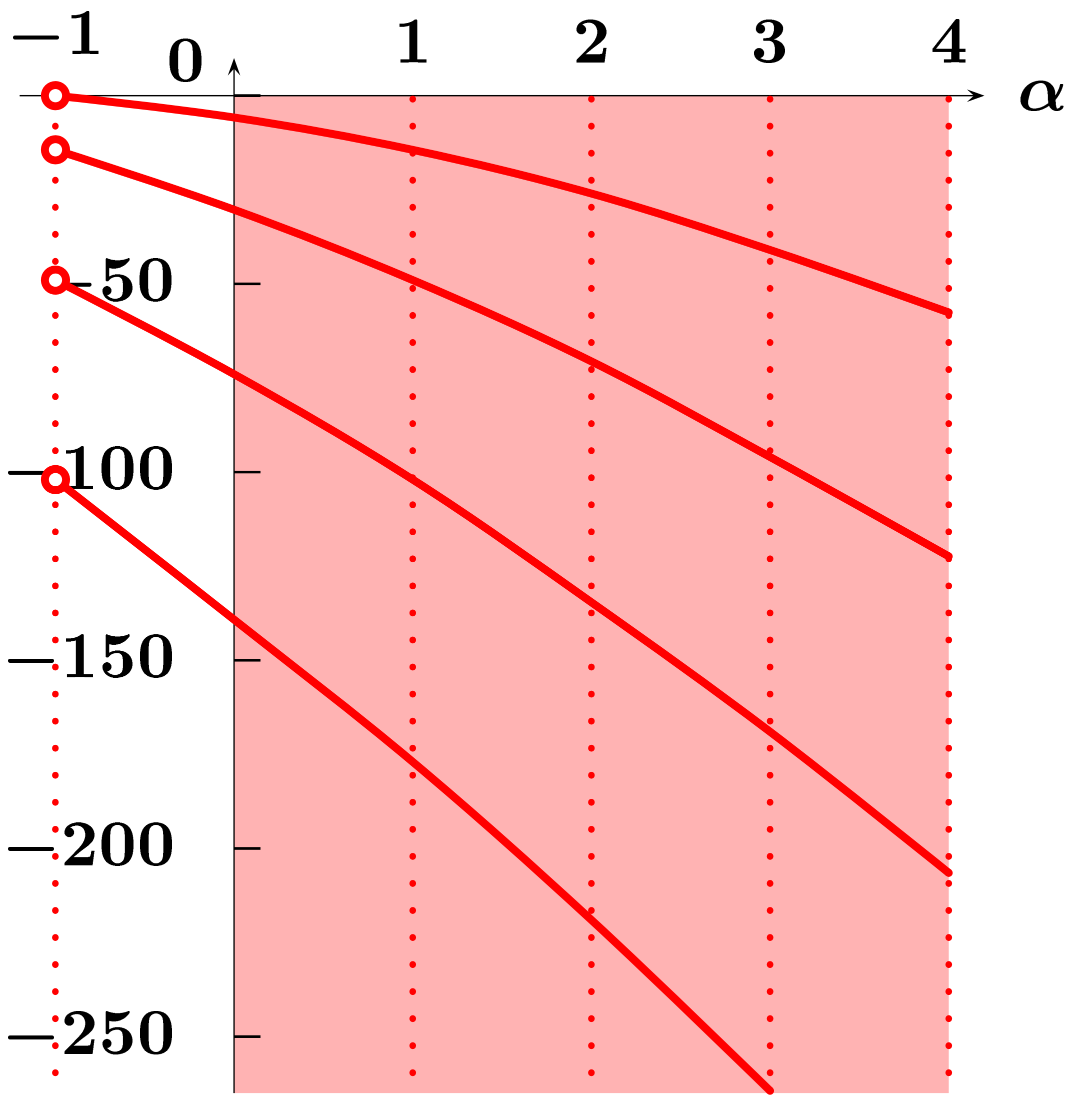
The eigenfunction for the eigenvalue $-\frac{z(-\alpha,n)^2}{4a}$ is $x^{-\alpha}B(1-\alpha;-\frac{z(-\alpha,n)^2}{4a}x))$.

