Non-symmetric diffusions on Riemannian manifolds and the ultracontractivity

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1. Introduction

Let \((X_t)\) be a diffusion on a compact Riemannian manifold \(M\) generated by \(\frac{1}{2} \Delta + b\). Its has a transition probability density \(p(t, x, y)\). We can see that \(p(t, x, y)\) converges to an invariant measure \(\nu(dx) = \rho(x) \text{vol}(dx)\).

We are interested in the convergence rate \(\gamma\):

\[
\gamma = - \lim_{t \to \infty} \frac{1}{t} \log \sup_{x, y} |p(t, x, y) - \rho(x)|.
\]

We give a lower bound of \(\gamma\).

Our main tool is the ultracontractivity of the semigroup.
Ultracontractivity

A semigroup \( \{T_t\} \) is called ultracontractive if \( T_t : L^1 \to L^\infty \) is bounded for all \( t > 0 \).

It is well-known that the following three conditions are equivalent for a symmetric Markovian semigroup. Let \( \mu > 0 \) be given.

(i) \( \exists c_1 > 0, \forall f \in L^1 : \)

\[
\| T_t f \|_\infty \leq c_1 t^{-\mu/2} \| f \|_1, \quad \forall t > 0.
\]

(ii) \( \exists c_2 > 0, \forall f \in \text{Dom}(\mathcal{E}) \cap L^\infty : \)

\[
\| f \|_2^{2+4/\mu} \leq c_2 \mathcal{E}(f, f) \| f \|_1^{4/\mu}.
\]

(iii) \( \mu > 2, \exists c_3 > 0, \forall f \in \text{Dom}(\mathcal{E}) : \)

\[
\| f \|_2^{2\mu/(\mu-2)} \leq c_3 \mathcal{E}(f, f).
\]

We extend this result for non-symmetric Markovian semigroups.
2. Non-symmetric Markovian semigroups

We give a framework in general Hilbert space scheme.

- $H$: a Hilbert space
- $\{T_t\}$: a contraction $C_0$ semigroup
- $\{T_t^*\}$: the dual semigroup
- $\mathfrak{A}, \mathfrak{A}^*$: the generators of $\{T_t\}$ and $\{T_t^*\}$

A natural bilinear form $\mathcal{E}$ is defined by

$$\mathcal{E}(u, v) = -(\mathfrak{A}u, v).$$

We do not assume the sector condition and so we can not use this bilinear form.
We introduce a symmetric bilinear form. For this, we assume the following condition:

(A.1) $\text{Dom}(A) \cap \text{Dom}(A^*)$ is dense in $\text{Dom}(A)$ and $\text{Dom}(A^*)$.

Under this condition, we define a symmetric bilinear form $\tilde{E}$ by

$$\tilde{E}(u, v) = -\frac{1}{2} \{(Au, v) + (u, Av)\}, \quad u, v \in \text{Dom}(A) \cap \text{Dom}(A^*).$$

**Proposition 1.** Under the condition (A.1), $\tilde{E}$ is closable and its closure contains $\text{Dom}(A)$ and $\text{Dom}(A^*)$. 
Covex set preserving property

- $C$: a convex set of $H$.
- $Pu$: the shortest point from $u$ to $C$

$$(u - Pu, v - Pu) \leq 0, \quad \forall v \in C.$$

**Theorem 2.** If $\{T_t\}$ and $\{T_t^*\}$ preserve a convex set $C$, then $Pu \in \text{Dom}(\tilde{E})$ for any $u \in \text{Dom}(\tilde{E})$ and we have

$$\tilde{E}(Pu, u - Pu) \geq 0.$$
Markovian semigroup

- \((M, m)\): a measure space
- \(H = L^2(m)\): a Hilbert space
- \(\{T_t\}\): a Markovian semigroup

We assume that \(\{T_t^*\}\) is also a Markovian semigroup.

