On spectra of 1-dimensional diffusion operators

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1. Introduction

**Hermite polynomials**

Hermite polynomials are defined by

\[ H_n(x) = \frac{(-1)^n}{n!} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}, \quad n = 0, 1, \ldots \]

These are eigenfunctions of the Ornstein-Uhlenbeck operator

\[ \frac{d^2}{dx^2} - x \frac{d}{dx}. \]

We have

\[ \frac{d}{dx} H_n(x) = H_{n-1}(x). \]
In this talk, we give a general framework of this fact.
2. One dimensional diffusion processes

- $M = [0, \infty)$
- $a, p$: positive continuous functions on $(0, \infty)$

We consider the diffusion process generated by

\[
\mathcal{A}u = \frac{1}{p} (apu')'.
\]

This operator is regarded as a self-adjoint operator in $L^2(p)$. Here $p$ denotes a measure $p(x)dx$ on $[0, \infty)$.

By formal calculation, the associated Dirichlet form is

\[
\mathcal{E}(u, v) = \int_0^\infty u'v' ap \, dx.
\]
The speed measure and the scale function

- $dm = p(x)dx$: the speed measure.
- $s(x)$: the scale function.

\begin{align*}
(3) \quad m(x) &= \int_0^x p(y) \, dy \\
(4) \quad s(x) &= \int_0^x \frac{1}{a(y)p(y)} \, dy.
\end{align*}

We assume that $0$ is a regular boundary point (Feller’s classification). So we have $m(0) = 0$, $s(0) = 0$. We also assume that $\infty$ is not a regular boundary point, i.e.,

$$m(\infty) + s(\infty) = \infty.$$
The precise domain of $\mathcal{E}$ is given by

$$\text{Dom}(\mathcal{E}) = \{ u \in L^2(p); \ u \text{ is absolutely continuous on } (0, \infty) \text{ and } u' \in L^2(ap) \}.$$ 

**Proposition 1.** If $u \in \text{Dom}(\mathcal{E})$, then $u$ is a.c. (absolutely continuous) on $[0, \infty)$, i.e., $u(0+)$ exists and $u$ is a.c. on $[0, \infty)$ by defining $u(0) = u(0+)$. In this case, we have

$$|u(0)| \leq \frac{1}{m(x)^{1/2}} \|u\|_{L^2(p)} + \mathcal{E}(u, u)^{1/2} s(x)^{1/2}. \tag{5}$$

Moreover, if $s(\infty) < \infty$ then $u(\infty)$ exists and if $s(\infty) < \infty$, $m(\infty) = \infty$ then $u(\infty) = 0$. 
To show this, we use

\[ |u(y) - u(x)| \leq \mathcal{E}(u, u)^{1/2} (s(y) - s(x))^{1/2}. \]

By Proposition 1, \( u \mapsto u(0) \) is a continuous linear functional from \( \text{Dom}(\mathcal{E}) \) to \( \mathbb{R} \).

Now we define an operator \( V : L^2(p) \to L^2(ap) \) by

(6)

\[ V u = u' \]

Here \( \text{Dom}(V) = \text{Dom}(\mathcal{E}) \).

If we impose the Dirichlet boundary condition at 0, we set \( \text{Dom}(V) = \text{Dom}(\mathcal{E}) \cap \{ u : u(0) = 0 \} \).

**Proposition 2.** \( V : L^2(p) \to L^2(ap) \) is a closed operator.
The dual operator $V^*$

We give a characterization of $V^*$.

Proposition 3. Take any $\theta \in L^2(ap)$. If $ap\theta$ is a.c. on $(0, \infty)$ and $\frac{(ap\theta)'}{p} \in L^2(p)$, then $ap\theta$ is a.c. on $[0, \infty)$, i.e., $ap\theta(0+)$ exists and $ap\theta$ is a.c. on $[0, \infty)$ by defining $ap\theta(0) = ap\theta(0+)$. We also have

\begin{equation}
|ap\theta(0+)| \leq \frac{\|\theta\|_{L^2(ap)}}{s(x)^{1/2}} + \frac{\|\frac{(ap\theta)'}{p}\|_{L^2(p)} m(x)^{1/2}}
\end{equation}

Moreover, if $m(\infty) < \infty$ then $ap\theta(\infty)$ exists and if $m(\infty) < \infty$, $s(\infty) = \infty$ then $ap\theta(\infty) = 0$. 
To show this, we use

\[ |a p \theta(y) - a p \theta(x)| \leq \sqrt{\int_x^y \frac{(a p \theta)'^2}{p^2} p \, dt} (m(y) - m(x))^{1/2}. \]

**Dense domain**

We denote the set of all continuous functions on \([0, \infty)\) with compact support by \(C_0\).

**Proposition 4.** \(\text{Dom}(\mathcal{E}) \cap C_0\) is dense in \(\text{Dom}(\mathcal{E})\) and \(\text{Dom}(\mathcal{E}) \cap C_0 \cap \{u : u(0) = 0\}\) is dense in \(\text{Dom}(\mathcal{E}) \cap \{u : u(0) = 0\}\).

