Hodge-Kodaira operators in infinite dimensional spaces

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Contents

1. Introduction

$(\boldsymbol{B}, \boldsymbol{H}, \boldsymbol{\mu})$: an **abstract Wiener space**

- $\boldsymbol{B}$: a Banach space
- $\boldsymbol{H}$: a Hilbert space $\hookrightarrow \boldsymbol{B}$
- $\boldsymbol{\mu}$: the Wiener measure with

$$\int_{\boldsymbol{B}} e^{\sqrt{-1}\langle x, \varphi\rangle} \boldsymbol{\mu}(dx) = \exp\Big\{-\frac{1}{2}|\varphi|^2_{H^*}\Big\},$$

$$\varphi \in \boldsymbol{B}^* \subset \boldsymbol{H}^*.$$

p-forms: $\bigwedge^p \boldsymbol{H}^*$-valued functions.

We consider the following operators acting on differential forms $L^2(\mu; \bigwedge^p H^*)$.

- d: the exterior differentiation
- δ: the dual operator of d
- $\Box_p = -(d\delta + \delta d)$: the **Hodge-Kodaira operator**

Then

$$\sigma(-\Box_p) = \{p, p+1, p+2, \ldots\}$$

In particular, setting

$$\mathcal{H}_p = \{\omega \in L^2(\mu; \bigwedge^p H^*); \Box_p \omega = 0\}$$

we have

$$\mathcal{H}_p = \begin{cases} \{\text{constant functions}\}, & p = 0, \\ \{0\}, & p \geq 1. \end{cases}$$

Aim : To show a similar kind of vanishing theorem in **infinite dimensional space**.

2. Abstract Wiener space with weighted measure

Set

$$\boldsymbol{\mu_F = e^{-2F}\mu, \qquad \mu_F(B) = 1}$$

and consider

- $\boldsymbol{d_F = d}$: the exterior differentiation
- $\boldsymbol{\delta_F}$: the dual operator of $\boldsymbol{d_F}$ w.r.t. $\boldsymbol{\mu_F}$
- $\boldsymbol{\Box_F = -(d_F\delta_F + \delta_F d_F)}$: the Hodge-Kodaira operator

Hino [1997] proved the following:

$$e^{|\nabla F|^2} \in L^{2+} \Rightarrow$$

$$\mathcal{H}_p = \begin{cases} \{\text{constant functions}\}, & p = 0, \\ \{0\}, & p \geq 1. \end{cases}$$

$$e^{|\nabla F|^2} \in L^{4+} \Rightarrow \lambda_p = \inf \sigma(-\Box_{F,p}) > 0, \quad p \geq 1.$$

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Define $I\colon L^2(\mu;\bigwedge^p H^*) \longrightarrow L^2(\mu_F;\bigwedge^p H^*)$ by

$$I\omega = e^F\omega$$

Under this isomorphism, we have

$$\begin{array}{ccc} L^2(\mu;\bigwedge^p H^*) & \xrightarrow{I} & L^2(\mu_F;\bigwedge^p H^*) \\ \hat{d}_F \big\downarrow & & \big\downarrow d_F \\ L^2(\mu;\bigwedge^{p+1} H^*) & \xrightarrow{I} & L^2(\mu_F;\bigwedge^{p+1} H^*) \end{array}$$

where

$$\hat{d}_F\omega = I^{-1}\circ d_F\circ I\omega = e^{-F}d_F(e^F\omega) = d\omega + \mathrm{ext}(dF)\omega$$

Similarly

$$\hat{\delta}_F\omega = I^{-1}\circ \delta_F\circ I\omega = e^{-F}\delta_F(e^F\omega) = \delta\omega + \mathrm{int}(dF)\omega$$

The associated **Hodge-Kodaira operator** $\hat{\Box}_{F,p}$ is given by

$$\hat{\Box}_{F,p} = -(\hat{\delta}_F \hat{d}_F + \hat{d}_F \hat{\delta}_F)$$

and the associated **bilinear form** $\hat{\mathcal{E}}_{F,p}$ is given by

$$\hat{\mathcal{E}}_{F,p}(\omega, \eta) = E[(d\omega + \mathrm{ext}(dF)\omega | d\eta + \mathrm{ext}(dF)\eta) \\ + (\delta\omega + \mathrm{int}(dF)\omega | \delta\eta + \mathrm{int}(dF)\eta)]$$

Proposition 2.1. We have

$$\boldsymbol{E}[|\nabla\omega|^2_{\mathrm{HS}}] \le \frac{p!\,a}{(\sqrt{a}-\sqrt{2})^2}\hat{\mathcal{E}}_{F,p}(\omega,\omega) + \frac{p!}{\sqrt{2}(\sqrt{a}-\sqrt{2})}(\log \boldsymbol{E}[e^{a|dF|^2}] - p\sqrt{2a})\|\omega\|_2^2$$

$$\hat{\mathcal{E}}_{F,p}(\omega,\omega) \le \frac{1}{p!}\left(1+\sqrt{\frac{2}{a}}\right)^2 \boldsymbol{E}[|\nabla\omega|^2_{\mathrm{HS}}] + \left(1+\sqrt{\frac{2}{a}}\right)\left(p+\frac{\log \boldsymbol{E}[e^{a|dF|^2}]}{\sqrt{2a}}\right)\|\omega\|^2$$

To show $\lambda_p > 0$, we take an approximation method.

Define an approximating sequence $\{F_n\}$ as follows.