Under the assumption (A.1), we can define a symmetric bilinear form \(\tilde{E}\) and \(\tilde{E}\) is a Dirichlet form.
We have the following implications.

\[ \| T_t f \|_\infty \leq c_1 t^{-\mu/2} \| f \|_1, \quad \forall t > 0 \]

\[ \uparrow \quad \downarrow \quad \text{under (1)} \]

\[ \| f \|_2^{2+4/\mu} \leq c_2 \tilde{E}(f, f) \| f \|_1^{4/\mu} \]

\[ \uparrow \]

\[ \| f \|_{2^\mu/(\mu-2)}^2 \leq c_3 \tilde{E}(f, f) \quad (\mu > 2) \]

(1)

\[ (\mathbf{A}^2 f, f)_2 + (\mathbf{A} f, \mathbf{A} f)_2 \geq 0. \]

(1) holds if \( \mathbf{A} \) is normal, i.e. \( \mathbf{A} \mathbf{A}^* = \mathbf{A}^* \mathbf{A} \).
Moreover

\[ \|T_t f\|_\infty \leq c_1 t^{-\mu/2} \|f\|_1, \quad \forall t \in (0, 1] \]

\[ \uparrow \quad \downarrow \quad \text{under (2)} \]

\[ \|f\|_2^{2+4/\mu} \leq c_2 (\tilde{E}(f,f) + \|f\|_2^2) \|f\|_1^{4/\mu} \]

\[ \uparrow \]

\[ \|f\|_2^{2/(\mu/(\mu-2))} \leq c_3 (\tilde{E}(f,f) + \|f\|_2^2) \quad (\mu > 2) \]

There there exists a constant \( M > 0 \) so that for all \( f \in \text{Dom}(A^2) \)

\[ (A - M)^2 f, f)_{2} + ((A - M)f, (A - M)f)_{2} \geq 0. \]
3. Dirichlet forms satisfying the sector condition

From now on, we assume the sector condition for the Dirichlet form $\mathcal{E}$.

In this case, we have

\[
\|T_t f\|_\infty \leq c_1 t^{-\mu/2} \|f\|_1, \quad \forall t \in (0, 1]
\]

\[
\uparrow
\]

\[
\|f\|_2^{2+4/\mu} \leq c_2 (\tilde{\mathcal{E}}(f, f) + \|f\|_2^2) \|f\|_1^{4/\mu}
\]

\[
\uparrow
\]

\[
\|f\|_2^{2/(\mu-2)} \leq c_3 (\tilde{\mathcal{E}}(f, f) + \|f\|_2^2) \quad (\mu > 2)
\]

Key estimate:

\[
\tilde{\mathcal{E}}(T_s f, T_s f) \leq C \{\tilde{\mathcal{E}}(f, f) + \|f\|_2^2\}
\]
Theorem 3. \( \mu > 2 \). Suppose that there exists a constant \( c_1 \) so that for any \( f \in L^1 \)

\[
\| T_t f \|_\infty \leq c_1 t^{-\mu/2} \| f \|_1, \quad \forall t \in (0, 1].
\]

Then, for any \( \tilde{\mu} > \mu \), there exists a constant \( c_3 > 0 \) so that for all \( f \in \text{Dom}(\tilde{\mathcal{E}}) \)

\[
\| f \|_{2^{\tilde{\mu}/(\tilde{\mu}-2)}}^2 \leq c_3 (\tilde{\mathcal{E}}(f,f) + \| f \|_2^2)
\]

Key estimate: for \( s < \frac{1}{2} \),

\[
\| (1 - \mathcal{A})^s f \|_2^2 \leq C(\tilde{\mathcal{E}}(f,f) + \| f \|_2^2).
\]
4. Dirichlet forms having invariant measure

We continue to assume the sector condition. In addition, we assume

- \( m \) is an invariant probability measure.
  \[
  \int_M T_t f \, dm = \int_M f \, dm
  \]
- \( T_t 1 = 1 \) and \( A1 = 0 \).

The following inequality is called the Poincaré inequality

\[
\| f - m(f) \|_2^2 \leq \lambda^{-1} \tilde{E}(f, f) \tag{3}
\]

where

\[
m(f) = \int_M f(x) \, m(dx).
\]

This inequality is equivalent to

\[
\| T_t f - m(f) \|_2^2 \leq e^{-2\lambda t} \| f - m(f) \|_2^2.
\]
Theorem 4. $\mu > 0$. We consider the following two conditions.

