The dual ultracontractivity and its applications

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Contents

1. Introduction

(X_t, P_x): an m-symmetric Markov process on M

- $\{T_t\}$: the associated semigroup
- $\mathfrak{A}$: the generator $\{T_t\}$
- $\mathcal{E}$: the Dirichlet form

Ultracontractivity

$\{T_t\}$ is called ultracontractive if

$$\|T_t f\|_\infty \leq a_t \|f\|_1, \quad \forall f \in L^1(m),\ \forall t > 0.$$

Criteria for ultracontractivity

Let $\mu > 0$. The followings are equivalent to each other:

(i) There exists a constant c_1 so that

$$\|T_t f\|_\infty \le c_1 t^{-\mu/2} \|f\|_1, \quad \forall f \in L^1, \forall t > 0.$$

(ii) There exists a constant c_2 so that

$$\|f\|_2^{2+4/\mu} \le c_2\, \mathcal{E}(f,f)\, \|f\|_1^{4/\mu}, \quad \forall f \in \mathrm{Dom}(\mathcal{E}) \cap L^1.$$

If $\mu > 2$ the conditions above are equivalent to

(iii) There exists a constant c_3 so that

$$\|f\|^2_{2\mu/(\mu-2)} \le c_3\, \mathcal{E}(f,f), \quad \forall f \in \mathrm{Dom}(\mathcal{E}).$$

We are interested in the following property:

There exist constants b_t so that

$$\|T_t f\|_1 \le b_t \, \|f\|_\infty, \quad \forall f \in L^\infty(m), \, \forall t > 0.$$

This property called the **dual ultracontractivity**.

$$b_t = \|T_t\|_{\infty \to 1} = P_m(\zeta > t).$$

Ultracontractivity

$$\|T_t f\|_\infty \le a_t \, \|f\|_1, \quad \forall f \in L^1(m), \, \forall t > 0.$$

2. Dual ultracontractivity

Theorem 2.1. Let $\mu > 0$. The followings are equivalent to each other:

(i) There exists a constant c_1 so that

$$\|T_t f\|_1 \leq c_1 t^{-\mu/2} \|f\|_\infty, \quad \forall f \in L^\infty, \forall t > 0.$$

(ii) There exists a constant c_2 so that

$$\|f\|_2^{2+4/\mu} \leq c_2 \, \mathcal{E}(f, f) \, \|f\|_\infty^{4/\mu}, \ \forall f \in \mathrm{Dom}(\mathcal{E}) \cap L^\infty.$$

Proof. From (i)

$$\|T_{2t}f\|_1 \le C_1 t^{-\mu/2}\|f\|_\infty.$$

Hence, for $0 \le f \in \mathrm{Dom}(\mathcal{E}) \cap L^\infty$

$$\begin{aligned} C_1 t^{-\mu/2}\|f\|_\infty^2 &\ge \|T_{2t}f\|_1\|f\|_\infty \\ &\ge (T_{2t}f, f) \\ &= (f, f) + \int_0^t \frac{d}{ds}(T_s f, T_s f)\, ds \\ &= (f, f) - 2\int_0^t \mathcal{E}(T_s f, T_s f)\, ds \\ &\ge (f, f) - 2t\mathcal{E}(f, f). \end{aligned}$$

Therefore

$$\|f\|_2^2 \leq 2t\mathcal{E}(f,f) + C_1 t^{-\mu/2}\|f\|_\infty^2.$$

Choose

$$t = \left(\frac{\|f\|_\infty^2}{\mathcal{E}(f,f)}\right)^{2/(\mu+2)}$$

so that

$$\|f\|_2^2 \leq (2+C_1)\mathcal{E}(f,f)^{\mu/(\mu+2)}\|f\|_\infty^{4/(\mu+2)}.$$

Thus

$$\|f\|_2^{2+4/\mu} \leq (2+C_1)^{(\mu+2)/\mu}\mathcal{E}(f,f)\|f\|_\infty^{4/\mu},$$

which is (ii). $\square$

Theorem 2.2. Let $\mu > 2$. The followings are equivalent to each other:

(i) There exists a constant c_1 so that

$$\|T_t f\|_1 \le c_1 t^{-\mu/2} \|f\|_\infty, \quad \forall f \in L^\infty, \forall t > 0.$$

(ii) There exists a constant c_3 so that

$$\|f\|^2_{2\mu/(\mu+2)} \le c_3 \mathcal{E}(f, f), \quad \forall f \in \mathrm{Dom}(\mathcal{E}), \forall t > 0.$$

Remark. $1 < \frac{2\mu}{\mu+2} < 2$.

