MORSE HOMOTOPY AND ITS QUANTIZATION.

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Contents

§0 Introduction.

Morse theory, which is one of the roots of differential topology, is a method to study topology of manifolds using functions on it.

One of its main application is Morse inequality, which gives a relation between Betti number and the number of critical points of Morse function. The original proof (due to Morse) involves something more. That is it gives a way to find a homology group of a manifold using a Morse function. Variants of this idea appeared repeatedly in many important works by Thom, Smale, Milnor, etc. This point again appeared in Witten's famous paper [W1] and call attentions of many mathematicians. Floer [Fl] (generalizing [CZ]) uses it to built a new infinite dimensional homology theory. There the moduli space of gradient lines played an essential role.

One main point in [W1] is to regard Morse theory as a topological field theory. In fact, one may regard it as one dimensional topological field theory or quantum mechanics, while topological σ-model, Gauge field theory etc. are regarded as (topological) quantum field theories.

The purpose of this paper is two fold. One is to push forward this point of view and shows that we can regard various operators of algebraic topology as n-point functions. We use several functions for this purpose while in finding homology one use a single Morse function.

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The other is to show a relation of (finite dimensional) Morse theory to topological σ-model and two dimensional gravity. Namely we regard topological σ-model as a quantization of Morse theory. (Furthermore they are related also to Gauge theory as we discuss in [Fu2],[Fu3],[Fu4].) This construction generalize A-model [W3], which is introduced to study Mirror symmetry of Calabi-Yau manifold. There it was already realized that the construction of A-model is a quantization of the ring structure of cohomology. Here we quantize many other operators in algebraic topology. They are genus zero case, in the sense that it is related to the moduli space of pseudo holomorphic map from Rieman sphere or disk. We will also discuss some ideas how to handle higher genus case. Then, we find that the triangulation of Teichmüller space due to Mumford [Mu]-Strebel [Sr]-Harer [Ha]-Penner[Pe]-Kontsevitch [Ko1], arises naturally in Morse theory. It is remarkable that it appears even in the situation before we quantize the story. In other words, from this point of view we can find easily what is the natural way to quantize "higher genus Morse theory", though there is a serious analytic trouble to make those arguments rigorous.

Here is a bit more detailed outline and contents of each sections.

In §1 we consider cup product, Massey product and its higher analogue. There we study the moduli space of trees with metric such that the number of vertices with one edge is given. It is a classical result by Stasheff that such a moduli space is a cell. Then we consider a finitely many functions on manifolds. By using these functions (whose number is equal to the number of vertices of trees with one edge), we construct a moduli space of Feynman diagrams. (Here the edges are regarded as a gradient line of difference of two functions.) One can prove (using Stasheff's result) that this moduli space is a manifold. And one can find a dimension formula for it. Then using the case when the dimension is zero one finds several maps, which give cup product, Massey product, etc.

Since there are many kinds of trees one can use, it is not clear what this construction means. Then we introduce the notion of A^∞-category and find that we constructed an A^∞-category whose object is a function on M.

This construction is a "classical limit" of one discussed in [Fu2],[Fu3]. There we considered a symplectic manifold X such that the Chern class is proportional to the symplectic form and found an A^∞-category whose object is a Lagrangian or Bohr-Sommerfeld orbit of it. In fact they are related to each other as follows. We take a symplectic manifold M satisfying the above assumption. And put $X = M \times M$. Given a function f on M we find an exact perturbation of the diagonal $M \subset X$ by the Hamiltonian flow generated by f. Hence a smooth function corresponds to a Lagrangian. In this way we can find a relation between two constructions. However the rigorous proof that they really coincide to each other is not yet known. (Except for homology group itself (without considering product etc.) which is due to Floer. [Fl].) There is also a quantum correction that is related to the bubbling phenomenon in the moduli space of pseudo holomorphic curves. We recall these facts briefly in §3 and refer [Fu3] for further detail.