(i) There exists a constant $c_1$ so that for all $f \in L^1$

$$\|T_t f - m(f)\|_{\infty} \leq c_1 t^{-\mu/2} \|f\|_1, \quad \forall t \in (0, 1].$$

(ii) There exists a constant $c_2$ so that for all $f \in \text{Dom}(\tilde{\mathcal{E}}) \cap L^1(m)$

$$\|f - m(f)\|_{2+4/\mu}^2 \leq c_2 \tilde{\mathcal{E}}(f, f) \|f\|_1^{4/\mu}.$$ 

Then, (ii) is equivalent to (i) with the Poincaré inequality.

Under the condition (ii), there exists a constant $c_4 > 0$ so that for all $f \in L^1$

$$\|T_t f - m(f)\|_{\infty} \leq c_4 e^{-\lambda t} \|f\|_1, \quad \forall t \geq 1.$$ 

Here $\lambda$ is a constant appears in the Poincaré inequality (3).
Proof.

\[ \|T_t - m\|_{1 \to \infty} = \|(T_1 - m)(T_{t-2} - m)(T_1 - m)\|_{1 \to \infty} \]
\[ \leq \|T_1 - m\|_{2 \to \infty} \|T_{t-2} - m\|_{2 \to 2} \|T_1 - m\|_{1 \to 2} \]
\[ \leq \|T_1 - m\|_{2 \to \infty} e^{-\lambda(t-2)} \|T_1 - m\|_{1 \to 2} \]

Let us investigate the convergence rete. Set \( a_t = \|T_t - m\|_{1 \to \infty} \) and define \( \gamma \) by

\( \gamma = - \lim_{t \to \infty} \frac{1}{t} \log a_t. \) (4)

**Theorem 5.** We have

\[ \gamma \geq \lambda \]

and the equality holds if \( \mathfrak{A} \) is normal. Here \( \lambda \) is the spectral gap (3).
Theorem 6. \( \mu > 2 \). Assume that there exists a constant \( c_1 \) so that

\[
\|T_t f - m(f)\|_\infty \leq c_1 t^{-\mu/2} \|f\|_1, \quad \forall t > 0
\]

and the Poincaré inequality holds. Then, for any \( \tilde{\mu} > \mu \), there exists a constant \( c_3 > 0 \) so that for all \( f \in \text{Dom}(\tilde{\mathcal{E}}) \)

\[
\|f - m(f)\|^2_{2\tilde{\mu}/(\tilde{\mu} - 2)} \leq c_3 \tilde{\mathcal{E}}(f, f).
\]
5. Non-symmetric diffusions on Riemannian manifolds

- \((M, g)\): a complete connected Riemannian manifold
- \(m = \text{vol}\): the Riemannian volume
- \(b\): a smooth vector field

We consider a diffusion generated by

\[
\mathcal{A} = \frac{1}{2}\triangle + b.
\]

We regard it as an operator in \(L^2(m)\).

The dual operator is

\[
\mathcal{A}^* = \frac{1}{2}\triangle - b - \text{div} \, b.
\]

Associated symmetric bilinear form \(\tilde{\mathcal{E}}\) is

\[
\tilde{\mathcal{E}}(u, v) = \frac{1}{2} \int_M (\nabla u, \nabla v) \, dm + \frac{1}{2} \int_M uv \, \text{div} \, b \, dm.
\]
We have to show the existence of associated semigroups.

- $o \in M$: any fixed point
- $d$: the Riemannian distance
- $\rho(x) = d(o, x)$

We assume the following conditions:

(A.2) \[ \text{div} \ b \geq 0. \]

(A.3) There exists a non-increasing function $\kappa : [0, \infty) \to [0, \infty)$ with
\[ \int_0^{\infty} \kappa(x) \, dx = \infty \] so that $|\nabla_b \rho| \leq \frac{1}{\kappa(\rho)}$.

Typical example of $\kappa$ is $\kappa(x) = \frac{1}{cx}$.

**Theorem 7.** Under the conditions (A.2), (A.3), The closure of $(\mathfrak{A}, C_0^\infty(M))$ generates a $C_0$ semigroup in $L^2(m)$ and the semigroup is Markovian. The same is true for $(\mathfrak{A}^*, C_0^\infty(M))$. 
We denote the associated semigroups by \( \{T_t\} \) and \( \{T^*_t\} \).

