On spectra of 1-dimensional diffusion operators

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1 Introduction

It is well-known that the Hermite polynomials are eigenfunctions of Ornstein-Uhlenbeck operator. Here the Hermite polynomials are defined by

$$H_n(x) = \frac{(-1)^n}{n!} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}, \quad n = 0, 1, \ldots$$

(1)

They satisfy the following identity:

$$H'_n(x) = H_{n-1}(x).$$

This relation suggests that the differentiation gives rise to a correspondence between two families of eigenfunctions. In this talk, we will give a general framework of this fact in one dimensional case.

2 1-dimensional diffusion operators

We take $I = [0, \infty)$ as a state space. Suppose we are given two continuous functions $a, p$ on $I$. We assume that $a > 0$, $p > 0$ on $(0, \infty)$. We define a measure $\nu$ by $\nu = p dx$. To denote $L^2(\nu)$, we use $L^2(p)$ for simplicity. We consider an operator on $H = L^2(p)$ defined by

$$\mathfrak{D}u = \frac{1}{p} (apu')'.$$

(2)

The associated Dirichlet form is

$$\mathcal{E}(u, v) = \int_0^\infty u' v' ap \, dx.$$

(3)

This corresponds to the Neumann boundary condition. If we impose the Dirichlet boundary condition, we restrict the domain to functions with $u(0) = 0$

Further we introduce following functions:

$$m(x) = \int_0^x p(y) \, dy$$

(4)

$$s(x) = \int_0^x \frac{1}{a(y) p(y)} \, dy.$$

(5)

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The measure $dm$ is called a speed measure and $s$ is called a scale function. We assume the integrability of $p$ and $ap$ near 0 and so $m(0) = s(0) = 0$. At infinity, we assume $m(\infty) + s(\infty) = \infty$. Other case can be treated similarly.

We define a operator $V: L^2(p) \to L^2(ap)$ by

$$V u = u'.$$

(6)

Then the dual operator $V^*: L^2(ap) \to L^2(p)$ is given by

$$V^* \theta = - \left(\frac{ap\theta'}{p}\right)'.$$  

(7)

Therefore we have

**Theorem 1.**

$$\mathcal{A} u = -V^*V u = au'' + bu',$$

(8)

$$-VV^* \theta = a\theta'' + (a' + b)\theta' + b' \theta.$$  

(9)

Here $b = a' + a(\log p)'$.

Now we can apply the following well-known fact:

**Proposition 2.** Let $T$ be an closed operator on a Hilbert space $H$, then $T^*T$ and $TT^*$ have the same spectrum except $\{0\}$.

Thus $au'' + bu'$ and $a\theta'' + (a' + b)\theta' + b' \theta$ have the same eigenvalue and the correspondence is given by $V = \frac{d}{dx}$.

3 Examples

**Example 3.1.** If $a = 1$, $p = e^{-x^2/2}$, then $\mathcal{A} u = u'' - xu'$, $-VV^* \theta = \theta'' - x\theta' - \theta$. The eigenfunctions are Hermite polynomials.

**Example 3.2.** If $a = x$, $p = x^{\alpha-1}e^{-x}$, then $\mathcal{A} u = xu'' + (\alpha - x)u'$, $-VV^* \theta = \theta'' + (\alpha + 1 - x)\theta' - \theta$. The eigenfunctions are Laguerre polynomials.