Schrödinger operators on the Wiener space

Ichiro SHIGEKAWA

Kyoto University

November 26, 2007

Zhongshan University

URL: **http://www.math.kyoto-u.ac.jp/~ichiro/**

Contents

1. Essential self-adjointness

$(\boldsymbol{B}, \boldsymbol{H}, \boldsymbol{\mu})$: an **abstract Wiener space**

- $\boldsymbol{B}$: a Banach space
- $\boldsymbol{H}$: a Hilbert space $\hookrightarrow \boldsymbol{B}$
- $\boldsymbol{\mu}$: the Wiener measure with

$$\int_B e^{\sqrt{-1}\langle x, \varphi\rangle} \mu(dx) = \exp\left\{-\frac{1}{2}|\varphi|_{H^*}^2\right\},$$

$$\varphi \in B^* \subset H^*.$$

$\mathcal{F}C_0^\infty$: $f: B \longrightarrow \mathbb{R}$ such that

$$f(x) = F(\langle x, \varphi_1 \rangle, \ldots, \langle x, \varphi_n \rangle),$$
$$F \in C_0^\infty(\mathbb{R}^n),\ \varphi_1, \ldots, \varphi_n \in B^*.$$

$L - V$: **Schrödinger operator** on $L^2(\mu)$

L: the **Ornstein-Uhlenbeck operator**

V: a scalar potential

Question:

Is $L - V$ essentially self-adjoin on $\mathcal{F}C_0^\infty$?

$\| \ \|_2$: L^2-norm

$V_+ := \max\{V, 0\}$ (the positive part)

$V_- := \max\{-V, 0\}$ (the negative part)

Proposition 1.1. Assume

- $V_+ \in L^{2+} = \bigcup_{p>2} L^p$,
- there exist $0 < a < 1$, $b > 0$ such that

$$\|V_- f\|_2 \le a\|Lf\|_2 + b\|f\|_2.$$

Then $L - V$ is essentially selfadjoint on $\mathcal{F}C_0^\infty$.

What is sufficient for

$$\|V_- f\|_2 \leq a\|Lf\|_2 + b\|f\|_2 \ ?$$

(Defective) logarithmic Sobolev inequality

$$\int_B |f|^2 \log(|f|/\|f\|_2)\, d\mu \leq \alpha \mathcal{E}(f, f) + \beta \|f\|_2^2.$$

- (B, μ): a probability space
- $\mathcal{E}$: a Dirichlet form
- L: the associated generator

We assume

- $\boldsymbol{\mathcal{E}}$ admits a **square field operator** $\boldsymbol{\Gamma}$.
- $\boldsymbol{\mathcal{E}}$ has a **local property**.

Hence $\boldsymbol{\mathcal{E}}$ has the following form

$$\boldsymbol{\mathcal{E}}(\boldsymbol{f},\boldsymbol{g}) = \int_{\boldsymbol{B}} \boldsymbol{\Gamma}(\boldsymbol{f},\boldsymbol{g})\, \boldsymbol{d\mu} \tag{1.1}$$

and $\boldsymbol{\Gamma}$ has the derivation property.

E.g. On an abstract Wiener space:

- $\boldsymbol{\Gamma}(\boldsymbol{f},\boldsymbol{g}) = \boldsymbol{\nabla f} \cdot \boldsymbol{\nabla g}$, $\quad \boldsymbol{\nabla}$: the gradient operator

Theorem 1.2. Assume

$$\int_B |f|^2 \log(|f|/\|f\|_2)\, d\mu \le \alpha \mathcal{E}(f,f) + \beta \|f\|_2^2.$$

Then, for any $\varepsilon > 0$, there exist positive constants K_1, K_2 such that

$$\int_B f^2 \log_+^2 f\, d\mu \le \alpha^2 (1+\varepsilon) \|Lf\|_2^2 + K_1 + K_2 \|f\|_2^6.$$

cf. Feissner(1975), Bakry-Meyer(1982)

Hausdorff-Young inequality

Set

$$\Phi(x) = x \log_+^2 x, \quad \psi^{-1}(x) = \Phi'(x),$$

$$\psi(x) = e^{\sqrt{x+1}-1}.$$

Define the complimentary function

$$\Psi(x) = \int_0^x \psi(y) dy.$$

Hausdorff-Young inequality:

$$xy \leq \Phi(x) + \Psi(y) \leq x \log_+^2 x + 2\sqrt{y} e^{\sqrt{y}}$$

Theorem 1.3. Assume the logarithmic inequality

$$\int_B |f|^2 \log(|f|/\|f\|_2)\, d\mu \leq \alpha\mathcal{E}(f,f) + \beta\|f\|_2^2$$

and $v \geq 0$,

$$e^v \in L^{2\alpha+} = \bigcup_{p>2\alpha} L^p.$$

Then, there exist constants $0 < a < 1$ and $b \geq 0$ such that

$$\|vf\|_2 \leq a\|Lf\|_2 + b\|f\|_2. \tag{1.2}$$

We now return to an abstract Wiener space.

