Contemporary Mathematics

On Dirichlet spaces over convex sets in infinite dimensions

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Dedicated to Professor Leonard Gross on the occasion of his 70th birthday

Abstract. When U is a convex subset of a Banach space E, we prove that $W^{1,2}(E)|_U$ is dense in $W^{1,2}(U)$ under suitable conditions, where $W^{1,2}(X)$ is the domain of a canonical Dirichlet form on X.

1. Introduction

When U is a sufficiently regular set of $\mathbb{R}^n$, it is a classical result that the Sobolev spaces on U have an extension property in the sense that there is a bounded linear map $W^{r,p}(U) \ni f \mapsto \tilde{f} \in W^{r,p}(\mathbb{R}^n)$ so that $\tilde{f}|_U = f$. In particular, the function space $\{f|_U \mid f \in W^{r,p}(\mathbb{R}^n)\}$ coincides with $W^{r,p}(U)$. This property allows to reduce many problems on U to those on $\mathbb{R}^n$.

In this paper, we study its counterpart for Dirichlet spaces in infinite dimensions. We briefly give a framework without stating technical details. Let E be a separable Banach space and μ a Borel probability measure on E. When a suitable Hilbert space H is equipped with E and X is a good subset of E, we can define an H-valued gradient operator D and an associated Dirichlet space $(\mathcal{E}^X, W^{1,2}(X))$. We will prove that $\{f|_U \mid f \in W^{1,2}(E)\}$ is dense in $W^{1,2}(U)$ under several conditions, when U is convex. This property is much weaker than the extension property mentioned above, but still useful for the study of the domain of Dirichlet forms. For example, there are studies [**AK92, AKR90, Eb99**] to assure that $W^{1,2}(E)$ is maximal among the Markovian extensions of given set $\mathcal{C}$ of smooth functions. In addition to this, if the Markovian uniqueness holds, namely, $\mathcal{C}$ is dense in $W^{1,2}(E)$, then we can conclude that $\mathcal{C}|_U$ is dense in $W^{1,2}(U)$ and $W^{1,2}(U)$ deserves to be the 'canonical' domain extending $\mathcal{C}|_U$.

The convexity assumption in this study is rather technical and we expect that the claim of the main theorem is true for more general sets, like a set $\{\varphi > 0\}$ with smooth function φ. We would like to emphasize, however, that the boundary of

1991 *Mathematics Subject Classification.* Primary 31C25, 28C20; Secondary 60J60, 46G12.
Key words and phrases. Dirichlet space, convex set, infinite dimension.
The author was supported in part by Ministry of Education, Culture, Sports, Science and Technology, Grant-in-Aid for Encouragement of Young Scientists, 13740095, 2001.

convex sets U is not smooth in general and such objects are appearing recently in various contexts such as [**Za01, Ot01, FÜ00, FH01**].

The organization of this paper is as follows. In section 2, we present a framework and give a precise statement of the theorem. Section 3 consists of proofs. We give an example and remarks in the last section.

2. Notation and result

Let $(E, |\cdot|_E)$ be an infinite dimensional real separable Banach space and μ a Borel probability measure on E. The σ-field will be completed by μ. Let H be a real separable Hilbert space densely and continuously embedded in E. By means of the Riesz isometry, we can consider the inclusion $E^* \subset H^* = H \subset E$, where E^* (resp. H^*) is a topological dual space of E (resp. H). E^* is dense both in H and in E. We will denote the norm of H by $|\cdot|_H$. The inner product $\langle \cdot, \cdot \rangle$ on H will be also used to describe the pairing between H^* and H, and E^* and E.

For a separable Hilbert space S, the L^p space of all S-valued L^p-functions on a measure space $(X, \mathcal{F}, \nu)$ is denoted by $L^p(X \to S, \nu)$ with norm $\|\cdot\|_{L^p(X \to S, \nu)}$. When $S = \mathbb{R}$, it is omitted from the notations. When X is a measurable subset of E and $\nu = \mu|_X$, we will simply write $L^p(X \to S, \mu)$ or $L^p(X \to S)$. For a topological space A, $\mathcal{B}(A)$ will denote the Borel σ-field of A. We often use λ (or λ_n) to denote the Lebesgue measure on $\mathbb{R}^n$ or an n-dimensional subspace of H.

For a finite dimensional subspace G of E^*, let $G^\perp = \{z \in E \mid \langle h, z \rangle = 0 \text{ for every } h \in G\}$. Then E is decomposed as a direct sum $G^\perp \oplus G$. The canonical projection maps on E onto $G^\perp$ and G will be denoted by P_G and Q_G, respectively. That is, they are given by

$$P_G z = z - Q_G z, \quad Q_G z = \sum_{i=1}^n \langle h_i, z \rangle h_i$$

where $\{h_1, \dots, h_n\} \subset E^* \subset H \subset E$ is an orthonormal basis of G in H.

Let $\mu_{G^\perp}$ be the image measure of μ by P_G. Since $G^\perp$ is a closed subspace of E, there exists a kernel $\rho_G : G^\perp \times \mathcal{B}(G) \to [0,1]$ such that the disintegration formula

$$\int_E f(z)\, \mu(dz) = \int_{G^\perp} \int_G f(x+y)\, \rho_G(x, dy) \mu_{G^\perp}(dx) \tag{2.1}$$

holds for every bounded Borel function f on E. ρ_G is uniquely determined in the sense that $\rho_G(x, \cdot) = \tilde{\rho}_G(x, \cdot)$ for $\mu_{G^\perp}$-a.e. x if $\tilde{\rho}_G$ is another kernel satisfying the same relation as (2.1). When $G = \mathbb{R}h$ for some $h \in E^*$, we will write $h^\perp$, $\mu_{h^\perp}$, and ρ_h in place of $G^\perp$, $\mu_{G^\perp}$, and ρ_G, respectively.

Let K be a dense linear subspace of H consisting of elements in E^*. We impose the following assumption.

(K) For each $h \in K \setminus \{0\}$, for $\mu_{h^\perp}$-a.e. $x \in h^\perp$, $\rho_h(x, \cdot)$ is absolutely continuous with respect to the Lebesgue measure λ_1 on $\mathbb{R}h$, and the Radon-Nikodym derivative, denoted by $\rho_h(x, y)$, satisfies the following: for any finite interval I in $\mathbb{R}$, $\lambda_1\text{-}\operatorname{essinf}_{s \in I} \rho_h(x, sh) > 0$ holds for $\mu_{h^\perp}$-a.e. x.

Then, the support of μ is necessarily the whole space E.

