Exponential convergence of Markov Processes

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Contents

1. Introduction

- $(M, \mathcal{B}, m)$: a measure space with $m(M) = 1$
- $\{T_t\}$: a Markovian semigroup in $L^2(m)$

We assume

- $\{T_t^*\}$: a Markovian semigroup in $L^2(m)$
- $T_t 1 = T_t^* 1 = 1$

Then $\{T_t\}$ and $\{T_t^*\}$ define semigroups in $L^p(m)$ $(1 \le p \le \infty)$. For $f \in L^1$, we denote

$$\langle f \rangle = \int_M f \, dm.$$

We are interested in the following ergodicity:

$$T_t f \to \langle f \rangle \quad \text{as } t \to \infty$$

To be precise, define the index $\gamma_{p\to q}$ by

$$\gamma_{p\to q} = -\overline{\lim}\, \frac{1}{t} \log \|T_t - m\|_{p\to q}. \tag{1}$$

Here

- m : an operator $f \mapsto m(f) = \int_X f\, dm$
- $\|\ \|_{p\to q}$: the operator norm from L^p to L^q

We are interested in how $\gamma_{p\to q}$ depends on p and q.

From the Riesz-Thorin interpolation theorem, we have

$$s \mapsto \gamma_{1/s\to 1/s}$$

is concave. So if the semigroup is symmetric, $\gamma_{2\to 2}$ is the largest.

2. Hypercontractivity and the exponential convergence

Hyperboundedness

$\{T_t\}$ is called hyperbounded if there exist $K > 0$, $r \in (2, \infty)$ and $C \geq 1$ such that

$$\|T_K f\|_r \leq C\|f\|_2, \quad \forall f \in L^2(m).$$

Theorem 1. The followings are equivalent to each other:

(1) $\{T_t\}$ is hyperbounded.

(2) $\gamma_{p\to q} \geq 0$ for some $1 < p < q < \infty$.

(3) $\gamma_{p\to q} = \gamma_{2\to 2}$ for all $p, q \in (1, \infty)$.

Hypercontractivity

$\{T_t\}$ is called hypercontractive if there exist $K > 0$ and $r \in (2, \infty)$ such that

$$\|T_K f\|_r \leq \|f\|_2, \quad \forall f \in L^2(m). \tag{2}$$

Proposition 2. Under (2), we have

$$\|T_K f - \langle f\rangle\|_2 \le (r-1)^{-1/2}\|f\|_2, \quad \forall f \in L^2(m).$$

Furthermore, for any $t \ge 0$, we have

$$\|T_t f - \langle f\rangle\|_2 \le \sqrt{r-1}\,\exp\left\{-\frac{t}{K}\log\sqrt{r-1}\right\}\|f\|_2, \quad \forall f \in L^2(m).$$

Proposition 3. Let $r > 2$. Suppose that there exist positive constants K_0, K_1 such that

$$M_0 := \|T_{K_0}\|_{2\to r} < \infty$$

$$\rho := \sup\{\|T_{K_1}f - \langle f\rangle\|_2/\|f\|_2 : f \in L^2(m) \setminus \{0\}\} < 1.$$

Then the semigroup $\{T_t\}$ is hypercontractive.

Theorem 4. The followings are equivalent to each other:

(1) $\{T_t\}$ is hypercontractive.

(2) $\gamma_{p\to q} > 0$ for some $1 < p < q < \infty$.

(3) $\gamma_{p\to q} = \gamma_{2\to 2} > 0$ for all $p, q \in (1, \infty)$.

