The spectrum of non-symmetric operators and Markov processes

Ichiro Shigekawa (joint work with Yusuke Ikeno)

Kyoto University

September 9, 2011

5th ICSAA at Bonn University

URL: **http://www.math.kyoto-u.ac.jp/˜ichiro/**

1. Normal operators

General framework

- H: a complex Hilbert space
- T: a closed operator with domain $\mathrm{Dom}(T)$
- $\Theta(T)$: the numeriacl range of T defined by

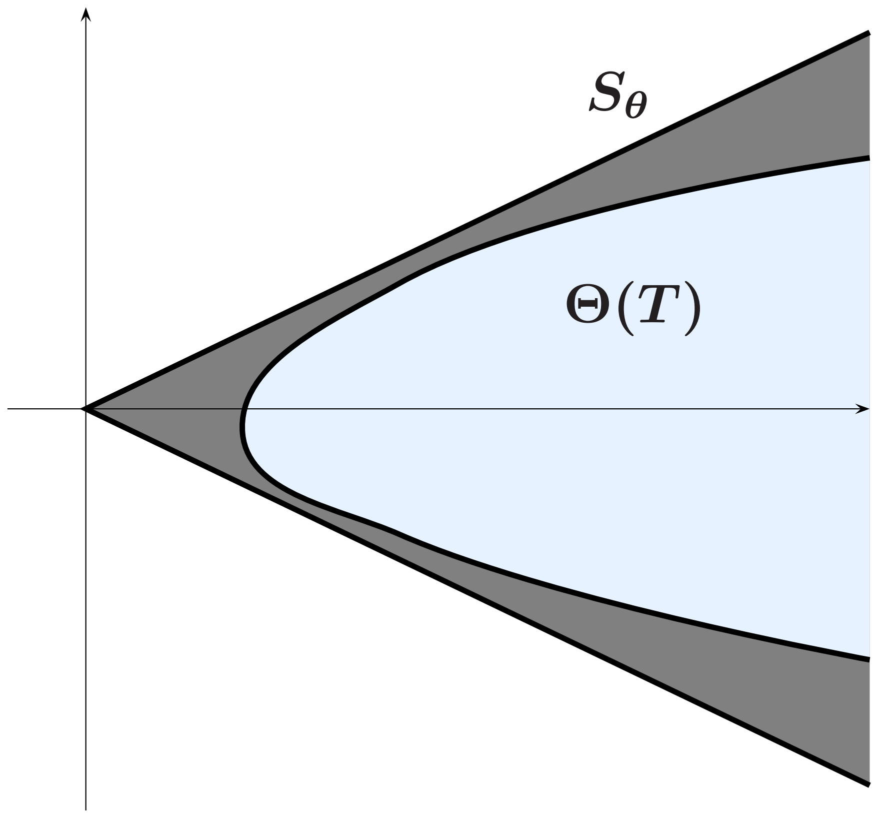
$$\Theta(T) := \{(Tu, u);\ u \in \mathrm{Dom}(T)\}.$$

- T is called accretive if

$$\Re(Tu, u) \geq 0, \quad \forall u \in \mathrm{Dom}(T)$$

- T is called m-accretive if $\mathrm{Ran}(T - \zeta) = H$ for some $\zeta \in \mathbb{C}$.

- T is called sectorial if $\Theta(T) \subseteq S_\theta$, $\theta \in [0, \frac{\pi}{2})$ where $S_\theta = \{z \in \mathbb{C}; \, |\arg z| \leq \theta\}$.



- T is called quasi-sectorial if $T + \gamma$ is sectorial for some $\gamma > 0$.

Normal operators

- A is called normal if

$$A^* A = A A^*$$

- A has an spectral decomposition:

$$A = \int_{\mathbb{C}} z E(dz)$$

- A^*: an adjoint operator of A.

$$A^* = \int_{\mathbb{C}} \overline{z} E(dz)$$

- $\overline{\Theta(A)} = \overline{\mathrm{co}}(\sigma(A))$

From now on, we assume that A is normal and m-accretive.

- $\sqrt{A}$ is defined by

$$\sqrt{A} = \int_{\mathbb{C}} \sqrt{z} E(dz)$$

  with

$$\mathrm{Dom}(\sqrt{A}) = \{u \in H;\ \int_{\mathbb{C}} |z|(u, E(dz)u) < \infty\}.$$

- $\mathrm{Dom}(\sqrt{A}) = \mathrm{Dom}(\sqrt{A^*})$
- a: a sesquilinear form associated with A is given by

$$a(u, v) = (Au, v), \quad u, v \in \mathrm{Dom}(A)$$

- A symmetric part of a is defied by

$$b(u, v) = \frac{(Au, v) + (A^*u, v)}{2}, \quad u, v \in \mathrm{Dom}(A).$$

- b can be written

$$b(u,v) = \int_{\mathbb{C}} \Re z (u, E(dz)v).$$

- $(b, \mathrm{Dom}(b))$ is closed where

$$\mathrm{Dom}(b) = \{u \in H; \int_{\mathbb{C}} \Re z (u, E(dz)u) < \infty\}.$$

- $\mathrm{Dom}(\sqrt{A}) \subseteq \mathrm{Dom}(b)$

Theorem 1. $\mathrm{Dom}(\sqrt{A}) = \mathrm{Dom}(b)$ if and only if $1 + \sigma(A) \subseteq S_\theta$ for some $\theta \in (0, \pi/2)$.