**Theorem 8.** Assume (A.2), (A.3) and that there exists a constant \( c_2 \) so that for all \( f \in \text{Dom}(\tilde{E}) \cap L^1(m) \)

\[
\|f\|^{2+4/\mu} \leq c_2 \tilde{E}(f, f) \|f\|^{4/\mu}. 
\]

Then, there exists a constant \( c_1 \) so that for all \( f \in L^1 \)

\[
(5) \quad \|T_t f\|_{\infty} \leq c_1 t^{-\mu/2} \|f\|_1, \quad \forall t > 0.
\]

**Remark 1.** Under the condition (A.2), we have

\[
\frac{1}{2} \int_M |\nabla u|^2 \, dm \leq \tilde{E}(u, u).
\]

If the Brownian motion satisfies (5), then the diffusion satisfies (5).
Case that $M$ is compact

If $M$ is compact, then there exists an invariant probability measure.

- $\nu$: an invariant probability measure
- $\nu = e^{-U}m$

We use the following notations

- $\nabla$: the covariant derivative
- $\nabla^*$: the dual operator of $\nabla$ w.r.t. $m$
- $\nabla_\nu^*$: the dual operator of $\nabla$ w.r.t. $\nu$
- $\omega_b$: 1-form corresponding to $b$

\[
\mathcal{A}f = \frac{1}{2}\triangle f + bf = \frac{1}{2}\triangle f + (\nabla f, \omega_b)
\]
We now change the reference measure to $\nu$. So our Hilbert space changes to $L^2(\nu)$.

Set

$$\mathcal{G}_\nu = \{ \mathcal{A} ; \mathcal{A} \text{ has an invariant measure } \nu \}. $$

We set

$$\tilde{b} = \frac{1}{2} (\nabla U)^\# + b,$$

$$\omega_{\tilde{b}} = \frac{1}{2} \nabla U + \omega_b.$$
Theorem 9. \( \mathcal{A} \in \mathcal{G}_\nu \) if and only if \( \nabla^*_\nu \omega \tilde{b} = 0 \). In this case,

\[
\mathcal{A} f = -\frac{1}{2} \nabla^*_\nu \nabla f + (\omega \tilde{b}, \nabla f)
\]

and

\[
\mathcal{A}^*_\nu f = -\frac{1}{2} \nabla^*_\nu \nabla f - (\omega \tilde{b}, \nabla f).
\]

Further the associated symmetric Dirichlet form is given by

\[
\tilde{\mathcal{E}}(f, h) = \frac{1}{2} \int_M (\nabla f, \nabla h) d\nu.
\]
Normal operator

Theorem 10. \( \mathcal{A} \) is normal if and only if \( \tilde{b} \) is a Killing field and \([\nabla U^\sharp, \tilde{b}] = 0\).

A vector field \( X \) is called a Killing field if \( L_X g = 0 \). It is known that \( X \) is a Killing field if and only if \( \nabla X \) is skew-symmetric. This is also equivalent to

\[
\text{div } X = 0,
\]
\[
\nabla^* \nabla X + \text{Ric}(X) = 0.
\]
Recall

\[ \mathcal{A} = \frac{1}{2} \triangle_{\nu} + \nabla \tilde{b}, \]
\[ \mathcal{A}^* = \frac{1}{2} \triangle_{\nu} - \nabla \tilde{b}. \]

Here

\[ \triangle_{\nu} = -\nabla^* \nabla = \nabla^* \nabla + \nabla U \cdot \nabla. \]

Then

\[ \mathcal{A} \mathcal{A}^* - \mathcal{A}^* \mathcal{A} = [\nabla \tilde{b}, \triangle_{\nu}]. \]

Moreover

\[ [\triangle_{\nu}, \nabla \tilde{b}] f = 2(\nabla \omega_{\tilde{b}}, \nabla^2 f) + (-\nabla^* \nabla \omega_{\tilde{b}} + \text{Ric}(\omega_{\tilde{b}}) + [\nabla U^d, \tilde{b}]^b, \nabla f). \]
$T_t$ has a density $p(t, x, y)$ with respect to $\nu$. Define

$$\gamma = - \lim_{t \to \infty} \frac{1}{t} \log \sup_{x, y \in M} |p(t, x, y) - 1|.$$ 

Let $\lambda$ be the spectral gap:

$$\|f - \nu(f)\|_\nu^2 \leq \lambda^{-1} \tilde{E}(f, f)$$

**Theorem 11.** We have

$$\gamma \geq \lambda.$$ 

The equality holds if $\mathfrak{A}$ is normal.
Thank you!