3. Sufficient condition for independence of L^p-spectrum

Normal opetator

- $\mathfrak{A}$: the generator of $\{T_t\}$
- $\mathfrak{A}^*$: the generator of $\{T_t^*\}$

We assume that $\mathfrak{A}$ is normal, i.e.,

$$\mathfrak{A}\mathfrak{A}^* = \mathfrak{A}^*\mathfrak{A}$$

Then $\mathfrak{A}$ has the spectral decomposition:

$$-\mathfrak{A} = \int_{\mathbb{C}} \lambda dE_\lambda$$

Multiplier

For any bounded function $\phi : \mathbb{C} \to \mathbb{C}$, define $\phi(-\mathfrak{A})$ by

$$\phi(-\mathfrak{A}) = \int_{\mathbb{C}} \phi(\lambda) dE_\lambda.$$

Theorem 5. Assume that $\{T_t\}$ is hyperbounded.
If $\phi(\lambda)$ is expresed as

$$\phi(\lambda) = h(1/\lambda)$$

for a bounded function h on $\mathbb{C}$ which is analytic near 0. Then $\phi(-\mathfrak{A})$ is bounded in $L^p(m)$.

Using this theorem, we can show that the boundedness of the resolvent is independent of p.

Theorem 6. Assume $\mathfrak{A}$ is normal. Then $\sigma(\mathfrak{A}_p)$, the spectrum of $\mathfrak{A}_p$, is independent of p $(1 < p < \infty)$.

4. Example of L^p-spectrum that depend on p

We give an example that the spectrum depends on p.

- $M = [0, \infty)$
- $m(dx) = \nu(dx) = e^{-x} dx$
- The Dirichlet form in $L^2(\nu)$:

$$\mathcal{E}(f, g) = \int_{[0,\infty)} f'(x) g'(x) \nu(dx)$$

with domain

$$\mathrm{Dom}(\mathcal{E}) = \{f \in L^2(\nu);\ f \text{ is absolutely continuous and } f' \in L^2(\nu)\}.$$

- The generator:

$$\mathfrak{A} = \frac{d^2}{dx^2} - \frac{d}{dx}$$

with domain

$$\mathbf{Dom}(\mathfrak{A}) = \{f \in \mathbf{Dom}(\mathcal{E});\ f' \text{ is absolutely continuous}$$
$$\text{and } f'' \in L^2(\nu) \text{ with } f'(0) = 0\}.$$

To see the spectrum of $\mathfrak{A}$, we introduce the following unitary transformation

$$If(x) = e^{-x/2} f(x) \tag{3}$$

and

$$I^{-1} f(x) = e^{x/2} f(x).$$

We note

$$I \circ \mathfrak{A} \circ I^{-1} f = -\frac{1}{4} f + \frac{d^2 f}{dx^2}$$

i.e., we have the following commutative diagram:

$$\begin{array}{ccc} L^2(\nu) & \xrightarrow{\mathfrak{A}} & L^2(\nu) \\ I \downarrow & & \downarrow I \\ L^2(dx) & \xrightarrow{\frac{d^2}{dx^2} - \frac{1}{4}} & L^2(dx) \end{array}$$

The boundary condition is involved as follows:

$$\begin{array}{c} \mathfrak{A} \quad \text{with } f'(0) = 0 \\ I \downarrow \\ \frac{d^2}{dx^2} - \frac{1}{4} \quad \text{with } \frac{1}{2} f(0) + f'(0) = 0 \end{array}$$

The corresponding Dirichlet form $\hat{\mathcal{E}}$:

$$\begin{aligned}\hat{\mathcal{E}}(f,g) &= \mathcal{E}(I^{-1}f, I^{-1}g) \\ &= \int_0^\infty f'g'\,dx + \frac{1}{4}\int_0^\infty fg\,dx - \frac{1}{2}f(0)g(0).\end{aligned}$$

We can show that $\hat{\mathcal{E}}$ is a compact perturbation of

$$\hat{\mathcal{E}}^{(0)}(f,g) = \int_0^\infty f'g'\,dx + \frac{1}{4}\int_0^\infty fg\,dx.$$

It is easy to see that spectrum corresponding to $\hat{\mathcal{E}}^{(0)}(f,g)$ is

$$[\frac{1}{4},\infty)$$

Now setting

$$A = \frac{d^2}{dx^2} - \frac{1}{4}$$

with $\mathbf{Dom}(A) = \{f, f'' \in L^2([0,\infty))$ with $\frac{1}{2}f(0) + f'(0) = 0\}$, we have