2. Nomal operators and generalized Dirichlet forms

Stannat (1994) introduced the generalized Dirichlet form.

We will show that Markovivan semigroup generated by a normal opetator can be formulated in the framework of generalized Dirichlet form.

- M: a Hausdorff topological space
- (M, m): σ-finite measure space
- $H = L^2(m)$
- $\mathfrak{A}$: a normal operator
- We assume that $\mathfrak{A}$ and $\mathfrak{A}^*$ is m-dissipative
  (i.e., $-\mathfrak{A}$ and $-\mathfrak{A}^*$ is m-accretive)

By spectral decomposition,

$$-\mathfrak{A} = \int_{\mathbb{C}} zE(dz). \tag{1}$$

We define

$$(2) \qquad -\boldsymbol{L} = \int_{\mathbb{C}} \Re z E(dz), \quad -\boldsymbol{\Lambda} = \int_{\mathbb{C}} i\Im z E(dz).$$

$\boldsymbol{L}$ and $i\boldsymbol{\Lambda}$ are seld-adjoint with domains

$$\mathbf{Dom}(\boldsymbol{L}) = \{f; \int_{\mathbb{C}} |\Re z|^2 (f, E(dz)f) < \infty\},$$
$$\mathbf{Dom}(\boldsymbol{\Lambda}) = \{f; \int_{\mathbb{C}} |\Im z|^2 (f, E(dz)f) < \infty\}.$$

$\boldsymbol{L}$ generates a semigroup. Symmetric bilinear form $\tilde{\mathcal{E}}$ is defined by

$$\tilde{\mathcal{E}}(f, g) = \int_{\mathbb{C}} \Re z (f, E(dz)g)$$

with the domain

$$\mathbf{Dom}(\tilde{\mathcal{E}}) = \{f; \int_{\mathbb{C}} |\Re z| (f, E(dz)f) < \infty\}.$$

We set $\mathcal{V} = \mathbf{Dom}(\tilde{\mathcal{E}})$.

Similarly, $\boldsymbol{\Lambda}$ generates a semigroup denoted by $\{U_t\}_{t \geq 0}$.

Proposition 2. $\{U_t\}$ is a C_0-semigroup in $\mathcal{V}$.

We regard $\Lambda\colon \mathbf{Dom}(\Lambda) \cap \mathcal{V} \to \mathcal{V}'$ as an operator from $\mathcal{V}$ to $\mathcal{V}'$. Its closure is denoted by $(\Lambda, \mathcal{F})$.

Proposition 3. $f \in \mathcal{F}$ if and only if

$$\int_{\mathbb{C}} \left(\frac{|\Im z|^2}{\Re z + 1} + \Re z \right) (f, E(dz) f) < \infty.$$

Similar argument can be done for the dual semigroup $\hat{U}_t$ of U_t. The generator is

$$\hat{\Lambda} = -\int_{\mathbb{C}} i \Im z E(dz).$$

Now we can apply the theory of generalized Dirichlet form. The Dirichlet form is defined by

$$\mathcal{E}(f,g) = \begin{cases} \tilde{\mathcal{E}}(f,g) - \langle \Lambda f, g\rangle, & \text{if } f \in \mathcal{F}, g \in \mathcal{V}, \\ \tilde{\mathcal{E}}(f,g) - \langle \hat{\Lambda} g, f\rangle, & \text{if } f \in \mathcal{V}, g \in \hat{\mathcal{F}}. \end{cases}$$

Assuming the Markovian property, we can define the capacity.

We assume the quasi-regularity of $\mathcal{E}$. Now applying the following theorem, we can get a Markov process associated with $\mathfrak{A}$.

Theorem 4. (**Stannat 1994**) Under the following condition (D3), there exists an m-thght special standard process.

(**D3**) There exists a linear subspace $\mathcal{Y} \subseteq L^2(m) \cap L^\infty(m)$ such that $\mathcal{Y} \cap \mathcal{F}$ is dense in $\mathcal{F}$, $\lim_{\alpha\to\infty} e_{\alpha G_\alpha u - u} = 0$ in H for all $u \in \mathcal{Y}$ and for the closure $\overline{\mathcal{Y}}$ of $\mathcal{Y}$ in $L^\infty(m)$ it follows that $u \wedge \alpha \in \overline{\mathcal{Y}}$ for $u \in \mathcal{Y}$ and $\alpha \geq 0$.

3. Criterion for nomal operators

- H: a Hilbert space
- A, B: accretive operators on $\mathcal{D}$
- Assume that $\overline{A}$, $\overline{B}$ are m-accretive

Theorem 5. Assume that $A\mathcal{D} \subseteq \mathcal{D}$, $B\mathcal{D} \subseteq \mathcal{D}$ and

$$AB = BA \quad \text{on } \mathcal{D},$$

$$(Au, v) = (u, Bv), \quad u, v \in \mathcal{D}.$$

Then $\overline{A}$ is normal and $\overline{A}^* = \overline{B}$.

