NON-MARKOV PROPERTY OF CERTAIN EIGENVALUE PROCESSES ANALOGOUS TO DYSON’S MODEL

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Abstract. It is proven that the eigenvalue process of Dyson’s random matrix process of size two becomes non-Markov if the common coefficient \( \frac{1}{\sqrt{2}} \) in the non-diagonal entries is replaced by a different positive number.

1. Introduction

Dyson [3] has introduced the matrix-valued stochastic process
\[
\Xi(t) = \begin{pmatrix}
B_{1,1}(t) & \frac{1}{\sqrt{2}}B_{1,2}(t) & \cdots & \frac{1}{\sqrt{2}}B_{1,N}(t) \\
\frac{1}{\sqrt{2}}B_{1,2}(t) & B_{2,2}(t) & \cdots & \frac{1}{\sqrt{2}}B_{2,N}(t) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{\sqrt{2}}B_{1,N}(t) & \frac{1}{\sqrt{2}}B_{2,N}(t) & \cdots & B_{N,N}(t)
\end{pmatrix}
\]
to model the dynamics of particles with the Coulomb type interactions, where \( B_{i,i} \)'s are real Brownian motions and \( B_{i,j} \)'s for \( i < j \) are complex Brownian motions all of which are mutually independent. He proved that the eigenvalue processes \( \lambda_1, \ldots, \lambda_N \) satisfy the (system of) stochastic differential equations
\[
d\lambda_i(t) = d\beta_i(t) + \frac{\beta}{2} \sum_{j \neq i} \frac{1}{\lambda_i(t) - \lambda_j(t)} dt
\]
with \( \beta = 2 \). It has been proven later that if the complex Brownian motions are replaced by real or quaternion Brownian motions, the eigenvalue processes satisfy similar stochastic differential equations with \( \beta = 1 \) or 4, respectively. (See [1, 4] for discussions based on the stochastic analysis.) These processes are now called Dyson’s Brownian motion models for GOE, GUE, and GSE when \( \beta = 1, 2, \) and 4, respectively. In any case, it is remarkable that the process \( \Lambda = (\lambda_1, \ldots, \lambda_N) \) is Markov.

We may ask the following question: “Does the process \( \Lambda \) remain Markov if we replace the common coefficient \( \sqrt{2} \) by a different positive number?” In this paper, we give the negative answer to this question when the matrix size \( N = 2 \).

Let \( c \geq 0 \) and \( \delta > 0 \). Consider the 2 \( \times \) 2-matrix-valued process
\[
\Xi^{c,\delta}(t) = \begin{pmatrix}
B_1(t) & \sqrt{c/2} \xi^\delta(t) \\
\sqrt{c/2} \xi^\delta(t) & B_2(t)
\end{pmatrix}
\]

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where $B_1$ and $B_2$ are two independent standard Brownian motions and $\xi$ is a Bessel process of dimension $\delta$ starting from 0 which is independent of $B_1$ and $B_2$. We see in Lemma 2.2 that $\Xi^{c,\delta}$ with $\delta = 1, 2, 4$ is unitarily equivalent in law to

$$\Xi^{c,\delta}(t) = \left( \frac{B_1(t)}{\sqrt{c/2} B_3(t)} \frac{\sqrt{c/2} B_3(t)}{B_2(t)} \right)$$

with $B_3$ a real, complex, or quaternion Brownian motion independent of $B_1$ and $B_2$, respectively. Let $\lambda_1(t)$ and $\lambda_2(t)$ for $t \geq 0$ denote the eigenvalues of the Hermitian matrix $\Xi^{c,\delta}(t)$ such that $\lambda_1(t) \geq \lambda_2(t)$. Define the two-dimensional process $\Lambda^{c,\delta} = (\lambda_1, \lambda_2)$.

When $c = 0$, $\lambda_1(t)$ and $\lambda_2(t)$ are nothing but the order statistics of $B_1(t)$ and $B_2(t)$, that is, $\lambda_1(t) = \max\{B_1(t), B_2(t)\}$ and $\lambda_2(t) = \min\{B_1(t), B_2(t)\}$. Hence it is obvious that the process $\Lambda^{0,\delta}$ is Markov.