Gross' logarithmic Sobolev inequality

$$\int_B |f|^2 \log(|f|/\|f\|_2)\, d\mu \le \int_B |\nabla f|^2\, d\mu$$

$$\Rightarrow \int_B f^2 \log_+^2 f\, d\mu \le (1+\varepsilon)\|Lf\|_2^2 + K_1 + K_2\|f\|_2^6.$$

Theorem 1.4. Assume

- $V_+,\ e^{V_-} \in L^{2+}$.

Then $L - V$ is essentially self-adjoint on $\mathcal{F}C_0^\infty$.

cf. Segal(1969), Glimm & Jaffe(1970), Simon(1973), Simon & Høegh-Krohn(1972)

2. Domain of Schrödinger operator

We consider a Schrödinger operator $\mathfrak{A} = L - V + W$ on an abstract Wiener space (B, H, μ).

Basic assumptions

(A.1) $V \geq 1$, $V \in L^{2+}$.

(A.2) $W \geq 0$ and there exists a constant $0 < \alpha < 1$ such that $e^W \in L^{2/\alpha}$.

$\Rightarrow \mathfrak{A} = L - V + W$ is essentially self-adjoint on $\mathcal{F}C_0^\infty$

Aim : To determine the domain,

i.e., $\mathbf{Dom}(\mathfrak{A}) = \mathbf{Dom}(\boldsymbol{L}) \cap \mathbf{Dom}(\boldsymbol{V})$

Main tools

- The **Lax-Milgram theorem**.
- The **intertwining property**, i.e.,

$$\sqrt{\boldsymbol{V}}\mathfrak{A} = \boldsymbol{A}\sqrt{\boldsymbol{V}}.$$

How to define an operator A?

We define a vector field b by

$$b = \frac{\nabla V}{2V} = \frac{1}{2}\nabla \log V.$$

and a bilinear form $\mathcal{E}_A$ by

$$\mathcal{E}_A(f, g) = (\nabla f, \nabla g) + (b \cdot \nabla f, g) \\ - (f, b \cdot \nabla g) + ((V - W - |b|^2) f, g).$$

By a formal computation, the associated generator is given by

$$A = L - 2b \cdot \nabla + (\nabla^* b - V + W + |b|^2). \tag{2.1}$$

Decompose $\mathcal{E}_A$ as

$$\mathcal{E}_A(f,g) = \underbrace{\hat{\mathcal{E}}_A(f,g)}_{\text{symmetric}} + \underbrace{\check{\mathcal{E}}_A(f,g)}_{\text{skew-symmetric}}$$

where

$$\hat{\mathcal{E}}_A(f,g) = (\nabla f, \nabla g) + ((V - W - |b|^2)f, g),$$
$$\check{\mathcal{E}}_A(f,g) = (b \cdot \nabla f, g) - (f, b \cdot \nabla g).$$

Moreover, we set

$$\hat{\mathcal{E}}_{A-\lambda}(f,g) = \hat{\mathcal{E}}_A(f,g) + \lambda(f,g).$$

The bilinear form associated to $L - V$ is

$$\mathcal{E}_{L-V}(f,g) = (\nabla f, \nabla g) + (Vf, g).$$

Clearly

$$\mathbf{Dom}(\mathcal{E}_{L-V}) = \mathbf{Dom}(\nabla) \cap \mathbf{Dom}(\sqrt{V}).$$

We will show that $\mathbf{Dom}(\hat{\mathcal{E}}_A) = \mathbf{Dom}(\mathcal{E}_{L-V})$.