Examples on a Riemannian manifold

- M: a complete Riemannian manifold
- m: the Riemannian volume

We take a function $U \in C^\infty(M)$ and define a measure ν by

$$\nu = e^{-U} m$$

Define an operator on $H = L^2(\nu)$ by

$$\mathfrak{A} = \frac{1}{2}\triangle_\nu + b$$

where $\triangle_\nu = -\nabla_\nu^* \nabla$. Then

$$\mathfrak{A}_\nu^* = \frac{1}{2}\triangle_\nu - b - \mathrm{div}_\nu\, b.$$

Here div_ν denotes the divergence with respect to ν.

We give a criterion for $\mathfrak{A} = \triangle_\nu + b$ being a normal operator.

Theorem 6. Assume that $\mathbf{div}_\nu\, b$ is bounded from below. Then $\mathfrak{A}$ is normal if and only if b is a Killing vector field and the following identies hold:

$$
\begin{aligned}
&(\frac{1}{2}\triangle_\nu + b)\,\mathbf{div}_\nu\, b = 0,\\
&[\nabla U, b] + \nabla\,\mathbf{div}_\nu\, b = 0.
\end{aligned}
$$

4. One-dimensional Brownian motion with a drift

We consider an operator $\mathfrak{A} = \frac{d^2}{dx^2} - c\frac{d}{dx}$ on $L^2(\mathbb{R}, \nu_1)$. Here ν_1 is a measure defined by

$$\nu_1(dx) = e^{-cx}dx. \tag{3}$$

Then $\mathfrak{A}$ is a self-adjoint operator with

$$(\mathfrak{A}f, g) = -\int_{\mathbb{R}} f'(x)g'(x)\,\nu_1(dx).$$

To investigate the spectrum of $\mathfrak{A}$, we use the following isometric map $I\colon L^2(\nu_1) \longrightarrow L^2(dx)$:

$$If(x) = e^{-cx/2}f(x).$$

We have

$$I \circ \mathfrak{A} \circ I^{-1} = \frac{d^2 f}{dx^2} - \frac{c^2}{4},$$

i.e., the following diagram is commutative:

$$\begin{array}{ccc} L^2(\nu_1) & \xrightarrow{\mathfrak{A}} & L^2(\nu_1) \\ I\downarrow & & \downarrow I \\ L^2(dx) & \xrightarrow{\frac{d^2}{dx^2}-\frac{c^2}{4}} & L^2(dx) \end{array}$$

Hence the spectrum $-\mathfrak{A}$ is

$$\sigma(-\mathfrak{A}) = [\frac{c^2}{4}, \infty). \tag{4}$$

We now consider an perturbation of $\mathfrak{A}$. Let b be an vector field defined by

$$b = k\frac{d}{dx}.$$

We consider an operator of the form $\mathfrak{A} + b$. We are interested in how the spectrum changes. b is clearly an Killing vector field. The divergence of b with respect to ν_1

$$\mathrm{div}_{\nu_1} b = -ck$$

and so it satisfies

$$(\mathfrak{A} + b)\,\mathrm{div}_\nu b = 0,$$
$$[(\nabla U)^\sharp, b] + \nabla\,\mathrm{div}_\nu b = 0.$$

Here $U(x) = cx$. By Theorem 6, $\mathfrak{A} + b$ is a normal operator. Under the transformation of I, we have

$$I \circ (\mathfrak{A} + b) \circ I^{-1} = \frac{d^2}{dx^2} + k\frac{d}{dx} - \frac{c(c-2k)}{4}.$$

It is enough to get the spectrum of $\frac{d^2}{dx^2} + k\frac{d}{dx}$. Recall the Fourier transform as

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\xi x}\, dx.$$

This gives an isometry from $L^2(dx)$ onto $L^2(d\xi)$. Note that

$$\int_{\mathbb{R}} (\frac{d^2}{dx^2} + k\frac{d}{dx}) f(x)\overline{g(x)}\, dx = \int_{\mathbb{R}} (-\xi^2 + ik\xi)\hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi$$

