

The dual ultracontractivity and its applications

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1. Introduction

(X_t, P_x) : an m -symmetric Markov process on M

- $\{T_t\}$: the associated semigroup
- \mathfrak{A} : the generator $\{T_t\}$
- \mathcal{E} : the Dirichlet form

Ultracontractivity

$\{T_t\}$ is called ultracontractive if

$$\|T_t f\|_\infty \leq a_t \|f\|_1, \quad \forall f \in L^1(m), \quad \forall t > 0.$$

Criteria for ultracontractivity

Let $\mu > 0$. The followings are equivalent to each other:

(i) There exists a constant c_1 so that

$$\|T_t f\|_\infty \leq c_1 t^{-\mu/2} \|f\|_1, \quad \forall f \in L^1, \forall t > 0.$$

(ii) There exists a constant c_2 so that

$$\|f\|_2^{2+4/\mu} \leq c_2 \mathcal{E}(f, f) \|f\|_1^{4/\mu}, \quad \forall f \in \text{Dom}(\mathcal{E}) \cap L^1.$$

If $\mu > 2$ the conditions above are equivalent to

(iii) There exists a constant c_3 so that

$$\|f\|_{2\mu/(\mu-2)}^2 \leq c_3 \mathcal{E}(f, f), \quad \forall f \in \text{Dom}(\mathcal{E}).$$

We are interested in the following property:

There exist constants b_t so that

$$\|\mathbf{T}_t f\|_1 \leq b_t \|f\|_\infty, \quad \forall f \in L^\infty(m), \quad \forall t > 0.$$

This property called the **dual ultracontractivity**.

$$b_t = \|\mathbf{T}_t\|_{\infty \rightarrow 1} = P_m(\zeta > t).$$

Ultracontractivity

$$\|\mathbf{T}_t f\|_\infty \leq a_t \|f\|_1, \quad \forall f \in L^1(m), \quad \forall t > 0.$$

2. Dual ultracontractivity

Theorem 2.1. Let $\mu > 0$. The followings are equivalent to each other:

(i) There exists a constant c_1 so that

$$\|T_t f\|_1 \leq c_1 t^{-\mu/2} \|f\|_\infty, \quad \forall f \in L^\infty, \forall t > 0.$$

(ii) There exists a constant c_2 so that

$$\|f\|_2^{2+4/\mu} \leq c_2 \mathcal{E}(f, f) \|f\|_\infty^{4/\mu}, \quad \forall f \in \text{Dom}(\mathcal{E}) \cap L^\infty.$$

Proof. From (i)

$$\|T_{2t}f\|_1 \leq C_1 t^{-\mu/2} \|f\|_\infty.$$

Hence, for $0 \leq f \in \text{Dom}(\mathcal{E}) \cap L^\infty$

$$\begin{aligned} C_1 t^{-\mu/2} \|f\|_\infty^2 &\geq \|T_{2t}f\|_1 \|f\|_\infty \\ &\geq (T_{2t}f, f) \\ &= (f, f) + \int_0^t \frac{d}{ds} (T_s f, T_s f) ds \\ &= (f, f) - 2 \int_0^t \mathcal{E}(T_s f, T_s f) ds \\ &\geq (f, f) - 2t \mathcal{E}(f, f). \end{aligned}$$

Therefore

$$\|f\|_2^2 \leq 2t\mathcal{E}(f, f) + C_1 t^{-\mu/2} \|f\|_\infty^2.$$

Choose

$$t = \left(\frac{\|f\|_\infty^2}{\mathcal{E}(f, f)} \right)^{2/(\mu+2)}$$

so that

$$\|f\|_2^2 \leq (2 + C_1) \mathcal{E}(f, f)^{\mu/(\mu+2)} \|f\|_\infty^{4/(\mu+2)}.$$

Thus

$$\|f\|_2^{2+4/\mu} \leq (2 + C_1)^{(\mu+2)/\mu} \mathcal{E}(f, f) \|f\|_\infty^{4/\mu},$$

which is (ii). □

Theorem 2.2. Let $\mu > 2$. The followings are equivalent to each other:

(i) There exists a constant c_1 so that

$$\|T_t f\|_1 \leq c_1 t^{-\mu/2} \|f\|_\infty, \quad \forall f \in L^\infty, \forall t > 0.$$

(ii) There exists a constant c_3 so that

$$\|f\|_{2\mu/(\mu+2)}^2 \leq c_3 \mathcal{E}(f, f), \quad \forall f \in \text{Dom}(\mathcal{E}), \forall t > 0.$$

Remark. $1 < \frac{2\mu}{\mu+2} < 2$.

