Ultracontractivity for non-symmetric Markovian semigroups

Ichiro Shigekawa

Kyoto University

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in Kyushu

URL: **http://www.math.kyoto-u.ac.jp/~ichiro/**

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1. Introduction

A semigroup $\{T_t\}$ is called ultracontractive if $T_t : L^1 \longrightarrow L^\infty$ is bounded for all $t > 0$.

It is well-known that the following three conditions are equivalent for a symmetric Markovian semigroup. Let $\mu > 0$ be given.

(i) $\exists c_1 > 0$, $\forall f \in L^1$:

$$\|T_t f\|_\infty \le c_1 t^{-\mu/2} \|f\|_1, \quad \forall t > 0.$$

(ii) $\exists c_2 > 0$, $\forall f \in \mathrm{Dom}(\mathcal{E}) \cap L^\infty$:

$$\|f\|_2^{2+4/\mu} \le c_2 \, \mathcal{E}(f, f) \, \|f\|_1^{4/\mu}.$$

(iii) $\mu > 2$, $\exists c_3 > 0$, $\forall f \in \mathrm{Dom}(\mathcal{E})$:

$$\|f\|_{2\mu/(\mu-2)}^2 \le c_3 \, \mathcal{E}(f, f).$$

We extend this result for non-symmetric Markovian semigroups.

2. Non-symmetric Markovian semigroups

We give a framework in generall Hilbert space scheme.

- H: a Hilbert space
- $\{T_t\}$: a contraction C_0 semigroup
- $\{T_t^*\}$: the dual semigroup
- $\mathfrak{A}$, $\mathfrak{A}^*$: the generators of $\{T_t\}$ and $\{T_t^*\}$

A natural bilinear form $\mathcal{E}$ is defined by

$$\mathcal{E}(u, v) = -(\mathfrak{A}u, v).$$

We do not assume the sector condition and so we can not use this bilinear form.

We introduce a symmetric bilinear form. For this, we assume the following condition:

$(\mathbf{A.1})$ $\mathbf{Dom}(\mathfrak{A}) \cap \mathbf{Dom}(\mathfrak{A}^*)$ is dense in $\mathbf{Dom}(\mathfrak{A})$ and $\mathbf{Dom}(\mathfrak{A}^*)$.

Under this condition, we define a symmetric bilinear form $\tilde{\mathcal{E}}$ by

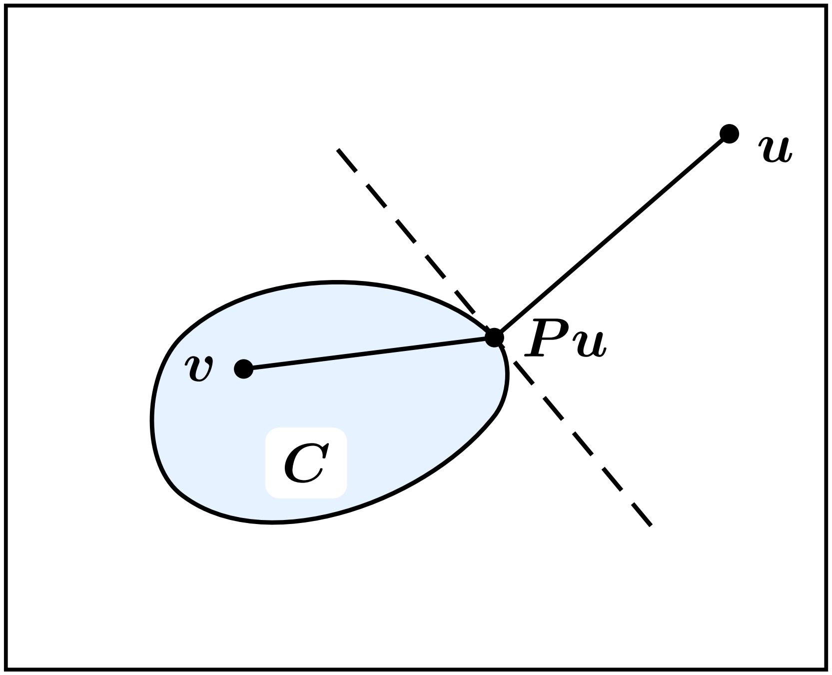
$$\tilde{\mathcal{E}}(u,v) = -\frac{1}{2}\{(\mathfrak{A}u,v) + (u,\mathfrak{A}v)\}, \quad u,v \in \mathbf{Dom}(\mathfrak{A}) \cap \mathbf{Dom}(\mathfrak{A}^*).$$

Proposition 1. Under the condition (A.1), $\tilde{\mathcal{E}}$ is closable and its closure contains $\mathbf{Dom}(\mathfrak{A})$ and $\mathbf{Dom}(\mathfrak{A}^*)$.