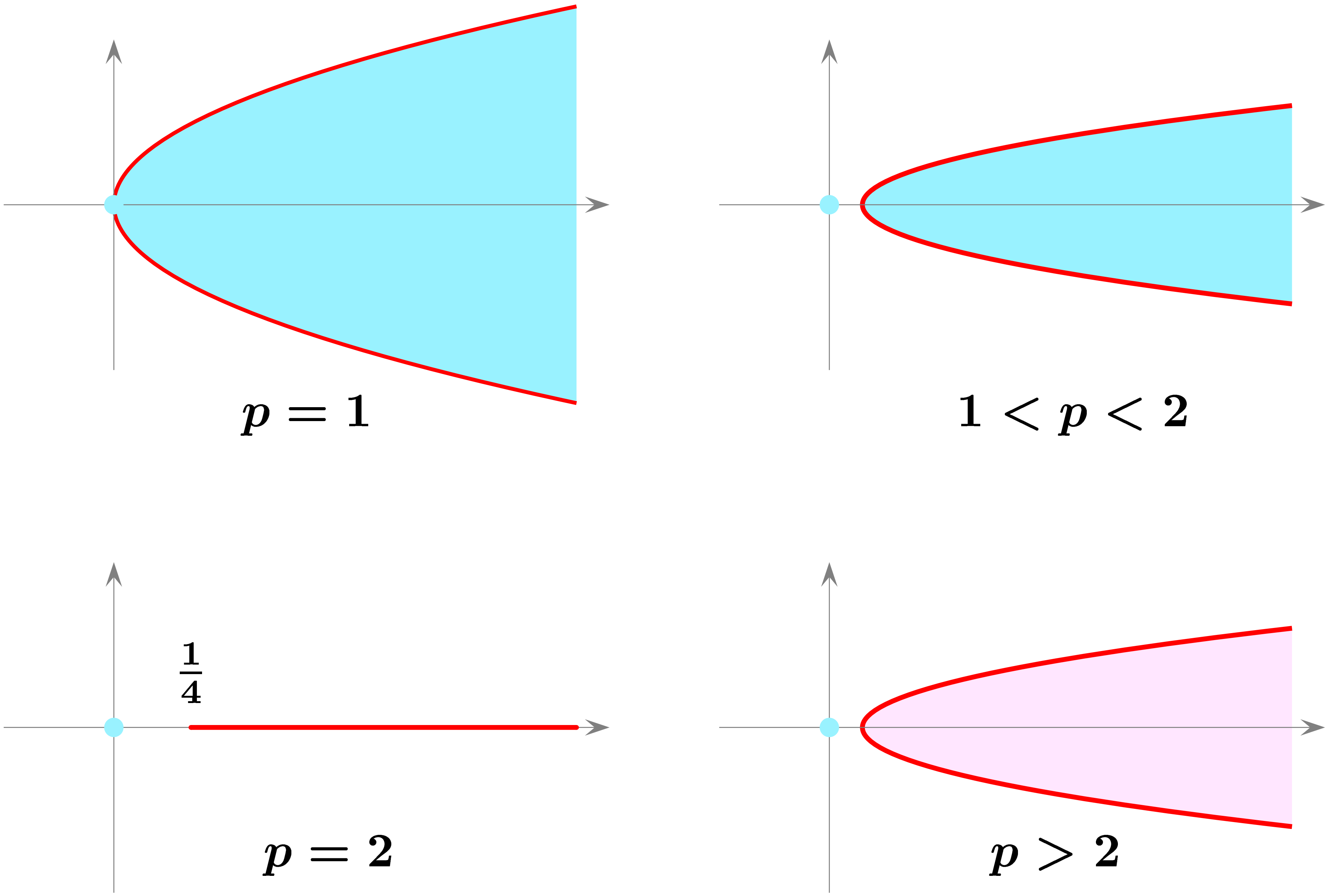
$$\sigma(-A) = \{0\} \cup [\frac{1}{4}, \infty).$$