which means that

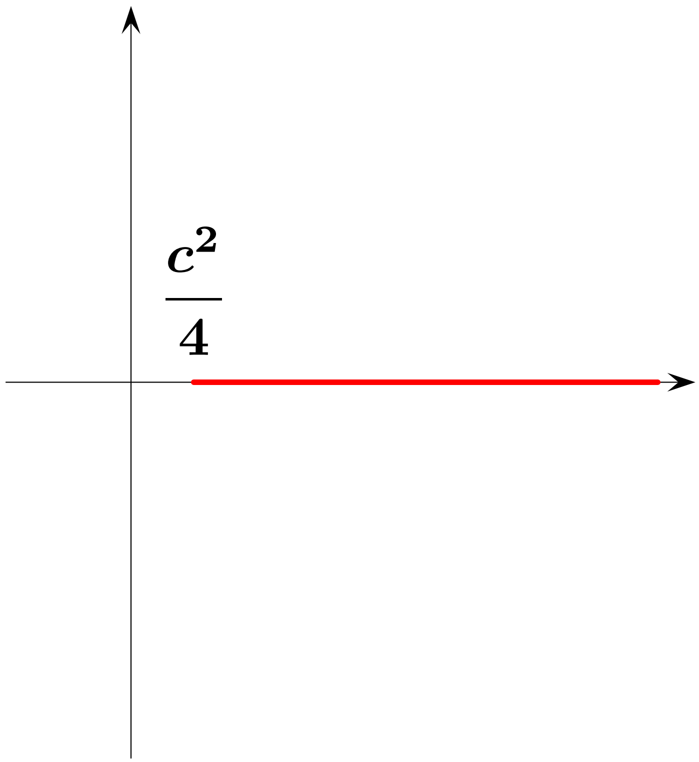
$$\sigma(\frac{d^2}{dx^2} + k\frac{d}{dx}) = \{-\xi^2 + ik\xi; \xi \in \mathbb{R}\}.$$

Theorem 7. We have

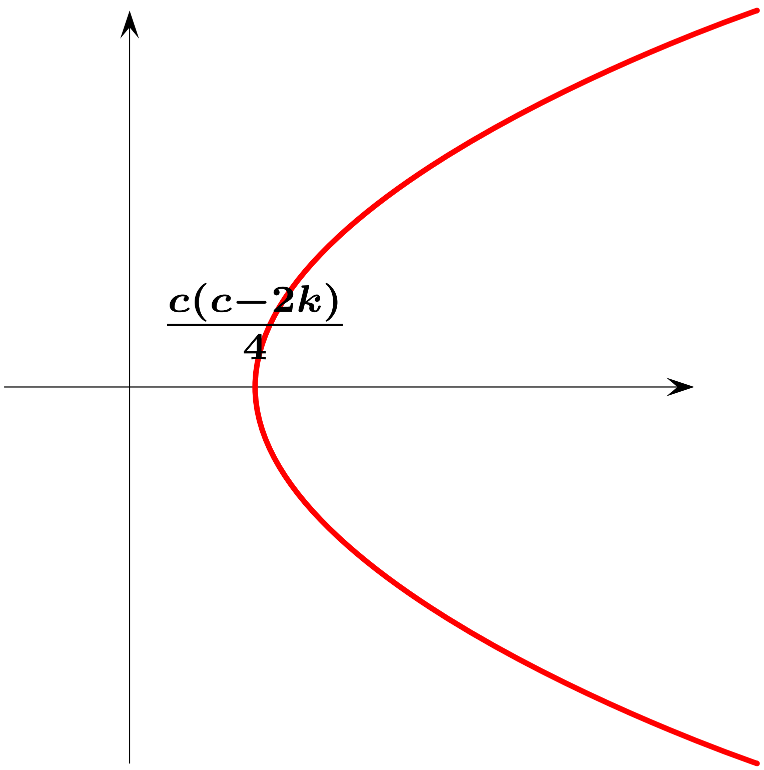
$$\sigma(-\mathfrak{A}) = [\frac{c^2}{4}, \infty)$$

and

$$\sigma(-\mathfrak{A} - b) = \{\frac{c(c-k)}{2} + \xi^2 + ik\xi;\ \xi \in \mathbb{R}\}.$$



$-\mathfrak{A}$



$-\mathfrak{A} - k\frac{d}{dx}$

Now we take a different point of view.

We fix an operator $\mathfrak{A} = \frac{d^2}{dx^2} - c\frac{d}{dx}$ but we change a reference measure. For $\theta \in [0,1]$, define

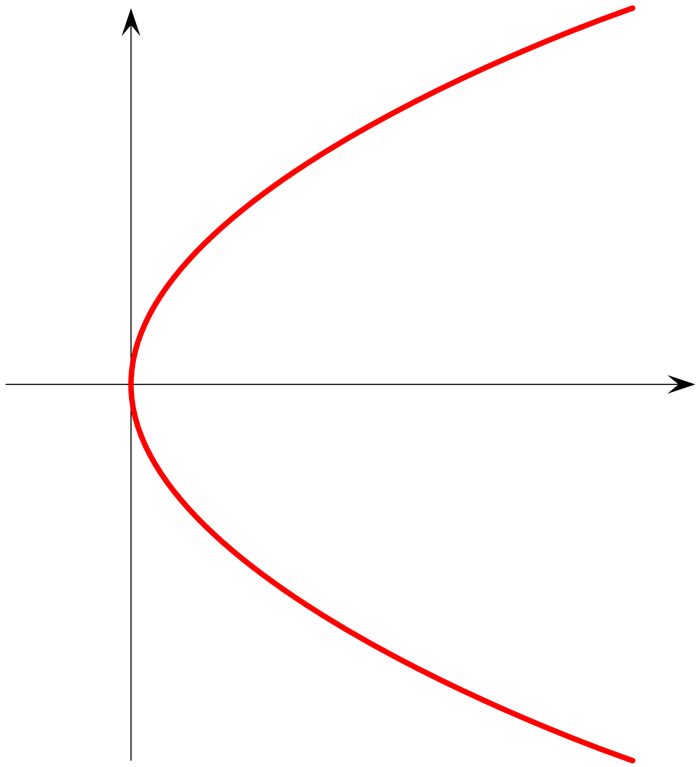
$$\nu_\theta(dx) = (1-\theta)dx + \theta e^{-cx}dx$$

ν_θ is an invariant measure for $\mathfrak{A}$. $\nu_0(dx) = dx$, $\nu_1(dx) = e^{-cx}\,dx$.