When $c = 1$, the process (1.1) is a time-dependent version of Dumitriu-Edelman’s matrix model for beta-ensembles (cf. [2]) and we see in Lemma 2.1 that the processes $\lambda_1(t)$ and $\lambda_2(t)$ satisfy Dyson’s stochastic differential equations with index $\beta = \delta$ given by

$$d\lambda_1(t) = d\beta_1(t) + \frac{\delta}{2(\lambda_1(t) - \lambda_2(t))} dt,$$

$$d\lambda_2(t) = d\beta_2(t) + \frac{\delta}{2(\lambda_2(t) - \lambda_1(t))} dt$$

for two independent Brownian motions $\beta_1(t)$ and $\beta_2(t)$. In particular, the process $\Lambda^{1,\delta}(t)$ is Markov.

**Theorem 1.1.** The process $\Lambda^{c,\delta}$ is Markov if and only if $c \in \{0, 1\}$.

We prove this theorem by reducing it to the following.

**Theorem 1.2.** Let $\delta_1, \delta_2 > 0$. Let $X^{\delta_1}$ and $Y^{\delta_2}$ be two independent squared Bessel processes starting from 0 of dimension $\delta_1$ and $\delta_2$, respectively. Then the process $Z(t) = cX^{\delta_1}(t) + Y^{\delta_2}(t)$ for $c \geq 0$ is Markov if and only if $c \in \{0, 1\}$.

Theorems 1.1 and 1.2 seem similar to Matsumoto-Ogura’s $cM - X$ theorem [6]. Let $X$ be a Brownian motion and set $M(t) = \sup_{0 \leq s \leq t} X(s)$. When $c = 0, 1, 2$, the process $cM - X$ is Markov; indeed, $-X$ is a Brownian motion, $M - X$ is a reflecting Brownian motion by Lévy’s theorem (see, e.g., [7, Thm.VI.2.3]), and $2M - X$ is a three-dimensional Bessel process by Pitman’s theorem (see, e.g., [7, Thm.VI.3.5]).

**Theorem 1.3** ([6]). The process $cM - X$ is Markov if and only if $c \in \{0, 1, 2\}$.

2. Non-Markov Property of the Eigenvalue Processes

**Proof of Theorem 1.1 provided Theorem 1.2 is justified.** An elementary calculation shows that $\lambda_1$ and $\lambda_2$ are given by

$$\lambda_1(t) = \frac{1}{2} \left\{ B_1(t) + B_2(t) + \sqrt{(B_1(t) - B_2(t))^2 + 2c\xi^2(t)} \right\},$$

$$\lambda_2(t) = \frac{1}{2} \left\{ B_1(t) + B_2(t) - \sqrt{(B_1(t) - B_2(t))^2 + 2c\xi^2(t)} \right\}.$$
Set $B_3(t) = \{B_1(t) + B_2(t)\}/\sqrt{2}$, $X^1(t) = \{B_1(t) - B_2(t)\}^2/2$ and $Y^\delta(t) = \xi^\delta(t)^2$. Then $B_3$ is a real Brownian motion, $X^1$ and $Y^\delta$ are squared Bessel processes of dimension 1 and $\delta$, respectively. Moreover, $B_3$, $X^1$, and $Y^\delta$ are mutually independent. It follows that

$$
\lambda_1(t) = \frac{1}{\sqrt{2}} \left\{ B_3 + \sqrt{X^1(t) + cY^\delta(t)} \right\},
$$

$$
\lambda_2(t) = \frac{1}{\sqrt{2}} \left\{ B_3 - \sqrt{X^1(t) + cY^\delta(t)} \right\}.
$$

It is obvious that the two dimensional process $\Lambda^{c,\delta} = (\lambda_1, \lambda_2)$ is Markov if and only if so is the process $(\lambda_1 + \lambda_2, \lambda_1 - \lambda_2)$. Since

$$
\lambda_1 + \lambda_2 = \sqrt{2}B_3,
$$

$$
\lambda_1 - \lambda_2 = \sqrt{2}\sqrt{X^1 + cY^\delta}
$$

and they are independent, for the process $\Lambda^{c,\delta}$ to be Markov it is necessary and sufficient that the process $X^1 + cY^\delta$ is Markov. This is equivalent to $c = 0$ or 1 by Theorem 1.2.

