The dual ultracontractivity and its applications

Ichiro SHIGEKAWA

KYOTO UNIVERSITY

December 13, 2007

Kumamoto University

URL: **http://www.math.kyoto-u.ac.jp/~ichiro/**

Contents

1. Introduction

(X_t, P_x): an m-symmetric Markov process on M

- $\{T_t\}$: the associated semigroup
- $\mathfrak{A}$: the generator $\{T_t\}$
- $\mathcal{E}$: the Dirichlet form

Ultracontractivity

$\{T_t\}$ is called ultracontractive if

$$\|T_t f\|_\infty \leq a_t \|f\|_1, \quad \forall f \in L^1(m), \forall t > 0.$$

Criteria for ultracontractivity

Let $\mu > 0$. The followings are equivalent to each other:

(i) There exists a constant c_1 so that

$$\|T_t f\|_\infty \le c_1 t^{-\mu/2} \|f\|_1, \quad \forall f \in L^1, \forall t > 0.$$

(ii) There exists a constant c_2 so that

$$\|f\|_2^{2+4/\mu} \le c_2 \, \mathcal{E}(f,f) \, \|f\|_1^{4/\mu}, \quad \forall f \in \mathrm{Dom}(\mathcal{E}) \cap L^1.$$

If $\mu > 2$ the conditions above are equivalent to

(iii) There exists a constant c_3 so that

$$\|f\|_{2\mu/(\mu-2)}^2 \le c_3 \, \mathcal{E}(f,f), \quad \forall f \in \mathrm{Dom}(\mathcal{E}).$$

We are interested in the following property:

There exist constants b_t so that

$$\|T_t f\|_1 \leq b_t \|f\|_\infty, \quad \forall f \in L^\infty(m), \forall t > 0.$$

This property called the **dual ultracontractivity**.

$$b_t = \|T_t\|_{\infty \to 1} = P_m(\zeta > t).$$

Ultracontractivity

$$\|T_t f\|_\infty \leq a_t \|f\|_1, \quad \forall f \in L^1(m), \forall t > 0.$$

2. Dual ultracontractivity

DUC	$\|T_t f\|_1 \le c_1 t^{-\mu/2} \|f\|_\infty, \quad t > 0,\ f \in L^\infty$ $\|f\|_2^{2+4/\mu} \le c_2\, \mathcal{E}(f,f)\, \|f\|_\infty^{4/\mu}$ $\|f\|_{2\mu/(\mu+2)}^2 \le c_3\, \mathcal{E}(f,f) \quad (\mu > 2)$
UC	$\|T_t f\|_\infty \le c_1 t^{-\mu/2} \|f\|_1, \quad t > 0,\ f \in L^1$ $\|f\|_2^{2+4/\mu} \le c_2\, \mathcal{E}(f,f)\, \|f\|_1^{4/\mu}$ $\|f\|_{2\mu/(\mu-2)}^2 \le c_3\, \mathcal{E}(f,f) \quad (\mu > 2)$

DUC	$\|T_t f\|_1 \le c_1 t^{-\mu/2} \|f\|_\infty, \quad t \in (0,1],\ f \in L^\infty$ $\|f\|_2^{2+4/\mu} \le c_2 \left(\mathcal{E}(f,f) + \|f\|_2^2\right) \|f\|_\infty^{4/\mu}$ $\|f\|_{2\mu/(\mu+2)}^2 \le c_3 \left(\mathcal{E}(f,f) + \|f\|_2^2\right) \quad (\mu > 2)$
UC	$\|T_t f\|_\infty \le c_1 t^{-\mu/2} \|f\|_1, \quad t \in (0,1],\ f \in L^1$ $\|f\|_2^{2+4/\mu} \le c_2 \left(\mathcal{E}(f,f) + \|f\|_2^2\right) \|f\|_1^{4/\mu}$ $\|f\|_{2\mu/(\mu-2)}^2 \le c_3 \left(\mathcal{E}(f,f) + \|f\|_2^2\right) \quad (\mu > 2)$

3. One dimensional diffusion processes

$D = (l_1, l_2)$.

$\{(X_t), P_x\}$: a (minimal) diffusion on D (Dirichlet boundary condition)

$s(x)$: the sclae function. We assume that $s(x) = x$.

dm : the speed measure, $m(y) - m(x) = \int_{(x,y]} dm$

ζ : the explosion time

$\frac{d}{dm}\frac{d}{ds}$: the generator

Dirichlet form: $\mathcal{E}(f, g) = \int_D \frac{df}{ds}\frac{dg}{ds} ds$

D	**DUC**
$(0,l)$	$\sup_{0<x<l/2} x^{\frac{\mu}{\mu+2}} m([x,l/2)) < \infty$ $\sup_{l/2<x<l} (l-x)^{\frac{\mu}{\mu+2}} m((l/2,x]) < \infty$
$(0,\infty)$	$\sup_{x>0} x^{\frac{\mu}{\mu+2}} m([x,\infty)) < \infty$
$\mathbb{R}$	$m(\mathbb{R}) < \infty$

D	**UC**
$(0,l)$	$\sup_{0<x<l/2} x^{\frac{\mu}{\mu-2}} m([x,l/2)) < \infty$ $\sup_{l/2<x<l} (l-x)^{\frac{\mu}{\mu-2}} m((l/2,x]) < \infty$
$(0,\infty)$	$\sup_{x>0} x^{\frac{\mu}{\mu-2}} m(([x,\infty)) < \infty$
$\mathbb{R}$	$\sup_{x\geq 1} x^{\frac{\mu}{\mu-2}} m([x,\infty)) < \infty$ $\sup_{x\geq 1} x^{\frac{\mu}{\mu-2}} m((-\infty,-x]) < \infty$

Asymptotics of $P_x(\zeta > t)$

$$p(t,x,y) = \sum_{j=0}^{\infty} e^{-\lambda_j t}\varphi_j(x)\varphi_j(y).$$

$$P_x[\zeta > t] = \int_D p(t,x,y)\, dm(y)$$

$$\sim e^{-\lambda_0 t}\varphi_0(x) \int_D \varphi_0(y)\, dm(y).$$

To ensure $\int_D \varphi_0(y)\, dm(y) < \infty$, we need the dual ultracontractivity:

$$\varphi_0 \in L^2(m) \Rightarrow \varphi_0 \in L^1(m).$$