INDICES OF DIRICHLET FORMS

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Abstract. This article presents recent progress on the theory of Dirichlet forms, and, in particular, on the indices and related concepts of strong-local Dirichlet forms. We discuss the relation to geometric structures of the underlying spaces and local structures of the associated diffusion processes. Some results on quantitative estimates of indices for typical fractal sets are also explained.

1. Introduction

This article surveys the indices and related concepts of strong-local Dirichlet forms as an example of recent progress in the theory of Dirichlet forms. In particular, we discuss how such concepts are related with geometric structures of the state spaces and local structures of the associated diffusion processes.

A fundamental problem in the theory of stochastic processes is how to obtain infinitesimal behaviors of diffusion processes. For state spaces equipped with some differential structures, if diffusion processes are characterized by stochastic differential equations, or if their generators are expressed as differential operators, then we have basic information on such processes. However, in nonsmooth state spaces, such as fractal sets, difficulty immediately arises at this point. Another useful method for understanding stochastic processes is the study of structures of the filtration associated with stochastic processes, and study of the space of martingales (or martingale additive functionals) with respect to the filtration has attracted particular interest. Such studies have been done in various frameworks ([MW64, Sk66, KW67, DV74]) and in a series of studies by Yor in the 1970s.[1] This formulation does not impose particular structures on state spaces, which is convenient for developing general theories on anomalous spaces. In one of the earliest papers discussing diffusion processes on self-similar fractals, Barlow and Perkins [BP88] asked how many dimensions the filtrations generated by the Brownian motions on Sierpinski gaskets have in the sense of Davis–Varaiya [DV74]. Kusuoka [Kus89] gave a remarkable answer to this problem by proving that the martingale dimension with respect to additive functionals is 1 for the Brownian motion on d-dimensional Sierpinski gaskets $\mathrm{SG}(d)$ for any $d \geq 2$. This unexpected result contrasts sharply with the Hausdorff dimension of $\mathrm{SG}(d)$, which is $\log_2(d+1)$ and can be arbitrarily large. The proof is based on characterization of the martingale dimension by an analytic index

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[1] See also the exposition [AIW09] for the theory of filtrations and noises on the basis of the theory of operator algebras, which is a different approach than the approach discussed in this article.

and determination of its value by concrete estimates. After a period of inactivity on this topic in the literature, the author obtained quantitative estimates for self-similar diffusions on more general self-similar fractals, such as nested fractals and Sierpinski carpets [Hi08, Hi13a]. The proof in [Hi13a] is based on a new concept of indices of strong-local regular Dirichlet forms and the fact that the index coincides with the martingale dimension associated with the diffusion process, which is a natural generalization of the results by Kusuoka [Kus89]. Notwithstanding the original motivation for studying diffusions on fractals, the definition of indices of Dirichlet forms is also useful for general state spaces. A general theory along this line has been developed in [Hi13b]; in particular, it was shown that the pointwise index represents the effective dimension of the virtual tangent space and that there exists a family of functions that play the role of coordinate maps and define a kind of Riemannian structure on the state space. These facts may be helpful in the study of "measurable Riemannian structures" proposed by Kigami (cf. [Ki08]).

This survey introduces such recent studies on the local structures of state spaces arising from stochastic analysis. Related studies have investigated this topic from the viewpoint of analysis on metric measure spaces, leading to study by the introduction of Sobolev spaces, differential structures, and functional inequalities on the basis of upper gradients [Ch99, Sh00], and studies related to curvatures of metric measure spaces by Sturm and Lott–Villani [KoZ12]. An interesting topic for future study is clarifying the relation between these studies and the subject of this article.

The organization of this article is as follows. Section 2 provides some typical examples of Dirichlet forms that we keep in mind for subsequent arguments. Section 3 introduces the concept of indices for general strong-local regular Dirichlet forms. We also justify interpreting the (pointwise) index as the dimension of virtual tangent spaces of the state space by noting that Dirichlet forms introduce a "measurable Riemannian structure" to the state space and "differentiation" is defined for all functions in the domain of the Dirichlet forms. Furthermore, as an interpretation in terms of stochastic analysis, we will show that the index coincides with the martingale dimension with respect to the space of additive functionals associated with the diffusion processes corresponding to the Dirichlet forms. Section 4 discusses quantitative estimates of the martingale dimensions in the case where the underlying spaces are self-similar fractals. Section 5 introduces an ongoing study that is expected to be involved with the concept of index. Section 6 presents some open problems as concluding remarks.

Many propositions stated in this article are valid for general structures of state spaces, but the most interesting examples are state spaces with anomalous structures, such as fractal sets. This is why we also give an exposition of analysis on self-similar fractals, though this is somewhat classical content.

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2. Examples of strong-local regular Dirichlet forms

We first recall several concepts of Dirichlet forms, following [FOT11]. Let K be a locally compact Hausdorff space with a countable basis (hereinafter, an LCCB space) and m a positive Radon measure on K with full support. A symmetric bilinear form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, m)$ with a dense subspace $\mathcal{F}$ of $L^2(K, m)$ as a domain is called a *Dirichlet form* if the following conditions are satisfied:

(i) (Nonnegativity) $\mathcal{E}(f,f) \geq 0$ for every $f \in \mathcal{F}$.
(ii) The space $\mathcal{F}$ equipped with an inner product $\mathcal{E}_1(f,g) := \mathcal{E}(f,g) + \int_K fg\,dm$ is a Hilbert space.
(iii) (Markov property) For every $f \in \mathcal{F}$, both $g := \min(\max(0,f),1) \in \mathcal{F}$ and $\mathcal{E}(g,g) \leq \mathcal{E}(f,f)$.

Let $C_0(K)$ denote the set of all continuous functions on K with compact support and equip it with the uniform norm. A Dirichlet form $(\mathcal{E},\mathcal{F})$ is called *regular* if $\mathcal{F} \cap C_0(K)$ is dense in both $C_0(K)$ and $\mathcal{F}$. Then the capacity Cap associated with $(\mathcal{E},\mathcal{F})$ is defined by

$$\mathrm{Cap}(A) = \inf\{\mathcal{E}_1(f,f) \mid f \in \mathcal{F} \text{ and } f \geq 1 \text{ } m\text{-a.e. on some open set including } A\}$$

for subsets A of K. We say that a property depending on $x \in K$ holds quasi-everywhere (q.e.) if the set of x for which the property does not hold has a null capacity. A function f on K is called *quasi-continuous* if, for every $\varepsilon > 0$, there exists an open subset G of K such that $\mathrm{Cap}(G) < \varepsilon$ and $f|_{K\setminus G}$ is continuous. Every $f \in \mathcal{F}$ has a quasi-continuous m-version $\tilde{f}$. The *energy measure* $\mu_{\langle f\rangle}$ of a bounded function f in $\mathcal{F}$ is a finite Borel measure on K that is characterized by the relation

$$\int_K \varphi\,d\mu_{\langle f\rangle} = 2\mathcal{E}(f,f\varphi) - \mathcal{E}(f^2,\varphi) \qquad \text{for all } \varphi \in \mathcal{F} \cap C_0(K).$$

For general $f \in \mathcal{F}$, $\mu_{\langle f\rangle}$ is defined by $\mu_{\langle f\rangle} = \sup_n \mu_{\langle \min(\max(-n,f),n)\rangle}$. Note that energy measures do not essentially depend on the underlying measure m. For f and g in $\mathcal{F}$, the signed measure $\mu_{\langle f,g\rangle}$ on K is defined by $\mu_{\langle f,g\rangle} = (\mu_{\langle f+g\rangle} - \mu_{\langle f\rangle} - \mu_{\langle g\rangle})/2$. We say that $(\mathcal{E},\mathcal{F})$ is *strong local* if $\mathcal{E}(f,g) = 0$ for all $f,g \in \mathcal{F}$ such that $\mathrm{supp}[f\cdot m]$ and $\mathrm{supp}[g\cdot m]$ are compact and f is constant on some open neighborhood of $\mathrm{supp}[g\cdot m]$. Throughout this article, we assume without exception that $(\mathcal{E},\mathcal{F})$ is a strong-local regular Dirichlet form. From the general theory of Dirichlet forms, $(\mathcal{E},\mathcal{F})$ has an associated m-symmetric diffusion process $\{X_t\}$ on K with no killing inside. Moreover, the identity $\mathcal{E}(f,g) = \mu_{\langle f,g\rangle}(K)/2$ holds for $f,g \in \mathcal{F}$.