[exit, Dirichlet] family

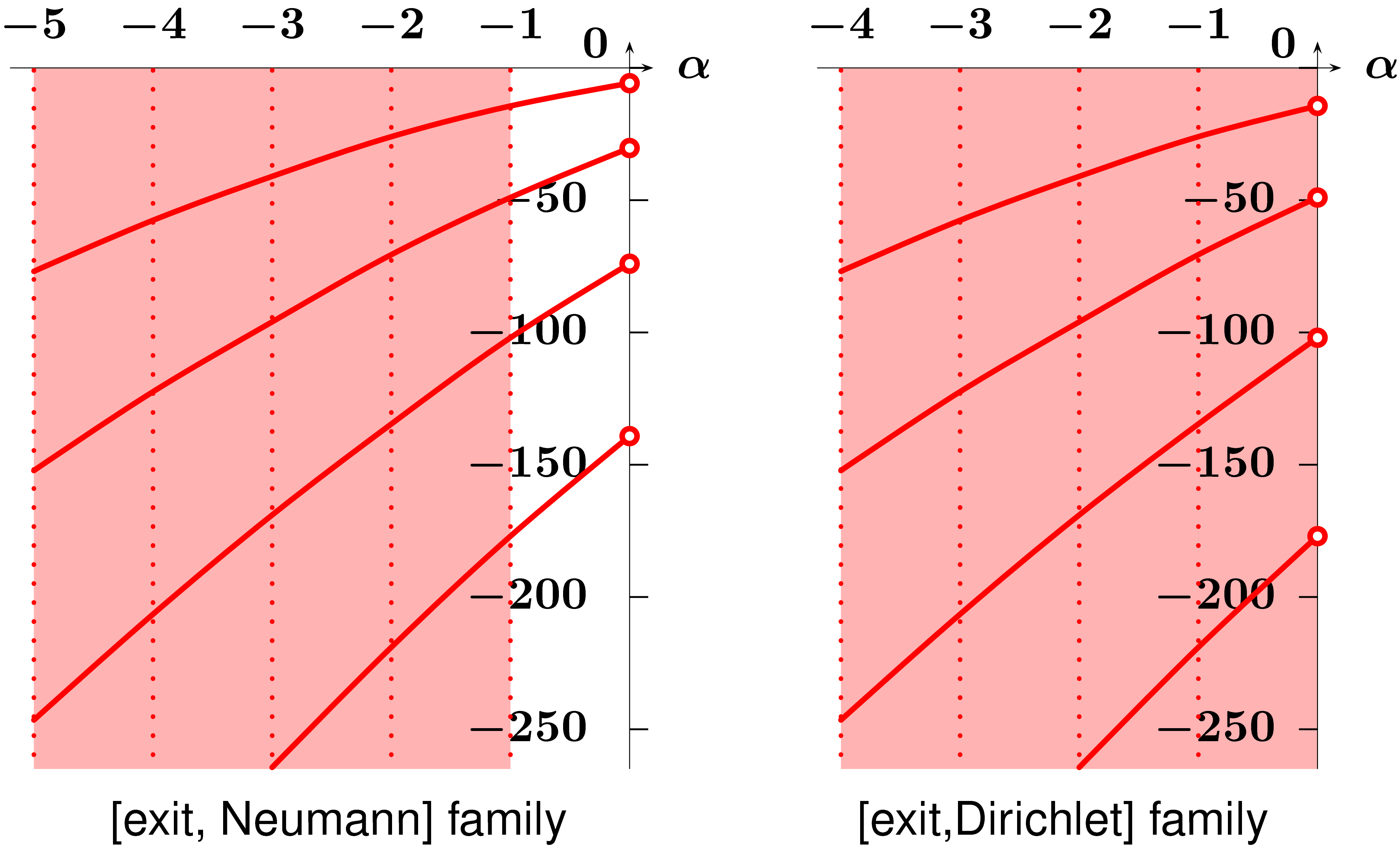
The eigenfunction for the eigenvalue $-\frac{z(1-\alpha,n)^2}{4a}$ is $x^{-\alpha}B(1-\alpha;-\frac{z(1-\alpha,n)^2}{4a}x)$.



[entrance, Neumann] family



[entrance, Dirichlet] family


−5
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−150
−200
−250
[exit, Neumann] family
−4
−3
−2
−1
0
α
−50
−100
−150
−200
−250
[exit,Dirichlet] family

4. Laguerre polynomials

We take $I = (0, \infty)$, $a = x$, $p = x^{\alpha} e^{-x}$. Then

$$\mathfrak{A}u = xu'' + (1 + \alpha - x)u', \quad \hat{\mathfrak{A}}u = xu'' + (2 + \alpha - x)u' - u.$$

These operators have the same spectrum except for 0.

To show the eigenfunction we need the hypergeometric functions defined by

$$(13) \qquad {}_1F_1(a; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n n!} x^n.$$

We set

$$(14) \qquad M(a, c; x) = {}_1F_1(a; c; x)$$

Theorem 3. We have

$$M' = \frac{a}{c} M(a+, c+), \tag{15}$$

$$xM' + (c-1)M = (c-1)M(c-), \tag{16}$$

$$c(M' - M) = (a-c)M(c+), \tag{17}$$

$$xM' + (1-c-x)M = (c-1)M(a-, c-). \tag{18}$$

(a) $\alpha > -1$ entrance family

The eigenfunction for the eigenvalue $-n$ $(n = 0, 1, \dots)$ is $M(-n, \alpha + 1; x)$ and

