Martingale dimensions of diffusion processes on fractal-like spaces

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Abstract The concept of martingale dimension has been defined for diffusion processes and more general stochastic processes, and is interpreted as the multiplicity of the associated filtration. This corresponds to the number of independent noises that the process possesses. Determining the martingale dimension is a difficult problem for diffusion processes on singular spaces such as fractal sets. The relationship between the martingale dimension and other kinds of dimensions is also not fully understood. In this paper, we give a survey of developments for this topic and discuss recent research on the case of diffusion processes on metric measure spaces.

1 Introduction

There are various kinds of dimensions in the study of metric measure spaces. One of the most fundamental is the Hausdorff dimension d_{H}, which has geometric implications. A less familiar one is the spectral dimension d_{s}, which appears as the exponent in the on-diagonal estimates of the heat kernel, and thus has analytic nature. The martingale dimension d_{m}, which is the main topic of this article, has a probabilistic meaning and describes the multiplicity of the filtration of a stochastic process on the underlying space.

The definitions of these dimensions are all different, so it is not straightforward to see the relations among them. Let us consider the most typical case, the d-dimensional Euclidean space ($d \in \mathbb{N}$). The Hausdorff dimension d_{H} is equal to d. The heat kernel $p_t(x, y)$, which is the transition density of the Brownian motion $\{B_t\}_{t\ge 0}$ on $\mathbb{R}^d$, is given by

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$$p_t(x,y) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x-y|_{\mathbb{R}^d}^2}{2t}\right).$$

Specifically, letting $y = x$ and taking the logarithm we obtain

$$d = \lim_{t\to 0} \frac{2\log p_t(x,x)}{-\log t}. \tag{1}$$

The right-hand side of (1) describes the small-time asymptotics of the on-diagonal heat kernel and is called the spectral dimension d_{s}. That is, $d_{\mathrm{s}} = d$ in this case.

To introduce the martingale dimension, let $\{\mathcal{F}_t\}_{t\ge 0}$ be the canonical filtration associated with $\{B_t\}_{t\ge 0}$. That is, each $\mathcal{F}_t$ is defined as the sub-σ-field of the probability space $(\Omega, \mathcal{F}, P)$ that is generated by $\{B_s \mid 0 \le s \le t\}$ and all P-null sets. Let $\mathcal{M}$ denote the totality of right-continuous square-integrable $\{\mathcal{F}_t\}_{t\ge 0}$-martingales starting at 0. That is, each element $M = \{M_t\}_{t\ge 0}$ in $\mathcal{M}$ is such that the sample path $\{M_t(\omega)\}_{t\ge 0}$ is right continuous for P-a.e. ω, M_t is $\mathcal{F}_t$-measurable and $E[M_t^2] < \infty$ for $t \ge 0$, $M_0 = 0$ P-a.s., and $E[M_t \mid \mathcal{F}_s] = M_s$ for $0 \le s \le t$. Then the martingale representation theorem states that every $M = \{M_t\}_{t\ge 0} \in \mathcal{M}$ is expressed as the sum of L^2-stochastic integrals in the form

$$M_t = \sum_{j=1}^{d} \int_0^t \varphi_s^{(j)}\, dB_s^{(j)}, \quad t \ge 0,\ P\text{-a.s.}$$

for some $\{\mathcal{F}_t\}_{t\ge 0}$-predictable processes $\{\varphi_t^{(j)}\}_{t\ge 0}$, $j = 1, \dots, d$, where $B_t^{(j)}$ is the jth component of $B_t = (B_t^{(1)}, \dots, B_t^{(d)})$ (see, e.g. [37, Chapter V, Proposition 3.2 and Theorem 3.4]). This means that $\{B_t^{(1)}\}_{t\ge 0}, \dots, \{B_t^{(d)}\}_{t\ge 0}$ form a kind of a basis of $\mathcal{M}$. The number d of these processes is interpreted as the multiplicity of the filtration $\{\mathcal{F}_t\}_{t\ge 0}$, or the number of independent noises in $\{\mathcal{F}_t\}_{t\ge 0}$, and is one definition of the martingale dimension d_{m}. In the next section, we introduce a slightly different definition of the martingale dimension, but it is in the same spirit as the one above.

In this example, d_{H}, d_{s}, and d_{m} coincide. The same is the case for Brownian motion on Riemannian manifolds. However, they can be different in general, especially when the state space has certain singularities. The first result for determining d_{m} in this direction was due to Kusuoka [30], who proved that the martingale dimension of the Brownian motion on the Sierpinski gasket of any dimension is one, solving the problem posed by Barlow and Perkins [5]. This may seem counter-intuitive. Since this result there have been a few contributions [16, 18] toward determining the martingale dimensions for fractals. The main difficulty lies in the fact that the diffusion process is not associated with a stochastic differential equation: the process is constructed by way of the scaling limit of a discrete approximation or the use of the theory of Dirichlet forms. In either case the infinitesimal behavior of the process is not at all clear, which makes it difficult to obtain useful information on the martingale dimension. In this article we focus on the quantitative estimate of the martingale dimension of diffusions associated with Dirichlet forms, and on the relations between other concepts of dimensions for the case where the state space is fractal-like, and we provide a survey of past [16, 18] and ongoing [20] research on this topic.

Note that this topic is part of the study of *local structures* of stochastic processes. The following is a brief list of studies of this kind.

- 1950s–1960s: Classification of 1-dimensional diffusion processes by Feller [8, 9], Itô–McKean [24], etc.
- 1960s–1970s: Structures of the space of martingales and martingale additive functionals associated with general Markov processes by Skorokhod [39], Motoo–Watanabe [34], Kunita–Watanabe [28], Ikeda–Watanabe [23], Davis–Varaiya [7], etc.
- 1990s–2000s: Study of noises by Tsirelson [40, 41] etc.

In this sense, our motivation lies in the infinitesimal analysis of diffusion processes on general state spaces, including singular spaces like fractals.

The rest of the paper is organized as follows. In Section 2, we introduce the concept of (additive functional-) martingale dimension in the framework of Dirichlet forms. In Section 3, we provide an overview of results for the martingale dimension of self-similar fractals. In Section 4, we discuss recent progress for more general metric measure spaces. In Section 5, we give some remarks on related problems for future studies.

2 Martingale dimension in the framework of Dirichlet forms

We follow Fukushima, Oshima, and Takeda's book [12] to set up the framework on the basis of Dirichlet form theory. See also [6, Appendix A.1] for a review of the general theory of Markov processes. Let K be a locally compact separable metrizable space and μ a positive Radon measure on K with full support. Let $C(K)$ denote the set of all real-valued continuous functions on K and $C_c(K)$ the set of all functions in $C(K)$ with compact support. Let $(\mathcal{E},\mathcal{F})$ be a strongly local regular Dirichlet form on $L^2(K,\mu)$, that is, $(\mathcal{E},\mathcal{F})$ satisfies the following conditions.