Let X be a subset of E with positive μ-measure. For $x \in E$ and $h \in E^*$, define a subset $I^X_{x,h}$ of $\mathbb{R}$ by

$$I^X_{x,h} = \{s \in \mathbb{R} \mid x + sh \in X\}.$$

We say that X is *moderate* if for each $h \in K$, the boundary of $I^X_{x,h}$ in $\mathbb{R}$ is a Lebesgue null set for μ-a.e. x. Suppose that X is moderate. For a function f on X, $x \in E$ and $h \in E^* \setminus \{0\}$, the function $f_h(x, \cdot)$ on $I^X_{x,h}$ will be defined by $f_h(x, s) = f(x + sh)$. For $h \in K \setminus \{0\}$, let $D(\mathcal{E}^X_h)$ be the space of all functions f in $L^2(X)$ such that for μ_h-a.e. $x \in h^\perp$, $f_h(x, \cdot)$ has an absolutely continuous version $\tilde{f}_h(x, \cdot)$ and there exists $\partial_h f \in L^2(X)$ so that

$$(\partial_h f)(x + sh) = \frac{\partial}{\partial s}\tilde{f}_h(x, s) \quad \lambda_1\text{-a.e. } s \in I^X_{x,h}, \quad \mu_{h^\perp}\text{-a.e. } x \in h^\perp.$$

Then, from [**AR90**, Theorem 3.2], the bilinear form $(\mathcal{E}^X_h, D(\mathcal{E}^X_h))$ on $L^2(X)$ defined by

$$\mathcal{E}^X_h(f, g) = \int_X (\partial_h f)(\partial_h g)\, d\mu, \quad f, g \in D(\mathcal{E}^X_h)$$

is a closed form.

With these notations, the space $W^{1,2}(X)$ on X is given by

$$W^{1,2}(X) = \left\{ f \in \bigcap_{h \in K\setminus\{0\}} D(\mathcal{E}^X_h) \;\middle|\; \begin{array}{l} \text{there exists } Df \in L^2(X \to H) \text{ such} \\ \text{that } \langle Df, h\rangle = \partial_h f \quad \mu\text{-a.e. on } X \\ \text{for every } h \in K \setminus \{0\} \end{array} \right\}.$$

We remark that it should correspond to a weak Sobolev space in the terminology of [**Eb99**]. By [**AR90**, Theorem 3.10], the bilinear form $(\mathcal{E}^X, W^{1,2}(X))$ on $L^2(X)$ defined by

$$\mathcal{E}^X(f, g) = \int_X \langle Df, Dg\rangle\, d\mu, \qquad f, g \in W^{1,2}(X)$$

is a Dirichlet form. Moreover, since D has a derivation property

$$D(\Phi(f)) = \Phi'(f)Df, \qquad f \in W^{1,2}(X),\ \Phi \in C^\infty_c(\mathbb{R}), \tag{2.2}$$

$(\mathcal{E}^X, W^{1,2}(X))$ is local in the sense of [**BH91**, Definition I.5.1.2]. We will enumerate its consequence for later use.

Proposition 2.1. *Let Φ be a Lipschitz function on $\mathbb{R}$ and let f and g be in $W^{1,2}(X)$. Then the following is true.*

1. *For any Lebesgue null set A of $\mathbb{R}$, $Df = 0$ on $f^{-1}(A)$ μ-a.e.*
2. *If $f = 0$ on a measurable set B, then $Df = 0$ on B μ-a.e.*
3. *$\Phi(f) \in W^{1,2}(X)$ and $D(\Phi(f)) = \Phi'(f)Df$ μ-a.e.*
4. *If $g \in L^\infty(X)$ and $|Dg|_H \in L^\infty(X)$, then $fg \in W^{1,2}(X)$ and $D(fg) = f(Dg) + g(Df)$ μ-a.e.*

Proof. The first and the second assertions follow from Theorem I.7.1.1 and Proposition I.7.1.4 in [**BH91**], respectively. The third one is proved by approximating Φ by smooth functions, using (2.2), and taking limits in view of (1). The proof of the fourth one is straightforward. □

We also note that if $f \in W^{1,2}(X)$ has a closed set A included in the interior of X so that $f = 0$ on $X \setminus A$, then $f \in W^{1,2}(E)$ by letting $f = 0$ on $E \setminus X$.

We equip $W^{1,2}(X)$ with the norm $\|\cdot\|_{W^{1,2}(X)}$ defined by

$$\|f\|_{W^{1,2}(X)} = \left(\mathcal{E}^X(f, f) + \|f\|^2_{L^2(X)}\right)^{1/2}.$$

The space $W^{1,2}(X)$ might depend on the choice of K. See, however, Remark 2.3 below.

When $\mathcal{A}$ is a function space on E, $\mathcal{A}|_X$ will denote the totality of functions on X obtained from functions in $\mathcal{A}$ by the natural restriction of the domain.

Now, our theorem is as follows.

THEOREM 2.2. *Let U be a subset of E. Assume the following conditions.*

(A1) *The embedding of H into E is compact.*

(A2) *In addition to the condition (K), $\rho_h(x,y)$ satisfies the following for each $h \in K \setminus \{0\}$: for any compact set A in $h^\perp$ and any $T > 0$,*

$$\mu_{h^\perp}\text{-}\operatorname*{esssup}_{x\in A}\left(\lambda_1\text{-}\operatorname*{essosc}_{|s|\le T}\log\rho_h(x,sh)\right) < \infty,$$

where λ_1 is the Lebesgue measure on $\mathbb{R}$.

(A3) *There exists an increasing sequence $\{G_n\}$ of finite dimensional subspaces of K such that $\bigcup_{n=1}^\infty G_n$ is dense in E^* in the weak* topology and*

$$\lim_{n\to\infty}\mu(\{|P_{G_n}z|_E > \varepsilon\}) = 0$$

for every $\varepsilon > 0$.

(A4) *U is a convex set with nonempty interior.*

Then, $W^{1,2}(E)|_U$ is dense in $W^{1,2}(U)$. In particular, if $\mathcal{C}$ is a dense subspace in $W^{1,2}(E)$, then $\mathcal{C}|_U$ is dense in $W^{1,2}(U)$.

Here, essosc is in general defined by

$$\operatorname*{essosc}_{y\in B}F(y) = \operatorname*{esssup}_{y\in B}F(y) - \operatorname*{essinf}_{y\in B}F(y),$$

and $\operatorname{esssup}\emptyset = -\infty$, $\operatorname{essinf}\emptyset = \infty$, and $(-\infty)-(-\infty) = \infty$ by definition.

REMARK 2.3. (1) Since (A4) assures that U is moderate, the space $W^{1,2}(U)$ is well-defined.

(2) When each h in a subset $\tilde{K}$ of E^* is well-(μ-)admissible in the sense of [**AKR90**], then for any subspace K of $\tilde{K}$ which is dense in H, $W^{1,2}(E)$ defines the same space by [**AKR90**, Theorem 3.4]. In such case, Theorem 2.2 concludes that $W^{1,2}(U)$ is also independent of the choice of K.

3. Proof

Firstly, we provide some notations. When A is a subset of E, A° and $\bar{A}$ denote the interior and the closure of A in E, respectively. For $a \in E$ and $r > 0$, we set $B_E(a,r) = \{z \in E \mid |z-a|_E < r\}$ and $B_H(a,r) = \{z \in E \mid z-a \in H \text{ and } |z-a|_H < r\}$. $\bar{B}_E(a,r)$ and $\bar{B}_H(a,r)$ are defined by replacing $<$ by $\le$ above. When $a = 0$, it is often omitted from the notation. The condition (A1) implies that $\bar{B}_H(a,r)$ is compact in E. For a subspace G of H and $r > 0$, we set $B^G(r) = \{y \in G \mid |y|_H < r\}$ and $\bar{B}^G(r) = \{y \in G \mid |y|_H \le r\}$. As usual, when A, $B \subset E$ and $z \in E$, we set $A+B = \{a+b \mid a \in A,\ b \in B\}$ and $z+B = \{z+b \mid b \in B\}$.

