The spectrum of non-symmetric operators and Markov processes

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September 9, 2011

5th ICSAA at Bonn University

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1. Normal operators

General framework

- $H$: a complex Hilbert space
- $T$: a closed operator with domain $\text{Dom}(T)$
- $\Theta(T)$: the numerical range of $T$ defined by
  \[ \Theta(T) := \{(Tu, u); \ u \in \text{Dom}(T)\}. \]
- $T$ is called accretive if
  \[ \Re(Tu, u) \geq 0, \ \forall u \in \text{Dom}(T) \]
- $T$ is called $m$-accretive if $\text{Ran}(T - \zeta) = H$ for some $\zeta \in \mathbb{C}$. 
- $T$ is called **sectorial** if $\Theta(T) \subseteq S_\theta$, $\theta \in [0, \frac{\pi}{2})$ where $S_\theta = \{ z \in \mathbb{C}; \ |\arg z| \leq \theta \}$.

- $T$ is called **quasi-sectorial** if $T + \gamma$ is sectorial for some $\gamma > 0$. 
Normal operators

- $A$ is called normal if

\[ A^* A = AA^* \]

- $A$ has an spectral decomposition:

\[ A = \int_{\mathbb{C}} zE(dz) \]


\[ A^* = \int_{\mathbb{C}} \overline{z}E(dz) \]

- $\Theta(A) = \overline{\operatorname{co}(\sigma(A))}$
From now on, we assume that $A$ is normal and $m$-accretive.

- $\sqrt{A}$ is defined by
  
  \[ \sqrt{A} = \int_{\mathbb{C}} \sqrt{z} E(dz) \]

  with
  
  \[ \text{Dom}(\sqrt{A}) = \{ u \in H; \int_{\mathbb{C}} |z|(u, E(dz)u) < \infty \} \]

- $\text{Dom}(\sqrt{A}) = \text{Dom}(\sqrt{A^*})$

- $a$: a sesquilinear form associated with $A$ is given by
  
  \[ a(u, v) = (Au, v), \quad u, v \in \text{Dom}(A) \]

- A symmetric part of $a$ is defined by
  
  \[ b(u, v) = \frac{(Au, v) + (A^*u, v)}{2}, \quad u, v \in \text{Dom}(A). \]
• $b$ can be written

$$b(u, v) = \int_{\mathbb{C}} \Re z(u, E(dz)v).$$

• $(b, \text{Dom}(b))$ is closed where

$$\text{Dom}(b) = \{u \in H; \int_{\mathbb{C}} \Re z(u, E(dz)u) < \infty\}.$$ 

• $\text{Dom}(\sqrt{A}) \subseteq \text{Dom}(b)$

**Theorem 1.** $\text{Dom}(\sqrt{A}) = \text{Dom}(b)$ if and only if $1 + \sigma(A) \subseteq S_\theta$ for some $\theta \in (0, \pi/2)$. 
2. Nomal operators and generalized Dirichlet forms

Stannat (1994) introduced the generalized Dirichlet form.

We will show that Markovivan semigroup generated by a normal operator can be formulated in the framework of generalized Dirichlet form.

- $M$: a Hausdorff topological space
- $(M, m)$: $\sigma$-finite measure space
- $H = L^2(m)$
- $A$: a normal operator
- We assume that $A$ and $A^*$ is $m$-dissipative (i.e., $-A$ and $-A^*$ is $m$-accretive)

By spectral decomposition,

\begin{equation}
-A = \int_C zE(dz).
\end{equation}
We define

\[ -L = \int_C \Re z E(dz), \quad -\Lambda = \int_C i \Im z E(dz). \tag{2} \]

\( L \) and \( i\Lambda \) are self-adjoint with domains

\[
\text{Dom}(L) = \{ f; \int_C |\Re z|^2 (f, E(dz) f) < \infty \},
\]

\[
\text{Dom}(\Lambda) = \{ f; \int_C |\Im z|^2 (f, E(dz) f) < \infty \}.\]

\( L \) generates a semigroup. Symmetric bilinear form \( \tilde{E} \) is defined by

\[
\tilde{E}(f, g) = \int_C \Re z (f, E(dz) g)
\]

with the domain

\[
\text{Dom}(\tilde{E}) = \{ f; \int_C |\Re z| (f, E(dz) f) < \infty \}.
\]

We set \( V = \text{Dom}(\tilde{E}) \).