- $\mathcal{F}$ is a dense linear subspace of $L^2(K,\mu)$.
- $\mathcal{E}\colon \mathcal{F}\times\mathcal{F}\to\mathbb{R}$ is a symmetric bilinear form on $\mathcal{F}$ such that $\mathcal{E}(f,f)\ge 0$ for all $f\in\mathcal{F}$.
- $(\mathcal{E},\mathcal{F})$ is a closed form, that is, $\mathcal{F}$ is a Hilbert space with respect to the inner product $(\cdot,\cdot)_{\mathcal{F}}$ that is defined as $(f,g)_{\mathcal{F}} := \mathcal{E}(f,g)+\int_K fg\,d\mu$.
- (Markov property) For any $f\in\mathcal{F}$, $\hat{f} := \min\{\max\{f,0\},1\}$ belongs to $\mathcal{F}$ and $\mathcal{E}(\hat{f},\hat{f})\le\mathcal{E}(f,f)$.
- (Regularity) The set $\mathcal{F}\cap C_c(K)$ is dense both in $\mathcal{F}$ and in $C_c(K)$. Here the topologies of $\mathcal{F}$ and $C_c(K)$ are the induced topology from $(\cdot,\cdot)_{\mathcal{F}}$ and the uniform topology, respectively.
- (Strong locality) For any $f,g\in\mathcal{F}$, $\mathcal{E}(f,g)=0$ as long as both $\operatorname{supp} f$ and $\operatorname{supp} g$ are compact and g is constant on a neighborhood of $\operatorname{supp} f$. Here, $\operatorname{supp} f$ denotes the support of the measure $|f|\cdot\mu$ on K.

Strong locality is an extra condition for the general theory of Dirichlet forms, but we always assume this condition in this paper.

The Dirichlet form $(\mathcal{E}, \mathcal{F})$ induces a non-positive self-adjoint operator L on $L^2(K, \mu)$ such that the domain of $\sqrt{-L}$ is equal to $\mathcal{F}$ and

$$\mathcal{E}(f, g) = \int_K (\sqrt{-L} f)(\sqrt{-L} g)\, d\mu \quad \text{for } f, g \in \mathcal{F}.$$

The operator L generates the semigroup $\{T_t\}_{t\geq 0}$ on $L^2(K, \mu)$ by defining $T_t = e^{tL}$. From the Markov property of $(\mathcal{E}, \mathcal{F})$, each T_t has the Markov property in the sense that $0 \leq T_t f \leq 1$ μ-a.e. for any $f \in L^2(K, \mu)$ with $0 \leq f \leq 1$ μ-a.e.

For an open set U of K, we define

$$\mathrm{Cap}(U) = \inf \left\{ \mathcal{E}(f, f) + \int_K f^2\, d\mu \;\middle|\; f \in \mathcal{F} \text{ and } f \geq 1 \ \mu\text{-a.e. on } U \right\},$$

where $\inf \emptyset = +\infty$. For general subsets A of K, let

$$\mathrm{Cap}(A) = \inf\{\mathrm{Cap}(U) \mid U \text{ is open and } U \supset A\}$$

and call it the *capacity* of A. A Borel measure λ on K is called *smooth* if the following conditions are satisfied.

- λ charges no set of zero capacity, that is, $\lambda(A) = 0$ for all Borel sets A of K with $\mathrm{Cap}(A) = 0$.
- There exists an increasing sequence $\{F_n\}_{n=1}^\infty$ of closed sets of K such that $\lambda(F_n) < \infty$ for all n and $\lim_{n\to\infty} \mathrm{Cap}(C \setminus F_n) = 0$ for any compact set C of K.

Let $K_\Delta = K \cup \{\Delta\}$ be the one-point compactification of K. The Borel σ-fields on K and K_Δ are denoted by $\mathcal{B}$ and $\mathcal{B}_\Delta$, respectively. From the general theory of Dirichlet forms, we can construct a diffusion process $\{X_t\}_{t\geq 0}$ on K associated with $(\mathcal{E}, \mathcal{F})$, defined on a filtered probability space $(\Omega, \mathcal{F}_\infty, \{\mathcal{F}_t\}_{t\geq 0})$ with a family of probability measures $\{P_x\}_{x\in K_\Delta}$ and shift operators $\{\theta_t\}_{t\in[0,+\infty]}$. More precisely, the following hold.

- $(\Omega, \mathcal{F}_\infty, \{\mathcal{F}_t\}_{t\geq 0})$ is a filtered probability space and the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is right continuous ($\bigcap_{t>s} \mathcal{F}_t = \mathcal{F}_s$ for all $s \geq 0$).
- For each $x \in K_\Delta$, P_x is a probability measure on $(\Omega, \mathcal{F}_\infty)$.
- For each $t \geq 0$, $X_t \colon \Omega \to K_\Delta$ is $\mathcal{F}_t/\mathcal{B}_\Delta$-measurable. We set $X_\infty(\omega) = \Delta$ for $\omega \in \Omega$.
- For each $t \geq 0$ and $A \in \mathcal{B}$, $P_x(X_t \in A)$ is $\mathcal{B}$-measurable in $x \in K$.
- For any $t \geq 0$, $P_\Delta(X_t = \Delta) = 1$.
- (Normality) For any $x \in K$, $P_x(X_0 = x) = 1$.
- $X_t(\omega) = \Delta$ for all $t \geq \zeta(\omega)$, where

$$\zeta(\omega) = \inf\{t \geq 0 \mid X_t(\omega) = \Delta\} \tag{2}$$

is the life time of $\{X_t\}_{t\ge 0}$.

- (Continuity of sample paths) For each $\omega \in \Omega$, the map $[0,\infty) \ni t \mapsto X_t(\omega) \in K_\Delta$ is continuous.
- Each θ_t is a measurable map from $(\Omega, \mathcal{F}_\infty)$ to itself. For each $s \ge 0$ and $t \in [0,+\infty]$, $X_s \circ \theta_t = X_{s+t}$.
- (Strong Markov property) For $A \in \mathcal{B}_\Delta$, $s \ge 0$, any $\{\mathcal{F}_t\}_{t\ge 0}$-stopping time σ, and any probability measure λ on $(K_\Delta, \mathcal{B}_\Delta)$,

$$P_\lambda(X_{\sigma+s} \in A \mid \mathcal{F}_\sigma) = P_{X_\sigma}(X_s \in A) \quad P_\lambda\text{-a.s.}$$

  Here, P_λ is the probability measure on $(\Omega, \mathcal{F}_\infty)$ defined as

$$P_\lambda(\Lambda) = \int_{K_\Delta} P_x(\Lambda)\,\lambda(dx), \qquad \Lambda \in \mathcal{F}_\infty. \tag{3}$$

- (Correspondence with Dirichlet forms) For any $t \ge 0$ and $f \in L^2(K,\mu)$ that are $\mathcal{B}$-measurable, $T_t f(x) = E_x[f(X_t)]$ holds for μ-a.e. $x \in K$. Here E_x denotes the expectation with respect to P_x, and f is extended to a function on K_Δ by setting $f(\Delta) = 0$.

Note that the continuity of sample paths results from the strong locality of the Dirichlet form $(\mathcal{E}, \mathcal{F})$. See [12, Theorem 4.5.3].