- $\{\varphi_i\} \subset B^* \subset H^*$: c.o.n.s of H^*
- $\mathcal{F}_n = \sigma\{\varphi_i; i = 1, \ldots, n\}$
- $F_n = E[F|\mathcal{F}_n]$

Then we have $F_n \longrightarrow F$ a.e. and

$$E[\exp(a|dF_n|^2)] \leq E[\exp(a|dF|^2)]$$

We show $\hat{\Box}_{F_n,p}$ converges to $\hat{\Box}_{F_n,p}$ **in norm resolvent sense**, i.e.,

$$G^{(n)} = (1 - \hat{\Box}_{F_n,p})^{-1}, \quad G = (1 - \hat{\Box}_{F,p})^{-1}$$

and

$$\|G^{(n)} - G\|_{\mathrm{op}} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Proposition 2.2. There exists a constant C so that

$$E[|\nabla G\omega|^2_{\mathrm{HS}}] \leq C\|\omega\|_2^2$$

$$E[|\nabla \hat{d}_F G\omega|^2_{\mathrm{HS}}] \leq C\|\omega\|_2^2$$

$$E[|\nabla \hat{\delta}_F G\omega|^2_{\mathrm{HS}}] \leq C\|\omega\|_2^2$$

$$E[|\nabla G^{(n)}\omega|^2_{\mathrm{HS}}] \leq C\|\omega\|_2^2$$

$$E[|\nabla \hat{d}_{F_n} G^{(n)}\omega|^2_{\mathrm{HS}}] \leq C\|\omega\|_2^2$$

$$E[|\nabla \hat{\delta}_{F_n} G^{(n)}\omega|^2_{\mathrm{HS}}] \leq C\|\omega\|_2^2$$

Proof of $\|G^{(n)} - G\|_{\mathrm{op}} \to 0$

$$(G\omega - G^{(n)}\omega|\eta)$$

$$= (\mathrm{ext}(dF - dF_n)G^{(n)}\omega, \hat{d}_F G\eta)$$

$$+ (\hat{d}_{F_n} G^{(n)}\omega, \mathrm{ext}(dF - dF_n)G\eta) + \text{terms of } \delta$$

$$= \left(\frac{\mathrm{ext}(dF - dF_n)}{\sqrt{|dF|^2 + |dF_n|^2 + 1}} G^{(n)}\omega, \sqrt{|dF|^2 + |dF_n|^2 + 1}\,\hat{d}_F G\eta \right)$$

$$+ \left(\sqrt{|dF|^2 + |dF_n|^2 + 1}\,\hat{d}_{F_n} G^{(n)}\omega, \frac{\mathrm{ext}(dF - dF_n)}{\sqrt{|dF|^2 + |dF_n|^2 + 1}} G\eta \right)$$

$$+ \text{terms of } \delta$$

3. Unbounded lattice spin system

Setting

- Configulation space: $\mathbb{R}^{\mathbb{Z}^d}$
- Gibbs measure:

$$\nu = Z^{-1} \exp\left\{ -2\mathcal{J} \sum_{\substack{i,j \in \mathbb{Z}^d \\ i \sim j}} (x^i - x^j)^2 - 2 \sum_{i \in \mathbb{Z}^d} U(x^i) \right\} \prod_{i \in \mathbb{Z}^d} dx^i.$$

- Tangent space: $\ell^2(\mathbb{Z}^d)$

Known results:

- B. Zegarlinski [1996] : logarithmic Sobolev inequality for $\boldsymbol{d = 1}$
- N. Yoshida [1999] : logarithmic Sobolev inequality for general $\boldsymbol{d}$
- T. Bodineau and B. Helffer [1999] : spectral gap for $\mathbf{1}$-fomrs

Instead of infinite volume Gibbs state, we consider finite volume Gibbs state:

Teke a finite set $\Lambda \subset \mathbb{Z}^d$ and a boundary condition η and define an Hamiltonian

$$\Phi_{\Lambda,\eta}(x) = \sum_{\substack{i,j\in\Lambda \\ i\sim j}} \mathcal{J}(x^i - x^j)^2 + \sum_{i\in Z^d} U(x^i) + 2 \sum_{\substack{i\in\Lambda, j\in\Lambda^c \\ i\sim j}} \mathcal{J}(x^i - \eta^j)^2$$

and a measure on $\mathbb{R}^\Lambda$

$$\nu_{\Lambda,\eta} = Z^{-1} e^{-2\Phi_{\Lambda,\eta}(x)} dx_\Lambda$$

The Hodge-Kodaira operator is defined by

$$\Box_p = -(d\delta + \delta d)$$

and set

$$\lambda_p = \inf \sigma(-\Box_p).$$

We are interested in the positivity of λ_p.

Theorem 3.1. Assume

- $U = V + W$, $V'' \geq c > 0$, W is bounded
- $W_{\sup} = \sup W$, $W_{\inf} = \inf W$
- $2(c + 8d\mathcal{J})e^{-2(W_{\sup} - W_{\inf})} - 16d\mathcal{J} > 0$

$$\Rightarrow \lambda_p \geq \{2(c + 8d\mathcal{J})e^{-2(W_{\sup} - W_{\inf})} - 16d\mathcal{J}\}p.$$

We also have the Hodge-Kodaira decomposition:

$$L^2(\nu_{\Lambda,\eta}; \bigwedge^p(\mathbb{R}^\Lambda)^*) = \mathrm{Ran}(d) \oplus \mathrm{Ran}(\delta), \quad p \geq 1$$

Theorem 3.2. Assume

- $U(t) = at^4 - bt^2$
- $\sqrt{3a} - b - 4d\mathcal{J} > 0$

$$\Rightarrow \lambda_p \geq 2(\sqrt{3a} - b - 4d\mathcal{J})p.$$

We also have the Hodge-Kodaira decomposition.

Thanks!