Spectra of Non-symmetric Operators

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Spectra of a Non-normal Operator

We discuss

$$A := -\frac{d^2}{dx^2} + c\frac{d}{dx},$$

acting on $L^2(\mathbf{R}, \nu_\theta(dx))$.

Here,

$$\nu_\theta(dx) := \{(1-\theta) + \theta e^{-cx}\}dx \qquad 0 \leq \theta \leq 1 \tag{1}$$

is the invariant measure of A.

- $\theta = 1$, A is self-adjoint.
- $\theta = 0$, A is normal.

$$A^* = -\frac{d^2}{dx^2} - c\frac{d}{dx}$$

- $0 < \theta < 1$, A is not even normal.

$$A^* = -\frac{d^2}{dx^2} - c\frac{d}{dx} - \frac{2\varphi'_\theta(x)}{\varphi_\theta(x)}\frac{d}{dx},$$

where $\varphi_\theta(x) := \{(1-\theta) + \theta e^{-cx}\}$ is the density of ν_θ

Normal case; with spectral representation

Normal (including self-adjoint) operators have spectral representation.

Theorem (Spectral representation)

Let A be a normal operator on a Hilbert space $\mathcal{H}$. Then there exists a partition of unity $\{E(z)\}$ such that

$$(Af, g) = \int_{\mathbf{C}} z d(E(z)f, g)_{\mathcal{H}}, \quad f \in \mathrm{Dom}(A), g \in \mathcal{H}. \tag{2}$$

Here the support of $d(E_z f, g)_{\mathcal{H}}$ is in the spectrum of A . Conversely we can construct a normal operator from a partition of unity $\{E(z)\}$.

Partitiion of Unity

Partition Unity of $E(z)$ satisfies

1. $E(x+iy)E(u+iv) = E((x\wedge u)+i(y\wedge v))$
2. $x_n \nearrow \infty, y_N \nearrow \infty, \Rightarrow E(x_n+iy_n) \to \mathrm{Id}_{\mathcal{H}}$
3. $x_n \searrow -\infty, y_n \searrow -\infty \Rightarrow E(x_n+iy_n) \to O$
4. $x_n \searrow x, y_n \searrow y$, then $E(x_n+iy_n) \to E(x+iy)$

By the Fourier transform, for $\theta = 1$

$$
\begin{aligned}
(Af, g)_{\nu_1} &= \int \mathcal{F}[f' e^{-\frac{cx}{2}}])\mathcal{F}[g' e^{-\frac{cx}{2}}]d\xi \\
&= \int \left(\xi^2 + \frac{c^2}{4}\right)\mathcal{F}[f]\left(\xi - \frac{ic}{2}\right)\overline{\mathcal{F}[g]\left(\xi - \frac{ic}{2}\right)}d\xi
\end{aligned}
$$

So we can construct a partition of unity for A acting on $L^2(\mathbf{R}, \nu_1)$.
Here $\mathcal{F} : L^2(\mathbf{R}, dx) \to L^2(\mathbf{R}, dx)$ is the Fourier transform.
Note that

$$\int |f|^2 e^{-cx} dx < \infty \Rightarrow \int |f e^{-\frac{cx}{2}}|^2 dx < \infty.$$

For $\theta = 0$

$$
\begin{aligned}
(Af, g)_{\nu_0} &= \int \{ f'g' + c f'g \} \, dx \\
&= \int \left\{ \mathcal{F}[f'] \overline{\mathcal{F}[g']} + c\mathcal{F}[f'] \overline{\mathcal{F}[g]} \right\} d\xi \\
&= \int \left(\xi^2 - c\xi i \right) \mathcal{F}[f] \overline{\mathcal{F}[g]} d\xi
\end{aligned}
$$

By this calculation, we can construct a partition of unity for A acting on $L^2(\mathbf{R}, \nu_0)$.

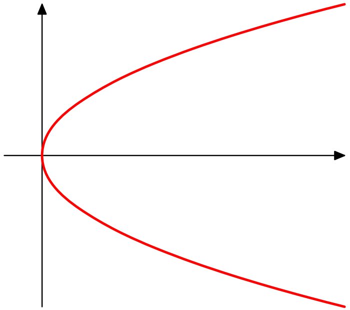
Theorem

Let σ_0, σ_1 *be spectra of* A *for* $\theta = 0, 1$, *respectively. Then*

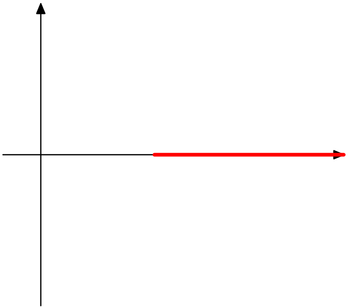
$$\sigma_0 = \left\{ z = x + iy \in \mathbf{C} \; ; \; c^2 x = y^2 \right\}, \tag{3}$$