$$[M(-n, \alpha + 1; x)]' = -\frac{n}{\alpha + 1} M(-n + 1, \alpha + 2; x).$$

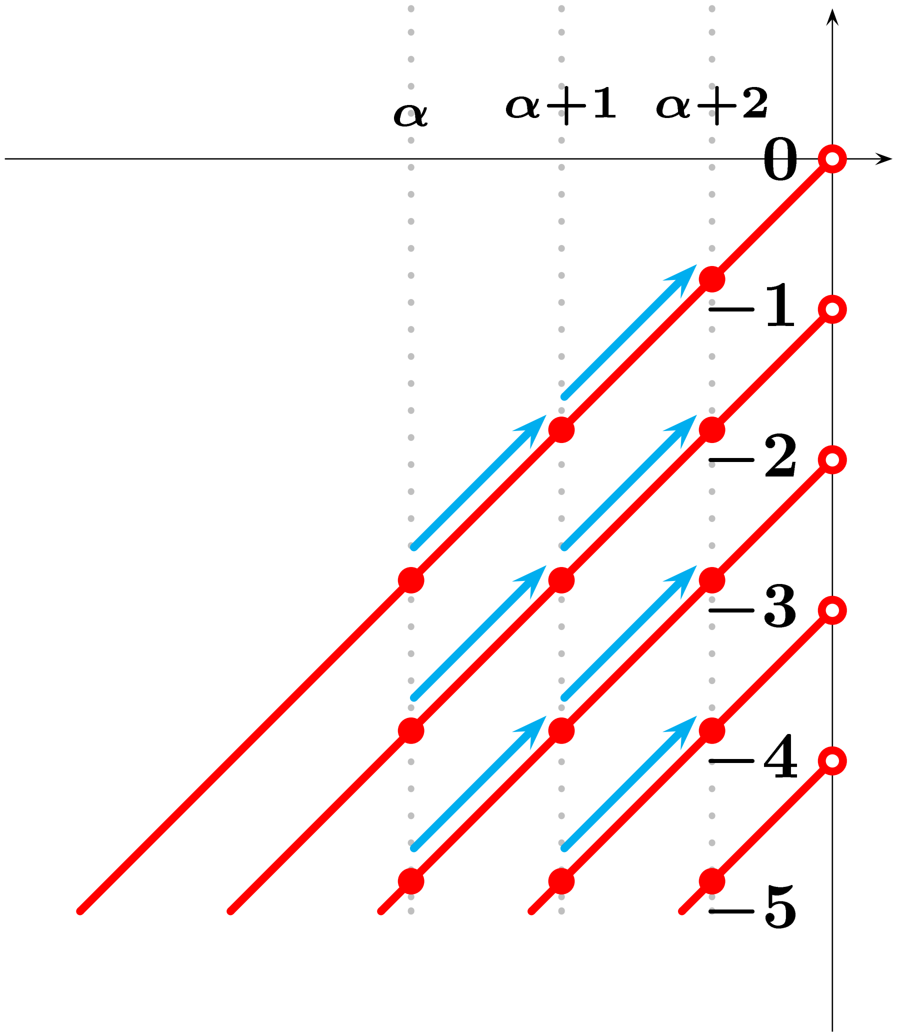
(b) $\alpha < 0$ exit family

The eigenfunction for the eigenvalue $-n + \alpha$ $(n = 0, 1, \dots)$ is $x^{-\alpha} M(-n, 1 - \alpha; x)$ and

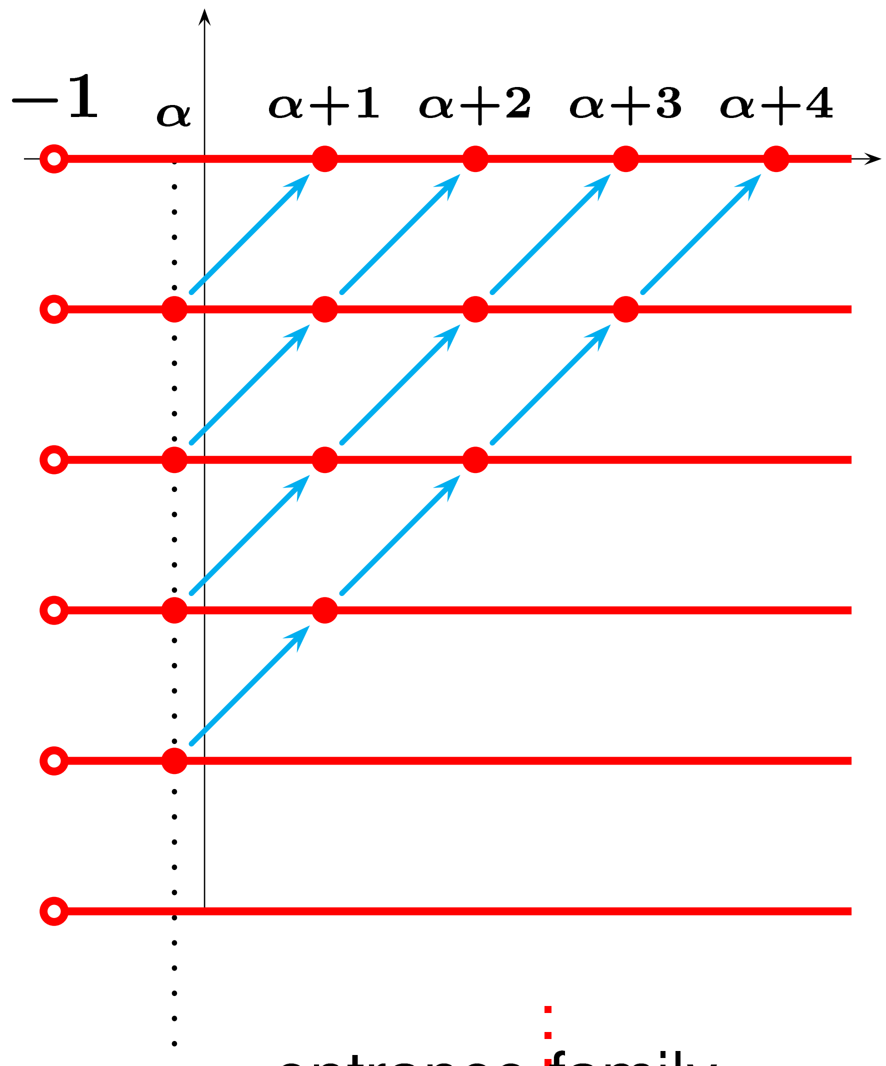
$$[x^{-\alpha} M(-n, 1 - \alpha; x)]' = -\alpha x^{-\alpha - 1} M(-n, -\alpha; x).$$

