One dimensional diffusions conditioned to be non-explosive

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1. Introduction

- $\{(X_t), P_x\}$: a diffusion on a state space D.
- ζ : the explosion time.

The diffusion conditioned to be non-explosive is defined as follows:

1. If $P_x[\zeta = \infty] > 0$,

$$P_x[\,\cdot \mid \zeta = \infty] = \frac{P_x[\,\cdot\, \cap \zeta = \infty]}{P_x[\zeta = \infty]}.$$

2. If $P_x[\zeta = \infty] = 0$,

$$\lim_{T \to \infty} P_x[\,\cdot \mid \zeta > T]. \tag{1.1}$$

The limit (1.1) is called a surviving diffusion.

We discuss the following issues:

1. When does the surviving diffusion exist?
2. Characterizasion of the surviving diffusion.

Strategy:

Since

$$\boldsymbol{E}_x[\,\cdot \mid \zeta > \boldsymbol{T}] = \boldsymbol{E}_x\left[\,\cdot\ \frac{\mathbf{1}_{\{\zeta > t\}}\boldsymbol{P}_{X_t}[\zeta > \boldsymbol{T} - \boldsymbol{t}]]}{\boldsymbol{P}_x[\zeta > \boldsymbol{T}]}\right],$$

our problem is reduce to show the existence of the limit

$$\text{(1.2)} \qquad \boldsymbol{M}_t = \lim_{T\to\infty} \frac{\mathbf{1}_{\{\zeta > t\}}\boldsymbol{E}_{X_t}[\zeta > \boldsymbol{T} - \boldsymbol{t}]}{\boldsymbol{P}_x[\zeta > \boldsymbol{T}]}$$

and to show that $(\boldsymbol{M}_t)$ is a martingale.

To do this, we show that there exist a φ with $-\frac{d}{dm}\frac{d}{ds}\varphi = \lambda\varphi$ so that

(1.3) $$\lim_{T\to\infty} \frac{P_y[\zeta > T - t]}{P_x[\zeta > T]} = \frac{\varphi(y)e^{\lambda t}}{\varphi(x)}$$

and

(1.4) $$M_t = 1_{\{\zeta > t\}}\varphi(X_t)e^{\lambda t}/\varphi(x).$$

The surviving diffusion is given by

$$\hat{E}_x[\,\cdot\,] = E_x\left[\,\cdot\,1_{\{\zeta>t\}}\frac{\varphi(X_t)e^{\lambda t}}{\varphi(x)}\right].$$

2. One dimensional diffusion processes

$D = (l_-, l_+)$.

$\{(X_t), P_x\}$: a (minimal) diffusion on D (Dirichlet boundary condition)

$s(x)$: the sclae function

$dm(x)$: the speed measure (standard measure)

ζ : the explosion time

$\frac{d}{dm}\frac{d}{ds}$: the generator

Dirichlet form $\mathcal{E}(f,g) = \int_D \frac{df}{ds}\frac{dg}{ds}ds$

From dm, we define a right continuous non-decreasing function m as

$$m(y) - m(x) = \int_{(x,y]} dm$$

Take any $a \in (l_-, l_+)$ and define

$$S(x) = \int_{(a,x]} \{m(y) - m(a)\} ds(y) = \int_{(a,x]} \{s(x) - s(u)\} dm(u),$$

$$M(x) = \int_{(a,x]} \{s(y) - s(a)\} dm(y) = \int_{(a,x]} \{m(x) - m(u)\} ds(u).$$

- $S(l_+) < \infty \Rightarrow l_+$ is called exit.
- $S(l_+) = \infty \Rightarrow l_+$ is called non-exit.
- $M(l_+) < \infty \Rightarrow l_+$ is called entrance.
- $M(l_+) = \infty \Rightarrow l_+$ is called non-entrance.