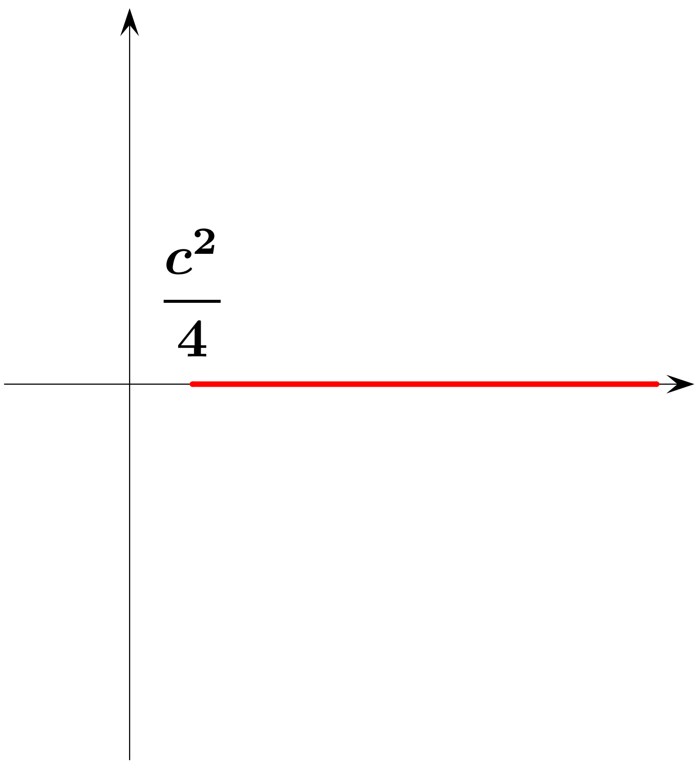
The computation above implies

$$\sigma(-\mathfrak{A}) = \{\xi^2 - ic\xi;\ \xi \in \mathbb{R}\} \quad \text{in } L^2(\nu_0)$$

$$\sigma(-\mathfrak{A}) = [\frac{c^2}{4}, \infty) \quad \text{in } L^2(\nu_1).$$



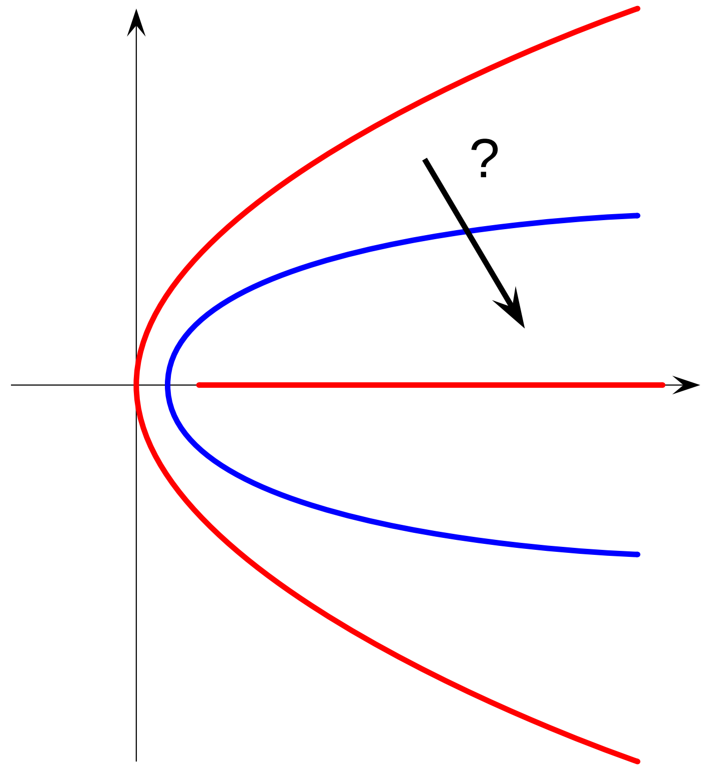
w.r.t. $\nu_0 = dx$



w.r.t. $\nu_1 = e^{-cx} dx$

What happens if we take the measure ν_θ?

