Exponential convergence of Markovian semigroups*

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1 Hypercontractivity and the exponential convergence

Let $(M, \mathscr{B}, m)$ be a measure space with $m(M) = 1$. Suppose we are given a Markovian semigroup $\{T_t\}$ in $L^2(m)$. We denote its dual semigroup $\{T_t^*\}$ and assume that $\{T_t^*\}$ is Markovian and $T_t 1 = 1$ and $T_t^* 1 = 1$. $\{T_t\}$ and $\{T_t^*\}$ define strongly continuous semigroups in $L^p(m)$ $(1 \le p < \infty)$ naturally.

We are interested in the following ergodicity:

$$T_t f \to \langle f \rangle \quad \text{as } t \to \infty$$

To be precise, define the index $\gamma_{p\to q}$ by

$$\gamma_{p\to q} = -\overline{\lim} \frac{1}{t} \log \|T_t - m\|_{p\to q}. \tag{1}$$

Here m denotes an operator $f \mapsto m(f) = \int_X f\, dm$ and $\|\ \|_{p\to q}$ denotes the operator norm from L^p to L^q.

We recall that $\{T_t\}$ is called hyperbounded if there exist $K > 0$, $r \in (2, \infty)$ and $C \ge 1$ such that

$$\|T_K f\|_r \le C \|f\|_2, \quad \forall f \in L^2(m).$$

Then we have

Theorem 1. *The followings are equivalent to each other:*

(1) $\{T_t\}$ *is hyperbounded.*

(2) $\gamma_{p\to q} \ge 0$ *for some* $1 < p < q < \infty$.

(3) $\gamma_{p\to q} = \gamma_{2\to 2}$ *for all* p, $q \in (1, \infty)$.

Also $\{T_t\}$ is called hypercontractive if there exist $K > 0$ and $r \in (2, \infty)$ such that

$$\|T_K f\|_r \le \|f\|_2, \quad \forall f \in L^2(m).$$

Then we have

Theorem 2. *The followings are equivalent to each other:*

(1) $\{T_t\}$ *is hypercontractive.*

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(2) $\gamma_{p\to q} > 0$ *for some* $1 < p < q < \infty$.

(3) $\gamma_{p\to q} = \gamma_{2\to 2} > 0$ *for all* p, $q \in (1,\infty)$.

Further if we assume that the generator $\mathfrak{A}$ of T_t is normal, we have the following p-independence of the spectrum.

Theorem 3. *Assume* $\mathfrak{A}$ *is normal. Then* $\sigma(\mathfrak{A}_p)$, *the spectrum of* $\mathfrak{A}_p$, *is independent of* p *(*$1 < p < \infty$*).*

2 Example of L^p-spectrum that depends on p

We give an example that the spectrum depends on p. Let $M = [0,\infty)$ and $m(dx) = \nu(dx) = e^{-x}dx$. The Dirichlet form in $L^2(\nu)$ is given by

$$\mathscr{E}(f,g) = \int_{[0,\infty)} f'(x)g'(x)\nu(dx).$$

The generator is

$$\mathfrak{A} = \frac{d^2}{dx^2} - \frac{d}{dx}$$

with boundary condition $f'(0) = 0$.

Theorem 4. *For* $p = 2$, *we have*

$$\sigma(-\mathfrak{A}) = \{0\} \cup [\frac{1}{4},\infty).$$

Theorem 5. *For* $1 \le p < 2$, *we have*

(i) $\sigma_{\mathrm{p}}(-\mathfrak{A}) = \{0\} \cup \{x+iy;\ x,y \in \mathbb{R}, y^2 < (\frac{2}{p}-1)^2(x - \frac{p-1}{p^2})\}$

(ii) $\sigma_{\mathrm{c}}(-\mathfrak{A}) = \{x+iy;\ x,y \in \mathbb{R}, y^2 = (\frac{2}{p}-1)^2(x - \frac{p-1}{p^2})\}$

(iii) $\rho(-\mathfrak{A}) = \{x+iy;\ x,y \in \mathbb{R}, y^2 > (\frac{2}{p}-1)^2(x - \frac{p-1}{p^2})\}$

Theorem 6. *For* $p > 2$, *we have*

(i) $\sigma_{\mathrm{p}}(-\mathfrak{A}) = \{0\}$

(ii) $\sigma_{\mathrm{r}}(-\mathfrak{A}) = \{x+iy;\ x,y \in \mathbb{R}, y^2 < (\frac{2}{p}-1)^2(x - \frac{p-1}{p^2})\}$

(iii) $\sigma_{\mathrm{c}}(-\mathfrak{A}) = \{x+iy;\ x,y \in \mathbb{R}, y^2 = (\frac{2}{p}-1)^2(x - \frac{p-1}{p^2})\}$

(iv) $\rho(-\mathfrak{A}) = \{x+iy;\ x,y \in \mathbb{R}, y^2 > (\frac{2}{p}-1)^2(x - \frac{p-1}{p^2})\}$