Now, by the unitary equivalence,

Theorem 7. We have

$$\sigma(-\mathfrak{A}) = \{0\} \cup [\frac{1}{4}, \infty).$$

Now we proceed to the L^p-spectrum. The result is



First we discuss the case $1 \leq p < 2$. We use the same mapping $If(x) = e^{-x/2}f(x)$ in (4) as

$$I\colon L^p(\nu) \longrightarrow L^p(\tilde{\nu})$$

where

$$\tilde{\nu}(dx) = e^{(p/2-1)x}dx.$$

Then I gives an isometry between $L^p(\nu)$ and $L^p(\tilde{\nu})$. Similarly as before, setting $\tilde{\mathfrak{A}} = I \circ \mathfrak{A} \circ I^{-1}$, we have

$$\tilde{\mathfrak{A}}f = \frac{d^2 f}{dx^2} - \frac{1}{4}f$$

with the boundary condition

$$f'(0) + \frac{1}{2}f(0) = 0.$$

Proposition 8. For $1 \leq p < 2$, we have

$$\sigma_{\mathrm{p}}(-\mathfrak{A}) = \{0\} \cup \{x + iy;\ x, y \in \mathbb{R}, y^2 < (\frac{2}{p} - 1)^2 (x - \frac{p-1}{p^2})\}$$

Proof. We solve the following differential equation:

$$\begin{cases} -u'' + \frac{1}{4}u = \lambda u, \\ u'(0) + \frac{1}{2}u(0) = 0. \end{cases}$$

The solution is given by

$$\begin{cases} u(x) = C_1 e^{x\sqrt{-\lambda+1/4}} + C_2 e^{-x\sqrt{-\lambda+1/4}}, \\ C_1(\frac{1}{2} + \sqrt{-\lambda + \frac{1}{4}}) + C_2(\frac{1}{2} - \sqrt{-\lambda + \frac{1}{4}}) = 0. \end{cases}$$

By checking the integrability, we get the desired result. □

Proposition 9. For $1 \leq p < 2$, we have

$$\rho(-\mathfrak{A}) \supseteq \{x + iy;\ x, y \in \mathbb{R}, y^2 > (\frac{2}{p} - 1)^2 (x - \frac{p-1}{p^2})\} \setminus \{0\}$$

Proof. For $\lambda \in \{z \in \mathbb{C};\ \Re\sqrt{-z + 1/4} > \frac{1}{p} - \frac{1}{2}\}$, define

$$\phi_\lambda(x) = \left(\frac{1}{2} - \sqrt{-\lambda + \frac{1}{4}}\right) e^{x\sqrt{-\lambda + 1/4}} - \left(\frac{1}{2} + \sqrt{-\lambda + \frac{1}{4}}\right) e^{-x\sqrt{-\lambda + 1/4}},$$

$$\psi_\lambda(x) = e^{-x\sqrt{-\lambda + 1/4}},$$

$$W_\lambda = -2\sqrt{-\lambda + \frac{1}{4}} \left(\frac{1}{2} - \sqrt{-\lambda + \frac{1}{4}}\right).$$

Further, define $g_\lambda \colon [0,\infty) \times [0,\infty) \to \mathbb{C}$ を

$$g_\lambda(x,y) = \begin{cases} \frac{1}{W_\lambda}\phi_\lambda(x)\psi_\lambda(y), & x \le y, \\ \frac{1}{W_\lambda}\phi_\lambda(y)\psi_\lambda(x), & x \ge y. \end{cases}$$

The possible Green operator G_λ is given by

$$G_\lambda f(x) = \int_0^\infty g_\lambda(x,y) f(y)\, dy.$$

For $f \in C_0^\infty([0,\infty) \to \mathbb{C})$, we have

$$(\lambda + \tilde{\mathfrak{A}})G_\lambda f = f, \quad \frac{1}{2}G_\lambda f(0) + (G_\lambda f)'(0) = 0.$$

