Semigroups that preserve a convex set in a Banach space

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July, 2008
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1. Introduction

The following properties are well-discussed:

(1) Positivity preserving
(2) Markovian
(3) Excessive function
(4) Invariant set

Aim: We give a unified method to prove them.
Positivity preserving property

\[ L^1 \int_{\{f<0\}} \mathcal{A} f(x) d\mu(x) \geq 0 \]

\[ L^p \int \mathcal{A} f(x) f^{p-1}_-(x) d\mu(x) \geq 0 \quad \text{as } p \to 1 \]

\[ C\infty \quad \mathcal{A} f(x_0) \geq 0, \quad x_0 : \text{maximum point of } f_- \]
2. Semigroups that preserve a convex set in a Banach space

- $B$: Banach space with a norm $\| \|$.
- $B^*$: the dual space of $B$.
- $F(x) =: \{ \varphi \in B^*; \langle x, \varphi \rangle = \| x \|^2 \}$ (conjugate mapping).
- $\{T_t\}$: a $(C_0)$-semigroup.
- $\mathcal{A}$: the generator.
- $\{G_\alpha\}$: the resolvent.
- $C$: a convex set in $B$. 
We are interested in the following property:

\[ T_t C \subseteq C, \quad \forall t \geq 0, \]

i.e., \( T_t \) preserves the convex set \( C \).

- \( d(x, C) = \inf \{ \| x - y \| ; y \in C \} \)
- \( P(x) = \{ y \in C ; d(x, y) = d(x, C) \} \)

We always assume that \( P(x) \neq \emptyset \).
Theorem 1. $\gamma \in \mathbb{R}$ is fixed.

Suppose that $\forall x \in \text{Dom}(A), \exists y \in P(x), \forall \varphi \in F(x - y) :$

$$\Re\langle Ax, \varphi \rangle \leq \gamma \|x - y\|^2,$$

then the semigroup $\{T_t\}$ preserves $C$.

Conversely, if $\{T_t\}$ preserves $C$ and $\{e^{-\gamma t}T_t\}$ is a contraction semigroup, then $\forall x \in \text{Dom}(A), \forall y \in P(x), \exists \varphi \in F(x - y)$, so that (1) holds.
Good selection

\((Q(x), G(x))\) : good selection

\[
\begin{align*}
&\iff \\
&\begin{cases}
(i) & Q(x) \in P(x), \quad G(x) \in F(x - Q(x)) \\
(ii) & \forall z \in C : \Re \langle z - Q(x), G(x) \rangle \leq 0
\end{cases}
\end{align*}
\]
Theorem 2. $\gamma \in \mathbb{R}$ is fixed.

Suppose that $\exists (Q(x), G(x))$: good selection so that $\forall x \in \text{Dom}(\mathbb{A})$:

$$\mathbb{R}\langle Ax, G(x) \rangle \leq \gamma \|x - Q(x)\|^2,$$

then the semigroup $\{T_t\}$ preserves $C$.

Conversely, if $\{T_t\}$ preserves $C$ and $\{e^{-\gamma t}T_t\}$ is a contraction semigroup, then for any good selection $(Q(x), G(x))$ (if it exists) (2) holds.

Remark 1. In Hilbert space case, $P(x)$ consists of one point and $F(x) = x$. In this case, the above theorem for $\gamma = 0$ is proved by Brezis-Pazy (1970).
3. Examples

Positivity preserving property

\[ C = \{ f; f \geq 0 \} \]

\[ Q(f) = f_+ \]

1. \( C_\infty(E) \)
   \[ G(f) = \| f^- \|_\infty \delta_{x_0}, \quad x_0: \text{maximaum point of } f_- \]
   \[ \mathcal{A}f(x_0) \geq \gamma f(x_0) \]

