Witten Laplacian for a lattice spin system

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1. Witten Laplacian in finite dimension

- Φ: a C^2 function on $\mathbb{R}^N$, (Hamiltonian)
- ν: a measure on $\mathbb{R}^N$ defined by

$$\nu(dx) = Z^{-1}e^{-2\Phi}dx, \qquad Z = \int_{\mathbb{R}^N} e^{-2\Phi}dx$$

- a Dirichlet form $\mathcal{E}$:

$$\mathcal{E}(f,g) = \int_{\mathbb{R}^N} (\nabla f, \nabla g)d\nu(x),$$

where $\nabla = (\partial_1, \ldots, \partial_N)$, $\partial_k = \frac{d}{dx_k}$.

- the dual of ∂_j is

$$\partial_j^* = -\partial_j + 2\partial_j\Phi.$$

- the generator $\mathfrak{A}$ is

$$\mathfrak{A}f = \sum_j (\partial_j^2 f - 2\partial_j \Phi \partial_j f) = \triangle f - 2(\nabla \Phi, \nabla f). \tag{1.1}$$

$\mathfrak{A}$ is essentially self-adjoint in $C_0^\infty(\mathbb{R}^N)$.

Witten Laplacian

We now define a Witten Laplacian. Let $I\colon L^2(dx) \longrightarrow L^2(\nu)$ be defined by

$$If(x) = e^{\Phi} f. \tag{1.2}$$

Lwt X_j be an operator defined by

$$X_j = e^{-\Phi} \partial_j e^{\Phi} = \partial_j + \partial_j \Phi.$$

Then the following is commutative:

$$\begin{array}{ccc} L^2(dx) & \xrightarrow{I} & L^2(\nu) \\ X_j \Big\downarrow & & \Big\downarrow \partial_j \\ L^2(dx) & \xrightarrow{I} & L^2(\nu) \end{array}$$

We use the convention that

- $*$ stands for the dual operator in $L^2(\nu)$
- $\tilde{}$ stands for the dual operator in $L^2(dx)$

$\tilde{X}_j$ has the following form:

$$\tilde{X}_j = -\partial_j + \partial_j \Phi = e^{-\Phi} \partial_j^* e^{\Phi}.$$

The operaotr A associated with the generator $\mathfrak{A} = -\sum_j \partial_j^* \partial_j$ is

$$A = e^{-\Phi} \mathfrak{A} e^{\Phi} = -\sum_j \tilde{X}_j X_j = \triangle + \triangle \Phi - |\nabla \Phi|^2.$$

Definition 1. $A = \triangle + \triangle\Phi - |\nabla\Phi|^2$ in $L^2(dx)$ is called a Witten Laplacian.

Proposition 1.1. In $L^2(\nu)$, we have

$$[\partial_i, \partial_j] = 0, \tag{1.3}$$

$$[\partial_i, \partial_j^*] = 2\partial_i\partial_j\Phi, \tag{1.4}$$

$$[\partial_i^*, \partial_j^*] = 0. \tag{1.5}$$

Further, in $L^2(dx)$, we have

$$[X_i, X_j] = 0, \tag{1.6}$$

$$[X_i, \tilde{X}_j] = 2\partial_i\partial_j\Phi, \tag{1.7}$$

$$[\tilde{X}_i, \tilde{X}_j] = 0. \tag{1.8}$$

2. Witten Laplacian acting on differential forms

differential forms

- tensor product

  $\boldsymbol{t}$: a $\boldsymbol{p}$-linear functional, $\boldsymbol{s}$: a $\boldsymbol{q}$-linear functional

  A tensor product $\boldsymbol{t} \otimes \boldsymbol{s}$ is defined by

$$\begin{aligned} \boldsymbol{t} \otimes \boldsymbol{s}(v_1, \ldots, v_p, v_{p+1}, \ldots, v_{p+q}) \\ = \boldsymbol{t}(v_1, \ldots, v_p)\boldsymbol{s}(v_{p+1}, \ldots, v_{p+q}). \end{aligned}$$

- the alternation mapping $\boldsymbol{A_p}$:

$$At(v_1, \ldots, v_p)) = \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} \operatorname{sgn} \sigma \, t(v_{\sigma(1)}, \ldots, v_{\sigma(p)}).$$

- $\boldsymbol{\theta}$ is called alternating if $\boldsymbol{A_p \theta = \theta}$.