Feller's criterion:

(X_t) is conservative $\Leftrightarrow S(l_+) = \infty$ and $S(l_-) = \infty$

h-transformation

Let v be a λ-harmonic function, i.e.,

$$\frac{d}{dm}\frac{d}{ds}v = \lambda v.$$

Define $d\hat{m} = v^2 dm$, $d\hat{s} = \frac{ds}{v^2}$. Then

$$\frac{1}{v}\left(\frac{d}{dm}\frac{d}{ds} - \lambda\right)(vf) = \frac{d}{d\hat{m}}\frac{d}{d\hat{s}}f. \tag{2.1}$$

$\frac{d}{d\hat{m}}\frac{d}{d\hat{s}}$ is the h-transform of $\frac{d}{dm}\frac{d}{ds} - \lambda$.

3. The case $P_x[\zeta = \infty] > 0$

Theorem 3.1. Let (X_t) be a diffusion process on $(0, l)$ with a natural scale $s(x) = x$ and a speed measure dm. Assume that 0 is exit and l is non-exit. Then $P_x[\zeta = \infty] > 0$ and the associated surviving diffusion has the scale $-1/x$ and the speed measure $x^2 dm$.

4. Exit - exit boundaries

$D = (0, l)$, the natural scale $s(x) = x$, the speed measure dm.

$$\int_0^{l/2} x\,dm(x) < \infty.$$

$$\int_{l/2}^{l} (l - x)\,dm(x) < \infty.$$

We assumet that there exists $\gamma > 0$ and M so that

$$\int_0^{y} x\,dm(x) \le My^{\gamma}.$$

$$\int_{l-y}^{l} (l - x)\,dm(x) \le My^{\gamma}.$$

In this case, the Green operator is of trace class. We define $\lambda_0 > 0$ to be a lowest eigenvalue of $-\frac{d}{dm}\frac{d}{ds}$ and φ_0 be its eigenfunction. φ_0 has the following asymptotics:

$$\varphi_0(x) \sim c_1 x \quad \text{as } x \longrightarrow 0$$
$$\varphi_0(x) \sim c_2(l - x) \quad \text{as } x \longrightarrow l.$$

Under these conditions,

Theorem 4.1.

$$\lim_{T\to\infty} e^{\lambda_0 T} P_x[\zeta > T] = \varphi_0(x) \int_D \varphi_0(y) dm(y).$$

In particular,

$$\lim_{T\to\infty} \frac{P_y[\zeta > T - t]}{P_x[\zeta > T]} = e^{\lambda_0 t} \frac{\varphi_0(y)}{\varphi_0(x)}.$$

The surviving diffusion exists and it has a scale $d\hat{s} = ds/\varphi_0^2$ and a speed measure $d\hat{m} = \varphi_0^2 dm$.

5. (exit & entrance) - (non-exit & non-entrance) boundaries

$D = (0, \infty)$, the natural scale $s(x) = x$, the speed measure dm. We assume

(5.1) $$m(x) \sim x^{1/\mu - 1} K(x) \quad \text{as } x \to \infty$$

where $0 < \mu < 1$ and K is a slowly varying function. Define a slowly varying function L so that the function $y \mapsto y^{\mu} L(y)$ is an inverse of the function $y \mapsto y^{1/\mu} K(y)$.

Under these conditions,

Theorem 5.1.

$$P_x[\zeta > t] \sim x\{\mu(1-\mu)\}^{\mu}\Gamma(1+\mu)^{-1}t^{-\mu}L(t)^{-1} \quad \text{as } t \to \infty.$$

In particular,

$$\lim_{T\to\infty} \frac{P_y[\zeta > T-t]}{P_x[\zeta > T]} = \frac{y}{x}.$$

The surviving diffusion exists and it has a scale $\hat{s}(x) = -1/x$ and a speed measure $d\hat{m} = x^2 dm$.