By using this, we have the following duality formula (integration by parts):
Proposition 5. For any \( u \in \text{Dom}(\mathcal{E}) \) and any \( \theta \in L^2(a p) \) satisfying \( \frac{(a p \theta)'}{p} \in L^2(p) \), we have

\[
\int_0^\infty u' \theta a p \, dt = -u(0) a p \theta(0+) - \int_0^\infty u(a p \theta)' \, dt
\]

Further we have \( u a p \theta(\infty) = 0 \).
Proposition 6. The dual operator $V^* : L^2(ap) \rightarrow L^2(p)$ of $V : L^2(p) \rightarrow L^2(ap)$ is given by

$$V^* \theta = -\frac{(ap\theta)'}{p}.$$  \hspace{1cm} (9)

Here

$$\text{Dom}(V^*) = \{\theta \in L^2(ap); \frac{(ap\theta)'}{p} \in L^2(p), ap\theta(0+) = 0\}$$  \hspace{1cm} (10)

for the Neumann boundary condition and

$$\text{Dom}(V^*) = \{\theta \in L^2(ap); \frac{(ap\theta)'}{p} \in L^2(p)\}$$  \hspace{1cm} (11)

for the Dirichlet boundary condition.
We define $\mathcal{A} = -V^*V$. We can give a characterization of $\text{Dom}(\mathcal{A})$ as follows:

**Theorem 7.** We have that $u \in \text{Dom}(\mathcal{A})$ if and only if

1. $u$ is a.c. on $(0, \infty)$ and $u' \in L^2(ap)$,
2. $apu'$ is a.c. on $(0, \infty)$ and $\frac{(apu')'}{p} \in L^2(p)$,
3. $apu'(0+) = 0$.

If $u$ satisfies these conditions, we have

$$\mathcal{A}u = -V^*Vu = \frac{(apu')'}{p}$$

If the boundary condition is Dirichlet, the third condition is replaced by $u(0) = 0$. 
Theorem 8. We have that $\theta \in \text{Dom}(VV^*)$ if and only if

1. $ap\theta$ is a.c. on $(0, \infty)$ and $\frac{(ap\theta)'}{p} \in L^2(p)$,

2. $\frac{(ap\theta)'}{p}$ is a.c. on $(0, \infty)$ and $\left(\frac{(ap\theta)'}{p}\right)' \in L^2(ap)$,

3. $ap\theta(0+) = 0$.

In this case we have

\begin{equation}
\hat{\mathcal{A}}\theta = -VV^*\theta = \left(\frac{(ap\theta)'}{p}\right)'
\end{equation}

If the boundary condition is Dirichlet, the third condition $ap\theta(0+) = 0$ should be omitted.
If we assume that $a$ and $p$ are $C^2$ functions, we have

**Corollary 9.** We have

\begin{align*}
(13) \quad \mathcal{A}u &= -V^*Vu = au'' + bu', \\
(14) \quad \widehat{\mathcal{A}}\theta &= -VV^*\theta = a\theta'' + (b + a')\theta' + b'\theta.
\end{align*}

Here $b = a' + a(\log p)'$ \quad (b + a' = a' + a(\log ap)')$.
3. Super symmetry and the spectrum

The super symmetry is an efficient machinery to investigate the spectrum, which depends on the following well-known fact:

Proposition 10. Let $T$ be a closed operator in a Hilbert space $H$. Then $T^*T$ and $TT^*$ have the same spectrum except for 0.

Let $x$ be an eigenvector for a point spectrum $\lambda$ of $T^*T$:

\begin{equation}
T^*Tx = \lambda x.
\end{equation}

Then

$$(TT^*)Tx = T(T^*T)x = T\lambda x = \lambda Tx$$

which shows that $Tx$ is an eigenvector for an eigenvalue $\lambda$ of $TT^*$. 
In the previous section, we took $T = V = \frac{d}{dx}$. So $\frac{d}{dx}$ give rise to a correspondence between eigenfunctions of the following operators:

\begin{align}
\mathcal{A}u &= -V^*Vu = au'' + bu', \\
\hat{\mathcal{A}}\theta &= -VV^*\theta = a\theta'' + (a' + b)\theta' + b'\theta.
\end{align}

Here $b = a' + a(\log p)'$.

This can be seen from the following computation. Assume $au'' + bu' = \lambda u$. Then

$$a'u'' + a'u''' + b'u' + bu'' = \lambda u'.$$

Hence

$$a(u')'' + (a' + b)(u') + b'u' = \lambda u'.$$
Corollary 11. Assume that \( b(x) \leq -c < 0 \), then \(-A\) has a spectral gap \( \geq c \).