Assume only (A2) for the present.

LEMMA 3.1. *Let G be a finite dimensional subspace of $K (\subset H)$. Then, for $\mu_{G^\perp}$-a.e. x, $\rho_G(x,dy)$ is absolutely continuous with respect to the Lebesgue measure λ on G, and the Radon-Nikodym derivative, denoted by $\rho_G(x,y)$, satisfies the following: for every compact set A in $G^\perp$ and any bounded subset B of G,*

$$\mu_{G^\perp}\text{-}\operatorname*{esssup}_{x\in A}\left(\lambda\text{-}\operatorname*{essosc}_{y\in B}\log\rho_G(x,y)\right) < \infty.$$

In particular, for $\mu_{G^\perp}$-a.e. x,

$$0 < \lambda\text{-}\operatorname*{essinf}_{y\in B} \rho_G(x,y) \le \lambda\text{-}\operatorname*{esssup}_{y\in B} \rho_G(x,y) < \infty.$$

Proof. When G is one-dimensional, the claim is nothing but (A2). Suppose that the claim is true for any $(N-1)$-dimensional subspaces. Let G be an N-dimensional subspace of K. Take an orthonormal basis $\{h_1,\dots,h_N\}$ of G in H. For each $i=1,\dots,N$, set $G_i=\{y\in G \mid \langle h_i,y\rangle = 0\}$. Then $h_i^\perp = G^\perp \oplus G_i$ and there exists a kernel $\tau_i : G^\perp \times \mathcal{B}(G_i) \to [0,1]$ such that for every bounded Borel function g on $h_i^\perp$,

$$\int_{h_i^\perp} g(u)\,\mu_{h_i^\perp}(du) = \int_{G^\perp}\int_{G_i} g(x+v)\,\tau_i(x,dv)\mu_{G^\perp}(dx).$$

Then, for a bounded Borel function f on E,

$$\begin{aligned}\int_E f(z)\,\mu(dz) &= \int_{h_i^\perp}\int_{\mathbb{R}h_i} f(u+w)\rho_{h_i}(u,w)\,\lambda_1(dw)\mu_{h_i^\perp}(du)\\ &= \int_{G^\perp}\int_{G_i}\int_{\mathbb{R}h_i} f(x+v+w)\rho_{h_i}(x+v,w)\,\lambda_1(dw)\,\tau_i(x,dv)\mu_{G^\perp}(dx).\end{aligned}$$

Comparing this with (2.1), we have for $\mu_{G^\perp}$-a.e. x,

$$\int_G \varphi(y)\,\rho_G(x,dy) = \int_{G_i}\int_{\mathbb{R}h_i} \varphi(v+w)\rho_{h_i}(x+v,w)\,\lambda_1(dw)\,\tau_i(x,dv) \tag{3.1}$$

for every bounded Borel function φ on G. From (K), this means that h_i is an admissible element of $(G,\rho_G(x,\cdot))$ in the sense of [**AR90**], for $\mu_{G^\perp}$-a.e. x. Then, for $\mu_{G^\perp}$-a.e. x, $\rho_G(x,\cdot)$ is absolutely continuous with respect to λ by [**AR90**, Theorem 5.2], which proves the first assertion. From the identity (3.1), the Radon-Nikodym derivative $\rho_G(x,y)$ satisfies the relation for each $i=1,\dots,N$ and for $\mu_{G^\perp}\otimes\lambda_1$-a.e. (x,w),

$$\tau_i(x,dv) = \frac{\rho_G(x,v+w)}{\rho_{h_i}(x+v,w)}\lambda_{N-1}(dv).$$

In particular, $\tau_i(x,dv)$ is absolutely continuous with respect to $\lambda_{N-1}(dv)$ for $\mu_{G^\perp}$-a.e. x and its Radon-Nikodym derivative $\tau_i(x,v)$ satisfies

$$\rho_G(x,v+w) = \rho_{h_i}(x+v,w)\tau_i(x,v) \quad \mu_{G^\perp}\otimes\lambda_N\text{-a.e. } (x,v,w). \tag{3.2}$$

On the other hand, $\rho_{G_i}(x',dv)$ is expressed by $\rho_{G_i}(x',v)\lambda_{N-1}(dv)$ and $\rho_{G_i}(x',v) > 0$ for $\mu_{G_i^\perp}\otimes\lambda_{N-1}$-a.e. (x',v) by the induction hypothesis. Comparing two expressions

$$\begin{aligned}&\int_E f(z)\,\mu(dz)\\ &\quad = \int_{G^\perp}\int_{\mathbb{R}h_i}\int_{G_i} f(x+v+w)\rho_{h_i}(x+v,w)\tau_i(x,v)\,\lambda_{N-1}(dv)\,\lambda_1(dw)\,\mu_{G^\perp}(dx)\end{aligned}$$

and

$$\int_E f(z)\,\mu(dz) = \int_{G^\perp\oplus\mathbb{R}h_i}\int_{G_i} f(x+v+w)\rho_{G_i}((x,w),v)\,\lambda_{N-1}(dv)\,\mu_{G_i^\perp}(dx\,dw),$$

we have $\tau_i(x,v)>0$ for $\mu_{G_i^\perp}\otimes\lambda_{N-1}$-a.e. (x,w,v). This means that $\tau_i(x,v)>0$ for $\mu_{G^\perp}\otimes\lambda_{N-1}$-a.e. (x,v).

Now, to prove the second assertion, we may assume that $B=\{\sum_{i=1}^N s_ih_i \mid s_i\in[-T,T],\ i=1,\dots,N\}$. Then,

$$\begin{aligned}
&\mu_{G^\perp}\text{- }\operatorname*{esssup}_{x\in A}\lambda\text{- }\operatorname*{essosc}_{y\in B}\log\rho_G(x,y)\\
&=\mu_{G^\perp}\text{- }\operatorname*{esssup}_{x\in A}\ \lambda_{2N}\text{- }\operatorname*{esssup}_{(s_1,\dots,s_N,t_1,\dots,t_N)\in[-T,T]^{2N}}\\
&\qquad\left(\log\rho_G\Big(x,\sum_{i=1}^N s_ih_i\Big)-\log\rho_G\Big(x,\sum_{j=1}^N t_jh_j\Big)\right)\\
&=\mu_{G^\perp}\text{- }\operatorname*{esssup}_{x\in A}\ \lambda_{2N}\text{- }\operatorname*{esssup}_{(s_1,\dots,s_N,t_1,\dots,t_N)\in[-T,T]^{2N}}\sum_{k=1}^N\\
&\qquad\left(\log\rho_G\Big(x,\sum_{i=1}^k s_ih_i+\sum_{j=k+1}^N t_jh_j\Big)-\log\rho_G\Big(x,\sum_{i=1}^{k-1} s_ih_i+\sum_{j=k}^N t_jh_j\Big)\right)\\
&\le\sum_{k=1}^N\mu_{G^\perp}\text{- }\operatorname*{esssup}_{x\in A}\ \lambda_{N\text{-}1}\text{- }\operatorname*{esssup}_{(s_1,\dots,s_{k-1},t_{k+1},\dots,t_N)\in[-T,T]^{N-1}}\\
&\qquad\lambda_1\text{- }\operatorname*{essosc}_{|s|\le T}\log\rho_G\Big(x,\sum_{i=1}^{k-1}s_ih_i+\sum_{j=k+1}^N t_jh_j+sh_k\Big)\\
&\le\sum_{k=1}^N\mu_{h_k^\perp}\text{- }\operatorname*{esssup}_{u\in A+Q_{G_k}(B)}\lambda_1\text{- }\operatorname*{essosc}_{|s|\le T}\log\rho_{h_k}(u,sh_k).\qquad\text{(by (3.2))}
\end{aligned}$$