Similarly, \( \Lambda \) generates a semigroup denoted by \( \{ U_t \}_{t \geq 0} \).
Proposition 2. \( \{ U_t \} \) is a \( C_0 \)-semigroup in \( \mathcal{V} \).

We regard \( \Lambda: \text{Dom}(\Lambda) \cap \mathcal{V} \to \mathcal{V}' \) as an operator from \( \mathcal{V} \) to \( \mathcal{V}' \). Its closure is denoted by \( (\Lambda, \mathcal{F}) \).

Proposition 3. \( f \in \mathcal{F} \) if and only if

\[
\int_{\mathbb{C}} \left( \frac{|\Im z|^2}{\Re z + 1} + \Re z \right) (f, E(dz)f) < \infty.
\]

Similar argument can be done for the dual semigroup \( \hat{U}_t \) of \( U_t \). The generator is

\[
\hat{\Lambda} = -\int_{\mathbb{C}} i\Im z E(dz).
\]
Now we can apply the theory of generalized Dirichlet form. The Dirichlet form is defined by

\[
\mathcal{E}(f, g) = \begin{cases} 
\tilde{\mathcal{E}}(f, g) - \langle \Lambda f, g \rangle, & \text{if } f \in \mathcal{F}, g \in \mathcal{V}, \\
\tilde{\mathcal{E}}(f, g) - \langle \hat{\Lambda} g, f \rangle, & \text{if } f \in \mathcal{V}, g \in \hat{\mathcal{F}}.
\end{cases}
\]

Assuming the Markovian property, we can define the capacity.

We assume the quasi-regularity of \( \mathcal{E} \). Now applying the following theorem, we can get a Markov process associated with \( \mathcal{A} \).

**Theorem 4.** (Stannat 1994) Under the following condition (D3), there exists an \( m \)-thght special standard process.

(D3) There exists a linear subspace \( \mathcal{Y} \subseteq L^2(m) \cap L^\infty(m) \) such that \( \mathcal{Y} \cap \mathcal{F} \) is dense in \( \mathcal{F} \), \( \lim_{\alpha \to \infty} e^{\alpha G_{\alpha}} u - u = 0 \) in \( H \) for all \( u \in \mathcal{Y} \) and for the closure \( \overline{\mathcal{Y}} \) of \( \mathcal{Y} \) in \( L^\infty(m) \) it follows that \( u \wedge \alpha \in \overline{\mathcal{Y}} \) for \( u \in \mathcal{Y} \) and \( \alpha \geq 0 \).
3. Criterion for normal operators

- \( H \): a Hilbert space
- \( A, B \): accretive operators on \( D \)
- Assume that \( A, B \) are \( m \)-accretive

**Theorem 5.** Assume that \( AD \subseteq D, BD \subseteq D \) and

\[
AB = BA \quad \text{on } D,
\]

\[
(Au, v) = (u, Bv), \quad u, v \in D.
\]

Then \( A \) is normal and \( A^* = B \).
Examples on a Riemannian manifold

- \( M \): a complete Riemannian manifold
- \( m \): the Riemannian volume

We take a function \( U \in C^\infty(M) \) and define a measure \( \nu \) by

\[
\nu = e^{-U} m
\]

Define an operator on \( H = L^2(\nu) \) by

\[
\mathfrak{A} = \frac{1}{2} \triangle_\nu + b
\]

where \( \triangle_\nu = -\nabla_\nu^* \nabla \). Then

\[
\mathfrak{A}^* = \frac{1}{2} \triangle_\nu - b - \text{div}_\nu b.
\]

Here \( \text{div}_\nu \) denotes the divergence with respect to \( \nu \).