Now it suffices to show that G_λ is a bounded operator. □

We can summarize as follows:

Theorem 10. For $\mathbf{1} \leq \boldsymbol{p} < \mathbf{2}$, we have

(i) $\boldsymbol{\sigma}_{\mathbf{p}}(-\mathfrak{A}) = \{\mathbf{0}\} \cup \{x + iy;\ x, y \in \mathbb{R}, y^2 < (\frac{2}{p} - 1)^2(x - \frac{p-1}{p^2})\}$

(ii) $\boldsymbol{\sigma}_{\mathbf{c}}(-\mathfrak{A}) = \{x + iy;\ x, y \in \mathbb{R}, y^2 = (\frac{2}{p} - 1)^2(x - \frac{p-1}{p^2})\}$

(iii) $\boldsymbol{\rho}(-\mathfrak{A}) = \{x + iy;\ x, y \in \mathbb{R}, y^2 > (\frac{2}{p} - 1)^2(x - \frac{p-1}{p^2})\}$

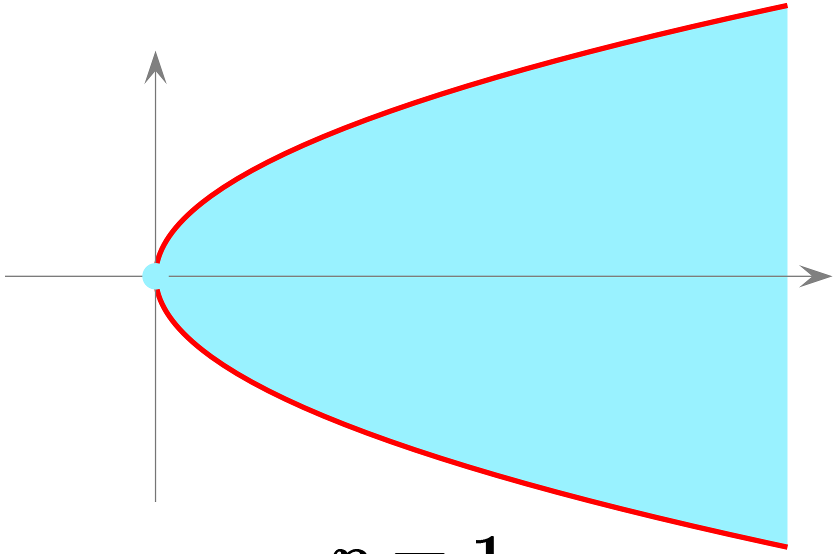
Theorem 11. For $\boldsymbol{p} > \mathbf{2}$, we have

(i) $\boldsymbol{\sigma}_{\mathbf{p}}(-\mathfrak{A}) = \{\mathbf{0}\}$

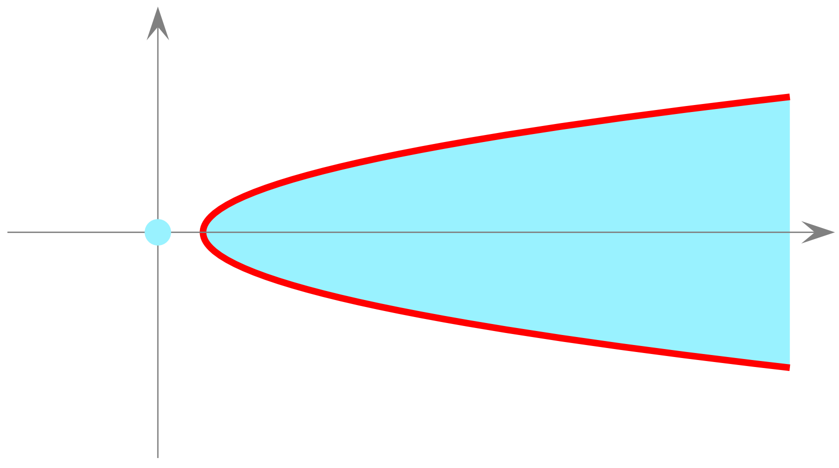
(ii) $\boldsymbol{\sigma}_{\mathbf{r}}(-\mathfrak{A}) = \{x + iy;\ x, y \in \mathbb{R}, y^2 < (\frac{2}{p} - 1)^2(x - \frac{p-1}{p^2})\}$