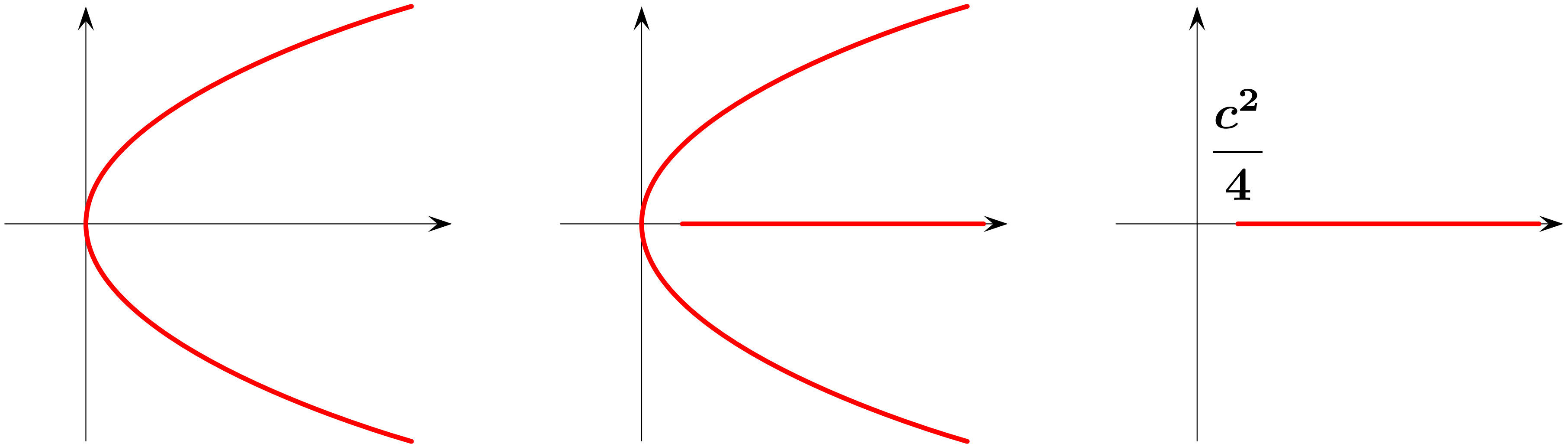
Does the spectrum chage continuously?



w.r.t. ν_θ

Theorem 8. For $\theta \in (0,1)$, $\sigma(-\mathfrak{A})$ in $L^2(\nu_\theta)$ is

$$\{\xi^2 - ik\xi; \xi \in \mathbb{R}\} \cup [\frac{c^2}{4}, \infty).$$



w.r.t. $\nu_0 = dx$ w.r.t. ν_θ w.r.t. $\nu_1 = e^{-cx}dx$

5. Perturbation by rotation

Laplacian on $\mathbb{R}^2$

Let $\mathfrak{A}$ be

$$\mathfrak{A} = \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y^2} + k\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \quad \text{on } L^2(\mathrm{R}^2, dxdy). \tag{5}$$

The spectrum of $-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ is $[0, \infty)$.

For the spectrum of $\mathfrak{A}$, we recall the Bessel functions:

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{l=0}^{\infty} \frac{(ix/2)^{2l}}{l!\Gamma(\nu+l+1)}, \quad \Re\nu > 0$$

which satisfies the following differential equation

$$I'' + \frac{1}{x}I' + \left(1 - \frac{\nu^2}{x^2}\right)I = 0$$

Since our space is $\mathbb{R}^2$, we only need the case that ν is a non-negative integer. We use the plar coordinate:

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \end{cases} \qquad r \geq 0,\ \theta \in [0, 2\pi)$$

Using this, $\mathfrak{A}$ can be written as

$$\mathfrak{A} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + k\frac{\partial}{\partial \theta}.$$

If $F = f(r)e^{in\theta}$, then

$$\mathfrak{A}F = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - n^2\frac{1}{r^2}\right)f(r)e^{in\theta} + iknf(r)e^{in\theta}.$$

Further

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - n^2\frac{1}{r^2}\right)J_n(\lambda r) = -\lambda^2 J_n(\lambda r).$$