A set A of K_Δ is called *nearly Borel* measurable if, for any probability measure λ on $(K_\Delta, \mathcal{B}_\Delta)$, there exist Borel sets A_1 and A_2 of K_Δ such that $A_1 \subset A \subset A_2$ and

$$P_\lambda(\{\text{There exists some } t \ge 0 \text{ such that } X_t \in A_2 \setminus A_1\}) = 0.$$

A subset N of K is called *exceptional* if there exists a nearly Borel set $\tilde{N}$ including N such that $P_\mu(\sigma_{\tilde{N}} < \infty) = 0$, where $\sigma_{\tilde{N}}(\omega) = \inf\{t > 0 \mid X_t(\omega) \in \tilde{N}\}$. A subset N of K is exceptional if and only if $\mathrm{Cap}(N) = 0$. If statements $P(x)$ depending on $x \in K$ hold for all $x \in K \setminus N$ for some exceptional set N, we say that $P(x)$ holds for *quasi-every* x (abbreviated q.e. x). A nearly Borel set N of K is called *properly exceptional* if $\mu(N) = 0$ and $K \setminus N$ is $\{X_t\}_{t\ge 0}$-invariant, that is, in our framework, $P_x(\{X_t \in K_\Delta \setminus N \text{ for all } t \ge 0\}) = 1$ for all $x \in K \setminus N$. Every properly exceptional set is an exceptional set, and any exceptional set is a subset of some properly exceptional set. The uniqueness of the diffusion processes associated with $(\mathcal{E}, \mathcal{F})$ holds in the sense that if $\{X'_t\}_{t\ge 0}$ with $(\Omega', \mathcal{F}'_\infty, \{\mathcal{F}'_t\}_{t\ge 0}, \{P'_x\}_{x\in K_\Delta})$ is another diffusion process associated with $(\mathcal{E}, \mathcal{F})$, then there exists a Borel set N of K such that N is properly exceptional for both processes and $P_x(X_t \in \cdot) = P'_x(X'_t \in \cdot)$ for every $x \in K \setminus N$ and $t \ge 0$. See [12, Sections 4.1 and 4.2] for further details in this paragraph.

We may, and do always, assume that $\{\mathcal{F}_t\}_{t\ge 0}$ is the minimum completed admissible filtration, that is, $\mathcal{F}_t$ $(t \ge 0)$ and $\mathcal{F}_\infty$ are defined as follows.

- Let $\mathcal{F}_t^0 = \sigma(\{X_s \mid 0 \le s \le t\})$ and $\mathcal{F}_\infty^0 = \sigma(\{X_s \mid s \ge 0\})$.
- Let $\mathcal{P}(K_\Delta)$ be the set of all probability measures on $(K_\Delta, \mathcal{B}_\Delta)$.
- For $\lambda \in \mathcal{P}(K_\Delta)$ let P_λ be the probability measure on $(\Omega, \mathcal{F}_\infty^0)$ defined as in (3).

- For $\lambda \in \mathcal{P}(K_\Delta)$, $\mathcal{F}_\infty^\lambda$ denotes the completion of $\mathcal{F}_\infty^0$ with respect to P_λ, and $\mathcal{F}_t^\lambda$ denotes the completion of $\mathcal{F}_t^0$ in $\mathcal{F}_\infty^\lambda$ with respect to P_λ. That is, $\mathcal{F}_t^\lambda = \{\Lambda \in \mathcal{F}_\infty^\lambda \mid \text{there exists } \Lambda' \in \mathcal{F}_t^0 \text{ such that } P_\lambda(\Lambda \vartriangle \Lambda') = 0\}$, where $\vartriangle$ denotes the symmetric difference.
- Let $\mathcal{F}_t = \bigcap_{\lambda \in \mathcal{P}(K_\Delta)} \mathcal{F}_t^\lambda$ and $\mathcal{F}_\infty = \bigcap_{\lambda \in \mathcal{P}(K_\Delta)} \mathcal{F}_\infty^\lambda$.

An *additive functional* (AF) $A_t(\omega)$, $t \ge 0$, $\omega \in \Omega$ is a $[-\infty, +\infty]$-valued function such that the following hold.

- For each $t \ge 0$, $A_t(\cdot)$ is $\mathcal{F}_t$-measurable.
- There exist a set $\Lambda \in \mathcal{F}_\infty$ and an exceptional set N such that $P_x(\Lambda) = 1$ for all $x \in K \setminus N$, $\theta_t(\Lambda) \subset \Lambda$ for all $t > 0$, and for each $\omega \in \Lambda$ the following hold.
  - $A_\cdot(\omega)$ is right continuous and has the left limit on $[0, \zeta(\omega))$, where ζ is the life time defined in (2).
  - $A_0(\omega) = 0$, $|A_t(\omega)| < \infty$ for $t < \zeta(\omega)$, and $A_t(\omega) = A_{\zeta(\omega)}(\omega)$ for $t \ge \zeta(\omega)$.
  - $A_{t+s}(\omega) = A_s(\omega) + A_t(\theta_s(\omega))$ for $t, s \ge 0$.

The sets Λ and N are called a *defining set* and an *exceptional set* of the AF $A_t(\omega)$, respectively. Two AFs $A^{(1)}$ and $A^{(2)}$ are called *equivalent* if $P_x(A_t^{(1)} = A_t^{(2)}) = 1$ for q.e. $x \in K$ for each $t > 0$. In this case, we can take a common defining set Λ and a common properly exceptional set N of $A^{(1)}$ and $A^{(2)}$ such that $A_t^{(1)}(\omega) = A_t^{(2)}(\omega)$ for all $t > 0$ and $\omega \in \Lambda$. We identify equivalent AFs.

An AF $A_t(\omega)$ is called a *positive* (resp. *continuous*) additive functional if there exists a defining set Λ such that $A_t(\omega) \in [0, +\infty]$ for all $t \in [0, \infty)$ (resp. $A_t(\omega)$ is continuous on $[0, \infty)$) for $\omega \in \Lambda$. A positive continuous additive functional is abbreviated as PCAF. Any PCAF A admits its *Revuz measure* μ_A, that is, μ_A is a smooth measure on K such that for any $t > 0$ and any non-negative Borel functions f and h on K,

$$\int_K h(x) E_x\left[\int_0^t f(X_s)\, dA_s\right] \mu(dx) = \int_0^t \int_K f(x) E_x[h(X_s)]\, \mu_A(dx)\, ds.$$

Such a measure μ_A exists uniquely. See [12, Section 5.1] for more information for these two paragraphs.

Let $\mathcal{M}$ be the family of AFs M such that the following hold.

- There exists a defining set Λ such that, for $\omega \in \Lambda$, $|M_t(\omega)| < \infty$ for all $t \in [0, \infty)$ and $M_t(\omega)$ is right continuous and has the left limit on $[0, \infty)$.
- For each $t > 0$, $E_x[M_t^2] < \infty$ and $E_x[M_t] = 0$ for q.e. x.

Note that, for $M \in \mathcal{M}$, $\{M_t\}_{t \ge 0}$ is an $\{\mathcal{F}_t\}_{t \ge 0}$-martingale with respect to P_x for q.e. x. An element M in $\mathcal{M}$ is called a martingale AF. Due to the assumption that $(\mathcal{E}, \mathcal{F})$ is strongly local, every $M \in \mathcal{M}$ is in fact a continuous additive functional (see [12, Lemma 5.5.1 (ii)]).

Each $M \in \mathcal{M}$ admits a unique PCAF $\langle M \rangle$ such that $E_x[\langle M \rangle_t] = E_x[M_t^2]$ for q.e. $x \in K$ for each $t > 0$. For $M, L \in \mathcal{M}$, let

$$\langle M, L\rangle_t = \frac{1}{2}(\langle M+L\rangle_t - \langle M\rangle_t - \langle L\rangle_t)$$

and

$$\mu_{\langle M,L\rangle} = \frac{1}{2}(\mu_{\langle M+L\rangle} - \mu_{\langle M\rangle} - \mu_{\langle L\rangle}).$$

The *energy* $e(M)$ of $M \in \mathcal{M}$ is defined as

$$e(M) = \sup_{t>0} \frac{1}{2t} E_\mu[M_t^2] \left(= \lim_{t\to 0} \frac{1}{2t} E_\mu[M_t^2]\right) \in [0, +\infty].$$

We set $\mathring{\mathcal{M}} = \{M \in \mathcal{M} \mid e(M) < \infty\}$. By the polarization

$$e(M, M') = \frac{1}{2}(e(M+M') - e(M) - e(M')), \quad M, M' \in \mathring{\mathcal{M}},$$

$(\mathring{\mathcal{M}}, e)$ becomes a Hilbert space. See [12, Section 5.2] for further details.