This is finite by the condition (A2). □

Let G be an N-dimensional subspace of K. Take a nonnegative and infinitely differentiable function Ψ on G with supp $\Psi\subset B^G(1)$ and $\int_G\Psi\,d\lambda_N=1$, and set $\Psi^\varepsilon(y)=\varepsilon^{-N}\Psi(y/\varepsilon)$ for each $\varepsilon>0$. Let S be a separable Hilbert space with norm $|\cdot|_S$ and suppose that a subset X of E is moderate. We consider $f\in L^2(X\to S)$ as an element of $L^2(E\to S)$ by letting $f=0$ on $E\setminus X$. Suppose also that $f=0$ outside a compact set. Then we can take a compact set $A\subset G^\perp$ and a bounded set $B\subset G$ so that $f=0$ on $E\setminus(A+B)$. Then, since

$$\begin{aligned}
\infty&>\int_A\int_B|f(x+y)|_S^2\rho_G(x,y)\,\lambda_N(dy)\mu_{G^\perp}(dx)\\
&\ge\int_A c(x)\int_B|f(x+y)|_S^2\,\lambda_N(dy)\mu_{G^\perp}(dx),
\end{aligned}$$

where $c(x)=\lambda_N\text{- }\operatorname{essinf}_{y\in B}\rho_G(x,y)>0$ for $\mu_{G^\perp}$-a.e. x, we obtain that $f(x+\cdot)\in L^2(G\to S,\lambda_N)$ for $\mu_{G^\perp}$-a.e. $x\in A$. We define for each $\varepsilon\in(0,1)$,

$$f^\varepsilon(z)=\int_G f(z-y)\Psi^\varepsilon(y)\,\lambda_N(dy),\quad z\in E. \tag{3.3}$$

Then, for $\mu_{G^\perp}$-a.e. $x\in A$,

$$\begin{aligned}
&\|f^\varepsilon(x+\cdot)\|^2_{L^2(G\to S,\rho_G(x,\cdot))}\\
&\quad=\int_G\left|\int_G f(x+y'-y)\Psi^\varepsilon(y)\,\lambda_N(dy)\right|_S^2\rho_G(x,y')\,\lambda_N(dy')
\end{aligned}\tag{3.4}$$

$$
\begin{aligned}
&\le \int_{B^G(1)} \Psi^\varepsilon(y)\left(\int_B |f(x+y'')|_S^2 \frac{\rho_G(x,y''+y)}{\rho_G(x,y'')}\rho_G(x,dy'')\right)\lambda_N(dy)\\
&\le c\int_{B^G(1)} \Psi^\varepsilon(y)\|f(x+\cdot)\|^2_{L^2(G\to S,\rho_G(x,\cdot))}\,\lambda_N(dy)\\
&= c\|f(x+\cdot)\|^2_{L^2(G\to S,\rho_G(x,\cdot))},
\end{aligned}
$$

where $c = \mu_{G^\perp}\text{-}\operatorname*{esssup}_{x\in A}\exp\left(\lambda_N\text{-}\operatorname*{essosc}_{y\in B+B^G(1)}\log\rho_G(x,y)\right) < \infty$. When $S=\mathbb{R}$, we know that $f^\varepsilon \in D(\mathcal{E}_h^X)$ for every $h\in G\setminus\{0\}$ by similar calculation.

Lemma 3.2. *Let f be as above.*

(1) *$f^\varepsilon \to f$ in $L^2(E\to S)$ as $\varepsilon\downarrow 0$.*
(2) *Suppose that $S=\mathbb{R}$ and a subset X' of X satisfies that $X'+B^G(r)\subset X$ for some $r>0$. If $f\in W^{1,2}(X)$ moreover, then $\partial_h f^\varepsilon \to \partial_h f$ in $L^2(X')$ as $\varepsilon\downarrow 0$ for every $h\in G\setminus\{0\}$.*

Proof. (1) Since $f^\varepsilon(x+\cdot)\to f(x+\cdot)$ in $L^2(G\to S,\rho_G(x,\cdot))$, we have by using the domination (3.4),

$$
\|f^\varepsilon - f\|^2_{L^2(E\to S)} = \int_A \|f^\varepsilon(x+\cdot)-f(x+\cdot)\|^2_{L^2(G\to S,\rho_G(x,\cdot))}\,\mu_{G^\perp}(dx)\to 0
$$

as $\varepsilon\downarrow 0$.

(2) It is easy to see that $\partial_h f^\varepsilon(z) = (\partial_h f)^\varepsilon(z)$ if $z+B^G(\varepsilon)\subset X$. Since $(\partial_h f)^\varepsilon\to\partial_h f$ in $L^2(E)$ as $\varepsilon\downarrow 0$ by (1), the assertion follows. □

Definition 3.3. For a subset A of E, we define

$$
\mathsf{d}_E(z,A) = \inf_{w\in A}|z-w|_E,\quad \mathsf{d}_H(z,A) = \inf_{w\in A\cap(z+H)}|z-w|_H,\qquad z\in E.
$$

The following theorem is proved in the same way as [**Kus82b**, Theorem 4.2].

Theorem 3.4. *Let f be a real measurable function on E. If f is square integrable and there is a constant c such that*

$$
|f(z+h)-f(z)|\le c|h|_H,\quad z\in E,\ h\in H,
$$

then $f\in W^{1,2}(E)$ and $|Df|_H\le c$ μ-a.e.

Denote by C the best constant so that $|h|_E\le C|h|_H$ holds for all $h\in H$.

Corollary 3.5. *For every $A\subset E$ and $T>0$, the function $f(z)=\mathsf{d}_E(z,A)\wedge T$ belongs to $W^{1,2}(E)$ and $|Df|_H\le C$ μ-a.e. If moreover A is a Borel set, then $g(z)=\mathsf{d}_H(z,A)\wedge T$ belongs to $W^{1,2}(E)$ and $|Dg|_H\le 1$ μ-a.e.*

Proof. What should be proved is only that g is measurable. For each $r>0$, $\{g\le r\}$ is expressed as $\bigcap_{n\in\mathbb{N}}(A+\bar{B}_H(r+1/n))$, which is a countable intersection of Suslin sets, in particular universally measurable. □

Next, we review a result from convex analysis (see e.g. [**Ro70**] for reference). Let G be a finite dimensional affine space of E. For a subset A of G, denote by $A^{\circ G}$ and $\overline{A}^G$, the interior and the closure of A with respect to the relative topology of G, respectively.