The following are some typical examples.

Example 2.1. Let $K = \boldsymbol{R}^d$ and $m = dx$ (the Lebesgue measure). We take a Lebesgue-measurable $\boldsymbol{R}^{d\times d}$-valued function $A(x) = \bigl(a_{ij}(x)\bigr)_{i,j=1}^d$ ($x \in \boldsymbol{R}^d$) such that $A(x)$ is symmetric for every x and there exists a constant $c \geq 1$ satisfying

$$c^{-1}|h|^2_{\boldsymbol{R}^d} \leq (A(x)h,h)_{\boldsymbol{R}^d} \leq c|h|^2_{\boldsymbol{R}^d}, \quad h \in \boldsymbol{R}^d,\ x \in \boldsymbol{R}^d.$$

Let $\mathcal{F}$ be the first-order L^2-Sobolev space $W^{1,2}(\boldsymbol{R}^d)$ and define

$$\mathcal{E}(f,g) = \frac{1}{2}\int_{\boldsymbol{R}^d} (A(x)\nabla f(x), \nabla g(x))_{\boldsymbol{R}^d}\,dx, \quad f,g \in \mathcal{F}.$$

Then, $(\mathcal{E},\mathcal{F})$ is a strong-local regular Dirichlet form on $L^2(\boldsymbol{R}^d, dx)$. By a concrete calculation, for each $f,g \in \mathcal{F}$, $\mu_{\langle f,g\rangle}$ can be expressed as

$$\mu_{\langle f,g\rangle}(dx) = (A(x)\nabla f(x), \nabla g(x))_{\boldsymbol{R}^d}\,dx.$$

Example 2.2 (superposition)**.** Let $K = \boldsymbol{R}^2$ and $m = dx\,dy$ (the 2-dimensional Lebesgue measure). For $f,g \in C_0^\infty(\boldsymbol{R}^2)$, we define

$$\mathcal{E}(f,g) = \frac{1}{2}\int_{\boldsymbol{R}^2} (\nabla f(x,y), \nabla g(x,y))_{\boldsymbol{R}^2}\,dx\,dy + \frac{1}{2}\int_{\boldsymbol{R}} \frac{\partial f}{\partial x}(x,0)\frac{\partial g}{\partial x}(x,0)\,dx.$$

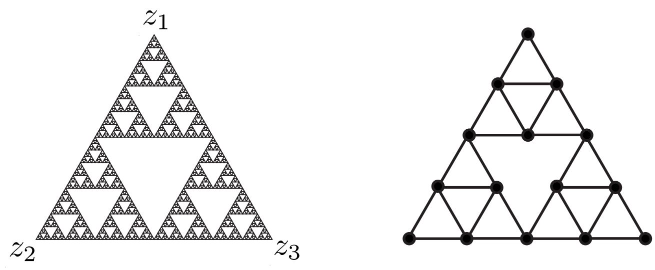

FIGURE 1. 2-dimensional Sierpinski gasket and a graph $(V_2, \overset{2}{\sim})$

Then, $(\mathcal{E}, C_0^\infty(\boldsymbol{R}^2))$ is closable on $L^2(K, m)$, and its closure, denoted by $(\mathcal{E}, \mathcal{F})$, is a strong-local regular Dirichlet form on $L^2(K, m)$. The energy measures are described as

$$\mu_{\langle f,g\rangle}(dx\,dy) = (\nabla f(x,y), \nabla g(x,y))_{\boldsymbol{R}^2}\,dx\,dy + \frac{\partial \tilde{f}}{\partial x}(x,0)\frac{\partial \tilde{g}}{\partial x}(x,0)\,dx \otimes \delta_0(dy)$$

for $f, g \in \mathcal{F}$, where $\tilde{f}$ and $\tilde{g}$ denote quasi-continuous modifications of f and g, respectively, and δ_0 denotes the delta measure at 0.

Examples of superposition have been studied from various viewpoints as typical examples of when the generators are not the usual differential operators (see, e.g., [Ik69, IW71, IW72, IW73, To80, HMO96]). The diffusion process associated with the Dirichlet form mentioned above behaves as 2-dimensional Brownian motion off the x-axis and gathers speed in the x-direction on the x-axis. More precisely, the diffusion process is described as $(B^{(1)}(t + \psi(t)), B^{(2)}(t))$, where $(B^{(1)}, B^{(2)})$ is the 2-dimensional Brownian motion and $\psi(t)$ is the local time of $B^{(2)}$ at 0. A concrete expression of the associated transition density $p_t(z_1, z_2)$ is also known; in particular, it has non-Gaussian behavior on the x-axis as

$$\lim_{t\to 0} t^{1/3} \log p_t\big((x,0),(x,'0)\big) = -\frac{3}{2}|x - x'|^{4/3}, \quad x, x' \in \boldsymbol{R}.$$

See [HMO96] for further details in more general situations.

Example 2.3 (2-dimensional Sierpinski gasket (e.g., [Ki89, Kus89, FS92][2]))**.** Let $z_1 = (0, \sqrt{3}/2)$, $z_2 = (-1/2, 0)$, and $z_3 = (1/2, 0)$ be three points of $\boldsymbol{R}^2$. For $i = 1, 2, 3$, a contraction map ψ_i on $\boldsymbol{R}^2$ with a fixed point z_i is defined by $\psi_i(z) = (z - z_i)/2 + z_i$ $(z \in \boldsymbol{R}^2)$. The 2-dimensional Sierpinski gasket K is determined by the unique nonempty compact subset of $\boldsymbol{R}^2$ satisfying $K = \bigcup_{i=1}^3 \psi_i(K)$. The normalized Hausdorff measure of K is denoted by m. Then, the standard Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, m)$ is defined as follows. First, an increasing sequence of finite subsets $\{V_n\}_{n=0}^\infty$ of $\boldsymbol{R}^2$ is inductively defined by $V_0 = \{z_1, z_2, z_3\}$ and $V_n = \bigcup_{i=1}^3 \psi_i(V_{n-1})$ $(n = 1, 2, 3, \dots)$. Let V_* denote $\bigcup_{n=0}^\infty V_n$. Then the closure of V_* is equal to K. Each V_n is equipped with a natural graph structure (Fig. 1). If $x, y \in V_n$ are both ends of some bond, we write $x \overset{n}{\sim} y$. For functions f and g on V_n, define

$$Q_n(f,g) = \sum_{x,y \in V_n,\ x \overset{n}{\sim} y} (f(x) - f(y))(g(x) - g(y)).$$

[2]Note that there are prior studies [Kus87, Go87, BP88] concerning the construction of standard diffusion processes on the Sierpinski gasket in the manner of scaling limits of random walks.

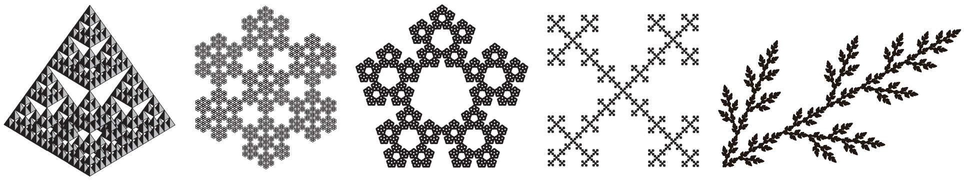

FIGURE 2. Examples of p. c. f. self-similar sets: 3-dimensional Sierpinski gasket, snowflake, Pentakun, Vicsek set, and Hata's tree-like set. All but the rightmost set are also nested fractals.