We give a criterion for \( \mathfrak{A} = \triangle_\nu + b \) being a normal operator.
Theorem 6. Assume that $\text{div}_\nu b$ is bounded from below. Then $\mathcal{A}$ is normal if and only if $b$ is a Killing vector field and the following identities hold:

$$\left(\frac{1}{2} \triangle_\nu + b\right) \text{div}_\nu b = 0,$$

$$[\nabla U, b] + \nabla \text{div}_\nu b = 0.$$
4. One-dimensional Brownian motion with a drift

We consider an operator $\mathcal{A} = \frac{d^2}{dx^2} - c \frac{d}{dx}$ on $L^2(\mathbb{R}, \nu_1)$. Here $\nu_1$ is a measure defined by

$$\nu_1(dx) = e^{-cx} dx.$$  

Then $\mathcal{A}$ is a self-adjoint operator with

$$\langle \mathcal{A}f, g \rangle = -\int_{\mathbb{R}} f'(x)g'(x) \nu_1(dx).$$

To investigate the spectrum of $\mathcal{A}$, we use the following isometric map $I : L^2(\nu_1) \longrightarrow L^2(dx)$:

$$If(x) = e^{-cx/2}f(x).$$

We have

$$I \circ \mathcal{A} \circ I^{-1} = \frac{d^2f}{dx^2} - \frac{c^2}{4},$$
i.e., the following diagram is commutative:

\[
\begin{array}{ccc}
L^2(\nu_1) & \overset{\mathfrak{A}}{\longrightarrow} & L^2(\nu_1) \\
I \downarrow & & \downarrow I \\
L^2(dx) & \overset{\frac{d^2}{dx^2} - \frac{c^2}{4}}{\longrightarrow} & L^2(dx)
\end{array}
\]

Hence the spectrum \(-\mathfrak{A}\) is

(4) \[\sigma(-\mathfrak{A}) = \left[ \frac{c^2}{4}, \infty \right).\]

We now consider an perturbation of \(\mathfrak{A}\). Let \(b\) be an vector field defined by

\[b = k \frac{d}{dx}.
\]

We consider an operator of the form \(\mathfrak{A} + b\). We are interested in how the spectrum changes. \(b\) is clearly an Killing vector field. The divergence of \(b\) with respect to \(\nu_1\)

\[\text{div}_{\nu_1} b = -ck\]
and so it satisfies

\[(\mathfrak{A} + b) \text{div}_\nu b = 0,\]

\[\left[ (\nabla U)^\sharp, b \right] + \nabla \text{div}_\nu b = 0.\]

Here \(U(x) = cx\). By Theorem 6, \(\mathfrak{A} + b\) is a normal operator. Under the transformation of \(I\), we have

\[I \circ (\mathfrak{A} + b) \circ I^{-1} = \frac{d^2}{dx^2} + k \frac{d}{dx} - \frac{c(c - 2k)}{4}.\]

It is enough to get the spectrum of \(\frac{d^2}{dx^2} + k \frac{d}{dx}\). Recall the Fourier transform as

\[\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-i\xi x} \, dx.\]

This gives an isometry from \(L^2(dx)\) onto \(L^2(d\xi)\). Note that

\[\int_{\mathbb{R}} \left( \frac{d^2}{dx^2} + k \frac{d}{dx} \right) f(x)g(x) \, dx = \int_{\mathbb{R}} (-\xi^2 + ik\xi) \hat{f}(\xi)\hat{g}(\xi) \, d\xi\]
which means that

$$\sigma\left(\frac{d^2}{dx^2} + k \frac{d}{dx}\right) = \{-\xi^2 + ik\xi; \xi \in \mathbb{R}\}.$$ 

Theorem 7. We have

$$\sigma(-\mathcal{A}) = \left[\frac{c^2}{4}, \infty\right)$$

and

$$\sigma(-\mathcal{A} - b) = \left\{\frac{c(c - k)}{2} + \xi^2 + ik\xi; \xi \in \mathbb{R}\right\}.$$
\[-A - k \frac{d}{dx}\]
Now we take a different point of view.

