Spectra of Non-symmetric Operators*

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1 Introduction

In this talk, we discuss spectra of non-symmetric operators. We computed the spectra of

1. generator of Brownian motion with drift,
2. Laplacian with rotation, and
3. Ornstein Uhlenbeck operator with rotation.

2 Spectrum of a non-normal operator

Let $A := -\frac{d^2}{dx^2} + c\frac{d}{dx}$ acting on $L^2(\mathbf{R}, \nu_\theta)$, where

$$\nu_\theta(dx) = \{(1-\theta) + \theta e^{-cx}\}dx, \tag{1}$$

and $c \geq 0$. These $\{\nu_\theta\}_{0\leq\theta\leq 1}$ are invariant measures of A.

A is a self-adjoint operator for $\theta = 1$, and normal one for $\theta = 0$. In these two cases, we can compute spectra of A as follows by the Fourier transform.

Theorem 1 *Let σ_0, σ_1 be spectra of A for $\theta = 0, 1$, respectively. Then*

$$\sigma_0 = \left\{ z = x + iy \in \mathbf{C} \ ; \ c^2 x = y^2 \right\}, \tag{2}$$

and

$$\sigma_1 = \left\{ \frac{c^2}{4} + t \ ; \ 0 \leq t \right\}. \tag{3}$$

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Since A is not a normal operator for $0 < \theta < 1$, we need another way to compute its spectrum. The relation

$$L^2(\mathbf{R}, \nu_\theta) = L^2(\mathbf{R}, \nu_0) \cap L^2(\mathbf{R}, \nu_1) \tag{4}$$

gives an idea to compute the spectrum. It is computed as follows.

Theorem 2 *For $0 < \theta < 1$, the spectrum of A is $\sigma_0 \cup \sigma_1$.*

3 Perturbation by rotation

3.1 Laplacian on $\mathbf{R}^2$

Let L be

$$-\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial y^2} + \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) \quad \text{on } L^2(\mathbf{R}^2, dxdy). \tag{5}$$

The spectrum of $-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ is $\mathbf{R}_{\geq 0}$. There are many eigenfunctions corresponding to a spectrum.

Using polar coordinate, we get;

Theorem 3 *The spectrum of L is*

$$\{\lambda + in \ ; \ \lambda \geq 0, \ n \in \mathbf{Z}\} \tag{6}$$

and the corresponding eigenfunction to $\lambda + in$ is $J_{|n|}(\sqrt{\lambda} r)e^{in\theta}$, where J_m is the Bessel functions of first kind of order m. Here (r, θ) is the usual polar coordinate.

3.2 Ornstein Uhlenbeck operator on $\mathbf{R}^2$

Let L_α be

$$-\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial y^2} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \alpha \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) \tag{7}$$

acting on $L^2(\mathbf{R}^2, e^{-\frac{x^2+y^2}{2}} dxdy)$.

The spectrum of Ornstein-Uhlenbeck operator L_0 is $\{0, 1, 2, \ldots\}$ and corresponding eigenfunctions can be represented by Hermite polynomials.

For $\alpha \neq 0$, the spectrum is clearly determined by complex Hermite polynomials

$$H_{p,q}(z, \bar{z}) := (-1)^{p+q} e^{\frac{z\bar{z}}{2}} \left(\frac{\partial}{\partial \bar{z}} \right)^p \left(\frac{\partial}{\partial z} \right)^q e^{-\frac{z\bar{z}}{2}}. \tag{8}$$

Here, we regard $\mathbf{R}^2$ as $\mathbf{C}$. Then we have;

Theorem 4 *The spectrum of L_α is*

$$\{(p+q)+(p-q)\alpha i\}_{p,q=0}^{\infty} \tag{9}$$

and corresponding eigenfunctions are $H_{p,q}$ respectively.

Let $V_n := \{L_0 f = nf\}$. Then by formulae of complex Hermite polynomials,

$$V_n = \bigoplus_{p+q=n} \mathbf{C} H_{p,q}.$$

This decomposition corresponds to a rotation group. $H_{n,n}$ is rotation invariant. Under the polar coordinate, $H_{n,n}$ satisfy a differential equation.

Theorem 5 *Complex Hermite polynomials $H_{n,n}$ are expressed as following;*

$$H_{n,n}(z,\bar{z}) = cL_n\left(\frac{|z|^2}{2}\right), \tag{10}$$

where $L_n = \frac{e^x}{n!}\frac{d^n}{dx^n}(e^{-x}x^n)$ are Laguerre polynomials and c is a constant.