**Lemma 2.1.** For $c = 1$ and $\delta > 0$, consider the $2 \times 2$-matrix-valued process $\Xi^{1,\delta}$ defined by (1.1). Then the corresponding eigenvalue processes satisfy the stochastic differential equations (1.3)–(1.4).

**Proof.** Set $\tilde{\lambda} = (\lambda_1 - \lambda_2)/\sqrt{2}$. Then, by (2.2) for $c = 1$ and by Shiga-Watanabe's theorem (see, e.g., [7, Thm.XI.1.2]), we see that the process $\tilde{\lambda}$ is a Bessel process of dimension $1 + \delta$. Hence we have

$$
d\tilde{\lambda}(t) = dB_4(t) + \frac{\delta}{2} \frac{1}{\tilde{\lambda}(t)} dt
$$

where $B_4$ is a real Brownian motion independent of $B_3$. If we set $\beta_1 = (B_3 + B_4)/\sqrt{2}$ and $\beta_2 = (B_3 - B_4)/\sqrt{2}$, then $\beta_1$ and $\beta_2$ are two independent real Brownian motions. Therefore, combining (2.3) with (2.1), we conclude that (1.3)–(1.4) hold.

**Lemma 2.2.** Let $c > 0$, $\delta = 1, 2,$ or 4, and $\Xi^{c,\delta}$ and $\tilde{\Xi}^{c,\delta}$ be the matrix-valued processes defined by (1.1) and (1.2), respectively. Then, there exists a unitary matrix-valued process $U_\delta(t)$ such that

$$
\left( \Xi^{c,\delta}(t) \right)_{t \geq 0} \text{ law } \left( U_\delta(t)\tilde{\Xi}^{c,\delta}(t)U_\delta^*(t) \right)_{t \geq 0}.
$$

In particular, eigenvalue processes associated with $\Xi^{c,\delta}$ and $\tilde{\Xi}^{c,\delta}$ have the same law.

**Proof.** We define

$$
U_\delta(t) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{|B_3(t)|} 1_{B_3(t) \neq 0} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} 1_{B_3(t) = 0}
$$

by using $B_3$ in (1.2). Then we have

$$
U_\delta(t)\tilde{\Xi}^{c,\delta}(t)U_\delta^*(t) = \begin{pmatrix} B_3(t) & \sqrt{c/2} |B_3(t)| \\ \sqrt{c/2} |B_3(t)| & B_2(t) \end{pmatrix},
$$

which shows the desired result since $|B_3| = \xi^\delta$. 

3. Transition probability density of squared Bessel processes

In this section, we recall some basic asymptotic estimates on the transition probability density \( p_\delta^t(x, y) \) of squared Bessel processes of dimension \( \delta \) which we shall use later. We first note that it has an expression

\[
p_\delta^t(x, y) = \frac{1}{2t^\frac{\delta}{2}} \frac{(\frac{y}{x})^{(\delta-2)/4}}{\Gamma(\frac{\delta}{2})} \exp\left(-\frac{x+y}{2t}\right) I_{(\delta-2)/2}\left(\frac{\sqrt{xy}}{t}\right)
\]

for \( x, y > 0 \), where \( I_\nu \) stands for the modified Bessel function of index \( \nu \) (see, e.g., [7, Cor.XI.1.4]). Now let us recall following two asymptotic estimates on the modified Bessel function (see, e.g., Sect. 5.16.4 of [5]):

\[
I_\nu(x) \sim \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu \quad \text{as} \quad x \downarrow 0,
\]

\[
I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}} \quad \text{as} \quad x \uparrow \infty.
\]