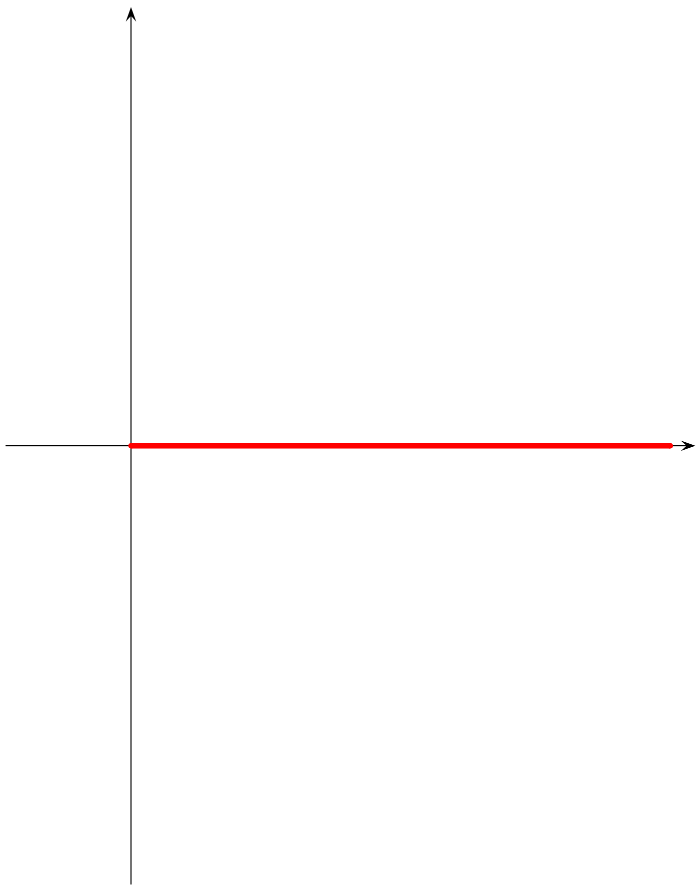
The spectral decomposition is given by

$$f(r,\theta) = \sum_{n\in\mathbb{Z}} \int \left\{ \int \left(\frac{1}{2\pi} \int f(\rho,\phi) e^{-in\phi}\, d\phi \right) J_{|n|}(\lambda\rho)\rho\, d\rho \right\} e^{in\theta} J_{|n|}(\lambda r)\lambda\, d\lambda.$$

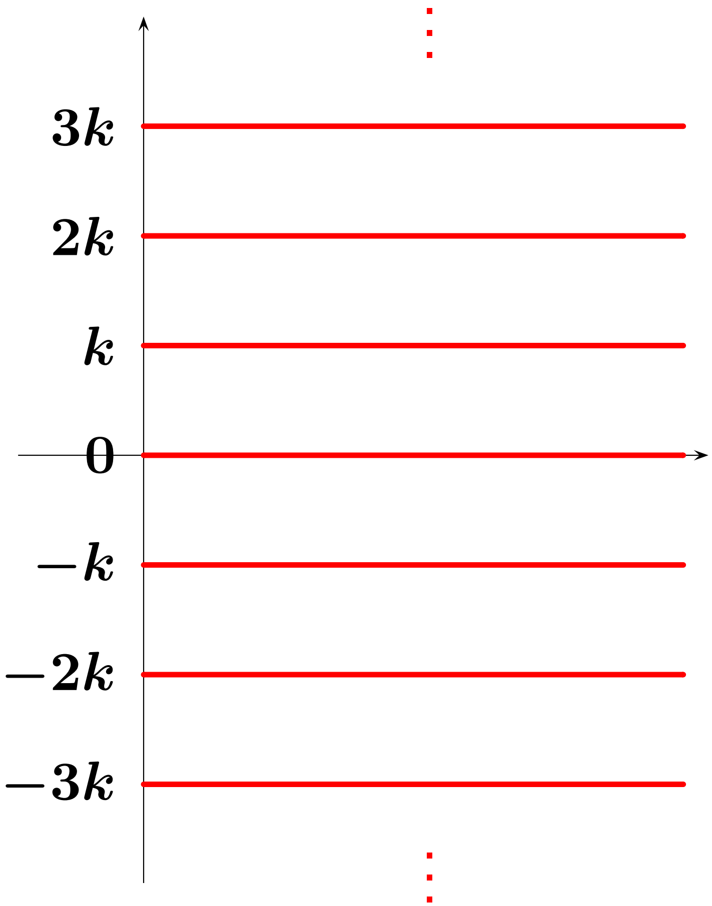
Theorem 9. The spectrum of $-\mathfrak{A}$ is

$$\{\lambda^2 - ikn\ ;\ \lambda \geq 0,\ n \in \mathbf{Z}\} \tag{6}$$

and the corresponding eigenfunction to $\lambda^2 - ikn$ is $J_{|n|}(\lambda r)e^{in\theta}$.



the spectrum of $-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$



the spectrum of $\mathfrak{A}$

Ornstein Uhlenbeck operator on $\mathbb{R}^2$

Let L_α be

$$L_\alpha = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + \alpha\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \tag{7}$$

acting on $L^2(\mathbb{R}^2, \frac{1}{2\pi}e^{-(x^2+y^2)/2}dxdy)$.

The spectrum of Ornstein-Uhlenbeck operator L_0 is $\{0, -1, -2, \dots\}$. In fact, define Hermite polynomials by

$$H_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2/2}.$$

Then

$$L_0 H_k(x)H_{n-k}(y) = -nH_k(x)H_{n-k}(y).$$

To get the spectram of L_α, we need the complex Hermite polynomials defined by

$$H_{p,q}(z,\bar{z}) = (-1)^{p+q}e^{\frac{z\bar{z}}{2}}\left(\frac{\partial}{\partial \bar{z}}\right)^p\left(\frac{\partial}{\partial z}\right)^q e^{-\frac{z\bar{z}}{2}}. \tag{8}$$

Here, we regard $\mathbb{R}^{\mathbf{2}}$ as $\mathbb{C}$ with $z = x + iy$. We denote

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

In the sequel, we write

$$\partial = \frac{\partial}{\partial z}, \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}}$$

for short. We have

$$\partial^* = -\bar{\partial} + \frac{z}{2}, \quad \bar{\partial}^* = -\partial + \frac{\bar{z}}{2} \tag{9}$$