Let $\mathcal{M}_1$ be the set of all elements $M \in \mathcal{M}$ such that $\mu_{\langle M\rangle}$ is a Radon measure, which is a linear space containing $\mathring{\mathcal{M}}$. According to [12, Theorem 5.6.1], given $M \in \mathcal{M}_1$ and $f \in L^2(K, \mu_{\langle M\rangle})$, there exists a unique element $f \bullet M \in \mathring{\mathcal{M}}$ such that

$$e(f \bullet M, L) = \frac{1}{2}\int_K f(x)\, \mu_{\langle M,L\rangle}(dx) \quad \text{for all } L \in \mathring{\mathcal{M}}.$$

We call $f \bullet M$ the *stochastic integral* of f with respect to M. This naming is justified by the following assertion.

Proposition 2.1 ([12, Lemma 5.6.2]) *Let $M \in \mathcal{M}_1$ and $f \in C_c(K)$. Then*

$$\lim_{|\Xi|\to 0} E_x[\{(f \bullet M)_t^{(\Xi)} - (f \bullet M)_t\}^2] = 0 \quad \textit{for any } t > 0 \textit{ for q.e. } x \in K,$$

where

$$(f \bullet M)_t^{(\Xi)} = \sum_{i=1}^n f(X_{t_{i-1}})(M_{t_i} - M_{t_{i-1}}),$$

and Ξ denotes a partition $0 = t_0 < t_1 < \cdots < t_n = t$ and $|\Xi| = \max_{1\le i\le n}(t_i - t_{i-1})$.

We now define the martingale dimension in this context.

Definition 2.2 ([17, Definition 3.3]) The *martingale dimension* d_{m} is defined as the smallest number D for which there exist D elements $M^{(1)}, \dots, M^{(D)} \in \mathring{\mathcal{M}}$ such that every $M \in \mathring{\mathcal{M}}$ is expressed as

$$M_t = \sum_{j=1}^D (\varphi^{(j)} \bullet M^{(j)})_t \quad \text{for any } t > 0\ P_x\text{-a.s. for q.e. } x \in K$$

for some $\varphi^{(j)} \in L^2(K, \mu_{\langle M^{(j)} \rangle})$ for $j = 1, \dots, D$. If such a D does not exist, d_{m} is defined as $+\infty$. If $\mathring{\mathcal{M}} = \{0\}$, d_{m} is defined as 0.

The definition of this martingale dimension, which is called the AF-martingale dimension in [16, 17, 18], is slightly different from that introduced in the previous section, but the general meaning is the same.

It is generally difficult to determine the value of d_{m} directly from the definition. We introduce a useful analytic interpretation of d_{m}. For each $f \in \mathcal{F}$ a smooth finite measure ν_f on K is defined as follows[1]. When f is bounded, ν_f is characterized by the identity

$$\int_K \varphi \, d\nu_f = 2\mathcal{E}(f\varphi, f) - \mathcal{E}(\varphi, f^2) \quad \text{for all } \varphi \in \mathcal{F} \cap C_c(K).$$

For general $f \in \mathcal{F}$, we define ν_f by $\nu_f(B) = \lim_{n\to\infty} \nu_{f_n}(B)$ for $B \in \mathcal{B}$, where $f_n = \min\{\max\{f, -n\}, n\}$. The measure ν_f is called the *energy measure* of f. For $f, g \in \mathcal{F}$, the mutual energy measure $\nu_{f,g}$, which is a signed Borel measure on K, is defined as

$$\nu_{f,g} = \frac{1}{2}(\nu_{f+g} - \nu_f - \nu_g).$$

For two σ-finite (or signed) Borel measures λ_1 and λ_2 on K, we write $\lambda_1 \ll \lambda_2$ if λ_1 is absolutely continuous with respect to λ_2. By following [17, Section 2], a σ-finite Borel measure ν on K is called a *minimal energy-dominant measure* of $(\mathcal{E}, \mathcal{F})$ if the following two conditions are satisfied.

(i) (Domination) For every $f \in \mathcal{F}$, $\nu_f \ll \nu$.
(ii) (Minimality) If another σ-finite Borel measure ν' on K satisfies condition (i) with ν replaced by ν', then $\nu \ll \nu'$.

In this situation, $\nu_{f,g} \ll \nu$ holds for any $f, g \in \mathcal{F}$. A minimal energy-dominant measure always exists and is realized by the energy measure of some function in $\mathcal{F}$.

We fix a minimal energy-dominant measure ν. The *index* p of $(\mathcal{E}, \mathcal{F})$ is defined as the smallest number satisfying the condition that for any $N \in \mathbb{N}$ and any $f_1, \dots, f_N \in \mathcal{F}$,

$$\operatorname{rank}\left(\frac{d\nu_{f_i, f_j}}{d\nu}(x)\right)_{i,j=1}^N \le p \quad \text{for } \nu\text{-a.e. } x.$$

If such a number does not exist, the index is defined as $+\infty$. We remark that the index is 0 if and only if $\mathcal{E}(f, f) = 0$ for every $f \in \mathcal{F}$. The following is a fundamental property.

Theorem 2.3 ([17, Theorem 3.4]) *The index of $(\mathcal{E}, \mathcal{F})$ coincides with the martingale dimension d_{m} of $\{X_t\}_{t\ge 0}$.*

We introduce a simple but typical example.

[1] See [12, pp. 122–123]. In [12], the symbol $\mu_{\langle f \rangle}$ is used in place of ν_f.

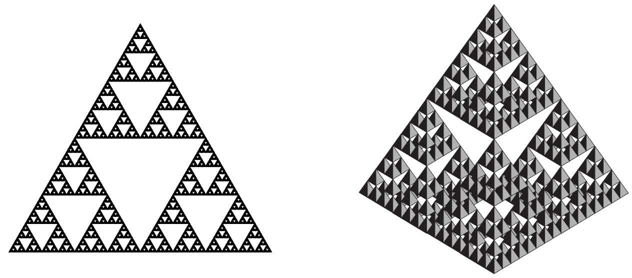

Fig. 1 Two- and three-dimensional Sierpinski gaskets.

Example 2.4 Let $K = \mathbb{R}^d$ and μ be the d-dimensional Lebesgue measure. Let $\mathrm{PSM}(d)$ denote the set of all non-negative definite symmetric matrices of size d. Suppose that a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$ satisfies the following.

- The set C defined as the totality of C^1-functions on K with compact support is dense in $\mathcal{F}$ in the topology of $(\cdot, \cdot)_{\mathcal{F}}$.
- $\mathcal{E}$ is given by

$$\mathcal{E}(f, g) = \frac{1}{2} \int_K (A(x) \nabla f(x), \nabla g(x))_{\mathbb{R}^d} \, \mu(dx), \quad f, g \in C, \tag{4}$$

  where A is a $\mathrm{PSM}(d)$-valued measurable function on K.