For every function f on V_*, the sequence $\{(5/3)^n Q_n(f|_{V_n}, f|_{V_n})\}_{n=0}^\infty$ is nondecreasing on n. Moreover, for any function f on V_0, there exists an extension of f to V_* such that this sequence is constant. Therefore, $5/3$ is the appropriate scaling constant in this example. Let

$$\mathcal{F}_* = \{f \mid f \text{ is a function on } V_* \text{ and } \lim_{n\to\infty}(5/3)^n Q_n(f|_{V_n}, f|_{V_n}) < \infty\}.$$

Since each $f \in \mathcal{F}_*$ is proved to be uniformly continuous (and, in fact, Hölder continuous) on V_*, it follows that f has a unique continuous extension to K. This extension is denoted by f again and the totality of such functions is denoted by $\mathcal{F}$. For $f, g \in \mathcal{F}$, we define $\mathcal{E}(f,g) = \lim_{n\to\infty}(5/3)^n Q_n(f|_{V_n}, g|_{V_n})$. Then, $(\mathcal{E},\mathcal{F})$ is a strong-local regular Dirichlet form on $L^2(K,m)$ that possesses a self-similar property

$$\mathcal{E}(f,g) = \sum_{i=1}^{3}\frac{5}{3}\mathcal{E}(f\circ\psi_i, g\circ\psi_i), \qquad f,g\in\mathcal{F}. \tag{2.1}$$

The associated diffusion process can properly be called the Brownian motion on K. From (2.1), the energy measures also have the following self-similar property:

$$\mu_{\langle f,g\rangle} = \sum_{i=1}^{3}\frac{5}{3}\mu_{\langle f\circ\psi_i, g\circ\psi_i\rangle}, \qquad f,g\in\mathcal{F}. \tag{2.2}$$

Kusuoka [Kus89] gave an expression of energy measures as a limit of products of random matrices; however, it is not easy to extract quantitative information from this expression. Since energy measures are singular with respect to m ([Kus89]), it is not possible to consider density functions of energy measures with respect to m.

Example 2.4 (Nested fractals and post-critically finite (p. c. f.) self-similar sets (e.g., [Li90, Kus89, Kus93, Ki93b])**.** Construction of the Dirichlet form in Example 2.3 is available for more general finitely ramified[3] self-similar sets. Nested fractals, which were introduced by Lindstrøm [Li90], are self-similar sets K in Euclidean spaces that have some nice properties such as symmetries (Fig. 2). Let $\{\psi_i\}_{i=1}^M$ denote the family of contraction maps determining K and m, the normalized Hausdorff measure on K. In a manner similar to Example 2.3, we can construct a nontrivial strong-local regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(K,m)$ that

[3]The property that removing a finite number of points can make the set disconnected.

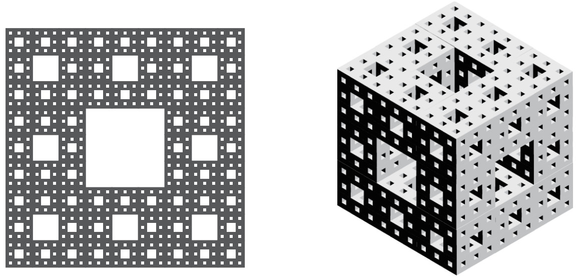

FIGURE 3. Examples of 2- and 3-dimensional Sierpinski carpets

satisfies the self-similar property

$$\mathcal{E}(f,g)=\sum_{i=1}^{M}\frac{1}{r_i}\mathcal{E}(f\circ\psi_i,g\circ\psi_i),\qquad f,g\in\mathcal{F}$$

for some constants $r_i\in(0,1)$ with $i=1,\dots,M$. The associated diffusion process can be called the Brownian motion on K. The concept of p. c. f. self-similar sets is a further abstract generalization that was introduced by Kigami [Ki93b]. Though energy measures are expressed as limits of products of random matrices, just as in Example 2.3, it is more difficult to obtain quantitative estimates. In many cases, energy measures are singular with respect to self-similar measures on K ([BST99, Hi05, HN06]).

Example 2.5 (Sierpinski carpets)**.** The following are typical examples of infinite ramified self-similar fractals. Let n be an integer greater than 1, and l an integer greater than 2. The unit cube $[0,1]^n$ in $\boldsymbol{R}^n$ is denoted by Q_0. A family of distinct contraction maps $\psi_1,\dots,\psi_M$ $(M<l^n)$ on $\boldsymbol{R}^n$ is assumed to satisfy the following: Each ψ_i is expressed as $\psi_i(x)=l^{-1}x+b_i$ for some $b_i\in\boldsymbol{R}^n$ and each coordinate of every b_i belongs to $\{0,1/l,2/l,\dots,(l-1)/l\}$. Then, the unique nonempty compact set K in $\boldsymbol{R}^n$ such that $K=\bigcup_{i=1}^M\psi_i(K)$ is called the (generalized) n-dimensional Sierpinski carpet associated with $\{\psi_i\}_{i=1}^M$ (Fig. 3). Let m be the normalized Hausdorff measure on K. Let $Q_1=\bigcup_{i=1}^M\psi_i(Q_0)$ and $\mathrm{Int}(A)$ denote the interior of a subset A of $\boldsymbol{R}^n$. Furthermore, we assume the following geometrical conditions, due to Barlow and Bass:

- (Symmetry) Q_1 is invariant under any isometric transformations on Q_0.
- (Connectivity) $\mathrm{Int}(Q_1)$ is connected and contains a path connecting hyperplanes $\{x_1=0\}$ and $\{x_1=1\}$.
- (Nondiagonality) Let $N\in\boldsymbol{N}$ and B be a cube in Q_0 with size $2/l^N$ that is expressed as $B=\prod_{j=1}^n\bigl[k_j/l^N,(k_j+2)/l^N\bigr]$ for $k_j\in\{0,1,\dots,l^N-2\}$, $j=1,\dots,n$. Then, $\mathrm{Int}(Q_1\cap B)$ is a connected set (perhaps empty).
- (Border included) $\{(x_1,0,\dots,0)\in\boldsymbol{R}^n\mid 0\le x_1\le 1\}\subset Q_1$.

Standard diffusion processes on Sierpinski carpets have been constructed and studied (e.g., in [BB89, KuZ92, BB99, HKKZ00, Os01]). The uniqueness of diffusions with natural symmetries has been proved recently by Barlow, Bass, Kumagai, and Teplyaev [BBKT10]. In particular, there exists a nontrivial strong-local regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(K,m)$ that is unique up to a constant multiple and satisfies the following properties:

- (Conservativeness) $1 \in \mathcal{F}$.
- (Symmetry) For any $f \in \mathcal{F}$ and any isometries Ψ on Q_0, it holds that $f \circ \Psi \in \mathcal{F}$ and $\mathcal{E}(f \circ \Psi, f \circ \Psi) = \mathcal{E}(f, f)$.
- (Self-similarity) There exists some constant $r \in (0,1)$ such that for every $f, g \in \mathcal{F}$,

$$\mathcal{E}(f,g) = \sum_{i=1}^{M} \frac{1}{r} \mathcal{E}(f \circ \psi_i, g \circ \psi_i).$$

Moreover, the associated transition density $p_t(x,y)$ has the following Aronson type estimate (cf. [BB99, BBK06]):

(2.3)
$$\begin{aligned} &c_1 t^{-d_\mathrm{s}/2} \exp\left(-c_2(|x-y|_{\boldsymbol{R}^n}^{d_\mathrm{w}}/t)^{1/(d_\mathrm{w}-1)}\right) \\ &\quad \le p(t,x,y) \le c_3 t^{-d_\mathrm{s}/2} \exp\left(-c_4(|x-y|_{\boldsymbol{R}^n}^{d_\mathrm{w}}/t)^{1/(d_\mathrm{w}-1)}\right), \ t \in (0,1], \ x, y \in K, \end{aligned}$$

where $d_\mathrm{s} = (2\log M)/\log(M/r) > 1$ (spectral dimension), $d_\mathrm{w} = \log(M/r)/\log l \ge 2$ (walk dimension), and $c_1, \dots, c_4$ are positive constants. By denoting the Hausdorff dimension of K by d_H $(= \log M/\log l)$, the relation

$$d_\mathrm{s} = \frac{2d_\mathrm{H}}{d_\mathrm{w}} \le d_\mathrm{H} \tag{2.4}$$

is satisfied. If $d_\mathrm{w} > 2$ (the sub-Gaussian case)[4], then the inequality $d_\mathrm{s} < d_\mathrm{H}$ holds. Although there is a characterization of r (or equivalently, d_s), and quantitative estimates are known, the exact value of r is unknown. There are other ways of constructing $(\mathcal{E}, \mathcal{F})$, such as construction from a diffusion process by a scaling limit of the Brownian motions on Lipschitz domains of $\boldsymbol{R}^n$ approximating K ([BB89, BB99]), or construction by a graph approximation as in Example 2.3 ([KuZ92, HKKZ00]). In no method, however, is there monotonicity of approximating sequences. This is in contrast to Examples 2.3 and 2.4, which results in analysis to show the existence of nontrivial limits being hard. A concrete expression of the Dirichlet form is also unknown. The same can be said regarding energy measures, so only limited information is available for analysis. Energy measures and the Hausdorff measures are mutually singular ([Hi05, BBK06]).