- $\bigwedge^p(\mathbb{R}^N)^*$: the set of all alternating functionals of degree p
- the exterior product $\theta \wedge \eta$ is defined by

$$\theta \wedge \eta = \frac{(p+q)!}{p!q!} A(\theta \otimes \eta), \quad \theta \in \bigwedge^p(\mathbb{R}^N)^*, \ \eta \in \bigwedge^q(\mathbb{R}^N)^*$$

- Taking an orthonomal basis $\theta_1, \dots, \theta_N$ in $(\mathbb{R}^N)^*$, the followings form a basin in $\bigwedge^p(\mathbb{R}^N)^*$

$$\theta_{i_1} \wedge \cdots \wedge \theta_{i_p} \tag{2.1}$$

- We define an inner product in $\bigwedge^p(\mathbb{R}^N)^*$ so that (2.1) become an o.n.b.
- $A^p(\mathbb{R}^N) = \mathbb{R}^N \times \bigwedge^p(\mathbb{R}^N)^*$ is an exterior bundle.
- A differential form: a section of $A^p(\mathbb{R}^N)$.

- $\boldsymbol{\Gamma(A^p(\mathbb{R}^N))}$: The set of all sections, identified with $\bigwedge^p(\mathbb{R}^N)^*$ valued functions.

Creation and anihilation operator

- **ext**$(\boldsymbol{\theta})$**:** $\bigwedge^p(\mathbb{R}^N)^* \longrightarrow \bigwedge^{p+1}(\mathbb{R}^N)^*$ is defined by

$$\mathbf{ext}(\boldsymbol{\theta})\boldsymbol{\omega} = \boldsymbol{\theta} \wedge \boldsymbol{\omega}$$

- **int**$(\boldsymbol{\theta})$**:** $\bigwedge^p(\mathbb{R}^N)^* \longrightarrow \bigwedge^{p-1}(\mathbb{R}^N)^*$ is defined by

$$\mathbf{int}(\boldsymbol{v})\boldsymbol{\omega}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{p-1}) = \boldsymbol{\omega}(\boldsymbol{v}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{p-1}).$$

- Taking a standard basis $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_N\}$ of $\mathbb{R}^N$ and its dual basis $\{\boldsymbol{\theta}^1, \ldots, \boldsymbol{\theta}^N\}$, we define

$$\boldsymbol{a}^i = \mathbf{int}(\boldsymbol{e}_i)$$
$$(\boldsymbol{a}^i)^* = \mathbf{ext}(\boldsymbol{\theta}^i).$$

They satisfy the following commutation relation:

$$[\boldsymbol{a}^i, \boldsymbol{a}^j]_+ = \mathbf{0} \tag{2.2}$$

$$[\boldsymbol{a}^i, (\boldsymbol{a}^j)^*]_+ = \boldsymbol{\delta}_{ij} \tag{2.3}$$

$$[(\boldsymbol{a}^i)^*, (\boldsymbol{a}^j)^*]_+ = \mathbf{0} \tag{2.4}$$

Here $[\boldsymbol{a}^i, \boldsymbol{a}^j]_+ = \boldsymbol{a}^i\boldsymbol{a}^j + \boldsymbol{a}^j\boldsymbol{a}^i$.

For differential forms, the covariant differentiation $\boldsymbol{\nabla}$ can be defined. More generaly, the covariant differentiation $\boldsymbol{\nabla}$ is defined for tensor fields as follows:

- the covariant differentiation $\boldsymbol{\nabla}$:

$$\boldsymbol{\nabla} \boldsymbol{t} = \sum_i \boldsymbol{\theta}^i \otimes \boldsymbol{\partial}_i \boldsymbol{t}.$$

- The dual operator of $\boldsymbol{\nabla}$:

$$\boldsymbol{\nabla}^* (\sum_i \boldsymbol{\theta}^i \otimes \boldsymbol{t}_i) = \sum_i \boldsymbol{\partial}_i^* \boldsymbol{t}_i.$$

- the covariant Laplacian

$$\boldsymbol{\nabla}^* \boldsymbol{\nabla} \boldsymbol{t} = \sum_i \boldsymbol{\partial}_i^* \boldsymbol{\partial}_i \boldsymbol{t} = -\sum_i (\boldsymbol{\partial}_i^2 - 2 \boldsymbol{\partial}_i \boldsymbol{\Phi} \boldsymbol{\partial}_i) \boldsymbol{t}.$$

- the exterior differentiation:

$$\boldsymbol{d} = \sum_i \mathbf{ext}(\boldsymbol{\theta}^i) \boldsymbol{\partial}_i = \sum_i (\boldsymbol{a}^i)^* \boldsymbol{\partial}_i.$$

- the dual operator of $\boldsymbol{d}$

$$\boldsymbol{d}^* = \sum_i \boldsymbol{a}^i \boldsymbol{\partial}_i^*.$$

- the Hodge-Kodaira Laplacian: $-(\boldsymbol{d}\boldsymbol{d}^* + \boldsymbol{d}^*\boldsymbol{d})$