Additional assumptions

We assume either

(B.1) $$e^{W+|b|^2} \in L^{2/\alpha}$$

or there exists a constant $C > 0$ such that

(B.2) $$|b|^2 \leq \alpha V + C.$$

Proposition 2.1. Assume (A.1), (A.2) and one of (B.1) and (B.2). Then there exists a constant $\boldsymbol{\beta}$ such that

$$\boldsymbol{(W + |b|^2 f, f) \leq \underset{\sim}{\alpha}\mathcal{E}_{L-V}(f, f) + \beta(f, f)}$$

appeared in (A.2)

and hence

$$\begin{aligned}\boldsymbol{(1-\alpha)\mathcal{E}_{L-V}(f, f)} &\boldsymbol{\leq \hat{\mathcal{E}}_A(f, f) + \beta(f, f)} \\ &\boldsymbol{\leq (1+\alpha)\mathcal{E}_{L-V}(f, f) + \beta(f, f).}\end{aligned}$$

Therefore

$$\mathbf{Dom}(\hat{\mathcal{E}}_A) = \mathbf{Dom}(\mathcal{E}_{L-V}) = \mathbf{Dom}(\nabla) \cap \mathbf{Dom}(\sqrt{V}).$$

Estimate of $\check{\mathcal{E}}_A$

Proposition 2.2. Assume (A.1), (A.2) and one of (B.1) and (B.2). Then, for sufficiently large λ, there exists a constant $K > 0$ such that

$$|\check{\mathcal{E}}_A(f, g)| \le K \hat{\mathcal{E}}_{A-\lambda}(f, f)^{1/2} \hat{\mathcal{E}}_{A-\lambda}(g, g)^{1/2}.$$

Therefore $\mathcal{E}_A$ satisfies the sector condition.

$\mathcal{E}_A = \hat{\mathcal{E}}_A + \check{\mathcal{E}}_A$ is a closed bilinear form.

Intertwining property

Instead of

$$\sqrt{V}\mathfrak{A} = A\sqrt{V},$$

we show

$$\mathcal{E}_{\mathfrak{A}}(f, \sqrt{V}g) = \mathcal{E}_A(\sqrt{V}f, g). \tag{2.2}$$

Proposition 2.3. Assume (A.1), (A.2) and one of (B.1) and (B.2). Then (2.2) holds for $f, g \in \mathcal{F}C_0^\infty$. Moreover, we have, for $f \in \mathbf{Dom}(\mathfrak{A})$, $g \in \mathbf{Dom}(A^*)$,

$$(\mathfrak{A}f, \sqrt{V}g) = (\sqrt{V}f, A^*g). \tag{2.3}$$

Domain of the Schrödinger operator

Theorem 2.4. Assume (A.1), (A.2) and one of (B.1) and (B.2). Then $\mathbf{Dom}(\mathfrak{A}) = \mathbf{Dom}(L) \cap \mathbf{Dom}(V)$. Moreover, for sufficiently large λ, there exist positive constants K_1, K_2 such that

$$\begin{aligned} K_1\|(\mathfrak{A}-\lambda)f\|_2 &\leq \|Lf\|_2 + \|Vf\|_2 \\ &\leq K_2\|(\mathfrak{A}-\lambda)f\|_2. \end{aligned}$$

Remark. K_1, K_2 depend only on constants in (A.1), (A.2), (B1), (B.2).

3. Spectral gap of Schrödinger operator

A Schrödinger operator $\mathfrak{A} = L - V + W$ on an abstract Wiener space (B, H, μ).

$\sigma(\mathfrak{A})$: the spectrum of $\mathfrak{A} = L - V + W$.

Bounded potential

Theorem 3.1. Assume V is bounded and $W = 0$. Then $l = \sup \sigma(\mathfrak{A})$ is a point spectrum of multiplicity one and the associated eigenfunction can be chosen to be positive. Moreover, the spectrum is discrete on $(l-1, l]$, i.e., it consists of point spectrums of finite multiplicity.

General potential

Theorem 3.2. Assume (A.1), (A.2) and one of (B.1) and (B.2). Then $\boldsymbol{l = \sup \sigma(\mathfrak{A})}$ is a point spectrum of multiplicity one and the associated eigenfunction can be chosen to be positive. Moreover, the spectrum is discrete on $\boldsymbol{(l-1, l}]$, i.e., it consists of point spectrums of finite multiplicity.

Proof of Theorem 3.1

Approximation method

$\{\varphi_i\}_{i=1}^{\infty} \subseteq \boldsymbol{B}^*$: a c.o.n.s of $\boldsymbol{H}^*$.

$\mathcal{F}_n := \sigma(\varphi_1, \varphi_2, \ldots, \varphi_n)$.