2. \( L^p(d\mu) \quad (1 < p < \infty) \)
   \[ G(f) = \| f^- \|_{p^{-1}}^{2-p} f_-^{p-1} \]
   \[ \int \mathcal{A}f(x) f_-^{p-1} \, d\mu(x) \geq -\gamma \| f_- \|_p^p \]
3. $L^1(\mu)$

$$G(f) = -\|f - 1\|_1 1\{f<0\}$$

$$\int_{\{f<0\}} \mathbb{A}f(x) \, d\mu(x) \geq -\gamma \|f - 1\|_1$$

Markovian property

$C = \{f; f \leq 1\}$ (or $\{f; 0 \leq f \leq 1\}$), $Q(f) = f \wedge 1 = \min\{f, 1\}$

1. $C_\infty(E)$

$$G(f) = \|(f - 1)_+\|_\infty \delta_{x_0}, \quad x_0: \text{positive maximum point of } f$$

$$\mathbb{A}f(x_0) \leq 0$$
2. $L^p(d\mu)$ \ (1 < p < \infty)

\[ G(f) = \| (f - 1)_+ \|^2 \! - \! p (f - 1)_+^{p-1} \]

\[ \int \mathcal{A}f(x) (f - 1)_+^{p-1} \, d\mu(x) \leq \gamma \| (f - 1)_+ \|^p \]

3. $L^1(\mu)$

\[ G(f) = -\| (f - 1)_+ \|_1 1\{f > 1\} \]

\[ \int_{\{f > 1\}} \mathcal{A}f(x) \, d\mu(x) \leq \gamma \| (f - 1)_+ \|_1 \]
**L¹** contraction

The dual notion of the Markovian property is **L¹**-contraction and positivity preserving. This time,

\[ C = \{ f; f \geq 0, \int f \, d\mu = 1 \} \]

\[ Q(f) = (f - c)_+ \quad \text{with} \quad \int (f - c)_+ \, d\mu = 1 \]

\[ L^p(d\mu) \quad (1 < p < \infty) \]

\[ G(f) = \| f \wedge c \|_p^{2-p} (f \wedge c)^{p-1} \]

\[ \int \mathcal{A} f(x) (f \wedge c)^{p-1} \, d\mu(x) \leq \gamma \| f \wedge c \|_p^p \]
**Excessive function**

A non-negative function $u$ is called excessive if

$$e^{-\alpha t} T_t u \leq u, \quad \forall t \geq 0.$$ 

We do not need to assume that $\{T_t\}$ is Markovian. If we assume that $\{T_t\}$ is positivity preserving, then the above condition is equivalent to the invariance of the convex set $C = \{f; f \leq u\}$ under $\{e^{-\alpha t} T_t\}$. So now

$$C = \{f; f \leq u\}, \quad Q(f) = f \wedge u = \min\{f, u\}.$$
1. $C_\infty(E), \quad G(f) = \|(f - u)_+\|_\infty \delta_{x_0}, \quad x_0: \text{positive maximum point of } f - u$

$(\mathcal{A} - \alpha)f(x_0) \leq \gamma(f(x_0) - u(x_0))$

2. $L^p(d\mu) \ (1 < p < \infty), \quad G(f) = \|(f - u)_+\|^{2-p}_p (f - u)^{p-1}_+\quad \int (\mathcal{A} - \alpha)f(x) (f(x) - u(x))^{p-1}_+ d\mu(x) \leq \gamma\|(f - u)_+\|^{p}_p$

3. $L^1(\mu), \quad G(f) = -\|(f - u)_+\|_1 1\{f > u\}$

$\int_{\{f > u\}} (\mathcal{A} - \alpha)f(x) d\mu(x) \leq \gamma\|(f - u)_+\|_1$
Invariant set

A set $K$ is called invariant if

$$1_{K^c} T_t 1_K = 0, \quad \forall t \geq 0.$$  

So now

$$C = \{ f; 1_{K^c} f = 0 \}, \quad Q(f) = 1_K f$$
1. $C_{\infty}(E)$, \quad $G(f) = \|1_{K^c} f\|_{\infty} \text{sgn}(f(x_0))\delta_{x_0}$, \quad $x_0$: positive maximum $|f|$ in $K^c$.

\[ \mathcal{A}f(x_0) \text{sgn}(f(x_0)) \leq \gamma |f(x_0)| \]