Theorem 2.1. We have the following identity.

$$\boldsymbol{d}\boldsymbol{d}^* + \boldsymbol{d}^*\boldsymbol{d} = \boldsymbol{\nabla}^*\boldsymbol{\nabla} + 2\sum_{i,j}(\boldsymbol{a}^i)^*\boldsymbol{a}^j\boldsymbol{\partial}_i\boldsymbol{\partial}_j\boldsymbol{\Phi}.$$

Unitary equivalent expression

By the isomorphism $\boldsymbol{I}\colon \boldsymbol{L^2(dx)} \longrightarrow \boldsymbol{L^2(\nu)}$, we can compute associated operators under the the Lebesgue measure.

$$\boldsymbol{D = e^{-\Phi} d e^{\Phi},}$$
$$\boldsymbol{\tilde{D} = e^{-\Phi} d^* e^{\Phi}.}$$

and the Hodge-Kodaira operator $\boldsymbol{\tilde{D}D + D\tilde{D}}$.

Theorem 2.2. We have the following identities:

$$\boldsymbol{\tilde{D}D + D\tilde{D} = \sum_i \tilde{X}_i X_i + 2\sum_{i,j} (a^i)^* a^j \partial_i \partial_j \Phi.}$$

3. Spectral gap for Witten Laplacian in a lattice spin system

A spin system

A spin system is characterized by a Gibbs measure on $\boldsymbol{X} = \mathbb{R}^{\mathbb{Z}^{\boldsymbol{d}}}$.

- Hamiltonian:

$$\boldsymbol{\Phi}(\boldsymbol{x}) = \sum_{\substack{i,j\in\mathbb{Z}^d \\ i\sim j}} \mathcal{J}(x^i - x^j)^2 + \sum_{i\in\mathbb{Z}^d} U(x^i).$$

  Here $i \sim j$ means that
  $|i-j|^2 = (i_1 - j_1)^2 + \cdots + (i_1 - j_1)^2 = 1$.

- a Gibbs measure:

$$\nu = Z^{-1} e^{-2\Phi(x)} dx, \quad \text{(formal expression)}$$

- Λ: a finite region, η: a boundary condition

$$\Phi_{\Lambda,\eta}(x) = \sum_{\substack{i,j\in\Lambda \\ i\sim j}} \mathcal{J}(x^i - x^j)^2 + \sum_{i\in\Lambda} U(x^i) + 2 \sum_{\substack{i\in\Lambda, j\in\Lambda^c \\ i\sim j}} \mathcal{J}(x^i - \eta^j)^2$$

- Define a measure $\nu_{\Lambda,\eta}$ on $\mathbb{R}^\Lambda$ by

$$\nu_{\Lambda,\eta} = Z^{-1} e^{-2\Phi_{\Lambda,\eta}(x)} dx_\Lambda$$

- the Gibbs measure is characterized by the following Dobrushin-Lanford-Ruelle equation:

$$E^\nu[\,\cdot\,|\omega_{\Lambda^c} = \eta_{\Lambda^c}] = \nu_{\Lambda,\eta}(d\omega_\Lambda) \otimes \delta_{\eta_{\Lambda^c}}(d\omega_{\Lambda^c})$$

Theorem 3.1. Assume

- $U = V + W$, $V'' \geq c > 0$, W is bounded.
- $W_{\sup}$: supremum of W, $W_{\inf}$: infimum of W satisfy $2(c + 8d\mathcal{J})e^{-2(W_{\sup} - W_{\inf})} > 16d\mathcal{J}$.

For $p \geq 1$, the bottom of the $\sigma(dd^* + d^*d)$ acting on p-forms is greater than $\{2(c + 8d\mathcal{J})e^{-2(W_{\sup} - W_{\inf})} - 16d\mathcal{J}\}p$ and so there is no harmonic forms.

Theorem 3.2. Assume $U(t) = at^4 - bt^2$ and $\sqrt{3a} - b - 4d\mathcal{J} > 0$, then the same conclusinon as Theorem 3.1 holds.

4. The Hodge-Kodaira decomposition

Theorem 4.1. The following Hodge-Kodaira decomposition holds: for $\boldsymbol{p = 0}$,

$$\boldsymbol{L^2(\nu) = \{ \text{ constant functions } \} \oplus \mathrm{Ran}(d^*).}$$

and for $\boldsymbol{p \geq 1}$,

$$\boldsymbol{L^2(\nu; \bigwedge^p(\mathbb{R}^N)^*) = \mathrm{Ran}(d) \oplus \mathrm{Ran}(d^*)}$$