A direct calculation implies that $\nu_{f,g}(dx) = (A(x) \nabla f(x), \nabla g(x))_{\mathbb{R}^d} \, \mu(dx)$ for $f, g \in C$. From this identity, we can prove that the index of $(\mathcal{E}, \mathcal{F})$ is equal to $\mu\text{-}\operatorname{esssup}_{x \in K} \operatorname{rank} A(x)$.

An expression similar to (4) is possible in an abstract way for general strongly local regular Dirichlet forms as discussed in [19], but knowing the exact value of $\operatorname{rank} A(x)$ is not at all straightforward in general, especially for Dirichlet forms on fractals.

3 Estimates of martingale dimensions for self-similar fractals

3.1 Case of Sierpinski gaskets

Here, we consider Sierpinski gaskets as the underlying space. Let d be an integer greater than 1. The d-dimensional Sierpinski gasket is constructed as follows. Fix a d-dimensional simplex in $\mathbb{R}^d$ and let $p_0, p_1, \ldots, p_d$ be its vertices. For each $j = 0, 1, \ldots, d$ define a contraction map ψ_j from $\mathbb{R}^d$ to itself by $\psi_j(x) = (x + p_j)/2$. Then, the d-dimensional Sierpinski gasket K is defined as the unique non-empty compact subset of $\mathbb{R}^d$ such that $K = \bigcup_{j=0}^d \psi_j(K)$ holds. This is one of the most typical self-similar fractals; see Fig. 1. From [33, Theorem III] (see also [22] for the

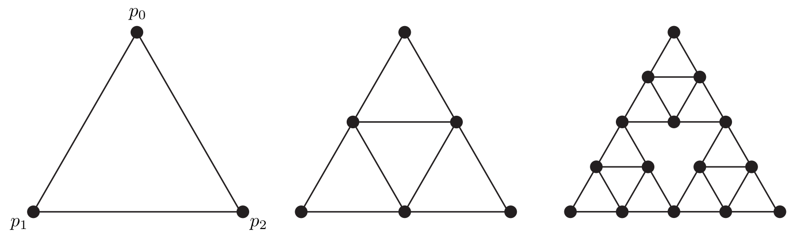

Fig. 2 Visualization of the graph (V_m, E_m) for $m = 0, 1, 2$ when $d = 2$.

general theory of self-similar sets), its Hausdorff dimension d_{H} is equal to $\{\log(d+1)\}/\log 2$. Let μ be the normalized Hausdorff measure on K, which is a self-similar measure on K that satisfies $\mu(\psi_j(A)) = (d+1)^{-1}\mu(A)$ for any $j = 0, 1, \dots, d$ and any Borel set A of K, with $\mu(K) = 1$.

The Brownian motion and canonical Dirichlet form can be defined on Sierpinski gaskets, which is due to [14, 29, 5, 25, 13]. Following [26, Sections 3.1–3.4], we construct the canonical Dirichlet form on $L^2(K, \mu)$ as follows. Let $V_0 = \{p_0, p_1, \dots, p_d\}$ and $E_0 = \{\{p_i, p_j\} \mid i, j = 0, 1, \dots, d,\ i \neq j\}$. Define $V_{m+1} = \bigcup_{j=0}^{d} \psi_j(V_m)$ and $E_{m+1} = \{\{\psi_j(p), \psi_j(q)\} \mid \{p, q\} \in E_m,\ j = 0, 1, \dots, d\}$, $m = 0, 1, 2, \dots$, inductively. Then, the pair (V_m, E_m) is a finite and non-oriented graph with vertex set V_m and edge set E_m for each $m = 0, 1, 2, \dots$. See Fig. 2 for visualization when $d = 2$.

For an at most countable set V, $l(V)$ denotes the set of all real-valued functions on V. Let $V_* = \bigcup_{m=0}^{\infty} V_m$, which is a dense subset of K. For each $m = 0, 1, 2, \dots$, let Q_m be a symmetric bilinear form on $l(V_m)$ defined as

$$Q_m(f, g) = \sum_{\{p,q\} \in E_m} (f(p) - f(q))(g(p) - g(q)), \quad f, g \in l(V_m).$$

It is not difficult to confirm that the sequence

$$\left\{ \left(\frac{d+3}{d+1} \right)^m Q_m(f|_{V_m}, f|_{V_m}) \right\}_{m=0}^{\infty}$$

is nondecreasing for every $f \in l(V_*)$, and that $(d+3)/(d+1)$ is the smallest positive constant satisfying this property. Define

$$Q(f, f) = \lim_{m \to \infty} \left(\frac{d+3}{d+1} \right)^m Q_m(f|_{V_m}, f|_{V_m}) \in [0, +\infty], \quad f \in l(V_*).$$

It is known that any $f \in l(V_*)$ with $Q(f, f) < \infty$ is uniformly continuous on V_* in K, so f extends to a continuous function on K. Keeping this fact in mind, we define $\mathcal{F} = \{f \in C(K) \mid Q(f|_{V_*}, f|_{V_*}) < \infty\}$ and

$$\mathcal{E}(f, g) = \frac{1}{2}\{Q((f+g)|_{V_*}, (f+g)|_{V_*}) - Q(f|_{V_*}, f|_{V_*}) - Q(g|_{V_*}, g|_{V_*})\}, \quad f, g \in \mathcal{F}.$$

Then, $(\mathcal{E}, \mathcal{F})$ becomes a strongly local regular Dirichlet form on $L^2(K, \mu)$ by regarding $C(K)$ as a subspace of $L^2(K, \mu)$. The corresponding diffusion process $\{X_t\}_{t\ge 0}$ on K is called the Brownian motion on K; indeed, it is invariant under isometries on K and has a scale invariance property. The associated semigroup $\{T_t\}_{t>0}$ has a transition density (heat kernel) $p_t(x, y)$ with respect to μ that is continuous in $(t, x, y) \in (0, \infty) \times K \times K$. It has the following sub-Gaussian estimate [5].

$$
\begin{aligned}
&c_1 t^{-d_{\mathrm{s}}/2} \exp\left\{-c_2\left(\frac{|x-y|_{\mathbb{R}^d}^{d_{\mathrm{w}}}}{t}\right)^{1/(d_{\mathrm{w}}-1)}\right\} \\
&\quad \le p_t(x, y) \le c_3 t^{-d_{\mathrm{s}}/2} \exp\left\{-c_4\left(\frac{|x-y|_{\mathbb{R}^d}^{d_{\mathrm{w}}}}{t}\right)^{1/(d_{\mathrm{w}}-1)}\right\}, \\
&\quad t \in (0, 1],\ x, y \in K, \qquad (5)
\end{aligned}
$$

where

$$
d_{\mathrm{s}} = \frac{2\log(d+1)}{\log(d+3)} \in (1, \min\{d_{\mathrm{H}}, 2\}), \quad d_{\mathrm{w}} = \frac{2d_{\mathrm{H}}}{d_{\mathrm{s}}} > 2,
$$

and $c_1, \dots, c_4$ are positive constants depending only on d. In particular,

$$
d_{\mathrm{s}} = \lim_{t\to 0} \frac{2\log p_t(x, x)}{-\log t}, \quad x \in K. \qquad (6)
$$

What exactly is the value of the martingale dimension d_{m} for the Brownian motion on the d-dimensional Sierpinski gasket? Kusuoka [30] proved the following result.