If $\mu < 2$, then $\frac{2\mu}{\mu+2} < 1$ and (ii) $\Rightarrow$ (i) holds.

DUC	$\|T_t f\|_1 \le c_1 t^{-\mu/2} \|f\|_\infty, \quad t > 0,\ f \in L^\infty$ $\|f\|_2^{2+4/\mu} \le c_2\, \mathcal{E}(f,f)\, \|f\|_\infty^{4/\mu}$ $\|f\|_{2\mu/(\mu+2)}^2 \le c_3\, \mathcal{E}(f,f) \quad (\mu > 2)$
UC	$\|T_t f\|_\infty \le c_1 t^{-\mu/2} \|f\|_1, \quad t > 0,\ f \in L^1$ $\|f\|_2^{2+4/\mu} \le c_2\, \mathcal{E}(f,f)\, \|f\|_1^{4/\mu}$ $\|f\|_{2\mu/(\mu-2)}^2 \le c_3\, \mathcal{E}(f,f) \quad (\mu > 2)$

DUC	$\\|T_t f\\|_1 \leq c_1 t^{-\mu/2} \\|f\\|_\infty, \quad t > 0,\ f \in L^\infty$ $\\|f\\|_2^{2+4/\mu} \leq c_2\, \mathcal{E}(f,f)\, \\|f\\|_\infty^{4/\mu}$ $\\|f\\|_{2\mu/(\mu+2)}^2 \leq c_3\, \mathcal{E}(f,f) \quad (\mu > 2)$
UC	$\\|T_t f\\|_\infty \leq c_1 t^{-\mu/2} \\|f\\|_1, \quad t > 0,\ f \in L^1$ $\\|f\\|_2^{2+4/\mu} \leq c_2\, \mathcal{E}(f,f)\, \\|f\\|_1^{4/\mu}$ $\\|f\\|_{2\mu/(\mu-2)}^2 \leq c_3\, \mathcal{E}(f,f) \quad (\mu > 2)$

DUC	$\|T_t f\|_1 \le c_1 t^{-\mu/2} \|f\|_\infty, \quad t \in (0,1],\ f \in L^\infty$ $\|f\|_2^{2+4/\mu} \le c_2 \left(\mathcal{E}(f,f) + \|f\|_2^2\right) \|f\|_\infty^{4/\mu}$ $\|f\|_{2\mu/(\mu+2)}^2 \le c_3 \left(\mathcal{E}(f,f) + \|f\|_2^2\right) \quad (\mu > 2)$
UC	$\|T_t f\|_\infty \le c_1 t^{-\mu/2} \|f\|_1, \quad t \in (0,1],\ f \in L^1$ $\|f\|_2^{2+4/\mu} \le c_2 \left(\mathcal{E}(f,f) + \|f\|_2^2\right) \|f\|_1^{4/\mu}$ $\|f\|_{2\mu/(\mu-2)}^2 \le c_3 \left(\mathcal{E}(f,f) + \|f\|_2^2\right) \quad (\mu > 2)$

3. One dimensional diffusion processes

$D = (l_1, l_2)$.

$\{(X_t), P_x\}$: a (minimal) diffusion on D (Dirichlet boundary condition)

$s(x)$: the sclae function. We assume that $s(x) = x$.

dm : the speed measure, $m(y) - m(x) = \int_{(x,y]} dm$

ζ : the explosion time

$\frac{d}{dm}\frac{d}{ds}$: the generator

Dirichlet form: $\mathcal{E}(f, g) = \int_D \frac{df}{ds}\frac{dg}{ds} ds$

We introduce some notations.