6. exit - (non-exit & entrance) boundaries

$D = (0, \infty)$, the natural scale $s(x) = x$, the speed measure dm. From the boundary condition,

$$\int_0^\infty x dm(x) < \infty.$$

We assumet that there exists $\gamma > 0$ and M so that

$$\int_0^y x dm(x) \leq M y^\gamma, \quad y > 0.$$

In this case, the Green operator is of trace class. We define $\lambda_0 > 0$ to be a lowest eigenvalue of $-\frac{d}{dm}\frac{d}{ds}$ and φ_0 be its eigenfunction.

$$\varphi_0(x) \sim c_1 x \quad \text{as } x \longrightarrow 0$$

$$\varphi_0(x) \sim c_2 \quad \text{as } x \longrightarrow \infty.$$

Under these conditions,

Theorem 6.1.

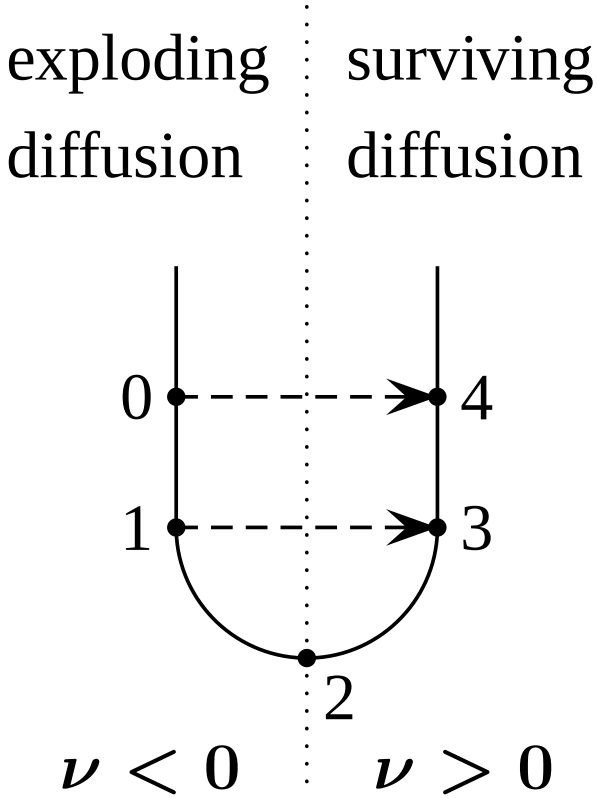
$$\lim_{T\to\infty} e^{\lambda_0 T} P_x[\zeta > T] = \varphi_0(x) \int_D \varphi_0(y)\,dm(y).$$

In particular,

$$\lim_{T\to\infty} \frac{P_y[\zeta > T - t]}{P_x[\zeta > T]} = e^{\lambda_0 t} \frac{\varphi_0(y)}{\varphi_0(x)}. \tag{6.1}$$

The surviving diffusion exists and it has a scale $d\hat{s} = ds/\varphi_0^2$ and a speed measure $d\hat{m} = \varphi_0^2 dm$.