Remark 3. M is essentially the Laguerre polynomial.

(19)
$$L_n^{(\alpha)}(x) = \frac{(\alpha + 1)_n}{n!} M(-n, \alpha + 1; x)$$



exit family



entrance family

5. Jacobi polynomials

We take $I = (0, 1)$, $a = x(1-x)$, $p = x^{\alpha}(1-x)^{\beta}$. Then

$$\mathfrak{A}u = x(1-x)u'' + ((\alpha+1)(1-x) - (\beta+1)x)u',$$

$$\hat{\mathfrak{A}}u = x(1-x)u'' + ((\alpha+2)(1-x) - (\beta+2)x)u' - (\alpha+\beta+2)u.$$

To get eigenfunctions, we need hypergeometric function defined by

$$ {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n. \tag{20}$$

We set

$$K(x) = K(\alpha, \beta, \gamma; x) = {}_2F_1(-\gamma, \alpha+\beta+\gamma+1; \alpha+1; x) \tag{21}$$

Remark 4. K is essentially the Jacobi polynomial.

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)} K(\alpha,\beta,n;\frac{1-x}{2}). \tag{22}$$

Proposition 4. We have

$$K' = -\frac{\gamma(\gamma+\alpha+\beta+1)}{\alpha+1} K(\alpha+,\beta+,\gamma-). \tag{23}$$

$$xK' + \alpha K = \alpha K(\alpha-,\beta+). \tag{24}$$

$$(1-x)K' - \beta K = -\frac{(\alpha+\gamma+1)(\beta+\gamma)}{\alpha+1} K(\alpha+,\beta-). \tag{25}$$

$$x(1-x)K' + (\alpha(1-x) - \beta x)K = \alpha K(\alpha-,\beta-,\gamma+). \tag{26}$$

(a) $\alpha > -1, \beta > -1$ [entrance,entrance] family

The eigenfunction for the eigenvalue $-n(n+\alpha+\beta+1)$ $(n = 0, 1, \ldots)$ is $K(\alpha, \beta, n)$ and

$$K'(\alpha, \beta, n) = -\frac{\gamma(\alpha+\beta+\gamma+1)}{\alpha+1} K(\alpha+1, \beta+1, n-1).$$