and

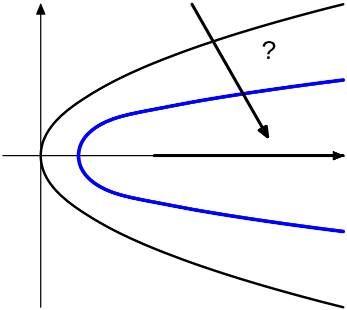
$$\sigma_1 = \left\{ \frac{c^2}{4} + t \; ; \; 0 \le t \right\}. \tag{4}$$



The figure of σ_0



The figure of σ_1



Does spectrum move along avobe arrow ?

Non-normal Case

Now we consider the case $0 < \theta < 1$.

Note that $\nu_\theta(dx) = (1-\theta)\nu_0(dx) + \theta\nu_1(dx)$, and

$$L^2(\mathbf{R}, \nu_\theta) = L^2(\mathbf{R}, \nu_0) \cap L^2(\mathbf{R}, \nu_1) \tag{5}$$

If linear operator $T : L^2(\nu_j) \to L^2(\nu_j)$ $(j = 0, 1)$ satisfies

$$||Tf||_{\nu_0} \leq C_0||f||_{\nu_0} \quad \forall f \in L^2(\mathbf{R}, \nu_0), \quad ||Tg||_{\nu_1} \leq C_1||g||_{\nu_1} \quad \forall g \in L^2(\mathbf{R}, \nu_1)$$

Then,

Non-normal Case

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Then,

$$\|Tf\|_{\nu_\theta} \le \sqrt{C_0^2 \vee C_1^2}\|f\|_{\nu_\theta} \qquad \forall f \in L^2(\mathbf{R}, \nu_\theta).$$

For $\alpha \in \mathbf{C} \setminus (\sigma_0 \cup \sigma_1)$, $(\alpha - A)^{-1}$ is bounded on $L^2(\mathbf{R}, \nu_0)$ and $L^2(\mathbf{R}, \nu_1)$

$\Longrightarrow \mathbf{C} \setminus (\sigma_0 \cup \sigma_1)$ is in the resolvent set of A on $L^2(\mathbf{R}, \nu_r)$

We check whether $\lambda \in \sigma_0 \cup \sigma_1$ is the spectrum of A.

Sufficient Condition to be a Spectrum

We use the following sufficient condition to be in the spectrum;

Theorem

Let L be a closed linear operator on a Banach $(X, \|\cdot\|_X)$ *and* $\lambda \in \mathbb{C}$*. If there exists a sequence* $\{f_n\} \subset \mathrm{Dom}(L)$ *such that* $\|f_n\|_X = 1$*and*

$$\lim_{n\to\infty} \|Lf_n - \lambda f_n\|_X = 0. \tag{6}$$

Then $\lambda \in \mathrm{Spec}(L)$*.*

Here, we call this $\{f_n\}$ a sequence of approximate-eigenfunctions for λ or approximate-eigenfunctions for λ.

For $\sigma_0 \ni \lambda_t^0 := \dfrac{t^2}{c^2} + it$, let $\alpha_t := -\frac{it}{2}$, and $e_t^0(x) := e^{\alpha_t x}$.
Then $Ae_t^0 = \lambda_t^0 e_t^0$.
Let $s \in C^\infty(\mathbf{R})$ satisfy

1. $|x| > 2 \implies s(x) = 0$
2. $1 < |x| < 2 \implies 0 < s(x) < 1$
3. $|x| \leq 1 \implies s(x) = 1$

and, $s_n(x) := \frac{1}{\sqrt{n}} s(\frac{x}{n})$. Then $e_t^0 s_n$ divided by its $\|\cdot\|_{\nu_0}$ satisfy the codition of approximate eigenfunctions for λ_t^0 in $L^2(\mathbf{R}, \nu_0)$.

Recipe of approximate eigenfunctions with ν_θ

Here is the recipe of approximate eigenfunctions for σ_0.
You can construct for σ_1 with almost same recipe.