**Hermite polynomials**

We take \( a = 1, p = e^{-x^2/2}, M = \mathbb{R} \). Then

\[
b = a' + a(\log p)' = (-x^2/2)' = -x.
\]

Hence

\[
Au = -V^*Vu = u'' - xu'
\]

\[
\hat{A}\theta = -VV^*\theta = \theta'' - x\theta' - \theta.
\]
$\mathfrak{A}$ is the Ornstein-Uhlenbeck operator. $u'' - xu'$ and $\theta'' - x\theta' - \theta$ have the same spectrum except for 0. This shows that the Ornstein-Uhlenbeck operator has eigenvalues $0, -1, -2, \ldots$. Eigenfunctions are Hermite polynomials

\begin{equation}
H_n(x) = \frac{(-1)^n}{n!} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}.
\end{equation}
\[
\begin{align*}
\text{eigenvalue} & \quad u'' - xu' & \quad \theta'' - x\theta' - \theta \\
-1 & \quad \begin{pmatrix} H_1(x) \end{pmatrix} & \quad \begin{pmatrix} H_0(x) \\ \vdots \\ H_n(x) \end{pmatrix} \\
-2 & \quad \begin{pmatrix} H_2(x) \\ \vdots \\ H_n(x) \end{pmatrix} & \quad \begin{pmatrix} H_1(x) \\ \vdots \\ H_n-1(x) \end{pmatrix} \\
\vdots & \quad \vdots & \quad \vdots \\
-n & \quad \begin{pmatrix} H_n(x) \\ \vdots \end{pmatrix} & \quad \begin{pmatrix} H_{n-1}(x) \\ \vdots \end{pmatrix} \\
\end{align*}
\]

\[
\frac{d}{dx} \rightarrow
\]
Laguerre polynomials

We take $a = x, \ p = x^{\alpha-1} e^{-x}, \ M = [0, \infty)$. Then

$$b = a' + a (\log p)' = 1 + x ((\alpha - 1) \log x - x)'$$

$$= 1 + x \left( \frac{\alpha - 1}{x} - 1 \right) = \alpha - x.$$ 

Hence

$$\mathbf{A} u = - V^* V u = xu'' + (\alpha - x) u'$$

$$\hat{\mathbf{A}} \theta = - V V^* \theta = x \theta'' + (\alpha + 1 - x) \theta' - \theta.$$ 

We call the operator $xu'' + (\alpha - x)u'$ as the Kummer operator.
Eigenvalues of the Kummer operator is 0, $-1, -2, \ldots$.

Eigenfunctions are Laguerre polynomials:

\begin{equation}
L_n^\alpha(x) = e^x \frac{x^{-\alpha}}{n!} \frac{d^n}{dx^n}(e^{-x}x^{n+\alpha}), \quad n = 0, 1, 2, \ldots
\end{equation}

We have

\[
\frac{d}{dx} L_n^{\alpha-1}(x) = -L_n^{\alpha}(x). 
\]
eigenvalue \[ xu'' + (\alpha - x)u' \]
\[ x\theta'' + (\alpha + 1 - x)\theta' - \theta \]

\[-1 \quad \begin{pmatrix} L_1^{\alpha-1}(x) \\ L_2^{\alpha-1}(x) \\ \vdots \\ L_n^{\alpha-1}(x) \end{pmatrix} \quad \frac{d}{dx} \quad \begin{pmatrix} -L_0^{\alpha}(x) \\ -L_1^{\alpha}(x) \\ \vdots \\ -L_n^{\alpha}(x) \end{pmatrix} \]
Laplacian

We take $a = 1$, $p = 1$, $M = \mathbb{R}$. Then $\mathfrak{A} = \frac{d^2}{dx^2}$.

Eigenfunctions are $e^{i\xi x}$. We have

$$\frac{d}{dx}e^{i\xi x} = i\xi e^{i\xi x}.$$
4. Logarithmic Sobolev inequality

For the Kummer operator, we have the following logarithmic Sobolev inequality.

Theorem 12. For $\mathcal{A}u = xu'' + (\alpha - x)u'$, we have

$$\int_0^\infty u^2 \log \left( \frac{u^2}{\|u\|_2^2} \right) p(x) \, dx \leq 4\mathcal{E}(u, u).$$
To show this, we use Bakry-Emery’s $\Gamma_2$ criterion.

$\mathcal{A}u = xu'' + (\alpha - x)u'$.

$$\Gamma(u, u) = \frac{1}{2} \{ \mathcal{A}(u^2) - 2u\mathcal{A}u \} = xu'^2$$

$$\Gamma_2(u, u) = \frac{1}{2} \{ \mathcal{A}\Gamma(u, u) - 2\Gamma(\mathcal{A}u, u) \}$$

$$= x^2(u'' + \frac{1}{2x}u')^2 + \frac{1}{2}(1 + \frac{2\alpha - 1}{2x})\Gamma(u, u).$$

So we have to assume that $\alpha \geq \frac{1}{2}$. 
Thanks!