Theorem 3.1 ([30]) $d_{\mathrm{m}} = 1$ *for any* $d \ge 2$.

Note that $d_{\mathrm{m}} < d_{\mathrm{s}} < d_{\mathrm{H}}$ holds in particular. It is remarkable that, as $d \to \infty$, d_{H} diverges to ∞ and d_{s} converges to 2, while d_{m} is always 1. The mechanism behind Theorem 3.1 is not obvious. Indeed, we had to wait until much later studies [16, 18] in which more examples were treated in a unified manner. We provide an overview of such results in the next section.

3.2 Case of more general self-similar fractals

Self-similar fractals are classified into two types, *finitely ramified* ones and *infinitely ramified* ones. Finitely ramified fractals can be disconnected by removing only finitely many points. Infinitely ramified fractals cannot, and this makes their analysis much harder. The first two fractals in Fig. 3 are typical examples of finitely ramified fractals. They are examples of *nested fractals* [32], which are compact self-similar sets in the Euclidean space that have some nice symmetries. Typical infinitely ramified fractals are Sierpinski carpets, the last two fractals in Fig. 3.

Their Hausdorff dimensions d_{H} are easily calculated from the general theory of self-similar sets [33]. In Fig. 3 d_{H} is equal to

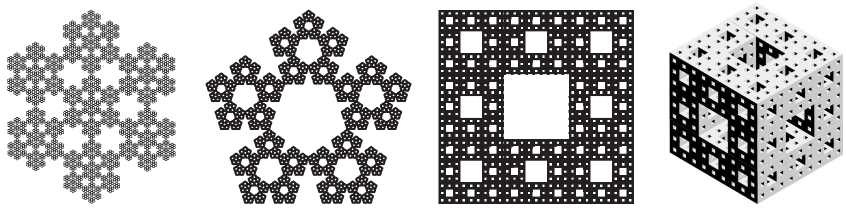

Fig. 3 Examples of self-similar fractals: snowflake, pentagasket, two-dimensional Sierpinski carpet, and three-dimensional Sierpinski carpet (Menger sponge).

$$\frac{\log 7}{\log 3}, \quad \frac{\log 5}{\log\{(3+\sqrt{5})/2\}}, \quad \frac{\log 8}{\log 3}, \quad \frac{\log 20}{\log 3},$$

respectively. The problems of the construction and uniqueness of Brownian motions and the estimates of heat kernels on such fractals have been extensively studied in, for example, [5, 2, 32, 11, 31, 27, 10, 38, 3, 4]. The following hold when K is a nested fractal or a (generalized) Sierpinski carpet.

- There exists a canonical strongly local regular Dirichlet form on $L^2(K,\mu)$, where μ is the normalized Hausdorff measure on K.
- The semigroup $\{T_t\}_{t>0}$ has a transition density $p_t(x,y)$ with respect to μ that is continuous in $(t,x,y) \in (0,\infty)\times K \times K$.
- For nested fractals, $\mathrm{Cap}(\{x\}) > 0$ for every $x \in K$ and the Brownian motion is point recurrent (see, e.g., [11, Theorem 2.3]). Moreover, an estimate similar to (5) holds with suitable d_{s} and d_{w}, where the Euclidean metric should be replaced by some suitable metric (see, e.g., [27, 10]). Moreover, $1 \le d_{\mathrm{s}} < 2$ holds. (This follows from, e.g., [1, Definition 8.2] and $\alpha \ge 1$ that is confirmed by [1, (8.2)] and the connectedness of K.)
- For Sierpinski carpets, the sub-Gaussian estimate (5) holds with suitable d_{s} and d_{w}, and (6) holds in particular. Moreover, $1 < d_{\mathrm{s}} \le d_{\mathrm{H}}$ holds ([3, Corollary 5.3]).

What are the values of the martingale dimension d_{m} for the Brownian motion on these fractals? The following is all that we know so far.

Theorem 3.2 ([16, 18]) *The following hold.*

(i) *For nested fractals*[2] $d_{\mathrm{m}} = 1$.
(ii) *For Sierpinski carpets,* $1 \le d_{\mathrm{m}} \le d_{\mathrm{s}}$. *In particular, if* $d_{\mathrm{s}} < 2$*, then* $d_{\mathrm{m}} = 1$.

We remark that the inequality $d_{\mathrm{m}} \le d_{\mathrm{s}}$ holds in both cases. We expect that this inequality is valid for more general situations, but the proof of Theorem 3.2 heavily depends on the self-similarity of the underlying space and the Dirichlet form.

Let us briefly explain an idea for the proof of Theorem 3.2 (ii), following [18]. The inequality $d_{\mathrm{m}} \ge 1$ is rather trivial (see [17, Proposition 2.11]), so we concentrate on the inequality $d_{\mathrm{m}} \le d_{\mathrm{s}}$. First, we consider the case $d_{\mathrm{s}} > 2$. Take any integer d greater

[2] In fact, [16, 18] treat more general self-similar fractals than nested fractals, specifically, *post-critically finite* self-similar sets [26]. We omit a detailed explanation here.

than d_s and assume that $d_\mathrm{m} \geq d$. A crucial step of the proof is constructing nice harmonic functions $h_1, \dots, h_d$ on K. Then, the map $\Phi = (h_1, \dots, h_d)$ is regarded as a nice harmonic map from K to $\mathbb{R}^d$. The push-forward $(\Phi_*\mathcal{E}, \Phi_*\mathcal{F})$ of the Dirichlet form $(\mathcal{E}, \mathcal{F})$ by Φ is a bilinear form on $L^2(\mathbb{R}^d, \Phi_*\mu)$. By comparing this with a standard energy form on $L^2(\mathbb{R}^d, dx)$ we can construct a sequence of functions that contradict the Sobolev inequality

$$\|f\|^2_{L^{2d_\mathrm{s}/(d_\mathrm{s}-2)}(K,\mu)} \leq C(\mathcal{E}(f,f) + \|f\|^2_{L^2(K,\mu)}), \qquad f \in \mathcal{F}.$$

This implies that $d_\mathrm{m} < d$. Thus, $d_\mathrm{m} \leq d_\mathrm{s}$ holds. For the case $d_\mathrm{s} = 2$ the same conclusion follows from the result when $d_\mathrm{s} > 2$.

In the case where $d_\mathrm{s} < 2$, $\mathrm{Cap}(\{x\}) > 0$ holds for every x. In this case it suffices to prove $d_\mathrm{m} < 2$. As in the argument above, by assuming $d_\mathrm{m} \geq 2$ we can construct a nice harmonic map $\Phi = (h_1, h_2)$ from K to $\mathbb{R}^2$. Then, using this map we make an argument similar to the case of $d_\mathrm{s} > 2$, but this time we can prove that $\Phi^{-1}(\{a\}) \neq \emptyset$ and $\mathrm{Cap}(\Phi^{-1}(\{a\})) = 0$ for some $a \in \mathbb{R}^2$. This is a contradiction, leading to the inequality $d_\mathrm{m} < 2$. Theorem 3.2 (i) can be proved in a similar way.

We use a kind of blow-up argument to obtain nice harmonic functions $\Phi = (h_1, \dots, h_d)$ in the proof, which requires the self-similarity of the underlying space and the Dirichlet form. Thus, it is not clear at the moment in how general a setting the inequality $d_\mathrm{m} \leq d_\mathrm{s}$ holds. In the next section we introduce a recent attempt of extension in this direction.