(b) $\alpha < 0, \beta > -1$ [entrance,exit] family

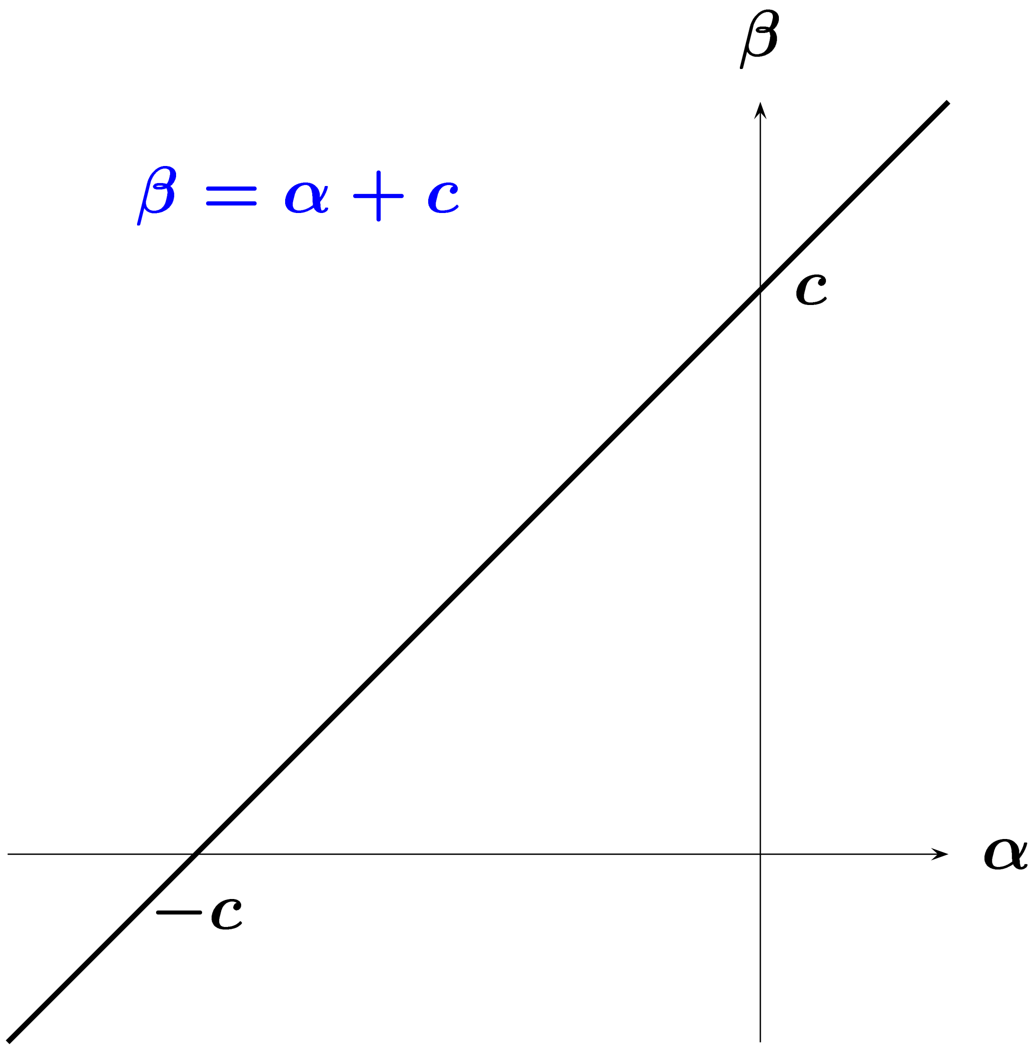
The eigenfunction for the eigenvalue $-(n-\alpha)(n+\beta+1)$ $(n = 0, 1, \ldots)$ is $x^{-\alpha} K(-\alpha, \beta, n)$ and

$$[x^{-\alpha} K(-\alpha, \beta, n)]' = -\alpha x^{-\alpha-1} K(-\alpha-1, \beta+1, n).$$