The construction we mentioned above corresponds open string of zero loop, since we consider pseudo holomorphic maps from 2 disk whose boundary condition is given by Lagrangians. In §2 we discuss it in a different way by using pseudo holomorphic spheres in place of pseudo holomorphic disks. There we work on a symplectic manifold M (and do not embed it to $M \times M$.) For this version it is obvious that its classical limit is the same as one discussed in §1. So we get quantum Massey product etc. What is not obvious and leads us a delicate singular bifurcation problem is the conjecture that this version is related to the special case of open string version we mentioned above. This conjecture is discussed at the end of §3 but the author do not have a proof of it yet.

Also it is easy to show that the quantum cup product we obtain in that way coincides with Gromov-Ruan invariant [Ru]. Using symmetries of graphs we also define a quantum Steenrod square in §2. This discussion depends much on the work by M.Betz-R.Cohen [BC].

Thus the discussion of §§1,2,3 concerns the case of tree (zero loop) or equivalently the case of Rieman surface of genus zero. §§4,5 are devoted to the case of higher genus. Here we need to borrow another important idea by M.Betz and R.Cohen. ([BC]) That is to introduce a graph which is not a tree. (However we do it in a way different from theirs.) See the remark at the end of this section the relation between their work and this paper.

In §4, we again consider the "classical limit". Namely Morse theory. Here we use the moduli of metric ribbon graph, that is a direct generalization of the construction of §1 and also closely related to the 2 dimensional (topological) gravity. But at this point we do not use any complex or symplectic structure on the manifold. Also the moduli theory of Rieman surface itself does not appear at this point. Our construction is based on quite elementary moduli spaces of ribbon graphs embedded to our manifold such that each edge is a gradient line of Morse function. We also introduce a variant of mapping class group as the symmetry of moduli space of metric ribbon graphs. So using cohomology classes of this group coupled with the cohomology classes of the manifold, we obtain various numbers (or maps.) They should coincide to the classical limit of the n point functions of the (open) string theory as we will discuss in §5. Since we use only elementary moduli space, the construction of §4 is rigorous and do not require nonlinear PDE etc. The author however do not know an appropriate algebraic machinery to describe what we obtain in that way. (We do have such an algebraic machinery in the case of trees, that is A^{∞}- category.) So the discussion of §4 is rather a preliminary sketch of the theory which we hope to develop in future.

§5 is devoted to a quantization of the higher genus Morse homotopy. Again there is two versions. But we discuss mainly open string version (one similar to §3.) Our discussion in §4 is organized so that it is quite immediate to see what we need to do for open string quantization. However we meet a serious trouble to make this argument mathematically rigorous. The major problem is that we do not know well about the compactification of the moduli space of pseudo holomorphic maps from Rieman surfaces of higher genus. (The author would like to thank Prof. Y. Ruan who explained him some essential points of this problem.) But it seems to the author that the problem is somewhat easier in this open string situation.

We also meet the same kind of singular bifurcation problems as we meet in genus zero case.

Because of these difficulties, we can not quantize higher genus Morse homotopy in mathematical level of rigor. So this section should be regarded as a collection of remarks than results.

Finally we remark here that related ideas are employed by M.Betz and R.Cohen [BC]. So the author mentions here which one he learned from their paper. First the idea that cup product is obtained by considering Feynman diagrams as in §1 is found independently by the author [Fu2], [Fu3] and them. (Although the proof of this fact is rather trivial the fact itself I believe is new and interesting.) (However Floer [Fl] may have known this fact already.) Next to consider the symmetry of graph to obtain Steenrod square is discovered by [BC]. Also they first considered the graph which is not necessary a tree. These two facts the author learned from their paper. The idea to use moduli parameter of the graph and to patch the moduli spaces of graphs with different combinatorial types in order to obtain secondary invariants is due to the author [Fu3]. Also we use a ribbon graph in place of a graph. This looks more natural from the point of quantization. Moreover the symmetry we obtain in that way (the mapping class group) looks more interesting than the symmetry of a graph. The observation that one can relate the Mumford-Strebel-Harer-Penner-Kontsevitch's triangulation of Teichmüller space to Morse theory in that way is discussed in this paper for the first time.

§1 Morse homotopy.

