Exponential convergence of Markov Processes

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1. Introduction

- \((M, \mathcal{B}, m)\) : a measure space with \(m(M) = 1\)

- \(\{T_t\}\) : a Markovian semigroup in \(L^2(m)\)

We assume

- \(\{T_t^*\}\) : a Markovian semigroup in \(L^2(m)\)

- \(T_t 1 = T_t^* 1 = 1\)

Then \(\{T_t\}\) and \(\{T_t^*\}\) define semigroups in \(L^p(m)\) \((1 \leq p \leq \infty)\). For \(f \in L^1\), we denote

\[
\langle f \rangle = \int_M f \, dm.
\]
We are interested in the following ergodicity:

\[ T_t f \to \langle f \rangle \quad \text{as} \quad t \to \infty \]

To be precise, define the index \( \gamma_{p\to q} \) by

\[
\gamma_{p\to q} = -\lim_{t \to \infty} \frac{1}{t} \log \|T_t - m\|_{p\to q}.
\]

Here

- \( m \): an operator \( f \mapsto m(f) = \int_X f \, dm \)
- \( \| \|_{p\to q} \): the operator norm from \( L^p \) to \( L^q \)
We are interested in how $\gamma_{p \rightarrow q}$ depends on $p$ and $q$.

From the Riesz-Thorin interpolation theorem, we have

$$ S \mapsto \gamma_{1/s \rightarrow 1/s} $$

is concave. So if the semigroup is symmetric, $\gamma_{2 \rightarrow 2}$ is the largest.
2. Hypercontractivity and the exponential convergence

Hyperboundedness

\( \{T_t\} \) is called hyperbounded if there exist \( K > 0, \ r \in (2, \infty) \) and \( C \geq 1 \) such that

\[
\|T_K f\|_r \leq C \|f\|_2, \quad \forall f \in L^2(m).
\]
Theorem 1. The followings are equivalent to each other:

(1) \( \{T_t\} \) is hyperbounded.

(2) \( \gamma_{p \rightarrow q} \geq 0 \) for some \( 1 < p < q < \infty \).

(3) \( \gamma_{p \rightarrow q} = \gamma_{2 \rightarrow 2} \) for all \( p, q \in (1, \infty) \).
Hypercontractivity

\{T_t\} is called **hypercontractive** if there exist \( K > 0 \) and \( r \in (2, \infty) \) such that

\[
\|T_K f\|_r \leq \|f\|_2, \quad \forall f \in L^2(m).
\]
Proposition 2. Under (2), we have

$$\|T_k f - \langle f \rangle\|_2 \leq (r - 1)^{-1/2} \|f\|_2, \quad \forall f \in L^2(m).$$

Furthermore, for any $t \geq 0$, we have

$$\|T_t f - \langle f \rangle\|_2 \leq \sqrt{r - 1} \exp \left\{ -\frac{t}{K} \log \sqrt{r - 1} \right\} \|f\|_2, \quad \forall f \in L^2(m).$$
Proposition 3. Let $r > 2$. Suppose that there exist positive constants $K_0, K_1$ such that

$$M_0 := \|T_{K_0}\|_{2 \to r} < \infty$$

$$\rho := \sup\{\|T_{K_1}f - \langle f\rangle\|_2/\|f\|_2 : f \in L^2(m) \setminus \{0\}\} < 1.$$ 

Then the semigroup $\{T_t\}$ is hypercontractive.

Theorem 4. The followings are equivalent to each other:

(1) $\{T_t\}$ is hypercontractive.

(2) $\gamma_{p \to q} > 0$ for some $1 < p < q < \infty$.

(3) $\gamma_{p \to q} = \gamma_{2 \to 2} > 0$ for all $p, q \in (1, \infty)$. 
3. Sufficient condition for independence of $L^p$-spectrum

Normal operator

- $\mathbf{A}$: the generator of $\{T_t\}$
- $\mathbf{A}^*$: the generator of $\{T^*_t\}$

We assume that $\mathbf{A}$ is normal, i.e.,

$$\mathbf{A}\mathbf{A}^* = \mathbf{A}^*\mathbf{A}$$

Then $\mathbf{A}$ has the spectral decomposition:

$$-\mathbf{A} = \int_C \lambda dE_\lambda$$
**Multiplier**

For any bounded function $\phi : \mathbb{C} \to \mathbb{C}$, define $\phi(-\mathcal{A})$ by

$$\phi(-\mathcal{A}) = \int_{\mathbb{C}} \phi(\lambda) dE_{\lambda}.$$

**Theorem 5.** Assume that $\{T_t\}$ is hyperbounded.
If $\phi(\lambda)$ is expressed as

$$\phi(\lambda) = h(1/\lambda)$$

for a bounded function $h$ on $\mathbb{C}$ which is analytic near 0. Then $\phi(-\mathcal{A})$ is bounded in $L^p(m)$. 
Using this theorem, we can show that the boundedness of the resolvent is independent of $p$.