2. $L^p(d\mu)$ ($1 < p < \infty$), \quad $G(f) = \|1_{K^c} f\|_{p}^{2-p} 1_{K^c} |f|^{p-1} \text{sgn} f$

\[ \int_{K^c} \mathcal{A}f(x) |f(x)|^{p-1} \text{sgn} f(x) \, d\mu(x) \leq \gamma \|1_{K^c} f\|_{p}^{p} \]

3. $L^1(\mu)$, \quad $G(f) = \|1_{K^c} f\|_{1} 1_{K^c} \text{sgn} f$

\[ \int_{K^c} \mathcal{A}f(x) \text{sgn} f(x) \, d\mu(x) \leq \gamma \|1_{K^c} f\|_{1} \]
4. Hilber space case

We can give an conditions for preserving a convex set in terms of bilinear form. This was done by Ouhabaz [1996] for contraction semigroups. Our aim is to clarify when we need the contraction property or not.
**Kown results**

\[ \mathcal{E}(f, f - f \wedge 1) \geq 0 \]

\{T_t\} is Markovian

require contraction

\[ \mathcal{E}(f \wedge 1, f - f \wedge 1) \geq 0 \]

Ma-Röckner: Dirchelet forms

semi-Dirichlet form \( \overset{\text{def}}{\iff} \mathcal{E}(f + f \wedge 1, f - f \wedge 1) \geq 0 \)
Main results

\[ \mathcal{E}(f, f - f \wedge 1) \geq 0 \]

\{ T_t \} \text{ is Markovian contraction} \quad \Rightarrow \quad \{ T_t \} \text{ is Markovian}

\[ \mathcal{E}(f \wedge 1, f - f \wedge 1) \geq 0 \]
**Main theorem**

**Theorem 3.** We fix $\gamma \in \mathbb{R}$ and $\theta \in [0,1]$. Let us consider the following conditions:

(i) For any $x \in \text{Dom}(\mathcal{E})$, $Px \in \text{Dom}(\mathcal{E})$ and

$$\Re \mathcal{E}((1-\theta)x + \theta Px, x - Px) \geq -(1-\theta)\gamma|x - Px|^2.$$  

(ii) $\{T_t\}$ preserves $C$.

(iii) $\mathcal{E}(P(x), x - P(x)) \geq 0$, $\forall x \in \text{Dom}(\mathcal{E})$.

Then, the implications (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) holds.

If $\{e^{-\gamma t}T_t\}$ is contractive, then the above three conditions are equivalent to each other.
If $\mathcal{E}$ is Hermitian, then the following condition (without the contraction property of $\{e^{-\gamma t}T_t\}$)

(iv) for any $x \in \text{Dom}(\mathcal{E})$, $P(x) \in \text{Dom}(\mathcal{E})$ and

$$\mathcal{E}(Px, Px) \leq \mathcal{E}(x, x) + \gamma|x - Px|^2, \quad \forall x \in \text{Dom}(\mathcal{E})$$

deduces (ii). In addition, if we assume that $\{e^{-\gamma t}T_t\}$ is contractive, then all conditions (i) – (iv) are equivalent to each other.
**Positivity preserving property**

**Theorem 4.** The following conditions are equivalent to each other:

(i) \( \{ T_t \} \) preserves the positivity.

(ii) For any \( f \in \text{Dom}(\mathcal{E}) \), \( |f| \in \text{Dom}(\mathcal{E}) \) and \( \mathcal{E}(f^+, f^-) \leq 0 \).

Further (i) or (ii) implies the following (iii):

(iii) For any \( f \in \text{Dom}(\mathcal{E}) \), \( |f| \in \text{Dom}(\mathcal{E}) \) and \( \mathcal{E}(|f|, |f|) \leq \mathcal{E}(f, f) \).

If, in addition, \( \mathcal{E} \) is symmetric, then all conditions are equivalent to each other.
Theorem 5. We fix $\gamma \in \mathbb{R}$ and $\theta \in [0, 1)$. The following two conditions are equivalent to each other:

(i) $\{e^{-\gamma t}T_t\}$ is a positivity preserving contraction semigroup.

(ii) For any $f \in \text{Dom}(\mathcal{E})$, $|f| \in \text{Dom}(\mathcal{E})$ and $\mathcal{E}((1 - \theta)f + \theta f_+, f - f_+) \geq -\gamma(1 - \theta)\|f - f_+\|_2^2$.