Using (3.2) in (3.1), we can derive

\[
p_\delta^t(0^+, y) = \frac{y^{(\delta-1)/2}}{(2t)^{\delta/2} \Gamma(\delta/2)} \exp\left(-\frac{y}{2t}\right)
\]

for \( t, y > 0 \) and

\[
\lim_{y \to 0^+} y^{1-\delta/2} p_\delta^t(x, y) = x^{1-\delta/2} p_\delta^t(0^+, x)
\]

\[
= \frac{1}{(2t)^{\delta/2} \Gamma(\delta/2)} \exp\left(-\frac{x}{2t}\right)
\]

for \( t, x > 0 \). On the other hand (3.3) together with (3.1) yields

\[
p_\delta^t(x, y) \sim \frac{1}{2t \sqrt{2\pi x}^{(\delta-1)/4}} \frac{y^{(\delta-3)/4}}{x^{\delta/4}} \exp\left(-\frac{x+y-2\sqrt{xy}}{2t}\right)
\]

as \( \sqrt{xy} \to \infty \).

4. Non-Markov property of weighted sums of two independent squared Bessel processes

For the proof of Theorem 1.2, we may restrict ourselves to \( 0 < c < 1 \); otherwise consider \( Z^c/c \) instead. We prove that \( Z^c \) is non-Markov by checking that the conditional law

\[
P \left( Z^c(2) \in dz_3 \mid Z^c(\varepsilon) = z_1, \ Z^c(1) = z_2 \right) \quad \text{for} \quad 0 < \varepsilon < 1
\]

does depend on \( (\varepsilon, z_1) \). This conditional law has the density

\[
P \left( Z^c(2) \in dz_3 \mid Z^c(\varepsilon) = z_1, \ Z^c(1) = z_2 \right) = \frac{q(z_2, z_3; \varepsilon, z_1)}{q(z_2; \varepsilon, z_1)} dz_3,
\]

where \( q(z_2, z_3; \varepsilon, z_1) \) and \( q(z_2; \varepsilon, z_1) \) are the densities of the joint laws of \( (Z^c(\varepsilon), Z^c(1), Z^c(2)) \) and \( (Z^c(\varepsilon), Z^c(1)) \), respectively. Thus it suffices to prove that the fraction \( q(z_2, z_3; \varepsilon, z_1)/q(z_2; \varepsilon, z_1) \) depends on \( (\varepsilon, z_1) \).
To this end, we shall use the integral expression
\[
q(z_2, z_3; \varepsilon, z_1) = \int_0^{z_1} dx_1 \int_0^{z_2} dx_2 \int_0^{z_3} dx_3 A_{1,1} A_{1,2} A_{1,3},
\]
\[
q(z_2; \varepsilon, z_1) = \int_0^{z_1} dx_1 \int_0^{z_2} dx_2 A_{1,1} A_{1,2},
\]
where
\[
A_{1,1} = p_1^{\delta_1}(0+, x_1) p_\varepsilon^{\delta_2}(0+, z_1 - cx_1),
\]
\[
A_{1,2} = p_1^{\delta_1}(x_1, x_2) p_\varepsilon^{\delta_2}(z_1 - cx_1, z_2 - cx_2),
\]
\[
A_{1,3} = p_1^{\delta_1}(x_2, x_3) p_\varepsilon^{\delta_2}(z_2 - cx_2, z_3 - cx_3).
\]

We divide the proof into several steps. First of all, we prove

**Lemma 4.1.** Let \( f(\lambda, \cdot) \) for \( \lambda > 0 \) be a bounded measurable function on \((0, 1)\).
Suppose that \( f(\lambda, x / \lambda) \) converges to a constant \( f(\infty, 0) \) for any \( x \in (0, 1) \) as \( \lambda \to \infty \).
Let \( \phi \in C^1((0, 1)) \) and suppose that \( \phi(0+) = a \in \mathbb{R}, \phi'(0+) = b > 0 \) and \( \phi'(x) > 0 \)
for \( x \in (0, 1) \). Let \( \nu > 0 \). Then

\[
(4.2) \quad \int_0^1 e^{-\lambda \phi(x)} f(\lambda, x)x^{\nu-1} dx \sim f(\infty, 0) \frac{\Gamma(\nu)}{b^\nu} \lambda^{-\nu} e^{-a\lambda} \quad \text{as} \quad \lambda \to \infty.
\]