Proposition 3.6. *Let B be a convex subset of E. If $B^\circ\cap G\ne\emptyset$, then $(B\cap G)^{\circ G} = B^\circ\cap G$ and $\overline{B\cap G}^G = \bar{B}\cap G$.*

Lastly, we need one more proposition.

PROPOSITION 3.7. *Assume (A1) and (A3). Then, by taking a subsequence of $\{G_n\}$ if necessary, there exist a Borel subspace E_0 of E including H and a norm $|\cdot|_{E_0}$ of E_0 satisfying the following.*

- $\mu(E_0) = 1$.
- $(E_0, |\cdot|_{E_0})$ *is a separable Banach space.*
- $(H, |\cdot|_H)$ *is compactly embedded in* $(E_0, |\cdot|_{E_0})$.
- $(E_0, |\cdot|_{E_0})$ *is continuously embedded in* $(E, |\cdot|_E)$.
- *All* P_{G_n} *and* Q_{G_n} *are contraction operators on* $(E_0, |\cdot|_{E_0})$.

PROOF. We mainly follow the argument in [**Kus82a**, section 3]. We may and will assume that the operator norm of the embedding map from H to E is 1. By (A3), $\bigcup_{n\in\mathbb{N}} G_n$ is dense in H in the weak topology. Therefore, it is also dense in H in the strong topology, which means that $P_{G_n}h \to 0$ in H as $n \to \infty$ for all $h \in H$. Combining this with (A1), P_{G_n} converges to 0 in the norm sense as operators from H to E. Then taking a subsequence, we may assume that for all $n \in \mathbb{N}$,

$$|P_{G_n}h|_E \le 2^{-n}|h|_H, \quad h \in H, \qquad \mu(\{|P_{G_n}z|_E \ge 2^{-n}\}) \le 2^{-n}.$$

Define

$$q(z) = \left\{\sum_{n=0}^{\infty} 2^{n-1}|Q_{G_{n+1}}z - Q_{G_n}z|_E^2\right\}^{1/2}, \quad z \in E,$$

where $Q_{G_0}z := 0$. Let $E_0 = \{z \in E \mid q(z) < \infty\}$. Then E_0 is a Borel set and $\mu(E_0) = 1$ by Borel-Cantelli's lemma. Define $|z|_{E_0} = q(z)$ for $z \in E_0$. $(E_0, |\cdot|_{E_0})$ becomes a normed space. Indeed, the condition that $|z|_{E_0} = 0$ implies $z = 0$ is assured by (A3).

For $h \in H$,

$$|Q_{G_{n+1}}h - Q_{G_n}h|_E = |P_{G_n}Q_{G_{n+1}}h|_H \le 2^{-n}|Q_{G_{n+1}}h|_H \le 2^{-n}|h|_H. \tag{3.5}$$

Therefore, $q(h) \le \left(\sum_{n=0}^{\infty} 2^{n-1}2^{-2n}|h|_H^2\right)^{1/2} = |h|_H$.

Next, for $z \in E_0$,

$$\begin{aligned} |Q_{G_k}z|_{E_0}^2 &= \sum_{n=0}^{\infty} 2^{n-1}|Q_{G_{n+1}}Q_{G_k}z - Q_{G_n}Q_{G_k}z|_E^2 \\ &= \sum_{n=0}^{k-1} 2^{n-1}|Q_{G_{n+1}}z - Q_{G_n}z|_E^2 \le |z|_{E_0}^2. \end{aligned}$$

Therefore, Q_{G_k} is a contraction operator on E_0. Similarly, we have

$$|P_{G_k}z|_{E_0}^2 = \sum_{n=k}^{\infty} 2^{n-1}|Q_{G_{n+1}}z - Q_{G_n}z|_E^2 \le |z|_{E_0}^2.$$

This means that P_{G_k} is contractive on E_0, and $P_{G_k}z$ converges to 0 in E_0 as $k \to \infty$ for every $z \in E_0$.

When $h \in H$,

$$|h|_E \le \sum_{n=0}^{\infty} |Q_{G_{n+1}}h - Q_{G_n}h|_E \le 2|h|_{E_0} \tag{3.6}$$

by the Schwarz inequality. When $z \in E_0$, $Q_{G_n}z \to z$ in E_0 as $n \to \infty$. In particular, $\{Q_{G_n}z\}_{n\in\mathbb{N}}$ is a $|\cdot|_{E_0}$-Cauchy sequence. Since $Q_{G_n}z \in H$, it is also a $|\cdot|_E$-Cauchy sequence. Let z' be the limit of $\{Q_{G_n} z\}_{n\in\mathbb{N}}$ in E. For every $h \in \bigcup_{n\in\mathbb{N}} G_n$, $\langle Q_{G_n} z, h\rangle \to \langle z', h\rangle$ as $n \to \infty$. On the other hand, $\langle Q_{G_n} z, h\rangle = \langle z, h\rangle$ for sufficiently large n. Therefore $\langle z' - z, h\rangle = 0$. From (A3), this means $z = z'$. Namely, $Q_{G_n} z \to z$ in E as $n \to \infty$. From (3.6), $|z|_E \le 2|z|_{E_0}$.

If $\{z_n\}_{n\in\mathbb{N}} \subset E_0$ is a $|\cdot|_{E_0}$-Cauchy sequence, then it is a $|\cdot|_E$-Cauchy sequence. Let z be the limit in E. By Fatou's lemma, $q(z) \le \liminf_{n\to\infty} q(z_n) < \infty$. Therefore, $z \in E_0$. By Fatou's lemma again, $|z_n - z|_{E_0} \le \liminf_{m\to\infty} |z_n - z_m|_{E_0}$. By letting $n \to \infty$, $|z_n - z|_{E_0} \to 0$. This implies that $(E_0, |\cdot|_{E_0})$ is complete.

Lastly, for $h \in H$, by using (3.5),

$$
\begin{aligned}
|h - Q_{G_k} h|_{E_0}^2 &\le \sum_{n=0}^{\infty} 2^{n-1} \left(2^{-n} |Q_{G_{n+1}}(h - Q_{G_k} h)|_H\right)^2 \\
&\le \sum_{n=k}^{\infty} 2^{-n-1} |h - Q_{G_k} h|_H^2 \le 2^{-k} |h|_H^2.
\end{aligned}
\tag{3.7}
$$

Since each $Q_{G_k} : H \to E_0$ is a compact operator and (3.7) implies that Q_{G_k} converges to the embedding operator in the sense of norm convergence, $(H, |\cdot|_H)$ is compactly embedded in $(E_0, |\cdot|_{E_0})$. □

Proof of Theorem 2.2. It is enough to prove the first claim. By considering E_0 in Proposition 3.7 in place of E, it suffices to prove the assertion under the conditions (A1), (A2), (A4), and

(A3)' There exists an increasing sequence $\{G_n\}$ of finite dimensional subspaces of K such that $\bigcup_{n=1}^{\infty} G_n$ is dense in H and P_{G_n} and Q_{G_n} are contractive as operators from E to itself.