7. Examples

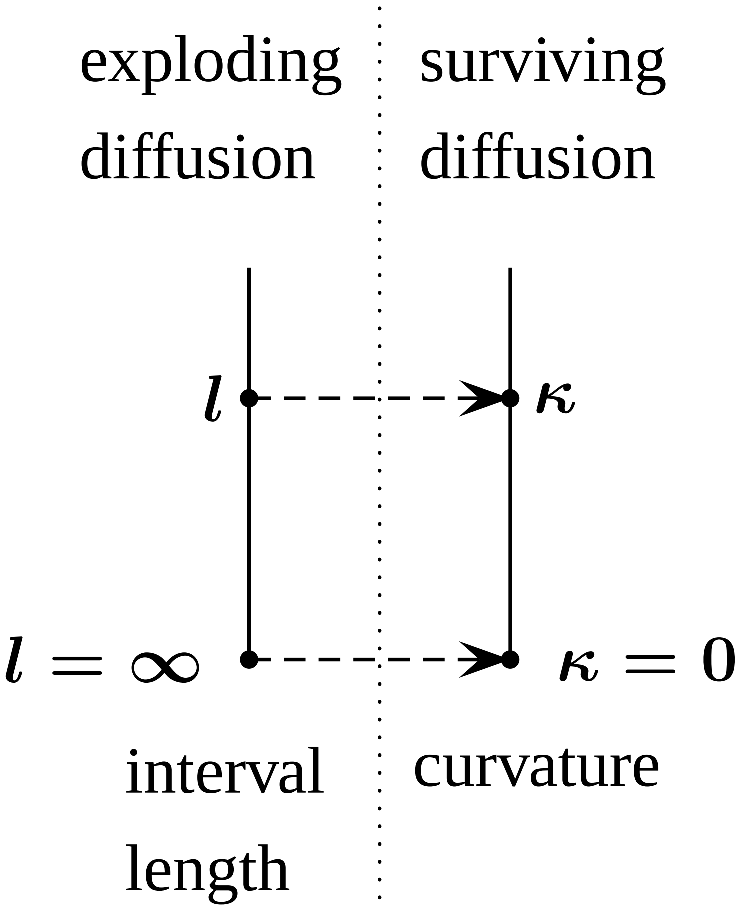


Bessel diffusions on $(0, \infty)$

$$\frac{d}{dm}\frac{d}{ds} = \frac{1}{2}\frac{d^2}{dx^2} - \frac{d-1}{2x}\frac{d}{dx}$$

$$d = \text{dimension}$$

$$\nu = \frac{d-2}{2}$$



Brownian motion on an interval $(0, l)$

$$\text{ground state}: \ \sin \frac{\pi}{l} x$$

The radial motion of the Brownian motion on a 3-dimensional sphere

$$\text{radial part of } \frac{1}{2}\triangle :$$

$$\frac{1}{2}\frac{d}{dx^2} + \sqrt{\kappa}\cot\sqrt{\kappa}x\frac{d}{dx}$$

$$\kappa = \frac{\pi^2}{l^2}$$

8. Proof of Theorem 4.1

Since the Green operator is compact, the transition function has the following expression

$$p(t, x, y) = \sum_{i=0}^{\infty} e^{-\lambda_i t} \varphi_i(x) \varphi_i(y)$$

Here λ_i are eigenvalues of $-\frac{d}{dm}\frac{d}{ds}$ and φ_i are eigenfunctions. The following estimate is crucial: there exist $C > 0$ and N so that

$$\int_0^l |\varphi_i(y)| dm(x) \leq C \lambda_i^N \left\{ \int_0^l \varphi_i(y)^2 dm(x) \right\}^{1/2}$$

9. Invariant function

$p(t, x, dy)$: a transition probability

φ is called a invariant function if

$$\varphi(x) = \int_D \varphi(y) p(t, x, dy), \quad \forall t \geq 0.$$

It is easy to see

φ is invariant $\Leftrightarrow$ h-transform by φ is conservative.

By the argument before, we can show that any one-dimensional (minimal) diffusion has a invariant function if the lowest eigenvalue is 0.

	left	right	$\boldsymbol{D}$	eigenvalue	$\boldsymbol{h}$-transform
case 1	exit $\longleftarrow$	exit $\longrightarrow$	$(\mathbf{0}, \boldsymbol{l})$	$\boldsymbol{\lambda}_\mathbf{0} > \mathbf{0}$	$\boldsymbol{\varphi}_\mathbf{0}(\boldsymbol{x})$
case 2	exit $\longleftarrow$	non-exit $\not\longrightarrow$ $\not\longleftarrow$ non-entrance	$\begin{cases} (\mathbf{0}, \infty) \\ (\mathbf{0}, \boldsymbol{l}) \end{cases}$	$\boldsymbol{\lambda}_\mathbf{0} \geq \mathbf{0}$	$\boldsymbol{s}(\boldsymbol{x}) = \boldsymbol{x}$
case 3	exit $\longleftarrow$	non-exit $\not\longrightarrow$ $\longleftarrow$ entrance	$(\mathbf{0}, \infty)$	$\boldsymbol{\lambda}_\mathbf{0} > \mathbf{0}$	$\boldsymbol{\varphi}_\mathbf{0}(\boldsymbol{x})$

Thanks!