**Proof.** Changing variables to \( u = \lambda x \), we find that the left hand side of (4.2) equals

\[
\lambda^{-\nu} e^{-a\lambda} \int_0^\lambda e^{-\lambda \{\phi(u / \lambda) - a\}} f(\lambda, u / \lambda) du.
\]

Note that \( \lambda \{\phi(u / \lambda) - a\} \geq K u \) for \( u \in (0, \lambda) \) and \( \lambda > 0 \) where \( K = \inf_{x \in (0, 1)} \{\phi(x) - \phi(0+)\} / x > 0 \). Hence we see that

\[
\lim_{\lambda \to \infty} \int_0^\lambda e^{-\lambda \{\phi(u / \lambda) - a\}} f(\lambda, u / \lambda) du = f(\infty, 0) \int_0^\infty e^{-bu} u^{\nu-1} du
\]

by the dominated convergence theorem. \( \square \)

Second, we take the limit as \( \varepsilon \to 0 \).

**Lemma 4.2.**

\[
\lim_{\varepsilon \to 0+} \frac{q(z_2, z_3; \varepsilon, z_1)}{q(z_2; \varepsilon, z_1)} = \frac{q(z_2, z_3; z_1)}{q(z_2; z_1)}
\]

with

\[
q(z_2, z_3; z_1) = \int_0^{z_1} dx_1 \int_0^{z_2} dx_2 \int_0^{z_3} dx_3 A_{2,1} A_{2,2}, \quad q(z_2; z_1) = \int_0^{z_1} dx_1 \int_0^{z_2} dx_2 A_{2,1}
\]

where \( A_{2,2} = A_{1,3} \) and

\[
A_{2,1} = A_{1,2} \Big|_{\varepsilon \to 0+, x_1 \to 0+} = p_1^{\delta_1}(0+, x_2) p_{1-\varepsilon}^{\delta_2}(z_1, z_2 - cx_2).
\]

**Proof.** We know that

\[
A_{1,1} = \frac{(x_1)^{(\delta_1/2) - 1} (z_1 - cx_1)^{(\delta_2/2) - 1}}{(2\varepsilon)^{(\delta_1 + \delta_2)/2} \Gamma(\delta_1/2) \Gamma(\delta_2/2)} \exp \left( -\frac{1}{2\varepsilon} \left\{ z_1 + (1 - c)x_1 \right\} \right)
\]
from (3.4). Now we can rewrite $q(z_2, z_3; \varepsilon, z_1)/q(z_2; \varepsilon, z_1)$ as $F_1/G_1$ with

$$F_1 = \int_0^{z_1} A_{1,4}(\varepsilon, x_1)x_1^{(\delta_2/2)-1}e^{-(\tilde{c}/\varepsilon)x_1}dx_1$$

(4.3)

$$G_1 = \int_0^{z_1} A_{1,5}(\varepsilon, x_1)x_1^{(\delta_2/2)-1}e^{-(\tilde{c}/\varepsilon)x_1}dx_1$$

(4.4)

where $\tilde{c} = (1 - c)/2$ and

$$A_{1,4}(\varepsilon, x_1) = (z_1 - cx_1)^{(\delta_2/2)-1}\int_0^{z_2} dx_2 \int_0^{z_3} dx_3 A_{1,2}A_{1,3},$$

$$A_{1,5}(\varepsilon, x_1) = (z_1 - cx_1)^{(\delta_2/2)-1}\int_0^{z_2} dx_2 A_{1,2}.$$

Using Lemma 4.1 in the integrals (4.3) and (4.4), we have

$$F_1 \sim \varepsilon^{\delta_1/2}\Gamma(\delta_1/2)\tilde{c}^{-\delta_1/2}A_{1,4}(0, 0),$$

$$G_1 \sim \varepsilon^{\delta_1/2}\Gamma(\delta_1/2)\tilde{c}^{-\delta_1/2}A_{1,5}(0, 0)$$

as $\varepsilon \to 0^+$. Here we have used the fact that $A_{1,4}(\varepsilon, x_1)$ and $A_{1,5}(\varepsilon, x_1)$ are continuous in $\varepsilon \in [0, \infty)$ and $x_1 \in [0, z_1]$. Therefore, $F_1/G_1$ approaches to $A_{1,4}(0, 0)/A_{1,5}(0, 0) = q(z_2, z_3; z_1)/q(z_2; z_1)$.