We fix an operator $\mathbf{A} = \frac{d^2}{dx^2} - c \frac{d}{dx}$ but we change a reference measure. For $\theta \in [0, 1]$, define

$$\nu_\theta(dx) = (1 - \theta)dx + \theta e^{-cx} dx$$

$\nu_\theta$ is an invariant measure for $\mathbf{A}$. $\nu_0(dx) = dx$, $\nu_1(dx) = e^{-cx} dx$.

The computation above implies

$$\sigma(-\mathbf{A}) = \{\xi^2 - ic\xi; \xi \in \mathbb{R}\} \text{ in } L^2(\nu_0)$$

$$\sigma(-\mathbf{A}) = \left[\frac{c^2}{4}, \infty\right) \text{ in } L^2(\nu_1).$$
w.r.t. $\nu_0 = dx$

$\frac{c^2}{4}$

w.r.t. $\nu_1 = e^{-cx} dx$

What happens if we take the measure $\nu_\theta$?
Does the spectrum change continuously?

w.r.t. $\nu_\theta$
Theorem 8. For \( \theta \in (0, 1) \), \( \sigma(-\mathcal{A}) \) in \( L^2(\nu_\theta) \) is

\[
\{ \xi^2 - ik\xi; \xi \in \mathbb{R} \} \cup \left[ \frac{c^2}{4}, \infty \right).
\]
5. Perturbation by rotation

**Laplacian on \( \mathbb{R}^2 \)**

Let \( \mathcal{A} \) be

\[
\mathcal{A} = \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y^2} + k \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \quad \text{on} \quad L^2(\mathbb{R}^2, dx dy).
\]

The spectrum of \(-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\) is \([0, \infty)\).

For the spectrum of \(\mathcal{A}\), we recall the Bessel functions:

\[
J_\nu(x) = \left( \frac{x}{2} \right)^\nu \sum_{l=0}^{\infty} \frac{(ix/2)^{2l}}{l!\Gamma(\nu + l + 1)}, \quad \Re \nu > 0
\]

which satisfies the following differential equation

\[
I'' + \frac{1}{x} I' + \left( 1 - \frac{\nu^2}{x^2} \right) I = 0
\]
Since our space is $\mathbb{R}^2$, we only need the case that $\nu$ is a non-negative integer. We use the polar coordinate:

\[
\begin{aligned}
    x &= r \cos \theta, \\
    y &= r \sin \theta,
\end{aligned}
\]

$r \geq 0$, $\theta \in [0, 2\pi)$

Using this, $\mathcal{A}$ can be written as

\[
\mathcal{A} = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + k \frac{\partial}{\partial \theta}.
\]

If $F = f(r)e^{in\theta}$, then

\[
\mathcal{A}F = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - n^2 \frac{1}{r^2} \right)f(r)e^{in\theta} + ikn f(r)e^{in\theta}.
\]

Further

\[
\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - n^2 \frac{1}{r^2} \right) J_n(\lambda r) = -\lambda^2 J_n(\lambda r).
\]
The spectral decomposition is given by

\[ f(r, \theta) = \sum_{n \in \mathbb{Z}} \int \left\{ \int \left( \frac{1}{2\pi} \int f(\rho, \phi) e^{-\imath n \phi} \, d\phi \right) J_{|n|}(\lambda \rho) \rho \, d\rho \right\} e^{\imath n \theta} J_{|n|}(\lambda r) \lambda \, d\lambda. \]

**Theorem 9.** The spectrum of \(-\mathcal{A}\) is

\[ \{ \lambda^2 - i kn ; \lambda \geq 0, \ n \in \mathbb{Z} \} \]

and the corresponding eigenfunction to \(\lambda^2 - i kn\) is \(J_{|n|}(\lambda r) e^{\imath n \theta}\).
the spectrum of $-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$

the spectrum of $\mathcal{A}$
Ornstein Uhlenbeck operator on $\mathbb{R}^2$

Let $L_\alpha$ be

$$L_\alpha = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} + \alpha \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)$$

acting on $L^2(\mathbb{R}^2, \frac{1}{2\pi} e^{-(x^2+y^2)/2} dx dy)$.