The transition densities $p_t(x,y)$ associated with the canonical Dirichlet forms on nested fractals in Example 2.4 have estimates of the same type as (2.3) and (2.4) (see, e.g., [Kum93, Ba98]), where the values of d_s, d_w, and d_H depend on the fractals. In the case of nested fractals and Sierpinski carpets, the domain $\mathcal{F}$ is described as a Besov space $\Lambda_{2,\infty}^{d_\mathrm{w}/2}$ ([Jo96, Kum00, Gr03]). In general, the functions in $\mathcal{F}$ are far from smooth in the classical sense. For example, in Example 2.3, the only functions in $\mathcal{F}$ that can extend to C^1-functions on $\boldsymbol{R}^2$ are constant functions.

3. Definition of index and its properties

This section introduces the concept of indices of Dirichlet forms and their properties, following [Hi10, Hi13b]. Henceforth, K is an LCCB space, m is a positive Radon measure on K with full support, and $(\mathcal{E}, \mathcal{F})$ is a strong-local regular Dirichlet form on $L^2(K,m)$.

[4] In [BB99] this inequality is proved under an additional assumption on K, and it is stated that the inequality is always true.

3.1. **Definition of index.** As seen in the examples in the last section, energy measures are not necessarily absolutely continuous with respect to the underlying measure m. Therefore, we introduce a new measure on K so that the energy measures have densities. A positive Radon measure ν on K is called a *minimal energy-dominant measure* if the following two conditions are satisfied.

(i) For every $f \in \mathcal{F}$, $\mu_{\langle f\rangle}$ is absolutely continuous with respect to ν ($\mu_{\langle f\rangle} \ll \nu$ in notation).
(ii) If another positive Radon measure ν' satisfies (i) with ν replaced by ν', then $\nu \ll \nu'$.

Two minimal energy-dominant measures are mutually absolutely continuous by definition, and there always exist minimal energy-dominant measures. In fact, the totality of $f \in \mathcal{F}$ such that $\mu_{\langle f\rangle}$ is a minimal energy-dominant measure of $(\mathcal{E}, \mathcal{F})$ is dense in $\mathcal{F}$. Hereafter we fix a minimal energy-dominant measure ν. The set of all nonnegative integers is denoted by $\boldsymbol{Z}_+$. The *pointwise index* $p(x)$ of $(\mathcal{E}, \mathcal{F})$ is defined as a $\boldsymbol{Z}_+ \cup \{+\infty\}$-valued ν-measurable function on K that is minimal in the ν-almost-everywhere (a.e.) sense among all functions satisfying the following: For any $n \in \boldsymbol{N}$ and any $f_1, \dots, f_n \in \mathcal{F}$,

$$\operatorname{rank}\left(\frac{d\nu_{f_i,f_j}}{d\nu}(x)\right)_{i,j=1}^{n} \le p(x) \quad \text{for } \nu\text{-a.e. } x \in K.$$

The *index* p of $(\mathcal{E}, \mathcal{F})$ is defined as $p = \nu\text{-ess sup}_{x\in K}\, p(x) \in \boldsymbol{Z}_+ \cup \{+\infty\}$. The pointwise index and the index do not depend on the choice of ν. $p(x)$ is determined uniquely up to ν-equivalence and represents the “dimension of the virtual tangent space at $x \in K$” as stated below. Also, as explained in Section 3.3, the index p has a probabilistic interpretation as the martingale dimension with respect to additive functionals associated with the diffusion process $\{X_t\}$. Prototypes of these concepts have been introduced by Kusuoka [Kus89] for a class of finitely ramified self-similar fractals.

The index has the following basic properties.

Proposition 3.1. *The index is* 0 *if and only if* $\mathcal{E} \equiv 0$.

Theorem 3.2 (Stability)**.** *Let* $(\mathcal{E}, \mathcal{F})$ *and* $(\mathcal{E},' \mathcal{F}')$ *be strong-local regular Dirichlet forms on* $L^2(K, m)$ *that are equivalent; that is,* $\mathcal{F} = \mathcal{F}'$ *and there exists some constant* $c \ge 1$ *such that*

$$c^{-1}\mathcal{E}(f,f) \le \mathcal{E}'(f,f) \le c\mathcal{E}(f,f), \qquad f \in \mathcal{F}.$$

Then, the pointwise indices of $(\mathcal{E}, \mathcal{F})$ *and* $(\mathcal{E}', \mathcal{F}')$ *coincide with each other.*

For some of the examples in Section 2, (pointwise) indices are easily determined.

- In Example 2.1, the Lebesgue measure dx can be taken as ν. Then,

$$\begin{aligned}\left(\frac{d\mu_{\langle f_i,f_j\rangle}}{d\nu}(x)\right)_{i,j=1}^{n} &= \Big(\big(A(x)\nabla f_i(x), \nabla f_j(x)\big)_{\boldsymbol{R}^d}\Big)_{i,j=1}^{n} \\ &= {}^{t}\Big(\sqrt{A(x)}B(x)\Big)\Big(\sqrt{A(x)}B(x)\Big),\end{aligned}$$

  where $B(x)$ is a $d\times n$-matrix in which the (k, l)-component is $(\partial f_l/\partial x_k)(x)$. This expression directly implies $p(x) \le d$ ν-a.e. By taking appropriate functions $\{f_i\}_{i=1}^{d}$, the reverse inequality is also proved. Thus $p(x) = d$ ν-a.e. and the index p is equal to the dimension d of the state space.

- In Example 2.2, the measure $dx \otimes (dy + \delta_0(dy))$ can be taken as ν. By a concrete calculation, the pointwise index $p(z)$ ($z = (x, y) \in \boldsymbol{R}^2$) and the index p are given by
$$p(z) = \begin{cases} 2 & \text{if } y \neq 0 \\ 1 & \text{if } y = 0 \end{cases} \quad \nu\text{-a.e.} \quad \text{and} \quad p = 2.$$
Though the x-axis is a negligible set with respect to the 2-dimensional Lebesgue measure, $p(z)$ reflects the difference between the behavior of the diffusion on the x-axis and that off the x-axis.

In Examples 2.3–2.5, it is not straightforward to obtain the concrete values of the indices because the energy measures do not have concise expressions. Further details will be discussed in Section 4.

3.2. **Interpretation of pointwise index as the dimension of "tangent space."** In this subsection only, the index p is assumed to be finite. We will introduce a kind of (measure theoretic-)Riemannian structure on the state space K and the concept of differentiation of functions in $\mathcal{F}$.

The set of all $(g_1, \dots, g_p) \in \mathcal{F}^p$ (the pth direct product of $\mathcal{F}$) satisfying the following is denoted by $\mathcal{G}$: For ν-a.e. x, the matrix $\bigl((d\mu_{\langle g_i, g_j\rangle}/d\nu)(x)\bigr)_{i,j=1}^{p(x)}$ of order $p(x)$ is invertible.