Third, we study the asymptotic behavior of the numerator $q(z_2, z_3; z_1)$ as $z_3 \to 0^+$.

**Lemma 4.3.**

$$\lim_{z_3 \to 0^+} z_3^{1-(\delta_1+\delta_2)/2}q(z_2, z_3; z_1) = C_1\tilde{q}(z_2; z_1)$$

with

$$C_1 = \int_0^1 u^{(\delta_1/2)-1}(1 - cu)^{(\delta_2/2)-1}du, \quad \tilde{q}(z_2; z_1) = \int_0^{z_2} dx_2 A_{3,1}A_{3,2}$$

where $A_{3,1} = A_{2,1}$ and

$$A_{3,2} = (x_2)^{1-\delta_1/2}(z_2 - cx_2)^{1-\delta_2/2}p_1(0+, x_2)p_1(0+, z_2 - cx_2).$$

**Proof.** Recall that

$$q(z_2, z_3; z_1) = \int_0^{z_3} dx_3 A_{2,3}(z_3, x_3)$$

(4.5)

where

$$A_{2,3}(z_3, x_3) = \int_0^{z_2} dx_2 A_{3,1}p_1(x_2, x_3)p_1^{\delta_2}(z_2 - cx_2, z_3 - cx_3).$$

Here we note that $A_{3,1}$ does not depend on $z_3$ nor $x_3$. If we take $x_3 = z_3u$ for $0 < u < 1$, we have

$$A_{2,3}(z_3, z_3u) = \int_0^{z_2} dx_2 A_{3,1}p_1(0+, x_2)u^{\delta_2}(z_2 - cx_2, z_3(1 - cu)).$$

Using (3.5), we have, as $z_3 \to 0^+$,

$$z_3^{2-(\delta_1+\delta_2)/2}A_{2,3}(z_3, z_3u) \to u^{(\delta_1/2)-1}(1 - cu)^{(\delta_2/2)-1}\int_0^{z_2} dx_2 A_{3,1}A_{3,2}. $$
Changing variables to \( u = x_3/z_3 \) in the integral (4.5), we obtain
\[
z_3^{-(\delta_1+\delta_2)/2} q(z_2, z_3; z_1) = z_3^{2-(\delta_1+\delta_2)/2} \int_0^1 du A_{2,3}(z_3, z_3 u),
\]
which converges to \( C \tilde{q}(z_2; z_1) \) as \( z_3 \to 0+ \).

Fourth, we study the asymptotic behaviors of \( \tilde{q}(z_2; z_1) \) and \( q(z_2; z_1) \) as \( z_2 \to \infty \). Recall that
\[
\tilde{q}(z_2; z_1) = \int_0^{z_2} dx_2 A_{3,1} A_{3,2}
\]
\[
= \int_0^{z_2} dx_2 x_2^{1-\delta_1/2} (z_2 - cx_2)^{1-\delta_2/2} p_1^\delta(0+, x_2)
\]
\[
\times p_1^{\tilde{\delta}_2}(z_1, z_2 - cx_2) p_1^{\tilde{\delta}}(0+, z_2 - cx_2)
\]
\[
= z_2^{-(\delta_1+\delta_2)/2} \int_0^1 du u^{1-\delta_1/2} (1-cu)^{1-\delta_2/2} p_1^\delta(0+, z_2 u)
\]
\[
\times p_1^{\tilde{\delta}_2}(z_1, z_2(1-cu)) p_1^{\tilde{\delta}}(0+, z_2 u) p_1^{\tilde{\delta}}(0+, z_2(1-cu))
\]
and that
\[
q(z_2; z_1) = z_2 \int_0^1 du p_1^\delta(0+, z_2 u) p_1^{\tilde{\delta}}(z_1, z_2(1-cu)).
\]