We first recall briefly the construction of Witten complex. Let M be a closed oriented manifold of finite dimension and f be a Morse function on it. Let $Cr(f)$ denote the set of all critical points of f, namely

$$Cr(f) = \{p \in M | df(p) = 0\}.$$

For $p, q \in Cr(f)$ we put

$$\mathcal{M}(p,q) = \left\{ \ell : (-\infty,\infty) \to M \,\middle|\, \begin{array}{l} \dfrac{d\ell}{dt} = -grad\ f, \\ \ell(-\infty) = p, \quad \ell(+\infty) = q. \end{array} \right\} / \sim .$$

Here $\ell \sim \ell'$ if and only if $\ell(t) = \ell'(t+c)$ for some constant c.

For $p \in Cr(f)$, let $\mu(f)$ be the Morse index of f. (Namely the number of negative eigenvalues of $Hess_p(f)$.) The following lemmas are proved by an easy transversality argument.

Lemma 1.1. *For a generic function f, the space $\mathcal{M}(p,q)$ is a smooth manifold of dimension $\mu(p) - \mu(q) - 1$.*

Lemma 1.2. *For a generic function f, the space $\mathcal{M}(p,q)$ is compactified to $\mathcal{CM}(p,q)$ such that*

$$\partial\mathcal{CM}(p,q) = \bigcup_r \mathcal{CM}(p,r) \times \mathcal{CM}(r,q).$$

Lemma 1.3. *For a generic function f, the space $\mathcal{M}(p,q)$ has an orientation such that they are compatible with respect to the compactification in Lemma 1.2*

Here orientation of a 0-dimensional manifold means that we assign $+1$ or -1 to each point. Then for a 0-dimensional oriented manifold X we can define $\sharp X$ as the number of point counted with sign. Now we define the Witten complex $C(M,f)$ for our Morse function f as follows.

$$C_k(M,f) = \bigoplus_{\mu(p)=k, p\in Cr(f)} \mathbf{Z}[p].$$

$$\partial[p] = \sum \sharp\mathcal{M}(p,q)[q].$$

Theorem 1.4. (Morse,Thom,Smale,Milnor,Witten,Floer,etc.) *$(C(M,f),\partial)$ is a chain complex. Its homology is canonically isomorphic to the homology of M.*

The proof of $\partial\partial = 0$ is based on Lemma 1.2. For the rest of this paper we shift to cohomology notation from homology. We can identify them by Poincaré duality. So we put

$$C^k(M,f) = C_{n-k}(M,f).$$

and $d = \partial$. Here n is the dimension of M. We hereafter write $\mu(p) = n -$ Morse index of p.

Our purpose here is to deduce more detailed structure on M from Morse theory. For this purpose we use several functions. First we define a moduli space of metric trees $\widetilde{\mathcal{T}}_{0,k}$. We consider the trees (one dimensional compact and simply connected simplicial complex). We assume that it has k vertices with one edges and no vertex with two edges. We let $\mathcal{V}_e$ the set of vertices with one edge. We say that an edge is *interior edge* if its boundary is disjoint to $\mathcal{V}_e$. Otherwise it is called an *exterior edge.* We fix an order of $\mathcal{V}_e = \{v_1, \cdots, v_k\}$. (Then an isotopy type of the embedding of our tree to $\mathbf{R}^2$ is automatically fixed.) We assign a positive real number to each interior edges. This number is regarded as the length of the edge. (We do *not* assign numbers to exterior edges. The length of them are regarded as infinity.)

We say such an object a *metric tree with k-exterior vertices.* Let $\widetilde{\mathcal{T}}_{0,k}$ be the set of such metric trees with k-exterior vertices. We can define a topology on this space in an obvious way. Now we recall the following :

Theorem 1.5. (Stasheff [St]) *$\widetilde{\mathcal{T}}_{0,k}$ is homeomorphic to $k-3$-dimensional (open) disk.*

See Figure 1 for the case when $k=5$. Compactification of $\widetilde{\mathcal{T}}_{0,k}$, (which is also essential for our discussion) is studied in [St]. The compactification is homeomorphic to a closed disk.