(c) $\alpha < 0, \beta < 0$ [exit,exit] family

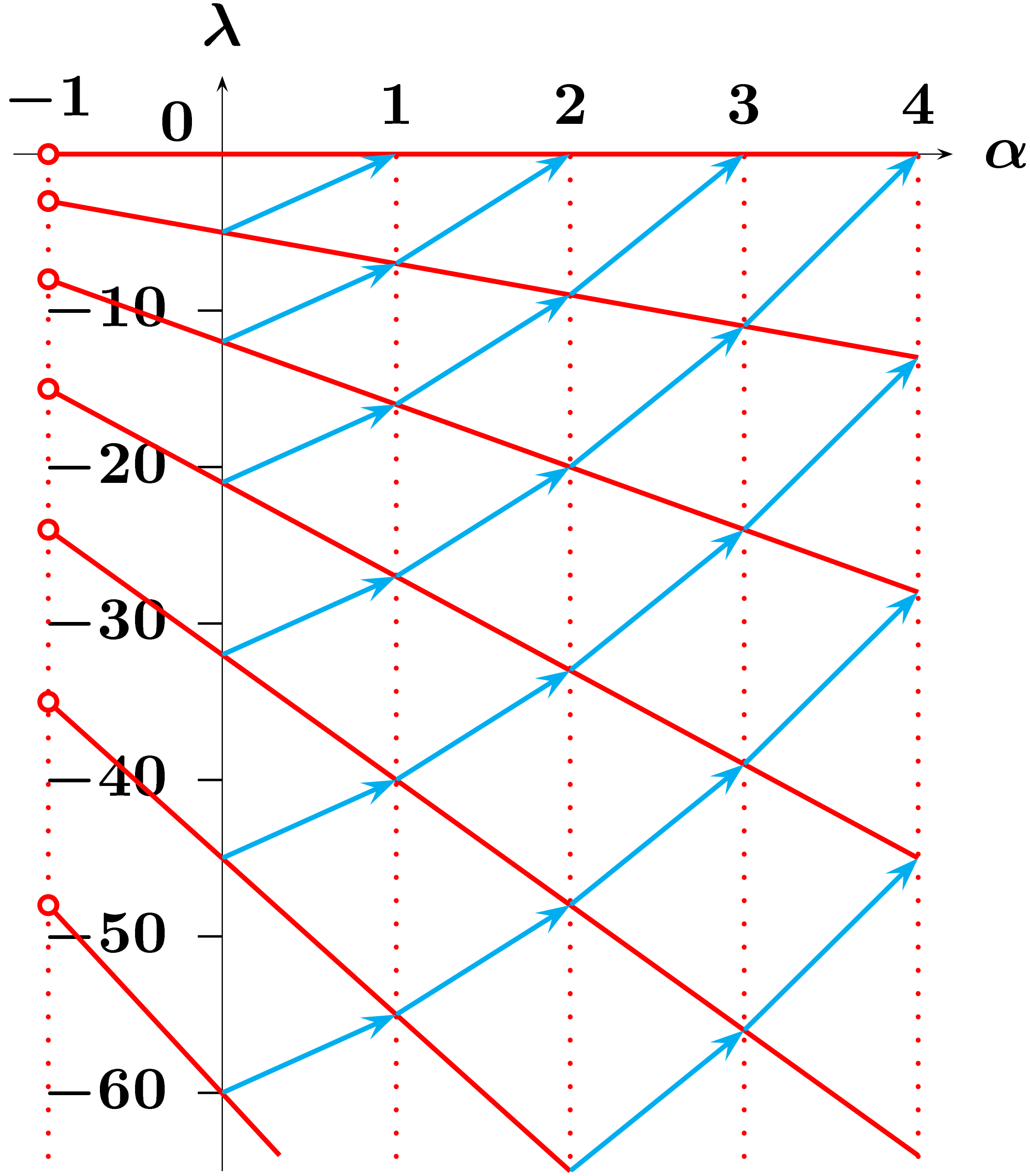
The eigenfunction for the eigenvalue $-(n+1)(n-\alpha-\beta)$ $(n = 0, 1, \ldots)$ is $x^{-\alpha}(1-x)^{-\beta}K(-\alpha, -\beta, \gamma)$ and

$$[x^{-\alpha}(1-x)^{-\beta}K(-\alpha, -\beta, n)]'$$
$$= -\alpha x^{-\alpha-1}(1-x)^{-\beta-1}K(-\alpha-1, -\beta-1, n+1).$$