Proposition 10. The following identities hold:

$$\partial H_{p,q} = \frac{p}{2} H_{p-1,q}, \quad \bar{\partial} H_{p,q} = \frac{q}{2} H_{p,q-1},$$
$$\partial^* H_{p,q} = H_{p+1,q}, \quad \bar{\partial}^* H_{p,q} = H_{p,q+1},$$
$$(2\partial\bar{\partial} - z\partial) H_{p,q} = -p H_{p,q}$$
$$(2\partial\bar{\partial} - \bar{z}\bar{\partial}) H_{p,q} = -q H_{p,q}$$
$$(z\partial - \bar{z}\bar{\partial}) H_{p,q} = (p-q) H_{p,q}$$

We can write

$$L_\alpha = (2\partial\bar{\partial} - z\partial) + (2\partial\bar{\partial} - \bar{z}\bar{\partial}) + \alpha i (z\partial - \bar{z}\bar{\partial}).$$

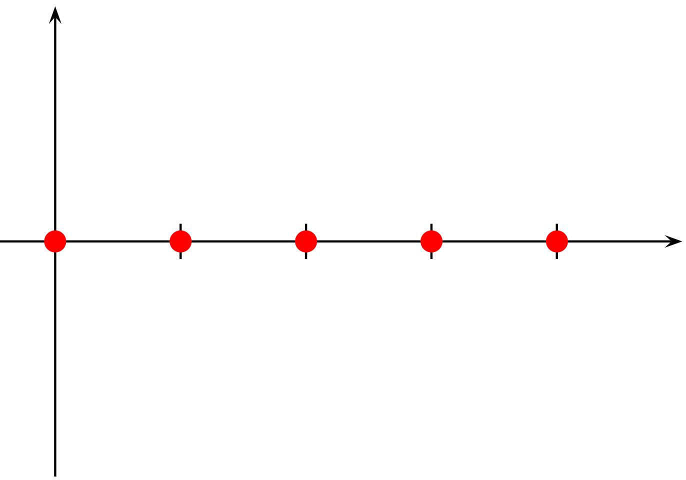
Hence

$$L_\alpha H_{p,q} = (-p - q + (p-q)\alpha i) H_{p,q}.$$

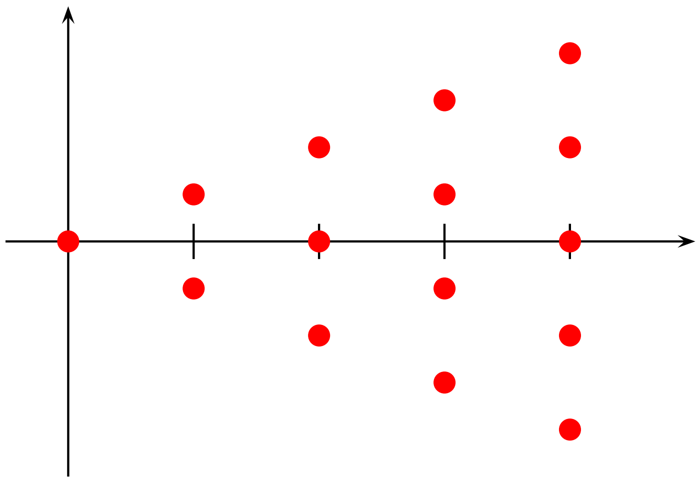
Theorem 11. The spectrum of $-L_\alpha$ is

$$\{(p+q)-(p-q)\alpha i\}_{p,q=0}^{\infty} \tag{10}$$

and corresponding eigenfunctions are $H_{p,q}$ respectively.



the spectrum of $-\boldsymbol{L}_{\mathbf{0}}$



the spectrum of $-\boldsymbol{L}_{\boldsymbol{\alpha}}$

Connection to the Laguerre polynomials

The eigenfunction $H_{n,n}$ for the eigenvalue $2n$, ($n \in \mathbb{Z}_+$) is rotation invariant since $\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) H_{n,n} = 0$. So $H_{n,n}$ is a function of $r = |z|$ and

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - r\frac{d}{dr}\right)H_{n,n} = -2nH_{n,n}.$$

Now, by the change of variable $r = \sqrt{2u}$, we have

$$\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - r\frac{d}{dr} = 2u\frac{d^2}{du^2} + 2(1-u)\frac{d}{du}.$$

$F(u) = H_{n,n}(r)$ satisfies

$$2u\frac{d^2}{du^2}F + 2(1-u)\frac{d}{du}F + nF = 0.$$

The Laguere polynomial satisfies this differential equation. Here the Laguere polynomial polynomial is defined by

$$L_n = \frac{e^x}{n!}\frac{d^n}{dx^n}(e^{-x}x^n) \tag{11}$$

Now we have

Theorem 12. Complex Hermite polynomials $H_{n,n}$ are expressed as following;

$$H_{n,n}(z,\bar{z}) = \frac{(-1)^n n!}{2^n} L_n\left(\frac{|z|^2}{2}\right) \tag{12}$$

where c is a constant.

Thanks a lot!