I. Let f_n be the approximate functions for A on $L^2(\mathbf{R}, \nu_0)$

II. $\tilde{f}_n := \tau_n f_n := f_n(\cdot - 2n^2)$, then

$$\|\tilde{f}_n\|_{\nu_0} = \|f_n\|_{\nu_0}, \qquad \|\tilde{f}_n\|_{\nu_1} = e^{-cn^2}\|f_n\|_{\nu_1}$$

III. τ_n is commutative with differential. Therefore,

$$\|A\tilde{f}_n - \lambda_t^0 \tilde{f}_n\|_{\nu_0} = \|Af_n - \lambda_t^0 f_n\|_{\nu_0} \to 0 \quad n \to \infty$$

$$\|A\tilde{f}_n - \lambda_t^0 \tilde{f}_n\|_{\nu_1} = e^{-cn^2}\|Af_n - \lambda_t^0 f_n\|_{\nu_1}$$

Recipe of approximate eigenfunctions with ν_θ

IV. since $\operatorname{supp} f_n \subset [-2n, 2n]$.

$$\|Af_n - \lambda_1 f_n\|_{\nu_1} \leq e^{2cn}\|Af_n - \lambda_1 f_n\|_{\nu_0},$$

Therefore

$$\|A\tilde{f}_n - \lambda_0 \tilde{f}_n\|_{\nu_1} \to 0$$

V. $\|\tilde{f}_n\|_{\nu_\theta} \geq (1-\theta)\|f_n\|_{\nu_0} = 1-\theta$

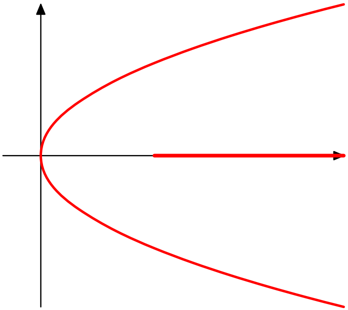
VI. Therefore

$$\frac{\tilde{f}_n}{\|\tilde{f}_n\|_{\nu_\theta}}$$

is a sequence of approximate-eigenfunctions for λ_t^0 in $L^2(\mathbf{R}, \nu_\theta)$.

Theorem

For $0 < \theta < 1$, *the spectrum of A is* $\sigma_0 \cup \sigma_1$.



The spectrum of A on $L^2(\mathbf{R}, \nu_\theta)$.

Laplacian with Rotation

Let L be

$$-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) \quad \text{on } L^2(\mathbf{R}, dxdy)$$

Laplacian with Rotation

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In polar coordinate,

$$-\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial}{\partial \theta} \quad \text{on } L^2([0,\infty)\times[0,2\pi), rdrd\theta)$$

For normal A,

$$A = \frac{A + A^*}{2} + i\frac{A - A^*}{2i},$$

and $\frac{A+A^*}{2}$, $\frac{A-A^*}{2i}$ are self-sdjoint.
So, the spectrum of A is contained

$$\{x + iy; x \in \mathrm{Spec}(\frac{A + A^*}{2}), y \in \mathrm{Spec}(\frac{A - A^*}{2i})\}$$

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$$\{x + iy; x \in \mathrm{Spec}(\frac{A + A^*}{2}), y \in \mathrm{Spec}(\frac{A - A^*}{2i})\}$$

Therefore the spectrum of L is contained in

$$\{\lambda + in; \lambda \geq 0, n \in \mathbf{Z}\}$$

Assume a smooth function $f = R(r)\Phi(\theta)$ satisfies for some $\lambda \geq 0, n \in \mathbf{Z}$,

$$\left(-\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial}{\partial\theta}\right)R\Phi = (\lambda + in)R\Phi \tag{7}$$

Dividing by $R\Phi$ and differetiating with θ

$$-\frac{1}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial}{\partial\theta}\frac{1}{\Phi}\frac{\partial\Phi}{\partial\theta} = 0$$

Then

$$\Phi'' = k_2\Phi, \qquad \Phi' = k_1\Phi.$$

Since Φ is periodic $\Phi = ce^{im\theta}$ for some $m \in \mathbf{Z}$.

Since the imaginaty part of the spectrum comes from $\frac{\partial}{\partial\theta}$, we take $m = n$.

Then (7) is

$$-\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial R}{\partial r} + \frac{n^2}{r^2}R = \lambda R. \tag{8}$$

Let $s = \sqrt{\lambda}r$. Then (8) becomes

$$\frac{d^2R}{ds^2} - \frac{1}{s}\frac{dR}{ds} + \left(1 - \frac{n^2}{s^2}\right)R = 0 \tag{9}$$

This is the difeerential equation which **Bessel function** satisfies.