We will assume these from now on.

Now, we introduce the following intermediate function spaces on U:

$$
\begin{aligned}
W_1(U) &= \{f \in W^{1,2}(U) \mid f \in L^\infty(U)\}, \\
W_2(U) &= \left\{ f \in W_1(U) \;\middle|\; \begin{array}{l} \text{there exist some } G \in \{G_n\}_{n\in\mathbb{N}}, \text{ a compact and} \\ \text{convex set } V \text{ in } G^\perp,\ a \in G \text{ and } s > 0 \text{ such that} \\ V + B_E(a,s) \subset U \text{ and } f = 0 \text{ on } U \setminus (V + G)\ \mu\text{-a.e.} \end{array} \right\}, \\
W_3(U) &= \left\{ f \in W_2(U) \;\middle|\; \begin{array}{l} \text{there exist some } G \in \{G_n\}_{n\in\mathbb{N}}, \text{ a compact and} \\ \text{convex set } V \text{ in } G^\perp,\ a \in G,\ s > 0, \text{ and } R > 0 \\ \text{such that } V + B_E(a,s) \subset U \text{ and } f = 0 \text{ on} \\ U \setminus (V + \bar{B}^G(R))\ \mu\text{-a.e.} \end{array} \right\}.
\end{aligned}
$$

It is enough to prove that the inclusions $W^{1,2}(U) \supset W_1(U) \supset W_2(U) \supset W_3(U)$ are all dense in $W^{1,2}(U)$ and every function in $W_3(U)$ can be approximated by functions in $W^{1,2}(E)|_U$.

1) $W_1(U)$ is dense in $W^{1,2}(U)$.

Proof. Let $f \in W^{1,2}(U)$. Since $(\mathcal{E}^U, W^{1,2}(U))$ is a Dirichlet form, $(f \wedge M) \vee (-M)$ belongs to $W_1(U)$ for every $M > 0$, and $(f \wedge M) \vee (-M) \to f$ in $W^{1,2}(U)$ as $M \to \infty$.

2) $W_2(U)$ is a dense subset of $W_1(U)$.

Proof. Let $f \in W_1(U)$ and $M = \|f\|_{L^\infty(U)}$. For any $\varepsilon > 0$, there exists some $\delta \in (0, \varepsilon)$ such that

$$\int_A |Df|_H^2 \, d\mu < \varepsilon \quad \text{for all } A \subset U \text{ with } \mu(A) < \delta. \tag{3.8}$$

Take $V_1 = B_E(a_0, s)$ so that $B_E(a_0, 3s) \subset U$. Since $\bigcup_{n=1}^\infty G_n$ is dense in E, we can take $G \in \{G_n\}_{n\in\mathbb{N}}$ such that $\mu(V_1 + G) > 1 - \delta/2$. Since μ is regular, there exists a compact set V_2 in $V_1 + G$ such that $\mu(V_2) > 1 - \delta$. Set $V_3 = P_G(V_2)$ and $a = Q_G(a_0)$. Let V_4 be a convex hull of V_3 and $V = V_4 + (G^\perp \cap \bar{B}_H(s/C))$. Here C has been defined before Corollary 3.5. Then both V_4 and V are compact in $G^\perp$, $V_4 + G \supset V_2$ and $V + B_E(a, s) \subset U$.

Define a function φ on E by $\varphi(z) = (1 - (2C/s)\mathsf{d}_H(z, V_4 + G)) \vee 0$. By Corollary 3.5, $\varphi \in W^{1,2}(E)$, $|D\varphi|_H \le 2C/s$ μ-a.e., $0 \le \varphi \le 1$, φ is equal to 1 on $V_4 + G$, and 0 on $E \setminus (V + G)$. we also have

$$\|D\varphi\|_{L^2(E)} \le \frac{2C}{s} \cdot \mu(E \setminus (V_4 + G))^{1/2} \le \frac{2C\delta^{1/2}}{s} \le \frac{2C\varepsilon^{1/2}}{s}.$$

Then $f\varphi \in W_2(U)$,

$$\|f - f\varphi\|_{L^2(U)} \le M\|1 - \varphi\|_{L^2(U)} \le M\mu(E \setminus (V_4 + G))^{1/2} \le M\delta^{1/2} \le M\varepsilon^{1/2},$$

and

$$\begin{aligned} \|D(f - f\varphi)\|_{L^2(U)} &\le \|(1 - \varphi)Df\|_{L^2(U)} + \|fD\varphi\|_{L^2(U)} \\ &\le \|Df\|_{L^2(U\setminus(V_4+G))} + M\|D\varphi\|_{L^2(U)} \\ &\le \varepsilon^{1/2} + \frac{2CM\varepsilon^{1/2}}{s}. \end{aligned}$$

Here, we used (3.8) in the third inequality. Therefore, f can be approximated by functions in $W_2(U)$.

3) $W_3(U)$ is a dense subset of $W_2(U)$.

Proof. Let $f \in W_2(U)$ and take G as in the definition of $W_2(U)$. For $R > 0$, take a smooth and non-increasing function Φ_R on $[0, \infty)$ such that $\Phi_R = 1$ on $[0, R/3]$, $\Phi_R = 0$ on $[R, \infty)$, and $|\Phi_R'| \le 2/R$. Define $\varphi_R(z) = \Phi_R(|Q_G(z)|_H)$. Then $f\varphi_R \in W_3(U)$ and

$$\begin{aligned} \|f - f\varphi_R\|_{L^2(U)} &\le \|f\|_{L^2(U)}\mu(\{|Q_G(\cdot)|_H > R/3\})^{1/2} \to 0 \quad \text{as } R \to \infty, \\ \|D(f - f\varphi_R)\|_{L^2(U)} &\le \|(1 - \varphi_R)Df\|_{L^2(U)} + \|fD\varphi_R\|_{L^2(U)} \\ &\le \|Df\|_{L^2(U)}\mu(\{|Q_G(\cdot)|_H > R/3\})^{1/2} + \frac{2}{R}\|f\|_{L^2(U)} \\ &\to 0 \quad \text{as } R \to \infty. \end{aligned}$$

Therefore, $W_3(U)$ is dense in $W_2(U)$.

4) Every function in $W_3(U)$ can be approximated by functions in $W^{1,2}(E)|_U$.

Proof. Let $f \in W_3(U)$ and $\varepsilon > 0$. Let G, V, a, s and R be as in the definition of $W_3(U)$ and N the dimension of G. By taking R larger if necessary, we may assume that $V + B_E(a, s) \subset G^\perp + \bar{B}^G(R)$.