Theorem 3.3 ([Hi13b])**.** *The set $\mathcal{G}$ is dense in $\mathcal{F}^p$.*

We fix $\boldsymbol{g} = (g_1, \dots, g_p) \in \mathcal{G}$ and denote $\bigl((d\mu_{\langle g_i, g_j\rangle}/d\nu)(x)\bigr)_{i,j=1}^{p}$ by $Z_{\boldsymbol{g}}(x)$. By the following theorem, the mapping $\boldsymbol{g}\colon K \to \boldsymbol{R}^p$ is regarded as a variant of local coordinates (though $\boldsymbol{g}$ is not injective in general), and $Z_{\boldsymbol{g}}(x)$ and $p(x)$ are regarded as the Riemannian metric and the dimension of the "tangent space" at $x \in K$, respectively.

Theorem 3.4 ([Hi13b], "differentiation" of functions in $\mathcal{F}$)**.** *For every $f \in \mathcal{F}$, there exists a ν-measurable $\boldsymbol{R}^p$-valued function $\nabla_{\boldsymbol{g}} f = {}^t(\partial^{(1)} f, \dots, \partial^{(p)} f)$ such that, for ν-a.e. x, $\partial^{(j)} f(x) = 0$ ($j > p(x)$), and in the representation*

$$\tilde{f}(y) - \tilde{f}(x) = \sum_{i=1}^{p(x)} \partial^{(i)} f(x)\bigl(\tilde{g}_i(y) - \tilde{g}_i(x)\bigr) + R_x(y), \; y \in K, \tag{3.1}$$

the property $(d\mu_{\langle R_x\rangle}/d\nu)(x) = 0$ follows. The function $\nabla_{\boldsymbol{g}} f$ is uniquely determined up to ν-equivalence. Moreover, the following identity follows:

$$\mathcal{E}(f, h) = \frac{1}{2}\int_K \bigl(Z_{\boldsymbol{g}}\, \nabla_{\boldsymbol{g}} f, \nabla_{\boldsymbol{g}} h\bigr)_{\boldsymbol{R}^p}\, d\nu, \quad f, h \in \mathcal{F}. \tag{3.2}$$

In the statement above, $\tilde{f}$ and $\tilde{g}$ represent quasi-continuous modifications of f and g, respectively. The precise meaning of $(d\mu_{\langle R_x\rangle}/d\nu)(x)$ is a substitution of y for x in the canonical version of $(d\mu_{\langle R_x\rangle}/d\nu)(y)$ that is defined so that (3.1) holds for fixed x, that is,

$$\frac{d\mu_{\langle f\rangle}}{d\nu}(x) - 2\sum_{i=1}^{p(x)} \partial^{(i)} f(x)\, \frac{d\mu_{\langle f, g^{(i)}\rangle}}{d\nu}(x) + \sum_{i=1}^{p(x)} \partial^{(i)} f(x)^2\, \frac{d\mu_{\langle g^{(i)}\rangle}}{d\nu}(x).$$

We should note that the integrand of the right-hand side of (3.2) is defined only ν-a.e. It is therefore more precise to call $Z_{\boldsymbol{g}}$ a "measure-theoretic Riemannian metric." As known in the general theory of Dirichlet forms, the strong local property of $(\mathcal{E},\mathcal{F})$ implies the derivation property of energy measures:

$$\mu_{\langle\varphi(f_1,\dots,f_n)\rangle}(dx)=\sum_{i,j=1}^{n}\frac{\partial\varphi}{\partial x_i}(\tilde f_1,\dots,\tilde f_n)\frac{\partial\varphi}{\partial x_j}(\tilde f_1,\dots,\tilde f_n)\,\mu_{\langle f_i,f_j\rangle}(dx),$$
$$\varphi\in C_b^1(\boldsymbol{R}^n),\ f_1,\dots,f_n\in\mathcal{F}.$$

It is thus not surprising to find this sort of differential structure. As stated in the final paragraph of Section 2, if K is a fractal set, then functions in $\mathcal{F}$ are usually nonsmooth in the classical sense. Theorem 3.4 implies that even in such a case the infinitesimal behaviors of functions in $\mathcal{F}$ can be described by those of representatives of $\mathcal{F}$.

As mentioned in Section 1, the concept of "differentiation" of functions on spaces without differential structures has been investigated from various viewpoints. In the framework of Dirichlet forms, Kusuoka [Kus89] has already obtained expressions similar to (3.2) for a class of finitely ramified fractals. Other studies have investigated expressions similar to (3.2) with respect to a family of canonical functions (with a degenerate "Riemannian metric" $Z_{\boldsymbol{g}}$) [Kus89, Kus93, Ki93a, Te00], and expressions similar to (3.1) with $p=1$ [PT08, Hi10]. In [Eb99], Hilbert spaces are considered as tangent spaces of general state spaces and expressions similar to (3.2) are obtained.[5] (In this case, the "Riemannian metric" is degenerate in most cases, too.) Theorems 3.3 and 3.4 are regarded as refinements of these results. The novelty here is that particular structures are not assumed on K and that a minimal function system is required for representing the differentials (in other words, roughly speaking, the Riemannian metric is nondegenerate). As a related topic, [CS03, CS09] studied the space of 1-differential forms under abstract settings, and [HRT13] pointed out that such a concept is essentially equivalent to the framework discussed in [Eb99], provided that it is associated with a strong-local Dirichlet form. A formulation of (stochastic) partial differential equations and existence theorems based on these differential-like structures are discussed in [HRT13] and the references therein.

3.3. **Probabilistic interpretation of index.** In this subsection, we introduce the concept of martingale dimensions corresponding to the "multiplicity of noises" and state a relation to the index. First, we introduce some notions following [FOT11]. The one-point compactification of K is denoted by K_Δ. From the general theory of Dirichlet forms, a diffusion process $\{X_t\}$ on K_Δ associated with $(\mathcal{E},\mathcal{F})$ is constructed on a filtered probability space $(\Omega,\mathcal{F}_\infty,P,\{P_x\}_{x\in K_\Delta},\{\mathcal{F}_t\}_{t\in[0,\infty)})$. We may assume that the shift operators $\theta_t\colon\Omega\to\Omega$ satisfying $X_s\circ\theta_t=X_{s+t}$ for any $s,t\ge0$ are also defined. The lifetime of the process $\{X_t(\omega)\}_{t\in[0,\infty)}$ is denoted by $\zeta(\omega)$, and integration with respect to the probability measure P_x is represented as E_x. A $[-\infty,+\infty]$-valued functional $A_t(\omega)$ $(t\in[0,\infty),\ \omega\in\Omega)$ is called an additive functional if the following conditions are satisfied:

- For every $t\ge0$, $A_t(\cdot)$ is $\mathcal{F}_t$-measurable.
- There exist $\Lambda\in\sigma(\mathcal{F}_t;t\ge0)$ and $N\subset K$ such that $\mathrm{Cap}(N)=0$, $P_x(\Lambda)=1$ for any $x\in K\setminus N$, $\theta_t\Lambda\subset\Lambda$ for every $t>0$, and for any $\omega\in\Lambda$, $A_\cdot(\omega)$ is

[5] After submitting the original exposition in Japanese, the author noticed that a related topic was also treated in [BH91, Exercise 5.9 in Chapter V].

càdlàg on $[0,\zeta(\omega))$, $A_0(\omega)=0$, $|A_t(\omega)|<\infty$ $(t<\zeta(\omega))$, $A_t(\omega)=A_{\zeta(\omega)}(\omega)$ $(t\ge\zeta(\omega))$, and

$$A_{t+s}(\omega)=A_s(\omega)+A_t(\theta_s\omega),\quad t,s\ge 0.$$

We note that the exceptional set N is allowed to exist, unlike usual additive functionals of Markov processes. A family of martingale additive functionals $\mathcal{M}$ is defined by

$$\mathcal{M}=\left\{ M \;\middle|\; \begin{array}{l} M \text{ is a real-valued additive functional, } M_{\cdot}(\omega) \text{ is} \\ \text{càdlàg on } [0,\infty) \text{ for } \omega\in\Lambda, \text{ and } E_x[M_t^2]<\infty \text{ and} \\ E_x[M_t]=0 \text{ for all } t>0 \text{ and q.e. } x\in K \end{array} \right\}.$$

In fact, any $M\in\mathcal{M}$ is a continuous additive functional, since we impose the strong local property on $(\mathcal{E},\mathcal{F})$.