(iii) $\boldsymbol{\sigma}_{\mathbf{c}}(-\mathfrak{A}) = \{x + iy;\ x, y \in \mathbb{R}, y^2 = (\frac{2}{p} - 1)^2(x - \frac{p-1}{p^2})\}$

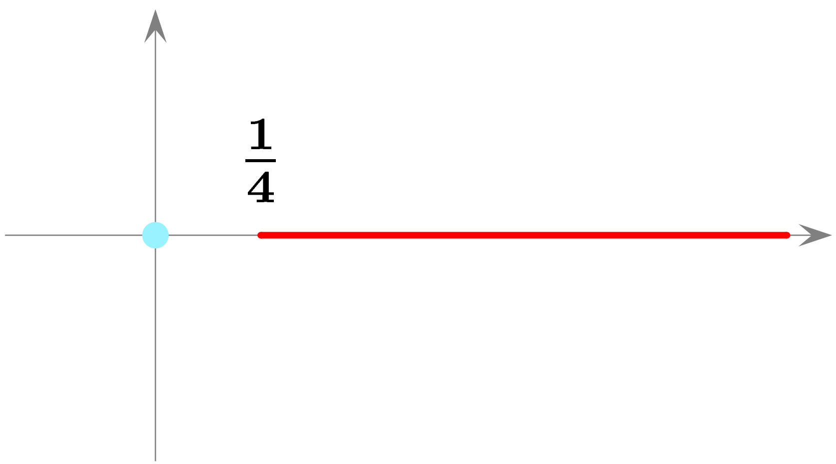
(iv) $\boldsymbol{\rho}(-\mathfrak{A}) = \{x + iy;\ x, y \in \mathbb{R}, y^2 > (\frac{2}{p} - 1)^2(x - \frac{p-1}{p^2})\}$



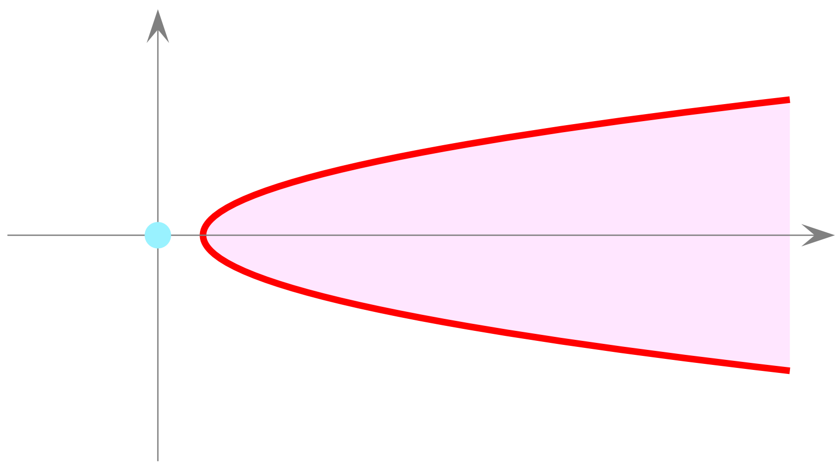
$p = 1$



$1 < p < 2$



$p = 2$



$p > 2$

By noting that

$$\inf\{\Re\lambda;\ \lambda \in \sigma(-\mathfrak{A}) \setminus \{0\}\} = -\lim_{t\to\infty} \frac{1}{t} \log \|T_t - m\|$$

we have

Theorem 12. For $1 \leq p < \infty$

$$\gamma_{p\to p} = \frac{p-1}{p^2}.$$

Thanks !