Let $\gamma \in (0, 1/2]$ and let us consider a homeomorphism T_γ on E given by

$$\begin{aligned} T_\gamma(z) &= P_G(z) + (1 - \gamma)Q_G(z) + \gamma a \\ &= z + \gamma(a - Q_G(z)), \qquad z \in E. \end{aligned}$$

For each $z \in E$, T_γ sends $z + G$ to itself and $T_\gamma|_{z+G}$ is a homothety centered at $P_G(z) + a$ with magnification ratio $1 - \gamma$. In particular, if $z \in V + \bar{B}_E(s/2)$, by Proposition 3.6,

$$
\begin{aligned}
(3.9)\qquad T_\gamma^{-1}(U^\circ) \cap (z+G) &= T_\gamma^{-1}(U^\circ \cap (z+G)) \\
&= T_\gamma^{-1}((U \cap (z+G))^{\circ(z+G)}) \\
&\supset \overline{U \cap (z+G)}^{(z+G)} = \bar{U} \cap (z+G).
\end{aligned}
$$

For any bounded Borel function f on E, we have

$$
\begin{aligned}
&\int_E f(z)\,\mu \circ T_\gamma^{-1}(dz) = \int_E f(T_\gamma(z))\,\mu(dz) \\
&= \int_{G^\perp}\int_G f(x + (1-\gamma)y + \gamma a)\rho_G(x,y)\lambda_N(dy)\mu_{G^\perp}(dx) \\
&= \int_{G^\perp}\int_G f(x+y')\rho_G\Big(x, \frac{y'-\gamma a}{1-\gamma}\Big)(1-\gamma)^{-N}\,\lambda_N(dy')\mu_{G^\perp}(dx) \\
&= \int_{G^\perp}\int_G f(x+y')\frac{\rho_G\Big(x, \frac{y'-\gamma a}{1-\gamma}\Big)}{\rho_G(x,y')}(1-\gamma)^{-N}\,\rho_G(x,dy')\mu_{G^\perp}(dx).
\end{aligned}
$$

Therefore, the image measure of μ under T_γ is mutually absolutely continuous with respect to μ, and its Radon-Nikodym density $\frac{d(\mu\circ T_\gamma^{-1})}{d\mu}$ is uniformly bounded on any compact set of E and in γ by Lemma 3.1.

Define

$$Y_\gamma = T_\gamma^{-1}((V + B_E(s/2) + G) \cap U^\circ).$$

Then, Y_γ's are moderate since they are convex, and $Y_\gamma \subset Y_\beta$ for every $\gamma < \beta$. Since $T_\gamma(Y_\gamma) \subset U$, we can define a measurable function $f_\gamma(z) := f(T_\gamma(z))$ on Y_γ. We will prove that $f_\gamma \in W^{1,2}(Y_{\gamma/2})$ and it converges to f appropriately as $\gamma \downarrow 0$. Since $f = 0$ on $U \setminus (V + \bar{B}^G(R))$, we have $f_\gamma = 0$ on $Y_\gamma \setminus T_\gamma^{-1}(V + \bar{B}^G(R))$, and therefore, there exists some $R' > R$ independent of $\gamma \in (0, 1/2]$ such that $f_\gamma = 0$ on $Y_\gamma \setminus (V + \bar{B}^G(R'))$. Set a relatively compact and convex set Y by

$$Y = (V + \bar{B}^G(R')) \cap U.$$

Then, $f_\gamma = 0$ on $(Y_\gamma \cap U) \setminus Y$ and $f = 0$ on $U \setminus Y$. We have

$$\int_{Y_\gamma} f_\gamma^2\,d\mu = \int_{T_\gamma(Y_\gamma)} f^2\,d(\mu\circ T_\gamma^{-1}) = \int_{T_\gamma(Y_\gamma)\cap Y} f^2 \frac{d(\mu\circ T_\gamma^{-1})}{d\mu}\,d\mu < \infty.$$

Therefore, $f_\gamma \in L^2(Y_\gamma)$. For $\varepsilon > 0$, we take f^ε as in (3.3). Then,

$$
\begin{aligned}
&\|f_\gamma - f\|_{L^2(Y)} \\
&\le \|f\circ T_\gamma - f^\varepsilon \circ T_\gamma\|_{L^2(Y)} + \|f^\varepsilon\circ T_\gamma - f^\varepsilon\|_{L^2(Y)} + \|f^\varepsilon - f\|_{L^2(Y)} \\
&\le \|f - f^\varepsilon\|_{L^2(Y)}\left(\left\|\frac{d(\mu\circ T_\gamma^{-1})}{d\mu}\right\|_{L^\infty(Y)}^{1/2} + 1\right) + \|f^\varepsilon\circ T_\gamma - f^\varepsilon\|_{L^2(Y)}.
\end{aligned}
$$

Since $f^\varepsilon \circ T_\gamma$ converges to f^ε pointwise on Y as $\gamma \downarrow 0$, the second term converges to 0 by the dominated convergence theorem. The first term converges to 0 as $\varepsilon \downarrow 0$ uniformly in γ by Lemma 3.2. Therefore, we conclude that $f_\gamma \to f$ in $L^2(Y)$ as $\gamma \downarrow 0$.

In the same manner, $Df \circ T_\gamma$ is proved to converge to Df in $L^2(Y \to H)$ as $\gamma \downarrow 0$.

Fix $h \in K \setminus \{0\}$. We denote by $\tilde{G}$ the linear space spanned by G and h. Let $\{h_1, \dots, h_N\}$ be an orthonormal basis of $\tilde{G}$ in H. For all $z \in E$, $T_\gamma(z + \cdot)$ maps $\tilde{G}$ to $\tilde{G} + T_\gamma(z)$ and it is a smooth map. Then, when f^ε is defined by (3.3) with G being replaced by $\tilde{G}$, $f^\varepsilon \circ T_\gamma$ is also differentiable along any directions in $\tilde{G}$ for μ-a.e. and by the chain rule,

$$\partial_h (f^\varepsilon \circ T_\gamma) = \sum_{j=1}^N \partial_{h_j} f^\varepsilon \circ T_\gamma \times \langle h - \gamma Q_G(h), h_j \rangle.$$

By Lemma 3.2, $\partial_{h_j} f^\varepsilon \to \partial_{h_j} f$ in $L^2(T_\gamma(Y_{\gamma/2}))$ as $\varepsilon \downarrow 0$. Then,

$$\begin{aligned}
(3.10) \qquad & \|\partial_{h_j} f^\varepsilon \circ T_\gamma - \partial_{h_j} f \circ T_\gamma\|^2_{L^2(Y_{\gamma/2})} \\
&= \int_{T_\gamma(Y_{\gamma/2})} (\partial_{h_j} f^\varepsilon - \partial_{h_j} f)^2 \, d(\mu \circ T_\gamma^{-1}) \\
&= \int_{T_\gamma(Y_{\gamma/2}) \cap (Y + B^R(1))} (\partial_{h_j} f^\varepsilon - \partial_{h_j} f)^2 \, \frac{d(\mu \circ T_\gamma^{-1})}{d\mu} \, d\mu \\
&\to 0 \qquad \text{as } \varepsilon \downarrow 0.
\end{aligned}$$

Therefore, when $\varepsilon \downarrow 0$, $\partial_h(f^\varepsilon \circ T_\gamma)$ converges in $L^2(Y_{\gamma/2})$ to

$$\sum_{j=1}^N \partial_{h_j} f \circ T_\gamma \times \langle h - \gamma Q_G(h), h_j \rangle = \langle Df \circ T_\gamma, h - \gamma Q_G(h) \rangle = \langle \Psi_\gamma, h \rangle,$$

where

$$\Psi_\gamma(z) = Df \circ T_\gamma(z) - \gamma Q_G \circ Df \circ T_\gamma(z).$$