Let μ_A denote the measure on K associated with nonnegative continuous additive functionals A in Revuz correspondence ([FOT11, pp. 228–230]). For additive functionals A, energy $e(A)$ is defined as $\lim_{t\to 0}(2t)^{-1}\int_K E_x[A_t^2]\,m(dx)$ if it exists in $[0,+\infty]$. For $M\in\mathcal{M}$, its quadratic variation is denoted by $\langle M\rangle$. The identity $e(M)=\mu_{\langle M\rangle}(K)/2$ holds. Let $\mathring{\mathcal{M}}=\{M\in\mathcal{M}\mid e(M)<\infty\}$. Then, $\mathring{\mathcal{M}}$ is a Hilbert space with the inner product $e(M,L):=(e(M+L)-e(M)-e(L))/2$. For $M,L\in\mathring{\mathcal{M}}$, let $\mu_{\langle M,L\rangle}$ denote $(\mu_{\langle M+L\rangle}-\mu_{\langle M\rangle}-\mu_{\langle L\rangle})/2$. For $M\in\mathring{\mathcal{M}}$ and $f\in L^2(K,\mu_{\langle M\rangle})$, the stochastic integral $f\bullet M$ denotes the unique element of $\mathring{\mathcal{M}}$ that is characterized by the identity

$$e(f\bullet M,L)=\frac{1}{2}\int_K f(x)\,\mu_{\langle M,L\rangle}(dx),\quad L\in\mathring{\mathcal{M}}.$$

If $f\in C_0(K)$, $(f\bullet M)_t$ is equal to $\int_0^t f(X_s)\,dM_s$ P_x-a.e. for q.e. $x\in K$.

Definition 3.5. The *AF-martingale dimension* of $\{X_t\}$ (or $(\mathcal{E},\mathcal{F})$) is the smallest number $d_{\mathrm{m}}\in\boldsymbol{Z}_+$ satisfying the following: There exist elements $\{M^{(k)}\}_{k=1}^{d_{\mathrm{m}}}$ of $\mathring{\mathcal{M}}$ such that every $M\in\mathring{\mathcal{M}}$ is represented as a stochastic integral

$$M_t=\sum_{k=1}^{d_{\mathrm{m}}}(h_k\bullet M^{(k)})_t,\quad t>0,\ P_x\text{-a.e. for q.e. } x\in K \tag{3.3}$$

for some $h_k\in L^2(K,\mu_{\langle M^{(k)}\rangle})$ $(k=1,\dots,d_{\mathrm{m}})$. If such d_{m} does not exist, the AF-martingale dimension d_{m} is defined as $+\infty$.

Even in the case $d_{\mathrm{m}}=+\infty$, an expression like (3.3) is possible if we take countable elements of $\mathring{\mathcal{M}}$ suitably. Informally speaking, the AF-martingale dimension represents the "multiplicity of noises" included in the diffusion process $\{X_t\}$.

Theorem 3.6 ([Hi10])**.** *The index of $(\mathcal{E},\mathcal{F})$ coincides with the AF-martingale dimension.*

This theorem is a natural generalization of the result for a class of finitely ramified self-similar fractals by Kusuoka [Kus89]. The definition of the index is different in [Kus89], where Kusuoka proved that his index and another index defined similarly to ours are consistent and that the latter is equal to the AF-martingale dimension. Theorem 3.6 is a generalization of the second claim, the proof of which is based on Kusuoka's, though the situation is more technically involved. When the index is finite, each $M^{(k)}$ in Definition 3.5 can be taken as a martingale additive functional $M^{[f_k]} \in \overset{\circ}{\mathcal{M}}$ appearing in the Fukushima decomposition of some $f_k \in \mathcal{F}$, provided that the sum of stochastic integrals in (3.3) is interpreted as a stochastic integral with respect to $\boldsymbol{R}^{d_m}$-valued functions and $\boldsymbol{R}^{d_m}$-valued martingale additive functionals ([Hi13b]).

In Definition 3.5, AF stands for "additive functional." We can consider some variations of martingale dimensions. For example, for (not necessarily symmetric) diffusion processes on K, let

$$\mathfrak{M} = \left\{ M = \{M_t\}_{t\in[0,\infty)} \,\middle|\, \begin{array}{l} \text{For every } x \in K,\ M \text{ is a square integrable} \\ P_x\text{-martingale and } M_0 = 0 \end{array} \right\},$$

and denote the quadratic variation process of $M \in \mathfrak{M}$ by $\langle M \rangle$. Let $L(\langle M \rangle)$ denote the totality of all progressively measurable processes $\varphi(t,\omega)$ such that

$$E_x\left[\int_0^t \varphi(s)^2 \, d\langle M\rangle_s\right] < \infty$$

for any $x \in K$ and $t > 0$. Then, the martingale dimension concerning $\mathfrak{M}$ can be defined as the smallest number q such that the following condition holds: There exist $M^{(1)}, \dots, M^{(q)} \in \mathfrak{M}$ such that every $M \in \mathfrak{M}$ is expressed as

$$M_t = \sum_{k=1}^q \int_0^t \varphi_k(s)\, dM_s^{(k)}, \quad t > 0,\ P_x\text{-a.e. for all } x \in K$$

for some $\varphi_k \in L(\langle M^{(k)} \rangle)$ $(k = 1, \dots, q)$.[6] Since we treat only AF-martingale dimensions introduced in Definition 3.5, we omit "AF-" from the notation after this.

4. Quantitative estimates of index

As stated in the previous section, for self-similar Dirichlet forms on self-similar fractals, determining the value of index is a highly nontrivial problem because energy measures do not have simple expressions for quantitative analysis. This problem was studied in [Kus89, Hi08, Hi13a], with different ideas introduced in each of those papers. In [Kus89, Hi08] the state spaces are assumed to be finitely ramified self-similar fractals, and the arguments essentially depend on the fact that energy measures can be expressed as limits of products of matrices. In this section, we introduce some results and ideas of the proof in [Hi13a], which could be applied to more general situations.

Let K be an LCCB space, m a finite Borel measure on K, and $(\mathcal{E}, \mathcal{F})$ a strong-local regular Dirichlet form on $L^2(K,m)$. We fix a closed subset K^∂ of K and set

$$\begin{aligned} \mathcal{F}_0 &= \{f \in \mathcal{F} \mid \operatorname{supp}[f\cdot m] \cap K^\partial = \emptyset\}, \\ \mathcal{H} &= \{h \in \mathcal{F} \mid \mathcal{E}(h,h) \le \mathcal{E}(h+f, h+f) \text{ for every } f \in \mathcal{F}_0\}. \end{aligned}$$

Functions in $\mathcal{H}$ are called *harmonic functions* (with respect to the boundary K^∂). We assume that $(\mathcal{E}, \mathcal{F})$ satisfies not only the following condition (A0), but also condition (A1) or (A1').

[6] This is essentially as discussed in [KW67].

(A0) (Nontriviality and conservativeness) $\mathcal{E} \not\equiv 0$. Moreover, $1 \in \mathcal{F}$ and $\mathcal{E}(1,1) = 0$.

(A1) (Sobolev's inequality) There exist some $d_{\mathrm{s}} > 2$ and $C > 0$ such that

$$\|f\|^2_{L^{2d_{\mathrm{s}}/(d_{\mathrm{s}}-2)}(K,m)} \le C\mathcal{E}_1(f,f), \qquad f \in \mathcal{F}.$$

(A1') Every one point set of K has a positive capacity.

The number d_{s} in (A1) is consistent with d_{s} in equation (2.3). Let d_{m} denote the index of $(\mathcal{E},\mathcal{F})$, and further assume the following:

(A2) For any natural numbers d with $d \le d_{\mathrm{m}}$, there exist harmonic functions $\{h_i\}_{i=1}^d$ on K such that
- $\mu_{\langle h_1\rangle} = \cdots = \mu_{\langle h_d\rangle}$ and this measure is not a null measure,
- If $i \ne j$, $\mu_{\langle h_i,h_j\rangle}$ is a null measure.

