Non-symmetric diffusions on Riemannian manifolds
and the ultracontractivity*

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1 Non-symmetric diffusions on Riemannian manifolds

Let (M, g) be a complete Riemannian manifold. We deonte the Riemannian volume by $m = \mathrm{vol}$. We consider a diffusion generated by

$$\mathfrak{A} = \frac{1}{2}\triangle + b. \tag{1}$$

Here $\triangle$ is the Laplace-Beltrami operator and b is a vector field on M. We regard it as an operator in $L^2(m)$. The dual operator is

$$\mathfrak{A}^* = \frac{1}{2}\triangle - b - \operatorname{div} b.$$

Associated symmetric bilinear form $\tilde{\mathscr{E}}$ is

$$\tilde{\mathscr{E}}(u, v) = \frac{1}{2}\int_M (\nabla u, \nabla v)\, dm + \frac{1}{2}\int_M uv \operatorname{div} b\, dm.$$

We take a point $o \in M$ and define $\rho(x) = d(o, x)$ where d is the Riemannian distance. We assume the following conditions:

(A.1) $\operatorname{div} b \geq 0$.

(A.2) There exists a non-increasing function $\kappa\colon [0, \infty) \to [0, \infty)$ with $\int_0^\infty \kappa(x)\, dx = \infty$ so that $|\nabla_b \rho| \leq \frac{1}{\kappa(\rho)}$.

Theorem 1. *Under the conditions (A.1), (A.2), the closure of $(\mathfrak{A}, C_0^\infty(M))$ generates a C_0 semigroup in $L^2(m)$ and the semigroup is Markovian.*

The same is true for $(\mathfrak{A}^*, C_0^\infty(M))$.

We denote the associated semigroups by $\{T_t\}$ and $\{T_t^*\}$.

Theorem 2. *Assume (A.1), (A.2) and that there exists a constant c_2 so that for all $f \in \mathrm{Dom}(\tilde{\mathscr{E}}) \cap L^1(m)$*

$$\|f\|_2^{2+4/\mu} \leq c_2\, \tilde{\mathscr{E}}(f, f)\, \|f\|_1^{4/\mu}.$$

Then, there exists a constant c_1 so that for all $f \in L^1$

$$\|T_t f\|_\infty \leq c_1 t^{-\mu/2} \|f\|_1, \quad \forall t > 0. \tag{2}$$

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Remark 1. Under the condition (A.2), we have

$$\frac{1}{2}\int_M |\nabla u|^2 \, dm \le \tilde{\mathscr{E}}(u,u).$$

If the Brownian motion satisfies (2), then the diffusion satisfies (2).

2 Non-symmetric diffusions on compact Riemannian manifolds

If M is compact, then there exists an invariant probability measure. We denote it by ν. We now change the reference measure to ν. The operator $\mathfrak{A}$ of the form (1) can be written as

$$\mathfrak{A}f = -\frac{1}{2}\nabla_\nu^* \nabla f + (\tilde{b}, \nabla f)$$

where $\tilde{b}$ is a vector field with $\mathrm{div}_\nu \tilde{b} = 0$. Here ∇_ν^* is the dual operator of ∇ with respect to ν. div_ν is defined similarly.

The generator of the dual semigroup is

$$\mathfrak{A}_\nu^* g = -\frac{1}{2}\nabla_\nu^* \nabla g - (\omega_{\tilde{b}}, \nabla g).$$

Further the associated symmetric Dirichlet form is given by

$$\tilde{\mathscr{E}}(f,g) = \frac{1}{2}\int_M (\nabla f, \nabla g) d\nu.$$

By Using these, we have

Theorem 3. *The semigroup $\{T_t\}$ generated by $\mathfrak{A}$ has a density $p(t,x,y)$ with respect to ν and there exists a constant C so that*

$$\sup_{x,y} |p(t,x,y) - 1| \le Ce^{-\lambda t}, \quad \forall t \ge 1.$$

Here λ is the spectral gap of $\tilde{\mathscr{E}}$.