Covex set preserving property

- C: a convex set of H.
- Pu: the shortest point from u to C

$$(u - Pu, v - Pu) \le 0, \quad \forall v \in C.$$



Theorem 2. If $\{T_t\}$ and $\{T_t^*\}$ preserve a convex set C, then $Pu \in \mathrm{Dom}(\tilde{\mathcal{E}})$ for any $u \in \mathrm{Dom}(\tilde{\mathcal{E}})$ and we have

$$\tilde{\mathcal{E}}(Pu, u - Pu) \ge 0.$$

Markovian semigroup

- (M, m): a measure space
- $H = L^2(m)$: a Hilbert space
- $\{T_t\}$: a Markovian semigroup

We assume that $\{T_t^*\}$ is also a Markovian semigroup.

Under the assumption (A.1), we can define a symmetric bilinear form $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{E}}$ is a Dirichlet form.

Then the following implications are easily obtained.

$$\|T_t f\|_\infty \le c_1 t^{-\mu/2} \|f\|_1$$

$$\Uparrow$$

$$\|f\|_2^{2+4/\mu} \le c_2\, \tilde{\mathcal{E}}(f,f)\, \|f\|_1^{4/\mu}$$

$$\Uparrow$$

$$\|f\|_{2\mu/(\mu-2)}^2 \le c_3\, \tilde{\mathcal{E}}(f,f) \quad (\mu > 2)$$

To show this, set $u(t) = \|T_t f\|_2^2$. Then

$$-\frac{du}{dt} = -2(\mathfrak{A} T_t f, T_t f) \ge 2\|T_t f\|_2^{2+4/\mu} / (c_2 \|T_t f\|_1^{4/\mu}) \ge 2u^{1+2/\mu} / (c_2 \|f\|_1^{4/\mu}).$$

Hence

$$\frac{d}{dt}(u^{-2/\mu}) \ge \frac{4}{c_2 \mu \|f\|_1^{4/\mu}}.$$

The rest is the same as the symmetric case.

Theorem 3. Assume that there exists a constant c_1 so that for all $f \in L^1$

$$\|T_t f\|_\infty \le c_1 t^{-\mu/2} \|f\|_1, \quad \forall t > 0.$$

Further we assume that for all $f \in \mathrm{Dom}(\mathfrak{A}^2)$

$$(\mathfrak{A}^2 f, f)_2 + (\mathfrak{A} f, \mathfrak{A} f)_2 \ge 0. \tag{1}$$

Then there esists a constant c_2 so that for all $f \in \tilde{\mathcal{E}} \cap L^1$

$$\|f\|_2^{2+4/\mu} \le c_2 \, \tilde{\mathcal{E}}(f,f) \, \|f\|_1^{4/\mu}.$$

(1) holds if $\mathfrak{A}$ is normal, i.e. $\mathfrak{A}\mathfrak{A}^* = \mathfrak{A}^*\mathfrak{A}$.

Corollary 4. $\mu > 0$. Assume that there exists a constant c_2 so that for all $f \in \mathbf{Dom}(\tilde{\mathcal{E}}) \cap L^1_+$

$$\|f\|_2^{2+4/\mu} \leq c_2(\tilde{\mathcal{E}}(f) + \|f\|_2^2)\|f\|_1^{4/\mu}.$$

Then there exists a constant c_1 so that for all $f \in L^1$

$$\|T_t f\|_\infty \leq c_1 t^{-\mu/2}\|f\|_1, \quad \forall t \in (0, 1].$$

Corollary 5. $\mu > 0$. Assume that there exists a constant c_1 so that for all $f \in L^1$

$$\|T_t f\|_\infty \le c_1 t^{-\mu/2} \|f\|_1, \quad \forall t \in (0,1].$$

Further we assume that there exists a constant $M > 0$ so that for all $f \in \mathbf{Dom}(\mathfrak{A}^2)$

$$((\mathfrak{A} - M)^2 f, f)_2 + ((\mathfrak{A} - M) f, (\mathfrak{A} - M) f)_2 \ge 0.$$

Then, there exists a constant c_2 so that for all $f \in \mathbf{Dom}(\mathcal{E}) \cap L^1_+$

$$\|f\|_2^{2+4/\mu} \le c_2 (\tilde{\mathcal{E}}(f) + \|f\|_2^2) \|f\|_1^{4/\mu}.$$

3. Dirichlet forms satisfying the sector condition

From now on, we assume the sector condition for the Dirichlet form $\mathcal{E}$.