Since we can prove that $f^\varepsilon \circ T_\gamma \to f_\gamma$ in $L^2(Y_\gamma)$ as $\varepsilon \downarrow 0$ by the same way as (3.10), we have $f_\gamma \in D(\mathcal{E}_h^{Y_{\gamma/2}})$ and $\partial_h f_\gamma = \langle \Psi_\gamma, h \rangle$. Since h is arbitrary and $\Psi_\gamma \in L^2(Y_\gamma \to H)$, we obtain that $f_\gamma \in W^{1,2}(Y_{\gamma/2})$ and $Df_\gamma = \Psi_\gamma$. Moreover, it is easy to see that Df_γ converges to Df in $L^2(Y)$ as $\gamma \downarrow 0$. Therefore, for any $\varepsilon > 0$, we can take $\gamma > 0$ so that $\|f_\gamma - f\|_{W^{1,2}(Y)} < \varepsilon$.

Let

$$U_1 = (V + \bar{B}^G(R')) \cap \bar{U}, \quad U_2 = (V + B_E(s/4) + B^G(R'+1)) \cap T_{\gamma/4}^{-1}(U^\circ).$$

Then U_1 is compact, U_2 is open and $Y \subset U_1 \subset U_2 \subset \bar{U}_2 \subset Y^\circ_{\gamma/2} = Y_{\gamma/2}$ by taking (3.9) into account. Define

$$\rho(z) = \frac{\mathsf{d}_E(z, E \setminus U_2)}{\mathsf{d}_E(z, U_1) + \mathsf{d}_E(z, E \setminus U_2)}$$

and

$$g(z) = \begin{cases} f_\gamma(z)\rho(z) & \text{if } z \in U_2 \\ 0 & \text{otherwise} \end{cases}, \qquad z \in E.$$

Since ρ is bounded and Lipschitz continuous and vanishes outside U_2, $g \in W^{1,2}(E)$. Also, $g = f_\gamma$ on U_1 and $g = f = 0$ on $U \setminus Y$. Therefore, $\|g - f\|_{W^{1,2}(U)} = \|f_\gamma - f\|_{W^{1,2}(Y)} < \varepsilon$. This completes the proof. $\square$

4. Example

Let (E, H, μ) be an abstract Wiener space. Namely, μ is the Wiener measure on E given by

$$\int_E \exp(\sqrt{-1}\langle z, h\rangle)\,\mu(dz) = \exp(-|h|_H^2/2) \quad \text{for every } h \in E^* \subset H.$$

For each $h \in E^* \setminus \{0\}$, we have

$$\rho_h(x, dy) = \frac{1}{\sqrt{2\pi}} e^{-|y|_H^2/2}\,\lambda_1(dy), \quad x \in h^\perp, y \in \mathbb{R}h,$$

where λ_1 is the Lebesgue measure on $\mathbb{R}h$. Therefore, for any subspace K of E^* that is dense in H, the condition (A2) in Theorem 2.2 holds. The conditions (A1) and (A3) are also satisfied. (See e.g. [**Kuo75**].) Let $\mathcal{F}C_b^\infty(K)$ be defined by

$$\mathcal{F}C_b^\infty(K) = \left\{ f : E \to \mathbb{R} \;\middle|\; \begin{array}{l} f(z) = \Phi(\langle z, h_1\rangle, \dots, \langle z, h_n\rangle), \\ h_1, \dots, h_n \in K,\ \Phi \in \mathcal{F}C_b^\infty(\mathbb{R}^n) \end{array} \right\}$$

and L the Ornstein-Uhlenbeck operator acting on $\mathcal{F}C_b^\infty(K)$. Then it is known that $(L, \mathcal{F}C_b^\infty(K))$ is essentially self-adjoint, which implies that $W^{1,2}(E)$ does not depend on the choice of K and $\mathcal{F}C_b^\infty(K)$ is dense in $W^{1,2}(E)$.

Let U be a convex set of E with nonempty interior. Then by Theorem 2.2, $W^{1,2}(E)|_U$, hence $\mathcal{F}C_b^\infty(K)|_U$, is dense in $W^{1,2}(U)$. In particular, $W^{1,2}(U)$ does not depend on K.

D. Feyel and A. S. Üstünel [**FÜ00**] proved the following logarithmic Sobolev inequality with $d\mu_U = \frac{1}{\mu(U)} d\mu|_U$:

$$\int_U \varphi^2 \log\left(\varphi^2 \Big/ \int_U \varphi^2\, d\mu_U\right) d\mu_U \le 2\int_U |D\varphi|_H^2\, d\mu_U \tag{4.1}$$

for any cylindrical Wiener functional φ. The result above implies that (4.1) is true for all $\varphi \in W^{1,2}(U)$. (In fact, (4.1) is proved for more general class of sets U including H-convex sets in [**FÜ00**]. The author does not know at present whether the conclusion of Theorem 2.2 is true for such sets.)

Next, we consider a Markov process associated with $(\mathcal{E}^U, W^{1,2}(U))$. From [**Fu00**, Theorem 2.1] (see also [**RS92**]), the closure of $(\mathcal{E}^U, \mathcal{F}C_b^\infty(E^*))$, which is now equal to $(\mathcal{E}^U, W^{1,2}(U))$, is a quasi-regular local Dirichlet form on $L^2(\bar{U}, \mu)$. Therefore, there is an associated diffusion process (X_t) on $\bar{U}$. If, moreover, the indicator function of U belongs to $BV(E)$ in [**FH01**], we have the following representation of Skorohod type ([**FH01**, Theorem 4.2]):

$$X_t(\omega) - X_0(\omega) = W_t(\omega) - \frac{1}{2}\int_0^t X_s(\omega)\, ds + \frac{1}{2}\int_0^t \sigma_U(X_s(\omega))\, dA_s^{\|D1_U\|}(\omega), \quad t \ge 0,$$

where σ_U is an H-valued function on E and $A^{\|D1_U\|}$ is an additive functional of $(\mathcal{E}^U, W^{1,2}(U))$, corresponding to the normal vector field of the 'boundary' ∂U of U and the functional induced by the surface measure of ∂U, respectively. See [**FH01**] for precise definitions. In [**Fu00, FH01**], (X_t) is called the *modified* reflecting Ornstein-Uhlenbeck process (for more general U), since the domain of the Dirichlet form is defined by the smallest extension of the space of smooth functions and it is not clear whether it is maximal or not. As long as U satisfies the condition (A4) in Theorem 2.2, (X_t) seems worth being called the *true* reflecting O-U process on $\bar{U}$. Strictly speaking, the maximality of $W^{1,2}(U)$ with respect to Markovian extensions of suitable cores consisting of smooth functions with Neumann boundary condition has not been proved yet, which will be left for future investigation.

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Errata: p. 3, line 5, $\cdots \tilde{f}_h(x,\cdot)$ $\underline{\text{on the interior of the closure of } I^X_{x,h}}$ and $\cdots$

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