$V_n = E[V|\mathcal{F}_n]$.

$$\Rightarrow \begin{cases} \sigma(L - V_n) \text{ is discrete on } (\lambda(V_n) - 1, \lambda(V_n)] \\ \text{where } \lambda(V_n) = \sup \sigma(L - V_n). \end{cases}$$

We set

$$\boldsymbol{G^{(n)} = (\lambda - L + V_n)^{-1},}$$
$$\boldsymbol{G = (\lambda - L + V)^{-1}.}$$

claim: $\boldsymbol{G^{(n)} \to G}$ in norm sense

$$\boldsymbol{G - G^{(n)} = G^{(n)}(V - V_n)G.}$$

We show $\|(\boldsymbol{V} - \boldsymbol{V_n})\boldsymbol{G}\|_{\mathbf{op}} \to \mathbf{0}$.
By the logarithmic Sobolev inequality and the Hausdorff-Young inequality $\boldsymbol{xy \leq x \log x - x + e^y}$

$$
\begin{aligned}
&\|(V - V_n)Gf\|_2^2 \\
&= E[(V - V_n)^2(Gf)^2] \\
&= \frac{1}{N}E[N(V - V_n)^2(Gf)^2] \\
&\leq \frac{1}{N}E[(Gf)^2\log(Gf)^2 - (Gf)^2 + e^{N(V-V_n)^2}] \\
&\leq \frac{1}{N}\{2E[|\nabla Gf|^2] + \|Gf\|_2^2\log\|Gf\|_2^2 \\
&\qquad - \|Gf\|_2^2 + E[e^{N(V-V_n)^2}]\}.
\end{aligned}
$$

Now replacing $\boldsymbol{f}$ with $\boldsymbol{f}/\|\boldsymbol{G}\boldsymbol{f}\|_{\mathbf{2}}$,

$$
\begin{aligned}
&\|(\boldsymbol{V}-\boldsymbol{V}_{\boldsymbol{n}})\boldsymbol{G}\boldsymbol{f}\|_{\mathbf{2}}^{\mathbf{2}} \\
&\leq \frac{\mathbf{1}}{\boldsymbol{N}}\{\mathbf{2}\boldsymbol{E}[|\nabla \boldsymbol{G}\boldsymbol{f}|^{\mathbf{2}}]+\boldsymbol{E}[e^{\boldsymbol{N}(\boldsymbol{V}-\boldsymbol{V}_{\boldsymbol{n}})^{\mathbf{2}}}-\mathbf{1}]\|\boldsymbol{G}\boldsymbol{f}\|_{\mathbf{2}}^{\mathbf{2}}\} \\
&\leq \frac{\mathbf{1}}{\boldsymbol{N}}\{\boldsymbol{E}[\boldsymbol{f}^{\mathbf{2}}]+\boldsymbol{E}[(\boldsymbol{G}\boldsymbol{f})^{\mathbf{2}}]+\boldsymbol{E}[|\boldsymbol{V}|(\boldsymbol{G}\boldsymbol{f})^{\mathbf{2}}] \\
&\qquad\qquad +\boldsymbol{E}[e^{\boldsymbol{N}(\boldsymbol{V}-\boldsymbol{V}_{\boldsymbol{n}})^{\mathbf{2}}}-\mathbf{1}]\|\boldsymbol{f}\|_{\mathbf{2}}^{\mathbf{2}}\} \\
&\leq \frac{\mathbf{1}}{\boldsymbol{N}}\{(\mathbf{2}+\|\boldsymbol{V}\|_{\infty})\|\boldsymbol{f}\|_{\mathbf{2}}^{\mathbf{2}}+\boldsymbol{E}[e^{\boldsymbol{N}(\boldsymbol{V}-\boldsymbol{V}_{\boldsymbol{n}})^{\mathbf{2}}}-\mathbf{1}]\|\boldsymbol{f}\|_{\mathbf{2}}^{\mathbf{2}}\}.
\end{aligned}
$$

Hence

$$
\|(\boldsymbol{V}-\boldsymbol{V}_{\boldsymbol{n}})\boldsymbol{G}\|_{\mathbf{op}}^{\mathbf{2}} \leq \frac{\mathbf{1}}{\boldsymbol{N}}\{\mathbf{2}+\|\boldsymbol{V}\|_{\infty}+\boldsymbol{E}[e^{\boldsymbol{N}(\boldsymbol{V}-\boldsymbol{V}_{\boldsymbol{n}})^{\mathbf{2}}}-\mathbf{1}]\}
$$

Now letting $n \to \infty$ and then letting $N \to \infty$, we have

$$\lim_{n\to\infty} \|(V - V_n)G\|_{\mathrm{op}} = 0.$$

This completes the proof.