Let $\boldsymbol{h}$ denote $(h_1,\dots,h_d)$, a mapping from K to $\boldsymbol{R}^d$. We may initially assume that each h_i is quasi-continuous. Let $\mu_{\boldsymbol{h}}$ denote $(1/d)\sum_{i=1}^d \mu_{\langle h_i\rangle}$. For every $i,j = 1,\dots,d$, $\mu_{\langle h_i,h_j\rangle}$ is absolutely continuous with respect to $\mu_{\boldsymbol{h}}$. When we write $\Phi_{\boldsymbol{h}}(x) = \bigl((d\mu_{\langle h_i,h_j\rangle}/d\mu_{\boldsymbol{h}})(x)\bigr)_{i,j=1}^d$, the constraints in (A2) are satisfied if and only if $\mu_{\boldsymbol{h}} \ne 0$ and $\Phi_{\boldsymbol{h}}$ is the identity matrix $\mu_{\boldsymbol{h}}$-a.e. We remark that we do not assume that $\mu_{\boldsymbol{h}}$ is a minimal energy-dominant measure.

In general, for a quasi-continuous function f in $\mathcal{F}$, the image measure of $\mu_{\langle f\rangle}$ by f is absolutely continuous with respect to the 1-dimensional Lebesgue measure (energy image density property, [BH91]). If $d \ge 2$, however, the absolute continuity is not expected in general for the image measure of $\sum_{i=1}^d \mu_{\langle f_i\rangle}$ by $(f_1,\dots,f_d) \in \mathcal{F}^d$. Sufficient conditions are provided in [BH91] under rather strong assumptions, which is not effective in our situation. Here, under another strong restriction (A2), we have the following claim on absolute continuity.

Proposition 4.1. *Let f be a quasi-continuous function belonging to $\mathcal{F}_0\cap\mathcal{F}_b$. Under* (A2)*, the image measure $\boldsymbol{h}_*(f^2\cdot\mu_{\boldsymbol{h}})$ of $f^2\cdot\mu_{\boldsymbol{h}}$ by $\boldsymbol{h}$ is absolutely continuous with respect to the d-dimensional Lebesgue measure, and the Radon–Nikodym density ξ satisfies $\sqrt{\xi} \in W^{1,2}(\boldsymbol{R}^d)$.*

Taking this proposition into consideration, we further assume the following:

(A3) In the notation of (A2), the image measure $\boldsymbol{h}_*\mu_{\boldsymbol{h}}$ of $\mu_{\boldsymbol{h}}$ by $\boldsymbol{h}$ is absolutely continuous with respect to the d-dimensional Lebesgue measure $\mathscr{L}^d$. Moreover, for the Radon–Nikodym density ρ, there exists a function ξ on $\boldsymbol{R}^d$ such that $\rho \le \xi$ $\mathscr{L}^d$-a.e. and $\sqrt{\xi} \in W^{1,2}(\boldsymbol{R}^d)$.

Under these assumptions, we can provide some estimates of the index as follows:

Theorem 4.2 ([Hi13a])**.** *Let d_{m} denote the index (or the martingale dimension) of $(\mathcal{E},\mathcal{F})$.*

(i) *Under conditions* (A0), (A1), (A2), *and* (A3), *the inequality* $1 \le d_{\mathrm{m}} \le d_{\mathrm{s}}$ *holds.*

(ii) *Under conditions* (A0), (A1'), (A2), *and* (A3), d_{m} *is equal to* 1.

The basic strategy for obtaining the quantitative estimates is to consider a bilinear form on $\boldsymbol{R}^d$ that is generated by pushing the Dirichlet form $(\mathcal{E},\mathcal{F})$ forward by the harmonic map $\boldsymbol{h}$ and comparing it with the canonical Dirichlet form on $\boldsymbol{R}^d$. Considering good harmonic coordinates to simplify the situation is a natural idea in differential geometry, and is also useful for making the generators of diffusion processes canonical. This idea has been used in the context of probability theory since at least the 1960s ([Sk63]; see also [Ik69] and the references therein). The introduction of $\boldsymbol{h}$ here is similar in spirit, but $\boldsymbol{h}$ is not a usual coordinate map, as it is not necessarily injective.

In Theorem 4.2, we do not require any geometric structures of the state spaces. To apply this theorem to concrete examples, it is necessary to find a family of functions $\boldsymbol{h} = (h_1, \dots, h_d)$ that satisfies conditions (A2) and (A3). At the moment, when the state spaces do not have differential structures, it is successful only in the case of Dirichlet forms on typical self-similar sets (p. c. f. self-similar sets and Sierpinski carpets), as described in Section 2:

Theorem 4.3 ([Hi13a])**.** *The following claims hold.*

(i) *For the self-similar Dirichlet forms on Sierpinski carpets in Example 2.5,* $1 \le d_{\mathrm{m}} \le d_{\mathrm{s}}$. *Here,* d_{s} *is the spectral dimension appearing in* (2.3)*. In particular,* $d_{\mathrm{m}} = 1$ *if* $d_{\mathrm{s}} < 2$.

(ii) *For the self-similar Dirichlet forms on p. c. f. self-similar sets in Example 2.4,* $d_{\mathrm{m}} = 1$.

In Theorem 4.3 (i), we remark that $d_{\mathrm{s}} \le d_{\mathrm{H}}$ (the Hausdorff dimension of K) and that the inequality $d_{\mathrm{s}} < 2$ holds if and only if the associated diffusion process is point recurrent. The claim of Theorem 4.3 (ii) covers the results on determination of martingale dimensions in [Kus89, Hi08, Hi10]. In Fig. 3, $d_{\mathrm{m}} = 1$ for the left-hand set. Concerning the right-hand set, d_{m} is either 1 or 2, since $2 < d_{\mathrm{s}} < 3$, but it has not been determined which.

We will sketch an outline of the proof. In case (i), we take $K \cap \partial([0,1]^n)$ as K^{∂}; in case (ii), we take a finite set V_0 in Example 2.3 as K^{∂}. To construct $\boldsymbol{h} = (h_1, \dots, h_d)$, first we take $\boldsymbol{h}'$ such that $\operatorname{rank} \Phi_{\boldsymbol{h}'} = d$ on a set with $\mu_{\boldsymbol{h}'}$-positive measure, and then we perform a kind of blowup, of the type often seen in geometric measure theory, with the help of the following observation: For Lebesgue integrable functions f on the Euclidean space $\boldsymbol{R}^n$, almost every point $x \in \boldsymbol{R}^n$ is a Lebesgue point, that is,

$$\lim_{r \to 0} \fint_{\{z \in \boldsymbol{R}^n \,|\, |z-x|_{\boldsymbol{R}^n} \le r\}} |f(y) - f(x)|\, dy = 0.$$

In other words, the pullback $\psi_r^* f$ of f by the contraction map ψ_r with center $x \in \boldsymbol{R}^n$ and contraction ratio r converges to a constant function taking the value $f(x)$ as $r \to 0$ in a certain sense. We apply this operation to a matrix-valued function $\Phi_{\boldsymbol{h}'}$ on K; for a composition ψ of (contraction) mappings defining K, $\Phi_{\psi^* \boldsymbol{h}'}$ is well understood by a property similar to (2.2) that is based on the self-similarity of the Dirichlet form. By taking a suitable sequence of mappings $\{\psi_n\}_{n=1}^{\infty}$, it is proved that the normalization of $\psi_n^* \boldsymbol{h}'$ by the norm of $\mathcal{F}$ converges to some nontrivial limit $\boldsymbol{h}$ in $\mathcal{F}$, and $\Phi_{\boldsymbol{h}}$ is a constant matrix $\mu_{\boldsymbol{h}}$-a.e. To justify this argument, we need self-similarities of both the state spaces and the Dirichlet forms, though it is not known whether these are needed for non-technical reasons. That the space of harmonic functions is infinite dimensional for Sierpinski carpets, while that for p. c. f. self-similar sets is finite dimensional, makes a significant difference in the difficulty of proving some technical propositions.