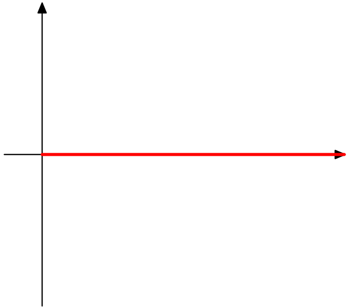
Theorem

The spectrum of L is

$$\{\lambda + in\ ;\ \lambda \geq 0,\ n \in \mathbf{Z}\} \tag{10}$$

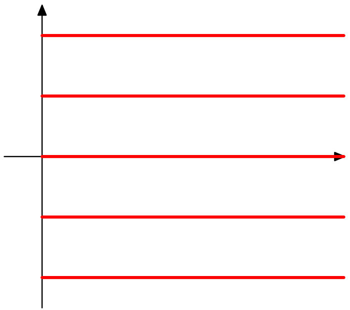
and the corresponding eigenfunction to $\lambda + in$ is $J_{|n|}(\sqrt{\lambda} r)e^{in\theta}$, where J_m is the Bessel functions of first kind of order m. Here (r, θ) is the usual polar coordinate.

The spectrum of $-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ is;



$\{e^{i\xi\cdot\mathbf{x}} | |\xi|^2 = \lambda\}$ corresnponds to $\lambda \geq 0$

The spectrum of L is;



$J_{|n|}(\sqrt{\lambda}r)e^{in\theta}$ corresnponds to $\lambda + in \quad (\lambda \geq 0 n \in \mathbf{Z}$

For Laplacian, we know Fourier inversion formula;

$$f(\mathbf{x}) = \int \left(\int f(\mathbf{u}) e^{-i\mathbf{u}\cdot\xi} d\mathbf{u} \right) e^{i\xi\cdot\mathbf{x}} d\xi \tag{11}$$

By "Fourier Bessel integral" ,

$$g(r) = \int \rho J_l(r\rho) \left(\int \lambda g(\lambda) J_l(\rho\lambda) d\lambda \right) d\rho.$$

For Laplacian, we know Fourier inversion formula;

$$f(\mathbf{x}) = \int \left(\int f(\mathbf{u}) e^{-i\mathbf{u}\cdot\xi} d\mathbf{u} \right) e^{i\xi\cdot\mathbf{x}} d\xi \tag{12}$$

By "Fourier Bessel integral" , we have the following formula for L;

$$f(r,\theta) = \sum_n \int \left\{ \int \left(\int f(\lambda,\phi) e^{-in\phi} d\phi \right) J_{|n|}(\rho\lambda)\lambda d\lambda \right\} J_{|n|}(r\rho) e^{in\theta} \rho d\rho$$

under the polar coordinate.

Ornstein-Uhlenbeck Operator with Rotation

Let L_α be

$$-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \alpha\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

acting on $L^2(\mathbf{R}, e^{-\frac{x^2+y^2}{2}}dxdy)$.

Ornstein-Uhlenbeck Operator with Rotation

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$$-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \alpha\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

acting on $L^2(\mathbf{R}, e^{-\frac{x^2+y^2}{2}}dxdy)$.
Ornstein-Uhlenbeck operator L_0 is understood by Hermite polynomials,

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}.$$

These polynomials satisfy,

$$L_0 H_k(x) H_{n-k}(y) = n H_k(x) H_{n-k}(y).$$

$\{\sum_{k=0}^n H_k(x) H_{n-k}(y) | c_n \in \mathbf{C}\}$ is the eigenspace corresponding to n.

Ornstein-Uhlenbeck Operator with Rotation

$$L_\alpha = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \alpha\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

acting on $L^2(\mathbf{R}, e^{-\frac{x^2+y^2}{2}} dxdy)$.

$\{\sum_{k=0}^{n} H_k(x)H_{n-k}(y)|c_n \in \mathbf{C}\}$ is also invariant by L_α, but corresponding matrix is too complicated to determine its spectrum.

We use ”complex Hermite polynomials”,

$$H_{p,q}(z,\overline{z}) := (-1)^{p+q} e^{\frac{z\overline{z}}{2}} \left(\frac{\partial}{\partial \overline{z}}\right)^p \left(\frac{\partial}{\partial z}\right)^q e^{-\frac{z\overline{z}}{2}},$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

and $z = x + iy$.

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and $z = x + iy$.