**Theorem 6.** Assume $\mathcal{A}$ is normal. Then $\sigma(\mathcal{A}_p)$, the spectrum of $\mathcal{A}_p$, is independent of $p$ ($1 < p < \infty$).
4. Example of $L^p$-spectrum that depends on $p$

We give an example that the spectrum depends on $p$.

- $M = [0, \infty)$
- $m(dx) = \nu(dx) = e^{-x}dx$
- The Dirichlet form in $L^2(\nu)$:
  \[
  \mathcal{E}(f, g) = \int_{[0, \infty)} f'(x)g'(x)\nu(dx)
  \]

  with domain

  \[
  \text{Dom}(\mathcal{E}) = \{f \in L^2(\nu); \ f \text{ is absolutely continuous and } f' \in L^2(\nu)\}.
  \]
The generator:

$$\mathcal{A} = \frac{d^2}{dx^2} - \frac{d}{dx}$$

with domain

$$\text{Dom}(\mathcal{A}) = \{ f \in \text{Dom}(\mathcal{E}); \ f' \text{ is absolutely continuous and } f'' \in L^2(\nu) \text{ with } f'(0) = 0 \}. \quad (3)$$

To see the spectrum of $\mathcal{A}$, we introduce the following unitary transformation

$$I f(x) = e^{-x/2} f(x)$$

and

$$I^{-1} f(x) = e^{x/2} f(x).$$
We note

$$I \circ \mathcal{A} \circ I^{-1} f = -\frac{1}{4} f + \frac{d^2 f}{dx^2}$$

i.e., we have the following commutative diagram:

\[
\begin{array}{ccc}
L^2(\nu) & \xrightarrow{\mathcal{A}} & L^2(\nu) \\
\downarrow I & & \downarrow I \\
L^2(dx) & \xrightarrow{\frac{d^2}{dx^2} - \frac{1}{4}} & L^2(dx)
\end{array}
\]

The boundary condition is involved as follows:

$$\mathcal{A} \text{ with } f'(0) = 0$$

\[
\begin{array}{c}
\downarrow I \\
\frac{d^2}{dx^2} - \frac{1}{4} \text{ with } \frac{1}{2} f(0) + f'(0) = 0
\end{array}
\]
The corresponding Dirichlet form $\hat{E}$:

\[
\hat{E}(f, g) = \mathcal{E}(I^{-1} f, I^{-1} g)
\]

\[
= \int_0^\infty f' g' \, dx + \frac{1}{4} \int_0^\infty f g \, dx - \frac{1}{2} f(0)g(0).
\]

We can show that $\hat{E}$ is a compact perturbation of

\[
\hat{E}^{(0)}(f, g) = \int_0^\infty f' g' \, dx + \frac{1}{4} \int_0^\infty f g \, dx.
\]

It is easy to see that spectrum corresponding to $\hat{E}^{(0)}(f, g)$ is

\[
\left[ \frac{1}{4}, \infty \right)
\]
Now setting

\[ A = \frac{d^2}{dx^2} - \frac{1}{4} \]

with \( \text{Dom}(A) = \{ f, f'' \in L^2([0, \infty)) \text{ with } \frac{1}{2} f(0) + f'(0) = 0 \} \), we have

\[ \sigma(-A) = \{0\} \cup [\frac{1}{4}, \infty). \]

Now, by the unitary equivalence,

**Theorem 7.** We have

\[ \sigma(-A) = \{0\} \cup [\frac{1}{4}, \infty). \]
Now we proceed to the $L^p$-spectrum. The result is

\begin{align*}
  p = 1 & \quad \text{for } 1 < p < 2 \\
  p = 2 & \quad \text{for } p > 2
\end{align*}
First we discuss the case $1 \leq p < 2$. We use the same mapping $I f(x) = e^{-x^2/2} f(x)$ in (3) as

$$I : L^p(\nu) \rightarrow L^p(\tilde{\nu})$$

where

$$\tilde{\nu}(dx) = e^{(p/2-1)x} dx.$$  

Then $I$ gives an isometry between $L^p(\nu)$ and $L^p(\tilde{\nu})$. Similarly as before, setting $\tilde{\mathcal{A}} = I \circ \mathcal{A} \circ I^{-1}$, we have

$$\tilde{\mathcal{A}} f = \frac{d^2 f}{dx^2} \ - \frac{1}{4} f$$

with the boundary condition

$$f'(0) + \frac{1}{2} f(0) = 0.$$
Proposition 8. For $1 \leq p < 2$, we have

$$\sigma_p(-\mathfrak{A}) = \{0\} \cup \{x + iy; \ x, y \in \mathbb{R}, y^2 < \left(\frac{2}{p} - 1\right)^2(x - \frac{p - 1}{p^2})\}$$