5. Related topic

This section introduces a topic that is expected to be involved with the concept of index.

As in Section 2, let K be an LCCB space, m a positive Radon measure on K with full support, and $(\mathcal{E},\mathcal{F})$ a strong-local regular Dirichlet form on $L^2(K,m)$. For positive Radon measures μ and λ on K, we write $\mu \le \lambda$ if μ is absolutely continuous with respect to λ and the Radon–Nikodym derivative $d\mu/d\lambda$ is dominated by 1 λ-a.e. The intrinsic distance d on K is defined by

(5.1)
$$\mathsf{d}(x,y) = \sup\{f(y) - f(x) \mid f \in \mathcal{F}_{\mathrm{loc}} \cap C(K),\ \mu_{\langle f\rangle} \le m\} \in [0,+\infty], \quad x,y \in K$$

(see, e.g., [BM95, St95]). Here, $\mathcal{F}_{\mathrm{loc}}$ denotes the set of all functions f on K such that for every relatively compact open set O there exists a function in $\mathcal{F}$ that coincides with f on O. Under suitable conditions, the Gaussian behavior of the transition density associated with $(\mathcal{E},\mathcal{F})$ is described using d (see, e.g., [St95, St96, Ra01, No97]); in this sense, d is a fundamental concept. In the typical example where K is a complete Riemannian manifold, m is the Riemannian volume, $\mathcal{F}$ is the first-order L^2-Sobolev space, and the Dirichlet form is given by

$$\mathcal{E}(f,g) = \frac{1}{2}\int_K (\nabla f, \nabla g)\, dm, \quad f,g \in \mathcal{F},$$

the condition $\mu_{\langle f\rangle} \le m$ is equivalent to the condition $|\nabla f| \le 1$ m-a.e. In such a case, the intrinsic distance d coincides with the geodesic distance ρ on K that is defined by

$$\rho(x,y) = \inf\{\text{the length of continuous curves connecting } x \text{ and } y\}, \quad x,y \in K.$$

This is easily proved using the Rademacher-type theorem and the standard approximation of functions by mollifiers. If K is a fractal set, then for typical self-similar Dirichlet forms on self-similar fractals in Section 2, the energy measures are singular with respect to the Hausdorff measures (or more generally, self-similar measures). Thus, if we take the Hausdorff measure as m in such cases, d vanishes identically since only constant functions satisfy the condition $\mu_{\langle f\rangle} \le m$; this phenomenon is closely related to the fact that the transition densities have sub-Gaussian estimates in typical examples. Therefore, to obtain a nontrivial setting, we make the following modification: Let $N \in \boldsymbol{N}$ and $h_1,\dots,h_N$ be elements of $\mathcal{F}_{\mathrm{loc}} \cap C(K)$. We set $\boldsymbol{h} = (h_1,\dots,h_N)$, which is a continuous mapping from K to $\boldsymbol{R}^N$. Let $\nu_{\boldsymbol{h}}$ denote the measure $\sum_{i=1}^N \mu_{\langle h_i\rangle}$. We define the intrinsic distance $\mathsf{d}_{\boldsymbol{h}}$ based on $\boldsymbol{h}$ by the right side of (5.1), with m replaced by $\nu_{\boldsymbol{h}}$. In general, $\mathsf{d}_{\boldsymbol{h}}$ is a nontrivial (pseudo-)distance on K. Moreover, the geodesic distance $\rho_{\boldsymbol{h}}$ based on $\boldsymbol{h}$ is defined by

$$\rho_{\boldsymbol{h}}(x,y) = \inf\left\{ \begin{matrix}\text{the length of the}\\ \text{curve } \boldsymbol{h}\circ\gamma \text{ in } \boldsymbol{R}^N\end{matrix} \,\middle|\, \begin{matrix}\gamma \text{ is a continuous curve}\\ \text{connecting } x \text{ and } y\end{matrix} \right\}, \quad x,y \in K.$$

In Example 2.3, Kajino [Ka12] proved the identity $\mathsf{d}_{\boldsymbol{h}} = \rho_{\boldsymbol{h}}$ when $N = 2$ and h_1 and h_2 are harmonic functions that satisfy $\mathcal{E}(h_i,h_j) = \delta_{ij}$. In a different method, relations between $\mathsf{d}_{\boldsymbol{h}}$ and $\rho_{\boldsymbol{h}}$ were investigated in [Hi14] for slightly more general situations. By comparison, for the standard Dirichlet form on $\boldsymbol{R}^n$ corresponding to the Brownian motion, if $N = n$ and h_i $(i = 1,\dots,n)$ is defined as the x_i-coordinate function, then $\mathsf{d}_{\boldsymbol{h}} = \rho_{\boldsymbol{h}}$ provided that $\mathsf{d}_{\boldsymbol{h}}$ is redefined by replacing m by $n^{-1}\nu_{\boldsymbol{h}}$ in (5.1). (Even when $N > n$, the same is true under appropriate assumptions on $h_1,\dots,h_N$.) This example suggests that information about the dimension should be involved in general to identify the analytic distance $\mathsf{d}_{\boldsymbol{h}}$ with the geometric distance $\rho_{\boldsymbol{h}}$. Considering that the index is 1 for every example in which the identity $\mathsf{d}_{\boldsymbol{h}} = \rho_{\boldsymbol{h}}$

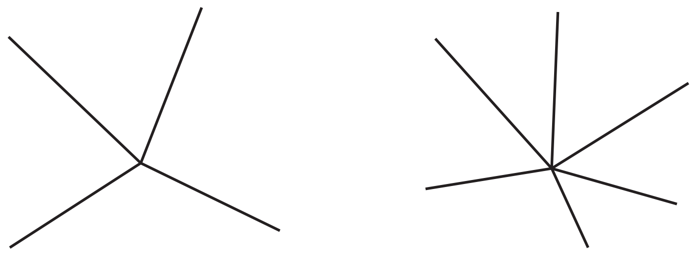

FIGURE 4. Examples of state spaces of spider martingales

is proved in [Ka12, Hi14], a more appropriate formulation of $\mathsf{d}_{\boldsymbol{h}}$ might be (5.1) with m replaced by $p(z)^{-1}\,\nu_{\boldsymbol{h}}(dz)$, where $p(z)$ is the pointwise index of $(\mathcal{E},\mathcal{F})$.

6. Concluding remarks

We list several problems as future issues, although some are not very explicit.

- In Theorem 4.2(i), to what extent is the upper estimate of the martingale dimension precise? Are there any nontrivial lower estimates?
- Is the inequality $d_{\mathrm{m}} \le d_{\mathrm{s}}$ valid for more general underlying spaces?
- Considering harmonic maps from K to Euclidean spaces is a key idea of the proof of Theorem 4.2. It would be useful to generalize the target space and construct the theory of harmonic maps on metric measure spaces. Though there are already some studies on this subject [St02a, St02b, St05], further development for quantitative analysis is desirable.
- From Theorem 3.4, the martingale dimensions are all 1 for canonical Dirichlet forms on p. c. f. self-similar sets and point recurrent Sierpinski carpets. However, it is expected that the "noises" generated by the diffusion processes are all different. How can a more detailed index be formulated to distinguish them from each other? The approach introduced in [AIW09] might be helpful.
- Is it possible to utilize the pointwise index, which is interpreted as the dimension of a tangent space, for the classification of diffusion processes? Since the pointwise index has only dimensional information and is defined only ν-a.e., more details are required for this purpose. For example, the Brownian-like diffusion processes on the state spaces in Fig. 4, which are called spider martingales or Walsh Brownian motions[7], the branching points play important roles for understanding the behavior of the diffusions, although the pointwise indices are not defined there. How is it possible to establish a general theory to extract such information on branching points (for example, the concept of splitting multiplicity by Barlow–Pitman–Yor [BPY89]) from the Dirichlet forms?

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[7] See also [CF12, Section 7.6 (3°)].

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