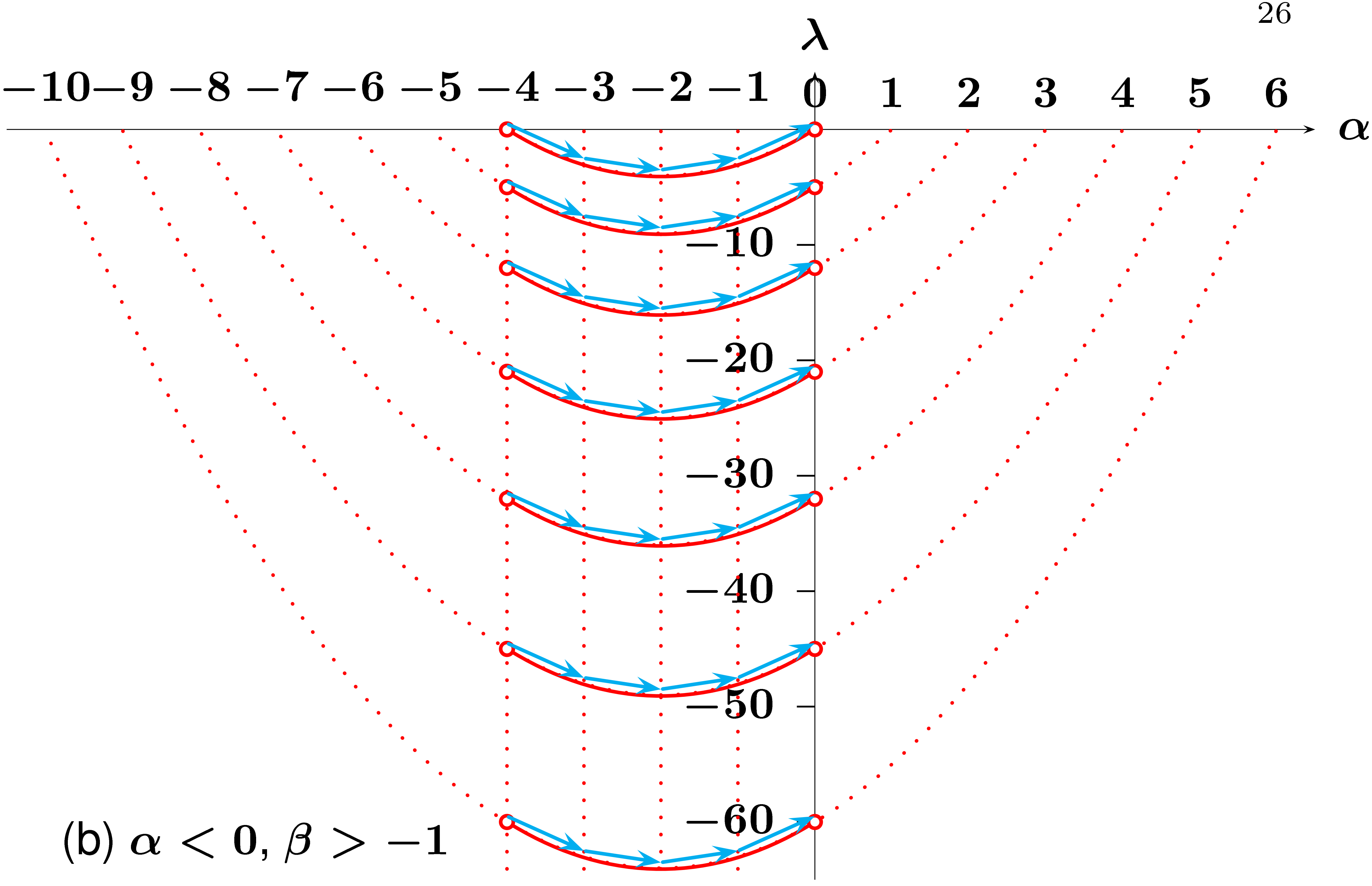


parameters

(a) $\boldsymbol{\alpha > -1, \beta > -1}$

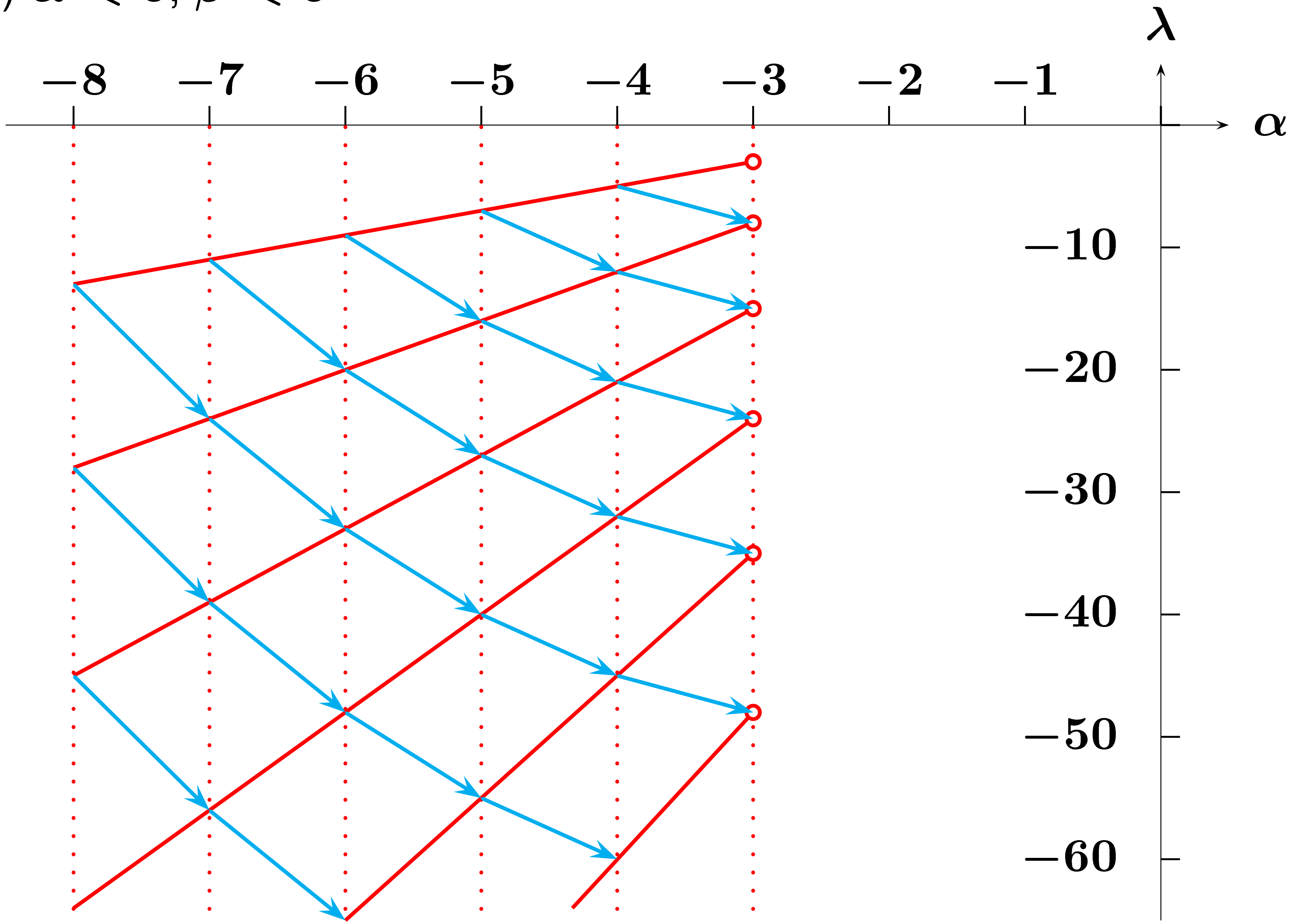


[entrance, entrance] family

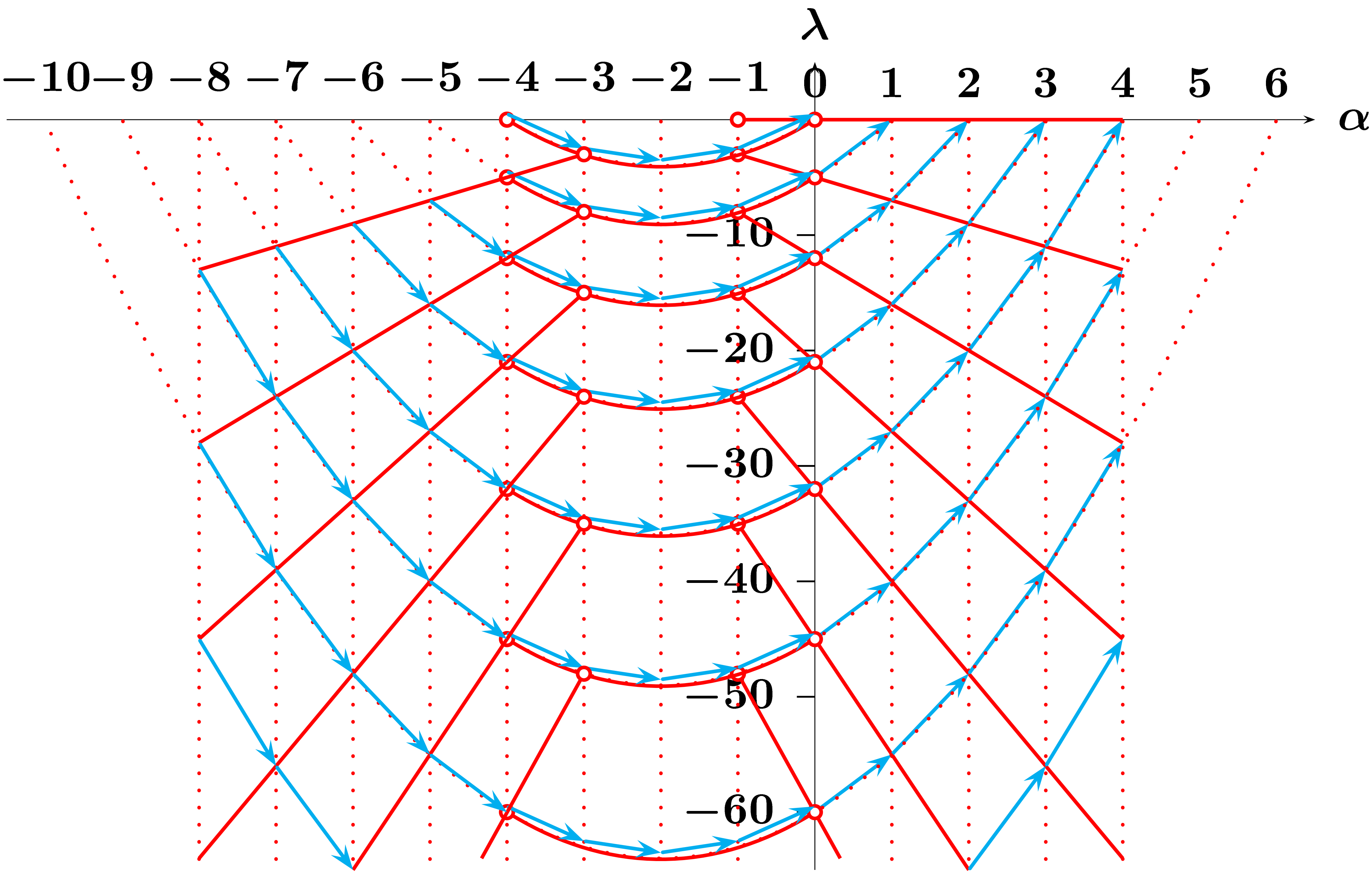


[exit, entrance] family

(c) $\alpha < 0, \beta < 0$



[exit, exit] family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−10
−20
−30
−40
−50
−60

Summary

$x\frac{d^2}{dx^2}+(\alpha+1)\frac{d}{dx}$	Bessel	${}_0F_1$
$x\frac{d^2}{dx^2}+(\alpha+1-x)\frac{d}{dx}$	Laguerre	${}_1F_1$
$x(1-x)\frac{d^2}{dx^2}+((\alpha+1)(1-x)-(\beta+1)x)\frac{d}{dx}$	Jacobi	${}_2F_1$

Thanks !