Proof. We solve the following differential equation:

$$\begin{cases}
-u'' + \frac{1}{4}u = \lambda u, \\
u'(0) + \frac{1}{2}u(0) = 0.
\end{cases}$$

The solution is given by

$$\begin{cases}
u(x) = C_1 e^{x\sqrt{-\lambda + 1/4}} + C_2 e^{-x\sqrt{-\lambda + 1/4}}, \\
C_1\left(\frac{1}{2} + \sqrt{-\lambda + \frac{1}{4}}\right) + C_2\left(\frac{1}{2} - \sqrt{-\lambda + \frac{1}{4}}\right) = 0.
\end{cases}$$

By checking the integrability, we get the desired result. \qed
Proposition 9. For $1 \leq p < 2$, we have

$$\rho(-\mathcal{A}) \supseteq \{x + iy; \, x, y \in \mathbb{R}, \, y^2 > \left(\frac{2}{p} - 1\right)^2(x - \frac{p - 1}{p^2})\} \setminus \{0\}$$

**Proof.** For $\lambda \in \{z \in \mathbb{C}; \, \Re \sqrt{-z + 1/4} > \frac{1}{p} - \frac{1}{2}\}$, define

$$\phi_\lambda(x) = \left(\frac{1}{2} - \sqrt{-\lambda + \frac{1}{4}}\right)e^{x\sqrt{-\lambda + 1/4}}$$

$$- \left(\frac{1}{2} + \sqrt{-\lambda + \frac{1}{4}}\right)e^{-x\sqrt{-\lambda + 1/4}},$$

$$\psi_\lambda(x) = e^{-x\sqrt{-\lambda + 1/4}},$$

$$W_\lambda = -2\sqrt{-\lambda + \frac{1}{4}}\left(\frac{1}{2} - \sqrt{-\lambda + \frac{1}{4}}\right).$$
Further, define $g_\lambda : [0, \infty) \times [0, \infty) \to \mathbb{C}$ to

$$g_\lambda(x, y) = \begin{cases} 
\frac{1}{W_\lambda} \phi_\lambda(x) \psi_\lambda(y), & x \leq y, \\
\frac{1}{W_\lambda} \phi_\lambda(y) \psi_\lambda(x), & x \geq y. 
\end{cases}$$

The possible Green operator $G_\lambda$ is given by

$$G_\lambda f(x) = \int_0^\infty g_\lambda(x, y) f(y) \, dy.$$

For $f \in C^\infty_0([0, \infty) \to \mathbb{C})$, we have

$$(\lambda + \tilde{A})G_\lambda f = f, \quad \frac{1}{2}G_\lambda f(0) + (G_\lambda f)'(0) = 0.$$ 

Now it suffices to show that $G_\lambda$ is a bounded operator.
We can summarize as follows:

**Theorem 10.** For $1 \leq p < 2$, we have

(i) $\sigma_p(-\mathcal{A}) = \{0\} \cup \{x + iy; \; x, y \in \mathbb{R}, y^2 < \left(\frac{2}{p} - 1\right)^2 (x - \frac{p-1}{p^2})\}$

(ii) $\sigma_c(-\mathcal{A}) = \{x + iy; \; x, y \in \mathbb{R}, y^2 = \left(\frac{2}{p} - 1\right)^2 (x - \frac{p-1}{p^2})\}$

(iii) $\rho(-\mathcal{A}) = \{x + iy; \; x, y \in \mathbb{R}, y^2 > \left(\frac{2}{p} - 1\right)^2 (x - \frac{p-1}{p^2})\}$

**Theorem 11.** For $p > 2$, we have

(i) $\sigma_p(-\mathcal{A}) = \{0\}$

(ii) $\sigma_r(-\mathcal{A}) = \{x + iy; \; x, y \in \mathbb{R}, y^2 < \left(\frac{2}{p} - 1\right)^2 (x - \frac{p-1}{p^2})\}$

(iii) $\sigma_c(-\mathcal{A}) = \{x + iy; \; x, y \in \mathbb{R}, y^2 = \left(\frac{2}{p} - 1\right)^2 (x - \frac{p-1}{p^2})\}$

(iv) $\rho(-\mathcal{A}) = \{x + iy; \; x, y \in \mathbb{R}, y^2 > \left(\frac{2}{p} - 1\right)^2 (x - \frac{p-1}{p^2})\}$
$p = 1$

$1 < p < 2$

$p = 2$

$p > 2$
By noting that

\[ \inf \{ R\lambda; \lambda \in \sigma(-A) \setminus \{0\} \} = - \lim_{t \to \infty} \frac{1}{t} \log \|T_t - m\| \]

we have

**Theorem 12.** For \( 1 \leq p < \infty \)

\[ \gamma_{p \to p} = \frac{p - 1}{p^2}. \]
Thanks !