4 Recent development: case of metric measure spaces

In this section we consider more general, not necessarily self-similar, sets as the state space. We keep the setting in Section 2. In particular, K is a locally compact separable metrizable space.

For an open set U of K, let $\mathcal{F}_U$ denote the closure of the set $\{f \in \mathcal{F} \mid f = 0\ \mu\text{-a.e. on some open set including } K \setminus U\}$ in $\mathcal{F}$. We call a function $h \in \mathcal{F}$ *harmonic* on U if $\mathcal{E}(h,h) \leq \mathcal{E}(h+f, h+f)$ for any $f \in \mathcal{F}_U$. For a subset V of U, we define

$$\begin{aligned}&\mathrm{Cap}(V;U)\\ &\ = \inf\{\mathcal{E}(f,f) \mid f \in \mathcal{F}_U \text{ and } f = 1\ \mu\text{-a.e. on some open set including } V\},\end{aligned}$$

where $\inf \emptyset = \infty$ by definition. For a Borel function f on K and a Borel set A of K, we define the μ-oscillation of f on A by

$$\mu\text{-}\operatorname*{osc}_A f := \mu\text{-}\operatorname*{esssup}_A f - \mu\text{-}\operatorname*{essinf}_A f.$$

We fix a minimal energy-dominant measure ν and introduce the following assumptions.

Assumption 4.1 (i) (Nontriviality) $\mathcal{E} \not\equiv 0$.

(ii) There is a sequence $\{U_k^{(1)}\}_{k\in\Lambda_1}, \{U_k^{(2)}\}_{k\in\Lambda_2}, \{U_k^{(3)}\}_{k\in\Lambda_3}, \dots$ of families of subsets of K such that the following hold.

(a) Each $U_k^{(n)}$ is a relatively compact open subset of K.

(b) For each n, $\{U_k^{(n)}\}_{k\in\Lambda_n}$ are disjoint in k and $(\mu+\nu)\Bigl(K \setminus \bigsqcup_{k\in\Lambda_n} U_k^{(n)}\Bigr) = 0$.

(c) For each n, $\{U_k^{(n+1)}\}_{k\in\Lambda_{n+1}}$ is a refinement of $\{U_k^{(n)}\}_{k\in\Lambda_n}$ in the sense that for each $k \in \Lambda_{n+1}$, $U_k^{(n+1)} \subset U_{k'}^{(n)}$ for some $k' \in \Lambda_n$.

(d) The σ-field $\mathcal{B}_0$ generated by $\{U_k^{(n)} \mid n \in \mathbb{N},\ k \in \Lambda_n\}$ coincides with the Borel σ-field $\mathcal{B}(K)$ of K up to $(\mu+\nu)$-null sets. That is, for any $A \in \mathcal{B}(K)$ there exists $A' \in \mathcal{B}_0$ such that $A \mathbin{\triangle} A'$ is $(\mu+\nu)$-null.

(e) For any compact subset S of K, there exists $n \in \mathbb{N}$ such that

$$\mu\left(\bigsqcup_{k\in\Lambda_n;\, U_k^{(n)}\cap S\neq\emptyset} U_k^{(n)}\right) < \infty.$$

(iii) There exists $C \ge 1$ such that the following hold. For each $n \in \mathbb{N}$ and $k \in \Lambda_n$, there exists a closed set $V_k^{(n)}$ included in $U_k^{(n)}$ such that for any $h \in \mathcal{F}$ that is non-constant on $U_k^{(n)}$ and harmonic on $U_k^{(n)}$, we have the following.

(a) $\nu_h(U_k^{(n)}) \le C\nu_h(V_k^{(n)})$.

(b) $\operatorname{Cap}(V_k^{(n)}; U_k^{(n)}) \le C \dfrac{\nu_h(U_k^{(n)})}{\bigl(\mu\text{-osc}_{U_k^{(n)}} h\bigr)^2}$.

(c) $\operatorname{Cap}(\{x\}; U_k^{(n)}) \ge C^{-1} \dfrac{\nu_h(U_k^{(n)})}{\bigl(\mu\text{-osc}_{U_k^{(n)}} h\bigr)^2}$ for every $x \in V_k^{(n)}$.

The main theorem in this section is as follows.

Theorem 4.2 ([20]) *Under Assumption 4.1,* $d_{\mathrm{m}} = 1$.

Note that Assumption 4.1 does not impose any self-similarities or other specific structures of the underlying space but only some analytic homogeneity. Assumption (iii) (c) is rather restrictive and is expected to hold only in typical situations with $d_{\mathrm{s}} < 2$. In this sense, the claim of Theorem 4.2 is consistent with the inequalities $1 \le d_{\mathrm{m}} \le d_{\mathrm{s}}$.

The proof of Theorem 4.2 is similar to that of Theorem 3.2 in its general outline, but it differs significantly in some technical aspects. Simply stated, while we made a blow-up argument *before* pushing forward the Dirichlet form by the harmonic map in the proof of Theorem 3.2, we make a blow-up-like argument *at the same time* when pushing forward the Dirichlet form in the proof of Theorem 4.2. This change avoids assuming particular structures on the underlying space.

The canonical Dirichlet form on any nested fractal satisfies Assumption 4.1, and thus we recover Theorem 3.2 (i) from Theorem 4.2. As another example satisfying Assumption 4.1, we introduce inhomogeneous Sierpinski gaskets as follows.

Example 4.3 Let ν be an integer greater than 1, and write $N(\nu) = \nu(\nu+1)/2$. Let $\tilde{K}$ be an equilateral triangle in $\mathbb{R}^2$ including the interior and V_0 denote the set of vertices of $\tilde{K}$. Let $K_i^{(\nu)} \subset \tilde{K}$, $i = 1, 2, \dots, N(\nu)$, be the equilateral triangles including the interior that are obtained by dividing the sides of $\tilde{K}$ in ν, joining these points by the line segments which are parallel to one of the edges of $\tilde{K}$, and removing all the downward-pointing triangles. Let $\psi_i^{(\nu)}$, $i = 1, 2, \dots, N(\nu)$, be the contractive affine map from $\tilde{K}$ onto $K_i^{(\nu)}$ of type $\psi_i^{(\nu)}(z) = \nu^{-1}z + \alpha_i^{(\nu)}$ for some $\alpha_i^{(\nu)} \in \mathbb{R}^2$, which is uniquely determined.

We take a non-empty finite subset T of $\{\nu \in \mathbb{N} \mid \nu \geq 2\}$. For each $\nu \in T$, let $S^{(\nu)}$ denote the set of letters i^ν for $i = 1, 2, \dots, N(\nu)$. We set $S = \bigcup_{\nu\in T} S^{(\nu)}$ and $\Sigma = S^{\mathbb{N}}$.

For each $v \in S$ a shift operator $\sigma_v \colon \Sigma \to \Sigma$ is defined by $\sigma_v(\omega_1\omega_2\cdots) = v\omega_1\omega_2\cdots$. Let $W_0 = \{\emptyset\}$ and $W_m = S^m$ for $m \in \mathbb{N}$, and define $W_* = \bigcup_{m=0}^\infty W_m$. For $w = w_1\cdots w_m \in W_m$ and $w' = w'_1\cdots w'_n \in W_n$, $ww' \in W_{m+n}$ denotes $w_1\cdots w_m w'_1\cdots w'_n$. For $\omega = \omega_1\omega_2\cdots \in \Sigma$ and $n \in \mathbb{N}$ let $[\omega]_n$ denote $\omega_1\cdots\omega_n \in W_n$. By convention, $[\omega]_0 := \emptyset \in W_0$ for $\omega \in \Sigma$.