If, in addition, $\mathcal{E}$ is symmetric, then the above conditions are equivalent to the following:

(iii) For any $f \in \text{Dom}(\mathcal{E})$, $|f| \in \text{Dom}(\mathcal{E})$ and $\mathcal{E}(f_+, f_+) \leq \mathcal{E}(f, f) + \gamma|f - f_+|^2$.

(iv) For any $f \in \text{Dom}(\mathcal{E})$, $|f| \in \text{Dom}(\mathcal{E})$ and $0 \leq \mathcal{E}_\gamma(|f|, |f|) \leq \mathcal{E}_\gamma(f, f)$. 
Markovian property

Theorem 6. The following conditions are equivalent to each other:

(i) $\{T_t\}$ is a Markovian semigroup.

(ii) For any $f \in \text{Dom}(\mathcal{E})$, $f \wedge 1 \in \text{Dom}(\mathcal{E})$ and

$$\mathcal{E}(f \wedge 1, f - f \wedge 1) \geq 0.$$ 

Replacing $f \wedge 1$ with $f_+ \wedge 1$, we have the same result.

We may define that a bilinear form $\mathcal{E}$ is called semi-Dirichlet form if it satisfies the condition of (ii).
Theorem 7. We fix $\gamma \in \mathbb{R}$ and $\theta \in [0, 1)$. The following two conditions are equivalent to each other:

(i) $\{T_t\}$ is a Markovian semigroup and $\{e^{-\gamma t}T_t\}$ is contractive.

(ii) For any $f \in \text{Dom}(\mathcal{E})$, $f \land 1 \in \text{Dom}(\mathcal{E})$ and

$$\mathcal{E}((1-\theta)f + \theta(f \land 1), f - f \land 1) \geq -\gamma(1-\theta)\|f - f \land 1\|_2^2.$$ 

If, in addition, $\mathcal{E}$ is symmetric, (i) or (ii) is equivalent to the following:

(iv) For any $f \in \text{Dom}(\mathcal{E})$, $f \land 1 \in \text{Dom}(\mathcal{E})$ and

$$\mathcal{E}(f \land 1, f \land 1) \leq \mathcal{E}(f, f) + \gamma\|f - f \land 1\|_2^2.$$ 

Replacing $f \land 1$ with $f_+ \land 1$, we have the same result.
Excessive function

Theorem 8. We fix $\gamma \in \mathbb{R}$ and $\alpha \geq 0$. The following conditions are equivalent to each other:

(i) $u$ is $\alpha$-excessive and $\{T_t\}$ preserves the positivity.

(ii) For any $f \in \text{Dom}(\mathcal{E})$, $f \land u \in \text{Dom}(\mathcal{E})$ and $\mathcal{E}_\alpha(f \land u, f - f \land u) \geq 0$. 
Theorem 9. We fix $\gamma \in \mathbb{R}$, $\alpha \geq 0$ and $\theta \in [0, 1)$. The following conditions are equivalent to each other:

(i) $u$ is $\alpha$-excessive and $\{e^{-(\alpha+\gamma)t}T_t\}$ is a positivity preserving contraction semigroup.

(ii) For any $f \in \text{Dom}(\mathcal{E})$, $f \land u \in \text{Dom}(\mathcal{E})$ and

$$\mathcal{E}_\alpha((1-\theta)f + \theta(f \land u), f - f \land u) \geq -\gamma(1-\theta)\|f - f \land u\|^2.$$
**Invariant set**

**Theorem 10.** The following conditions are equivalent to each other:

(i) $B$ is invariant.

(ii) For any $f \in \text{Dom}(\mathcal{E})$, $1_B f \in \text{Dom}(\mathcal{E})$ and $\mathcal{E}(1_B f, 1_{B^c} f) \geq 0$.

(iii) For any $f \in \text{Dom}(\mathcal{E})$, $1_B f \in \text{Dom}(\mathcal{E})$ and $\mathcal{E}(1_B f, 1_{B^c} f) = 0$. 
Thanks!