If $\mu < 2$, then $\frac{2\mu}{\mu+2} < 1$ and (ii) \Rightarrow (i) holds.

DUC	$\ T_t f\ _1 \leq c_1 t^{-\mu/2} \ f\ _\infty, \quad t > 0, f \in L^\infty$ $\ f\ _2^{2+4/\mu} \leq c_2 \mathcal{E}(f, f) \ f\ _\infty^{4/\mu}$ $\ f\ _{2\mu/(\mu+2)}^2 \leq c_3 \mathcal{E}(f, f) \quad (\mu > 2)$
UC	$\ T_t f\ _\infty \leq c_1 t^{-\mu/2} \ f\ _1, \quad t > 0, f \in L^1$ $\ f\ _2^{2+4/\mu} \leq c_2 \mathcal{E}(f, f) \ f\ _1^{4/\mu}$ $\ f\ _{2\mu/(\mu-2)}^2 \leq c_3 \mathcal{E}(f, f) \quad (\mu > 2)$

DUC	$\ T_t f\ _1 \leq c_1 t^{-\mu/2} \ f\ _\infty, \quad t > 0, f \in L^\infty$ $\ f\ _2^{2+4/\mu} \leq c_2 \mathcal{E}(f, f) \ f\ _\infty^{4/\mu}$ $\ f\ _{2\mu/(\mu+2)}^2 \leq c_3 \mathcal{E}(f, f) \quad (\mu > 2)$
UC	$\ T_t f\ _\infty \leq c_1 t^{-\mu/2} \ f\ _1, \quad t > 0, f \in L^1$ $\ f\ _2^{2+4/\mu} \leq c_2 \mathcal{E}(f, f) \ f\ _1^{4/\mu}$ $\ f\ _{2\mu/(\mu-2)}^2 \leq c_3 \mathcal{E}(f, f) \quad (\mu > 2)$

DUC	$\ \mathbf{T}_t f\ _1 \leq c_1 t^{-\mu/2} \ f\ _\infty, \quad t \in (0, 1], f \in L^\infty$ $\ f\ _2^{2+4/\mu} \leq c_2 (\mathcal{E}(f, f) + \ f\ _2^2) \ f\ _\infty^{4/\mu}$ $\ f\ _{2\mu/(\mu+2)}^2 \leq c_3 (\mathcal{E}(f, f) + \ f\ _2^2) \quad (\mu > 2)$
UC	$\ \mathbf{T}_t f\ _\infty \leq c_1 t^{-\mu/2} \ f\ _1, \quad t \in (0, 1], f \in L^1$ $\ f\ _2^{2+4/\mu} \leq c_2 (\mathcal{E}(f, f) + \ f\ _2^2) \ f\ _1^{4/\mu}$ $\ f\ _{2\mu/(\mu-2)}^2 \leq c_3 (\mathcal{E}(f, f) + \ f\ _2^2) \quad (\mu > 2)$

3. One dimensional diffusion processes

$$D = (l_1, l_2).$$

$\{(X_t), P_x\}$: a (minimal) diffusion on D (Dirichlet boundary condition)

$s(x)$: the **sclae function**. We assume that $s(x) = x$.

dm : the **speed measure**, $m(y) - m(x) = \int_{(x,y]} dm$

ζ : the explosion time

$\frac{d}{dm} \frac{d}{ds}$: the **generator**

Dirichlet form: $\mathcal{E}(f, g) = \int_D \frac{df}{ds} \frac{dg}{ds} ds$

We introduce some notations.