They satisfy

$$\left(-2\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} + z\frac{\partial}{\partial z}\right) H_{p,q} = pH_{p,q} \quad \left(-2\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} + \overline{z}\frac{\partial}{\partial \overline{z}}\right) H_{p,q} = qH_{p,q}$$

and

$$\left(z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial \overline{z}}\right) H_{p,q} = (p-q)H_{p,q}.$$

Under complex coordinate

$$L_\alpha = \left(-4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} + z\frac{\partial}{\partial z} + \overline{z}\frac{\partial}{\partial \overline{z}}\right) + \alpha i\left(z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial \overline{z}}\right).$$

Under complex coordinate

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Then,

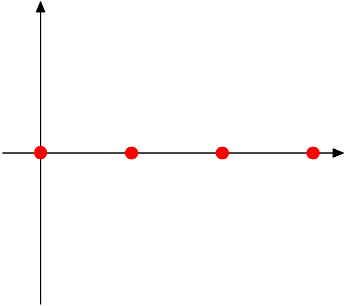
$$L_\alpha H_{p,q} = (p+q)H_{p,q} + (p-q)\alpha i H_{p,q}$$

Theorem

The spectrum of L_α *is*

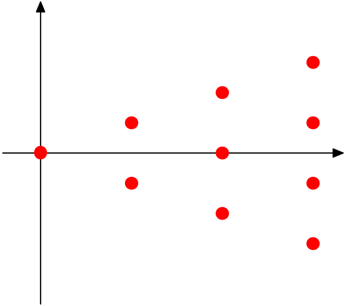
$$\{(p+q)+(p-q)\alpha i\}_{p,q=0}^{\infty} \tag{13}$$

and corresponding eigenfunctions are $H_{p,q}$ *respectively.*



Spectrum of Ornstein-Uhlenbeck operator

$\{H_{p,q}\}_{p+q=n}$ corresponds to each point n.



Spectrum of L_α

$H_{p,q}$ is the corresponding the eigenfunction of $(p+q)+(p-q)\alpha i$.

Since $\{H_{p,q}\}_{p,q}$ is dense in $L^2(\mathbf{R}^2, e^{-\frac{x^2+y^2}{2}}dxdy)$.

$$L^2(\mathbf{R}^2, e^{-\frac{x^2+y^2}{2}}dxdy) = \bigoplus_{n=0}^{\infty} \bigoplus_{p+q=n} \mathbf{C}H_{p,q}$$

This corresponds to the action of $U(1) = \{z \in \mathbf{C}; |z| = 1\}$.

Note;

$$H_{p,q}(\omega z, \overline{\omega z}) = \omega^{p-q} H_{p,q}(z, \overline{z}) \qquad \omega \in U(1)$$

Since $\{H_{p,q}\}_{p,q}$ is dense in $L^2(\mathbf{R}^2, e^{-\frac{x^2+y^2}{2}}dxdy)$.

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This corresponds to the action of $U(1) = \{z \in \mathbf{C}; |z| = 1\}$.

$H_{n,n}$ is invariant under rotation.

$$H_{n,n}(\omega z, \overline{\omega z}) = H_{p,q}(z, \overline{z}) \qquad \omega \in U(1)$$

Then,

$$\frac{\partial}{\partial\theta} H_{n,n} = 0,$$

Under the polar coordinate,

$$L_\alpha = -\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + r\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \alpha\frac{\partial}{\partial \theta} \tag{14}$$

So,

$$\left\{-\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} + r\frac{d}{dr}\right\}H_{n,n} = 2nH_{n,n},$$

Let $u := \frac{r^2}{2}$, then

$$u\frac{d^2H_{n,n}}{dr^2} + (1-u)\frac{dH_{n,n}}{dx} + nH_{n,n} = 0$$

Remind that **Laguerre** polynomial $L_m(u) := \frac{e^u}{m!}\frac{d^m}{dx^m}(e^{-u}u^m)$ satisfies

$$u\frac{d^2L_m}{du^2} + (1-u)\frac{dL_m}{du} + mL_m = 0$$

So

Theorem

Complex Hermite polynomials $H_{n,n}$ are expressed as following;

$$H_{n,n}(z,\bar{z}) = cL_n\left(\frac{|z|^2}{2}\right), \tag{15}$$

where $L_n = \frac{e^x}{n!}\frac{d^n}{dx^n}(e^{-x}x^n)$ are Laguerre polynomials and c is a constant.

$H_{n,n}$ is expressed by usual Hermite polynomials by

$$H_{n,n} = \frac{1}{4^n} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} H_{2k}(x) H_{2n-2k}(y).$$

Corollary

Laguerre polynomials L_n are expressed as following;

$$L_n\left(\frac{x^2+y^2}{2}\right) = \frac{c}{4^n} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} H_{2k}(x) H_{2n-2k}(y). \qquad (16)$$

where $H_n = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}})$ are Laguerre polynomials and c is a constant.

Thank You