Ultracontractivity:

$$\boldsymbol{R_\mu}: \qquad \|T_t f\|_\infty \le C t^{-\mu/2} \|f\|_1, \quad \forall t > 0,$$

$$\boldsymbol{R_\mu(0)}: \qquad \|T_t f\|_\infty \le C t^{-\mu/2} \|f\|_1, \quad \forall t \in (0, 1].$$

Dual ultracontractivity:

$$\boldsymbol{S_\mu}: \qquad \|T_t f\|_1 \le C t^{-\mu/2} \|f\|_\infty, \quad \forall t > 0,$$

$$\boldsymbol{S_\mu(0)}: \qquad \|T_t f\|_1 \le C t^{-\mu/2} \|f\|_\infty, \quad \forall t \in (0, 1].$$

The case $D = (0, l)$

Dual ultracontractivity:

$$\left.\begin{array}{l} \sup\limits_{0<x<l/2} x^{\frac{\mu}{\mu+2}} m([x, l/2)) < \infty \\ \\ \sup\limits_{l/2<x<l} (l-x)^{\frac{\mu}{\mu+2}} m((l/2, x]) < \infty \end{array}\right\} \Leftrightarrow S_\mu \Leftrightarrow S_\mu(0)$$

$$m((0, l)) < \infty \;\Leftrightarrow\; S_0$$

$S_\mu \;\Rightarrow\; P_x(\zeta > t)$ decays exponentially as $t \to \infty$

Asymptotics of $P_x(\zeta > t)$

- $0 < \lambda_0 < \lambda_1 \leq \cdots$: a discrete spectrum of $-\mathfrak{A}$.
- $\{\varphi_i\}$: c.o.n.s of eigen-functions
- $p(t, x, y)$: transition density function

$$p(t,x,y) = \sum_{j=0}^{\infty} e^{-\lambda_j t} \varphi_j(x) \varphi_j(y).$$

Then

$$\begin{aligned} P_x[\zeta > t] &= \int_D p(t,x,y)\, dm(y) \\ &\sim e^{-\lambda_0 t} \varphi_0(x) \int_D \varphi_0(y)\, dm(y). \end{aligned}$$

To ensure $\int_D \varphi_0(y)\, dm(y) < \infty$, we need the dual ultracontractivity:

$$\varphi_0 \in L^2(m) \Rightarrow \varphi_0 \in L^1(m).$$

Ultracontractivity: (Mao [2002] Acta Math. Sinica)

$$\left.\begin{aligned} \sup_{0<x<l/2} x^{\frac{\mu}{\mu-2}} m([x, l/2)) < \infty \\ \sup_{l/2<x<l} (l-x)^{\frac{\mu}{\mu-2}} m((l/2, x]) < \infty \end{aligned}\right\} \Leftrightarrow R_\mu$$

R_2 always holds.

$$\left.\begin{aligned} \int_{(0,l/2)} x m([x, l/2))^2 \, dx < \infty \\ \int_{(l/2,l)} (l-x) m([l/2, x])^2 \, dx < \infty \end{aligned}\right\}$$

$\Rightarrow p(t, x, y)$ decays exponentially as $t \to \infty$

The case $D = (0, \infty)$

Dual ultracontractivity:

$$\boxed{\begin{array}{l} \left.\begin{array}{c} \sup\limits_{0<x<1} x^{\mu/(\mu+2)} m([x,1)) < \infty \\[2em] \sup\limits_{x\geq 1} x^{\mu/(\mu+2)} m([x,\infty)) < \infty \end{array}\right\} \Leftrightarrow \ S_\mu \\[3em] \left.\begin{array}{c} \sup\limits_{0<x<1} x^{\mu/(\mu+2)} m([x,1)) < \infty \\[2em] m([1,\infty)) < \infty \end{array}\right\} \Leftrightarrow \ S_\mu(0) \end{array}}$$

$$\int_{(0,1)} x\,dm(x) = \infty \;\Rightarrow\; \text{DUC does not hold}$$

$$\left.\begin{array}{l} \displaystyle\sup_{0<x<1} x^{\mu/(\mu+2)} m([x,1)) < \infty \\ \displaystyle\int_{(0,\infty)} x m([x,\infty))^2 dx < \infty \end{array}\right\}$$

$$\Rightarrow P_x(\zeta > t) \text{ decays exponentially as } t \to \infty$$

Ultracontractivity:

$$\left.\begin{array}{l} \sup\limits_{0<x<1} x^{\mu/(\mu-2)} m([x,1)) < \infty \\[2em] \sup\limits_{x\geq 1} x^{\mu/(\mu-2)} m(([x,\infty)) < \infty \end{array}\right\} \Leftrightarrow \; R_\mu$$