Lemma 4.4. Let \( r > 0 \). Then
\[
\frac{\tilde{q}(z_2; z_2 r)}{q(z_2; z_2 r)} \sim C_2 D(r)^{-\delta_1/2} e^{-z_2/2} \quad \text{as } z_2 \to \infty
\]
where \( C_2 \) is some positive constant depending only on \( \delta_1 \) and \( \delta_2 \) and
\[
D(r) = 1 + \frac{1-c}{1-c + \sqrt{rc}}.
\]

Proof. If we express \( \tilde{q}(z_2; z_2 r) \) as
\[
r^{(1-\delta_2)/4} z_2^{(\delta_1-1)/2} \int_0^1 f_1(z_2, u) e^{-z_2 \phi_1(u)} u^{\delta_1/2-1} du
\]
using
\[
\phi_1(u) = b_1 u + \sqrt{r} \left\{ 1 - \sqrt{1-cu} \right\} + a_1
\]
with \( b_1 = 1-c \) and \( a_1 = (\sqrt{r}-1)^2/2 + 1/2 \), then \( f_1(z_2, \cdot) \) turns out to be a bounded continuous function such that \( f_1(z_2, u/z_2) \) converges to a constant depending only on \( \delta_1 \) and \( \delta_2 \) as \( z_2 \to \infty \), by (3.6). Since \( \phi_1 \) and \( f_1 \) satisfies the assumptions, we can use Lemma 4.1 and hence we obtain
\[
\tilde{q}(z_2; z_2 r) \sim C_2, r^{(1-\delta_1)/4} \phi_1'(0+) - \delta_1/2 z_2^{-1/2} e^{-a_1 z_2} \quad \text{as } z_2 \to \infty
\]
with some constant \( C_2, r \) depending only on \( \delta_1 \) and \( \delta_2 \).

We also have a similar expression
\[
r^{(1-\delta_2)/4} z_2^{(\delta_1-1)/2} \int_0^1 f_2(z_2, u) e^{-z_2 \phi_2(u)} u^{\delta_1/2-1} du
\]

\[
\quad
\]
for $q(z_2; z_2 r)$ using

$$
\phi_2(u) = b_2 u + \sqrt{r} \left\{ 1 - \sqrt{1 - c u} \right\} + a_2
$$

with $b_2 = (1 - c)/2$ and $a_2 = (\sqrt{r} - 1)^2/2$ and a function $f_2(z_2, \cdot)$ as before. Thus the same argument yields

$$
q(z_2; z_2 r) \sim C_2 2^{(1-\delta_2)/4} \phi_2'(0+)^{-\delta_1/2} z_2^{-1/2} e^{-a_2 z_2^2} \quad \text{as } z_2 \to \infty \quad (4.8)
$$

with some constant $C_2, 2$ depending only on $\delta_1$ and $\delta_2$.

Using (4.7) and (4.8) together with $\phi_1'(0+) = b_1 + \sqrt{r c}/2$ and $\phi_2'(0+) = b_2 + \sqrt{r c}/2$, we obtain (4.6).

Now we are in a position to prove Theorem 1.2.

**Proof of Theorem 1.2.** Let $0 < c < 1$. We combine Lemmas 4.2, 4.3 and 4.4 to obtain

$$
\lim_{z_2 \to \infty} e^{z_2/2} \lim_{z_3 \to 0^+} z_3^{-1(\delta_1 + \delta_2)/2} \lim_{\varepsilon \to 0^+} q(z_2, z_3; \varepsilon, z_2 r) = C_3 D(r)^{-\delta_1/2}
$$

for some constant $C_3$ which depends only on $\delta_1$, $\delta_2$ and $c$. Therefore we conclude that the conditional probability (4.1) does depend on $(\varepsilon, z_1)$, which proves that $Z^c$ is non-Markov.

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**References**