Theorem 6. $\mu > 0$. The following two conditions are equivalent:

(i) There esists a constant c_1 so that for all $f \in L^1$

$$\|T_t f\|_\infty \leq c_1 t^{-\mu/2} \|f\|_1, \quad \forall t \in (0,1].$$

(ii) There esists a constant c_2 so that for all $f \in \mathrm{Dom}(\tilde{\mathcal{E}}) \cap L^1(m)$

$$\|f\|_2^{2+4/\mu} \leq c_2 \left(\tilde{\mathcal{E}}(f,f) + \|f\|_2^2\right) \|f\|_1^{4/\mu}.$$

Key estimate:

$$\tilde{\mathcal{E}}(T_s f, T_s f) \leq C\{\tilde{\mathcal{E}}(f,f) + \|f\|_2^2\}$$

Theorem 7. $\mu > 2$. Suppose that there exists a constant c_1 so that for any $f \in L^1$

$$\|T_t f\|_\infty \leq c_1 t^{-\mu/2} \|f\|_1, \quad \forall t \in (0, 1].$$

Then, for any $\tilde{\mu} > \mu$, there exists a constant $c_3 > 0$ so that for all $f \in \mathrm{Dom}(\tilde{\mathcal{E}})$

$$\|f\|^2_{2\tilde{\mu}/(\tilde{\mu}-2)} \leq c_3(\tilde{\mathcal{E}}(f, f) + \|f\|^2_2)$$

Key estimate: for $s < \frac{1}{2}$,

$$\|(1 - \mathfrak{A})^s f\|^2_2 \leq C(\tilde{\mathcal{E}}(f, f) + \|f\|^2_2).$$

4. Dirichlet forms having invariant measure

We continue to assume the sector condition. In addition, we assume

- m is an invariant probability measure.

$$\int_M T_t f \, dm = \int_M f \, dm$$

- $T_t 1 = 1$ and $\mathfrak{A} 1 = 0$.

The following inequality is called the Poincaré inequality

$$\|f - m(f)\|_2^2 \le \lambda^{-1} \tilde{\mathcal{E}}(f, f) \tag{2}$$

where

$$m(f) = \int_M f(x) \, m(dx).$$

This inequality is equivalent to

$$\|T_t f - m(f)\|_2^2 \le e^{-2\lambda t} \|f - m(f)\|_2^2.$$

Theorem 8. $\mu > 0$. We consider the following two conditions.

(i) There exists a constant c_1 so that for all $f \in L^1$

$$\|T_t f - m(f)\|_\infty \le c_1 t^{-\mu/2} \|f\|_1, \quad \forall t > 0.$$

(ii) There exists a constant c_2 so that for all $f \in \mathrm{Dom}(\tilde{\mathcal{E}}) \cap L^1(m)$

$$\|f - m(f)\|_2^{2+4/\mu} \le c_2 \, \tilde{\mathcal{E}}(f, f) \, \|f\|_1^{4/\mu}.$$

Then, (ii) is equivalent to (i) with the Poincaré inequality.
Under the condition (ii), there exists a constant $c_4 > 0$ so that for all $f \in L^1$

$$\|T_t f - m(f)\|_\infty \le c_4 e^{-\lambda t} \|f\|_1, \quad \forall t \ge 1.$$

Here λ is a constant appears in the Poincaré inequality (2).

Proof.

$$
\begin{aligned}
\|T_t - m\|_{1\to\infty} &= \|(T_1 - m)(T_{t-2} - m)(T_1 - m)\|_{1\to\infty} \\
&\leq \|T_1 - m\|_{2\to\infty} \, \|T_{t-2} - m\|_{2\to 2} \, \|T_1 - m\|_{1\to 2} \\
&\leq \|T_1 - m\|_{2\to\infty} \, e^{-\lambda(t-2)} \, \|T_1 - m\|_{1\to 2}
\end{aligned}
$$

Theorem 9. $\mu > 2$. Assume that there exists a constant c_1 so that

$$\|T_t f - m(f)\|_\infty \leq c_1 t^{-\mu/2} \|f\|_1, \quad \forall t > 0$$

and the Poincaré inequality holds.
Then, for any $\tilde{\mu} > \mu$, there exists a constant $c_3 > 0$ so that for all $f \in \mathrm{Dom}(\tilde{\mathcal{E}})$

$$\|f - m(f)\|^2_{2\tilde{\mu}/(\tilde{\mu}-2)} \leq c_3 \tilde{\mathcal{E}}(f, f).$$

5. Non-symmetric diffusions on Riemannian manifolds

- (M, g): a complete connected Riemannian manifold
- $m = \mathrm{vol}$: the Riemannian volume
- b: a smooth vector field

We consider a diffusion generated by

$$\mathfrak{A} = \frac{1}{2}\triangle + b.$$

We regard it as an operator in $L^2(m)$.

