On spectra of 1-dimensional diffusion operators*

Ichiro SHIGEKAWA† (Kyoto University)

## 1 Introduction

It is well-known that the Hermite polynomials are eigenfunctions of Ornstein-Uhlenbeck operator. Here the Hermite polynomials are defined by

$$H_n(x) = \frac{(-1)^n}{n!} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}, \quad n = 0, 1, \dots \tag{1}$$

They satisfy the following identity:

$$H_n'(x) = H_{n-1}(x).$$

This relation suggests that the differentiation gives rise to a correspondence between two families of eigenfunctions. In this talk, we will give a general framework of this fact in one dimensional case.

## 2 1-dimensional diffusion operators

We take $I = [0, \infty)$ as a state space. Suppose we are given two continuous functions $a$, $p$ on $I$. We assume that $a > 0$, $p > 0$ on $(0, \infty)$. We define a measure $\nu$ by $\nu = p dx$. To denote $L^2(\nu)$, we use $L^2(p)$ for simplicity. We consider an operator on $H = L^2(p)$ defined by

$$\mathfrak{A} u = \frac{1}{p}(apu')'. \tag{2}$$

The associated Dirichlet form is

$$\mathscr{E}(u, v) = \int_0^\infty u'v'ap\, dx. \tag{3}$$

This corresponds to the Neumann boundary condition. If we impose the Dirichlet boundary condition, we restrict the domain to functions with $u(0) = 0$

Further we introduce following functions:

$$m(x) = \int_0^x p(y)\, dy \tag{4}$$

$$s(x) = \int_0^x \frac{1}{a(y)p(y)}\, dy. \tag{5}$$

*November 14-16, 2012, “Mathematical Quantum Field Theory and Related Topics” in Research Institute of Mathematical Sciences, Kyoto University

†E-mail: ichiro@math.kyoto-u.ac.jp URL: http://www.math.kyoto-u.ac.jp/~ichiro/

The measure $dm$ is called a speed measure and $s$ is called a scale function. We assume the integrability of $p$ and $ap$ near 0 and so $m(0) = s(0) = 0$. At infinity, we assume $m(\infty) + s(\infty) = \infty$. Other case can be treated similarly.

We define a operator $V\colon L^2(p) \to L^2(ap)$ by

$$Vu = u'. \tag{6}$$

Then the dual operator $V^*\colon L^2(ap) \to L^2(p)$ is given by

$$V^*\theta = -\frac{(ap\theta)'}{p}. \tag{7}$$

Therefore we have

**Theorem 1.**

$$\mathfrak{A}u = -V^*Vu = au'' + bu', \tag{8}$$

$$-VV^*\theta = a\theta'' + (a' + b)\theta' + b'\theta. \tag{9}$$

*Here* $b = a' + a(\log p)'$.

Now we can apply the following well-known fact:

**Proposition 2.** *Let $T$ be an closed operator on a Hilbert space $H$, then $T^*T$ and $TT^*$ have the same spectrum except* $\{0\}$.

Thus $au'' + bu'$ and $a\theta'' + (a' + b)\theta' + b'\theta$ have the same eigenvalue and the correspondence is given by $V = \frac{d}{dx}$.

## 3 Examples

**Example 3.1.** If $a = 1$, $p = e^{-x^2/2}$, then $\mathfrak{A}u = u'' - xu'$, $-VV^*\theta = \theta'' - x\theta' - \theta$. The eigenfunctions are Hermite polynomials.

**Example 3.2.** If $a = x$, $p = x^{\alpha-1}e^{-x}$, then $\mathfrak{A}u = xu'' + (\alpha - x)u'$, $-VV^*\theta = \theta'' + (\alpha + 1 - x)\theta' - \theta$. The eigenfunctions are Laguerre polynomials.