Ultracontractivity:

$$\mathbf{R}_\mu : \quad \|\mathbf{T}_t \mathbf{f}\|_\infty \leq C t^{-\mu/2} \|\mathbf{f}\|_1, \quad \forall t > 0,$$

$$\mathbf{R}_\mu(\mathbf{0}) : \quad \|\mathbf{T}_t \mathbf{f}\|_\infty \leq C t^{-\mu/2} \|\mathbf{f}\|_1, \quad \forall t \in (0, 1].$$

Dual ultracontractivity:

$$\mathbf{S}_\mu : \quad \|\mathbf{T}_t \mathbf{f}\|_1 \leq C t^{-\mu/2} \|\mathbf{f}\|_\infty, \quad \forall t > 0,$$

$$\mathbf{S}_\mu(\mathbf{0}) : \quad \|\mathbf{T}_t \mathbf{f}\|_1 \leq C t^{-\mu/2} \|\mathbf{f}\|_\infty, \quad \forall t \in (0, 1].$$

The case $D = (0, l)$

Dual ultracontractivity:

$$\left. \begin{array}{l} \sup_{0 < x < l/2} x^{\frac{\mu}{\mu+2}} m([x, l/2]) < \infty \\ \sup_{l/2 < x < l} (l - x)^{\frac{\mu}{\mu+2}} m((l/2, x]) < \infty \end{array} \right\} \Leftrightarrow S_\mu \Leftrightarrow S_\mu(\mathbf{0})$$

$$m((0, l)) < \infty \Leftrightarrow S_0$$

$$S_\mu \Rightarrow P_x(\zeta > t) \text{ decays exponentially as } t \rightarrow \infty$$

Asymptotics of $P_x(\zeta > t)$

- $0 < \lambda_0 < \lambda_1 \leq \dots$: a discrete spectrum of $-\mathfrak{A}$.
- $\{\varphi_i\}$: c.o.n.s of eigen-functions
- $p(t, x, y)$: transition density function

$$p(t, x, y) = \sum_{j=0}^{\infty} e^{-\lambda_j t} \varphi_j(x) \varphi_j(y).$$

Then

$$\begin{aligned} P_x[\zeta > t] &= \int_D p(t, x, y) dm(y) \\ &\sim e^{-\lambda_0 t} \varphi_0(x) \int_D \varphi_0(y) dm(y). \end{aligned}$$

To ensure $\int_D \varphi_0(\mathbf{y}) dm(\mathbf{y}) < \infty$, we need the dual ultracontractivity:

$$\varphi_0 \in L^2(m) \Rightarrow \varphi_0 \in L^1(m).$$

Ultracontractivity: (Mao [2002] Acta Math. Sinica)

$$\sup_{0 < x < l/2} x^{\frac{\mu}{\mu-2}} m([x, l/2)) < \infty$$

$$\sup_{l/2 < x < l} (l - x)^{\frac{\mu}{\mu-2}} m((l/2, x]) < \infty$$

$\Leftrightarrow R_\mu$

R_2 always holds.

$$\left. \begin{aligned} \int_{(0, l/2)} x m([x, l/2))^2 dx < \infty \\ \int_{(l/2, l)} (l - x) m((l/2, x])^2 dx < \infty \end{aligned} \right\}$$

$\Rightarrow p(t, x, y)$ decays exponentially as $t \rightarrow \infty$

The case $D = (0, \infty)$

Dual ultracontractivity:

$$\left. \begin{array}{l} \sup_{0 < x < 1} x^{\mu/(\mu+2)} m([x, 1)) < \infty \\ \sup_{x \geq 1} x^{\mu/(\mu+2)} m([x, \infty)) < \infty \end{array} \right\} \Leftrightarrow S_{\mu}$$
$$\left. \begin{array}{l} \sup_{0 < x < 1} x^{\mu/(\mu+2)} m([x, 1)) < \infty \\ m([1, \infty)) < \infty \end{array} \right\} \Leftrightarrow S_{\mu}(0)$$

$$\int_{(0,1)} x dm(x) = \infty \Rightarrow \text{DUC does not hold}$$

$$\sup_{0 < x < 1} x^{\mu/(\mu+2)} m([x, 1)) < \infty$$

$$\int_{(0,\infty)} x m([x, \infty))^2 dx < \infty$$

$\Rightarrow P_x(\zeta > t)$ decays exponentially as $t \rightarrow \infty$

Ultracontractivity:

$\sup_{0 < x < 1} x^{\mu/(\mu-2)} m([x, 1)) < \infty$ $\sup_{x \geq 1} x^{\mu/(\mu-2)} m([x, \infty)) < \infty$	$\Leftrightarrow R_\mu$
<p>Under $m([1, \infty)) < \infty$,</p> $\sup_{x \geq 1} x^{\mu/(\mu-2)} m([x, \infty)) < \infty \Leftrightarrow R_\mu(0)$	

$$\int_{(0,1)} x m([x, 1))^2 dx < \infty$$

$$\sup_{x \geq 1} x^{\mu/(\mu-2)} m([x, \infty)) < \infty$$

$\Rightarrow p(t, x, y)$ decays exponentially as $t \rightarrow \infty$

The case $D = (-\infty, \infty)$

DUC	$m(\mathbb{R}) < \infty \Leftrightarrow S_0$ $m(\mathbb{R}) = \infty \Leftrightarrow \text{DUC does not hold}$
UC	<p>Under $m(\mathbb{R}) < \infty$</p> $\left. \begin{aligned} \sup_{x \geq 1} x^{\frac{\mu}{\mu-2}} m([x, \infty)) < \infty \\ \sup_{x \geq 1} x^{\frac{\mu}{\mu-2}} m((-\infty, -x]) < \infty \end{aligned} \right\} \Leftrightarrow R_\mu(\mathbf{0})$ <p>$p(t, x, y)$ converges to $m(\mathbb{R})^{-1}$ exponentially</p>

Proof in the case $D = (0, \infty)$

Assume

$$\sup_{x>0} x^{\mu/(\mu+2)} m([x, \infty)) := M < \infty.$$

Define

$$If(x) = \int_0^x f(t) dt.$$

Then $I: L^1(dt) \rightarrow L^\infty(m)$ is bounded and

$$|If(x)| \leq x \|f\|_\infty, \quad f \in L^\infty(dt)$$

and hence

$$\begin{aligned}
 m(\{x; |If(x)| > \lambda\}) &\leq m(\{x; x\|f\|_\infty > \lambda\}) \\
 &= m((\lambda/\|f\|_\infty, \infty)) \\
 &\leq M \left(\frac{\lambda}{\|f\|_\infty} \right)^{-\mu/(\mu+2)}.
 \end{aligned}$$

This means that I is of **weak $(\infty, \frac{\mu}{\mu+2})$ -type**. By the interpolation theorem, $I: L^2(dt) \rightarrow L^{2\mu/(\mu+2)}(m)$ is bounded, which means ($g = If$)

$$\|g\|_{2\mu/(\mu+2)}^2 \leq C_1 \mathcal{E}(g, g).$$

D	DUC
$(0, l)$	$\sup_{0 < x < l/2} x^{\frac{\mu}{\mu+2}} m([x, l/2)) < \infty$ $\sup_{l/2 < x < l} (l - x)^{\frac{\mu}{\mu+2}} m((l/2, x]) < \infty$
$(0, \infty)$	$\sup_{x > 0} x^{\frac{\mu}{\mu+2}} m([x, \infty)) < \infty$
\mathbb{R}	$m(\mathbb{R}) < \infty$
D	UC
$(0, l)$	$\sup_{0 < x < l/2} x^{\frac{\mu}{\mu-2}} m([x, l/2)) < \infty$ $\sup_{l/2 < x < l} (l - x)^{\frac{\mu}{\mu-2}} m((l/2, x]) < \infty$
$(0, \infty)$	$\sup_{x > 0} x^{\frac{\mu}{\mu-2}} m([x, \infty)) < \infty$
\mathbb{R}	$\sup_{x \geq 1} x^{\frac{\mu}{\mu-2}} m([x, \infty)) < \infty$ $\sup_{x \geq 1} x^{\frac{\mu}{\mu-2}} m((-\infty, -x]) < \infty$