The dual operator is

$$\mathfrak{A}^* = \frac{1}{2}\triangle - b - \mathrm{div}\, b.$$

Associated symmetric bilinear form $\tilde{\mathcal{E}}$ is

$$\tilde{\mathcal{E}}(u, v) = \frac{1}{2}\int_M (\nabla u, \nabla v)\, dm + \frac{1}{2}\int_M uv \,\mathrm{div}\, b\, dm.$$

We have to show the existence of associated semigroups.

- $o \in M$: any fixed point
- d: the Riemannian distance
- $\rho(x) = d(o, x)$

We assume the following conditions:

(A.2) $\quad \operatorname{div} b \geq 0$.

(A.3) There exists a non-increasing function $\kappa : [0, \infty) \to [0, \infty)$ with $\int_0^\infty \kappa(x)\, dx = \infty$ so that $|\nabla_b \rho| \leq \frac{1}{\kappa(\rho)}$.

Typical example of κ is $\kappa(x) = \frac{1}{cx}$.

Theorem 10. Under the conditions (A.2), (A.3), The closure of $(\mathfrak{A}, C_0^\infty(M))$ generates a C_0 semigroup in $L^2(m)$ and the semigroup is Markovian.
The same is true for $(\mathfrak{A}^*, C_0^\infty(M))$.

We denote the associated semigroups by $\{T_t\}$ and $\{T_t^*\}$.

Theorem 11. Assume (A.2), (A.3) and that there exists a constant c_2 so that for all $f \in \mathrm{Dom}(\tilde{\mathcal{E}}) \cap L^1(m)$

$$\|f\|_2^{2+4/\mu} \leq c_2\, \tilde{\mathcal{E}}(f,f)\, \|f\|_1^{4/\mu}.$$

Then, there exists a constant c_1 so that for all $f \in L^1$

$$\|T_t f\|_\infty \leq c_1 t^{-\mu/2} \|f\|_1, \quad \forall t > 0. \tag{3}$$

Remark 1. Under the condition (A.2), we have

$$\frac{1}{2}\int_M |\nabla u|^2\, dm \leq \tilde{\mathcal{E}}(u,u).$$

If the Brownian motion satisfies (3), then the diffusion satisfies (3).

Case that M is compact

If M is compact, then there exists an invariant probability measure.

- ν: an invariant probability measure
- $\nu = e^{-U} m$

We use the following notations

- ∇: the covariant derivative
- ∇^*: the dual operator of ∇ w.r.t. m
- ∇^*_ν: the dual operator of ∇ w.r.t. ν
- ω_b: 1-form corresponding to b

$$\mathfrak{A} f = \frac{1}{2} \triangle f + b f = \frac{1}{2} \triangle f + (\nabla f, \omega_b)$$

We now change the reference measure to ν. So our Hilbert space changes to $L^2(\nu)$.

We set

$$\omega_{\tilde{b}} = \frac{1}{2}\nabla U + \omega_b.$$

Then

$$\mathfrak{A}f = -\frac{1}{2}\nabla_\nu^*\nabla f + (\omega_{\tilde{b}}, \nabla f)$$

and

$$\mathfrak{A}_\nu^* g = -\frac{1}{2}\nabla_\nu^*\nabla g - (\omega_{\tilde{b}}, \nabla g).$$

Further the associated symmetric Dirichlet form is given by

$$\tilde{\mathcal{E}}(f,g) = \frac{1}{2}\int_M (\nabla f, \nabla g)d\nu.$$

Since M is compact, $\tilde{\mathcal{E}}$ has a spectral gap:

$$\|f - \nu(f)\|_\nu^2 \leq \lambda^{-1}\tilde{\mathcal{E}}(f,f).$$

Theorem 12. The semigroup $\{T_t\}$ generated by $\mathfrak{A}$ has a density $p(t,x,y)$ with respect to ν and there exists a constant C so that

$$\sup_{x,y} |p(t,x,y) - 1| \leq Ce^{-\lambda t}, \quad \forall t \geq 1.$$

Thank you!