For $i^\nu \in S$ we define $\psi_{i^\nu} := \psi_i^{(\nu)}$. For $w = w_1w_2\cdots w_m \in W_m$, ψ_w denotes $\psi_{w_1} \circ \psi_{w_2} \circ \cdots \circ \psi_{w_m}$. Here $\psi_\emptyset$ is the identity map by definition. For $\omega \in \Sigma$, $\bigcap_{m=0}^\infty \psi_{[\omega]_m}(\tilde{K})$ is a one-point set, say $\{p\}$. The map $\Sigma \ni \omega \mapsto p \in \tilde{K}$ is denoted by π. The relation $\psi_v \circ \pi = \pi \circ \sigma_v$ holds for $v \in S$.

Now, we fix $L = \{L_w\}_{w\in W_*} \in T^{W_*}$. Namely, we assign $L_w \in T$ to each $w \in W_*$. We set $\tilde{W}_0 = \{\emptyset\}$ and

$$\tilde{W}_m = \bigcup_{w\in\tilde{W}_{m-1}} \left\{wv \,\middle|\, v \in S^{(L_w)}\right\}$$

for $m \in \mathbb{N}$, inductively. Define $\tilde{\Sigma} = \{\omega \in \Sigma \mid [\omega]_m \in \tilde{W}_m \text{ for all } m \in \mathbb{Z}_{\geq 0}\}$ and $K = \pi(\tilde{\Sigma})$. It holds that

$$K = \bigcap_{m=0}^\infty \bigcup_{w\in\tilde{W}_m} \psi_w(\tilde{K}).$$

We call K the *inhomogeneous Sierpinski gasket* generated by L, see Fig. 4.

Fix a finite Borel measure μ on K such that μ has full support and does not charge any one points. We can construct a canonical, strongly local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$ (see [15] and [12, p. 324], see also [21]). We can also confirm that this setting satisfies Assumption 4.1 by letting $\Lambda_n = \tilde{W}_n$ $(n \in \mathbb{N})$ and

$$U_w^{(n)} = K \cap \psi_w(\tilde{K} \setminus V_0), \quad w \in \Lambda_n,$$

and defining $V_w^{(n)}$ $(w \in \Lambda_n)$ suitably. Therefore, the martingale dimension d_{m} is one from Theorem 4.2.

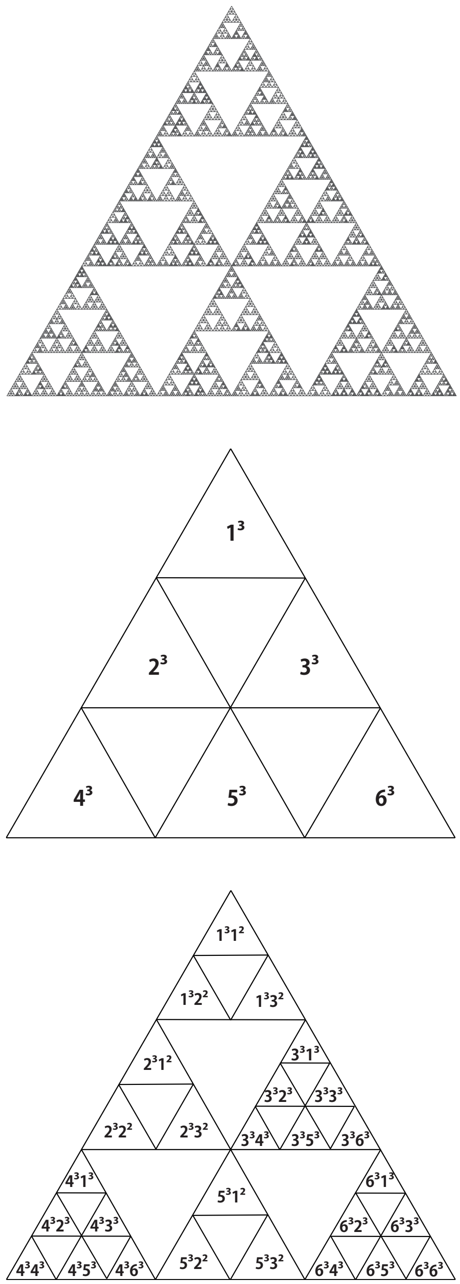

Fig. 4 An example of inhomogeneous Sierpinski gaskets with $T = \{2, 3\}$ (the top figure). Here, $L = \{L_w\}_{w \in W_*}$ is given by $L_\emptyset = 3$, $L_{1^3} = L_{2^3} = L_{5^3} = 2$, $L_{3^3} = L_{4^3} = L_{6^3} = 3$, $L_{1^3 1^2} = L_{1^3 2^2} = L_{1^3 3^2} = 2$, $L_{2^3 1^2} = 2$, $L_{2^3 2^2} = L_{2^3 3^2} = 3$, $L_{3^3 1^3} = L_{3^3 3^3} = L_{3^3 4^3} = L_{3^3 5^3} = L_{3^3 6^3} = 2$, $L_{3^3 2^3} = 3$, etc. The indices are indicated in the middle and bottom figures.

5 Concluding remarks

We will now discuss related problems for future studies.

(i) As described in Section 3.2, the main idea of the proof of Theorem 4.2, as well as that of Theorem 3.2, is summarized as introducing and studying harmonic maps from K to $\mathbb{R}^d$. This allows us to deduce an upper bound of d_{m}. Actually, what is proved in Theorem 4.2 is the inequality $d_{\mathrm{m}} < 2$. To obtain a detailed estimate of d_{m}, including the lower-side estimate, it may be useful to develop a suitable theory of harmonic maps between metric measure spaces fitting this context. A nontrivial lower bound of d_{m} other than $d_{\mathrm{m}} \geq 1$ has yet to be obtained for general fractal sets. For example, let us consider the three-dimensional Sierpinski carpet (see the rightmost figure in Fig. 3). Its spectral dimension d_{s} is known to satisfy $2 < d_{\mathrm{s}} < 3$ (see [3, (9.2)]), so that d_{m} is either 1 or 2 by Theorem 3.2 (ii). Which is the case remains unsolved. We would also like to draw attention to the very recent studies [35, 36] that have contributed to the study of martingale dimensions for metric measure spaces.

(ii) Theorem 4.2 can treat only the case of low spectral dimensions at the moment. A large part of the proof works for more general situations, but certain technical obstacles have not been overcome.

(iii) For many fractals the martingale dimension is one, which informally says that the space of noises is one-dimensional. We expect, however, the structures of noises are distinct for each fractal. How to distinguish them, in other words, what is a more minute indicator than the martingale dimension, is a natural problem to be investigated.

(iv) The problem of determining the martingale dimension d_{m} is not an isolated topic; the concept of martingale dimension appears in other subjects. Indeed, it was proved in [19] that general strongly local regular Dirichlet forms provide measurable Riemannian structures on the state space, and that the maximal dimension of the ‘‘(co-)tangent spaces’’ is equal to d_{m}. From the viewpoint of ‘‘differential geometry’’ and ‘‘(stochastic) differential calculus’’ on general state spaces like fractals, which will hopefully be developed well in the future, the martingale dimension may play fundamental roles.

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