The spectrum of Ornstein-Uhlenbeck operator $L_0$ is $\{0, -1, -2, \ldots \}$. In fact, define Hermite polynomials by

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}.$$

Then

$$L_0 H_k(x) H_{n-k}(y) = -n H_k(x) H_{n-k}(y).$$

To get the spectrum of $L_\alpha$, we need the complex Hermite polynomials defined by

$$H_{p,q}(z, \bar{z}) = (-1)^{p+q} e^{\frac{z\bar{z}}{2}} \left( \frac{\partial}{\partial \bar{z}} \right)^p \left( \frac{\partial}{\partial z} \right)^q e^{-\frac{z\bar{z}}{2}}.$$
Here, we regard $\mathbb{R}^2$ as $\mathbb{C}$ with $z = x + iy$. We denote
\[
\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).
\]
In the sequel, we write
\[
\partial = \frac{\partial}{\partial z}, \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}}
\]
for short. We have
\[
\partial^* = -\bar{\partial} + \frac{z}{2}, \quad \bar{\partial}^* = -\partial + \frac{\bar{z}}{2}
\]
Proposition 10. The following identities hold:

\[
\partial H_{p,q} = \frac{p}{2} H_{p-1,q}, \quad \bar{\partial} H_{p,q} = \frac{q}{2} H_{p,q-1},
\]

\[
\partial^* H_{p,q} = H_{p+1,q}, \quad \bar{\partial}^* H_{p,q} = H_{p,q+1},
\]

\[
(2\partial \bar{\partial} - z\partial) H_{p,q} = -p H_{p,q}
\]

\[
(2\partial \bar{\partial} - \bar{z}\bar{\partial}) H_{p,q} = -q H_{p,q}
\]

\[
(z\partial - \bar{z} \bar{\partial}) H_{p,q} = (p - q) H_{p,q}
\]

We can write

\[
L_\alpha = (2\partial \bar{\partial} - z\partial) + (2\partial \bar{\partial} - \bar{z}\bar{\partial}) + \alpha i(z\partial - \bar{z} \bar{\partial}).
\]

Hence

\[
L_\alpha H_{p,q} = (-p - q + (p - q)\alpha i) H_{p,q}.
\]
Theorem 11. The spectrum of $-L_\alpha$ is

\[(10) \quad \{(p + q) - (p - q)\alpha i\}_{p,q=0}^\infty\]

and corresponding eigenfunctions are $H_{p,q}$ respectively.
the spectrum of $-L_0$

the spectrum of $-L_\alpha$
Connection to the Laguerre polynomials

The eigenfunction $H_{n,n}$ for the eigenvalue $2n$, $(n \in \mathbb{Z}_+)$ is rotation invariant since

$$
\left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) H_{n,n} = 0. \quad \text{So } H_{n,n} \text{ is a function of } r = |z| \text{ and }
$$

$$
\left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - r \frac{d}{dr} \right) H_{n,n} = -2nH_{n,n}.
$$

Now, by the change of variable $r = \sqrt{2u}$, we have

$$
\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - r \frac{d}{dr} = 2u \frac{d^2}{du^2} + 2(1 - u) \frac{d}{du}.
$$

$F(u) = H_{n,n}(r)$ satisfies

$$
2u \frac{d^2}{du^2} F + 2(1 - u) \frac{d}{du} F + nF = 0.
$$

The Laguerre polynomial satisfies this differential equation. Here the Laguerre polynomial is defined by

$$
L_n = \frac{e^x}{n!} \frac{d^n}{dx^n}(e^{-x}x^n)
$$

(11)
Now we have

**Theorem 12.** Complex Hermite polynomials $H_{n,n}$ are expressed as following;

\[
H_{n,n}(z, \bar{z}) = (-1)^n n! \frac{1}{2^n} L_n \left( \frac{|z|^2}{2} \right)
\]

where $c$ is a constant.
Thanks a lot!