Under $m([1,\infty)) < \infty$,

$$\sup_{x\geq 1} x^{\mu/(\mu-2)} m(([x,\infty)) < \infty \;\Leftrightarrow\; R_\mu(0)$$

$$\left.\begin{array}{l} \displaystyle\int_{(0,1)} x\,m([x,1))^2\,dx < \infty \\[2ex] \displaystyle\sup_{x\geq 1} x^{\mu/(\mu-2)} m([x,\infty)) < \infty \end{array}\right\}$$

$$\Rightarrow\ p(t,x,y) \text{ decays exponentially as } t \to \infty$$

The case $D = (-\infty, \infty)$

DUC	$m(\mathbb{R}) < \infty \iff S_0$ $m(\mathbb{R}) = \infty \iff$ DUC does not hold
UC	Under $m(\mathbb{R}) < \infty$ $\left.\begin{array}{c} \sup_{x \geq 1} x^{\frac{\mu}{\mu-2}} m([x, \infty)) < \infty \\ \sup_{x \geq 1} x^{\frac{\mu}{\mu-2}} m((-\infty, -x]) < \infty \end{array}\right\} \iff R_\mu(0)$ $p(t, x, y)$ converges to $m(\mathbb{R})^{-1}$ exponentially

Proof in the case $D = (0, \infty)$

Assume

$$\sup_{x>0} x^{\mu/(\mu+2)} m([x, \infty)) := M < \infty.$$

Define

$$If(x) = \int_0^x f(t)\, dt.$$

Then $I\colon L^1(dt) \to L^\infty(m)$ is bounded and

$$|If(x)| \le x \|f\|_\infty, \quad f \in L^\infty(dt)$$

and hence

$$\begin{aligned}
m(\{x;\ |If(x)| > \lambda\}) &\le m(\{x;\ x\|f\|_\infty > \lambda\}) \\
&= m((\lambda/\|f\|_\infty, \infty)) \\
&\le M\left(\frac{\lambda}{\|f\|_\infty}\right)^{-\mu/(\mu+2)}.
\end{aligned}$$

This means that I is of weak $(\infty, \frac{\mu}{\mu+2})$-type. By the interpolation theorem, $I\colon L^2(dt) \to L^{2\mu/(\mu+2)}(m)$ is bounded, which means ($g = If$)

$$\|g\|^2_{2\mu/(\mu+2)} \le C_1\,\mathcal{E}(g, g).$$

D	**DUC**
$(0,l)$	$\sup\limits_{0<x<l/2} x^{\frac{\mu}{\mu+2}} m([x,l/2)) < \infty$ $\sup\limits_{l/2<x<l} (l-x)^{\frac{\mu}{\mu+2}} m((l/2,x]) < \infty$
$(0,\infty)$	$\sup\limits_{x>0} x^{\frac{\mu}{\mu+2}} m([x,\infty)) < \infty$
$\mathbb{R}$	$m(\mathbb{R}) < \infty$

D	**UC**
$(0,l)$	$\sup\limits_{0<x<l/2} x^{\frac{\mu}{\mu-2}} m([x,l/2)) < \infty$ $\sup\limits_{l/2<x<l} (l-x)^{\frac{\mu}{\mu-2}} m((l/2,x]) < \infty$
$(0,\infty)$	$\sup\limits_{x>0} x^{\frac{\mu}{\mu-2}} m(([x,\infty)) < \infty$
$\mathbb{R}$	$\sup\limits_{x\geq 1} x^{\frac{\mu}{\mu-2}} m([x,\infty)) < \infty$ $\sup\limits_{x\geq 1} x^{\frac{\mu}{\mu-2}} m((-\infty,-x]) < \infty$