$$Figure 1$$

For an element Γ of $\widetilde{\mathcal{T}}_{0,k}$ we embed it to $D^2 = \{x \in \mathbf{R}^2 | |x| \le 1\}$, such that $\partial D^2 \cap \Gamma = \mathcal{V}_e$. Then there are k connected components of $D^2 - \Gamma$. We number it so that the closure of i-th component contains v_i and v_{i+1}. (See Figure 2.) Its boundary is a subgraph of Γ, and is called the *i-th (exterior) circle.*

$$Figure 2$$

Now let us take k smooth functions $f_1, \cdots, f_k$ on a manifold M. Let $x_i \in Cr(f_i - f_{i-1})$. We define a moduli space $\mathcal{M}(M, 0, k; f_1, \cdots, f_k; x_1, \cdots, x_k)$ as follows. (Sometimes we write $\mathcal{M}(M; f_1, \cdots, f_k; x_1, \cdots, x_k)$ or $\mathcal{M}(x_1, \cdots, x_k)$ to save notations.) $\mathcal{M}(x_1, \cdots, x_k)$ is the set of all maps $I : \Gamma \to M$, where $\Gamma \in \widetilde{\mathcal{T}}_{0,k}$ such that the following holds.

(1.6.1) $I(v_i) = x_i$.

(1.6.2) Let e is an interior edge of length ℓ such that it is contained in i-th and j-th circles. We assume that its orientation is as in Figure 3. We identify $e \simeq [0, \ell]$. Then we assume that the restriction of I to e is a gradient line of $f_i - f_j$.

(1.6.3) Let e is the k-th exterior circle. We identify $e - \{v_i\} = (-\infty, 0)$. We then assume that $I_{e-\{v_i\}}$ is a gradient line of $f_i - f_{i-1}$

$$Figure 3$$

There is a map $\pi : \mathcal{M}(x_1, \cdots, x_k) \to \widetilde{\mathcal{T}}_{0,k}$. Now the following is a consequence of a standard transversality argument. (And hence its proof is omitted.)

Theorem 1.7. *For generic $f_1, \cdots, f_k$, the space $\mathcal{M}(M; f_1, \cdots, f_k; x_1, \cdots, x_k)$ is a manifold of dimension $n - \sum \mu(x_i) + k - 3$*

Here $\mu(x_i)$ is the Morse index of x_i as a critical point of $f_i - f_{i-1}$. (Note that we consider x_1 as a critical point of $f_1 - f_k$ and not one of $f_k - f_1$. (The Morse index depends on the sign of the Morse function.) We need also the compactification of our moduli space.

Theorem 1.8. *One can find a compactification $\mathcal{CM}(M; f_1, \cdots, f_k; x_1, \cdots, x_k)$ of $\mathcal{M}(M; f_1, \cdots, f_k; x_1, \cdots, x_k)$ which is a manifold with boundary and corner. The codimension one stratum (the boundary) of this compactification is the union of the following spaces.*

(1.8.1) $\mathcal{M}(M; f_1, \cdots, f_k; x_1, \cdots, x_i', \cdots, x_k) \times \mathcal{M}(x_i, x_i')$. *Here* $x_i' \in Cr(f_i - f_{i-1})$.

(1.8.2) $\mathcal{M}(M; f_1, \cdots, f_i, f_j, \cdots, f_k; x_1, \cdots, x_i, x, x_{j+1}, \cdots, x_k) \times \mathcal{M}(M; f_i, \cdots, f_j; x, x_{i+1} \cdots, x_j)$

This theorem is an immediate consequence of the proof of Lemma 1.2 and Stasheff's compactification of $\widetilde{\mathcal{T}}_{0,k}$. Roughly speaking (1.8.1) corresponds to the compactification of the fibre of π and (1.8.2) corresponds to the compactification of $\widetilde{\mathcal{T}}_{0,k}$.

Now we define a map

$$\eta_{k-1} : C^{a_1}(M; f_2 - f_1) \otimes \cdots \otimes C^{a_{k-1}}(M; f_k - f_{k-1}) \to C^{\sum a_i + k - 3}(M; f_k - f_1)$$

by

$$\eta_{k-1}([x_1] \otimes \cdots \otimes [x_{k-1}]) = \sum_{x_k} \sharp \mathcal{M}(M; f_1, \cdots, f_k; x_1, \cdots, x_k)[x_k].$$

(We write sometimes $\eta_{k-1}([x_1], \cdots, [x_{k-1}])$ in place of $\eta_{k-1}([x_1] \otimes \cdots \otimes [x_{k-1}])$.) A consequence of Theorem 1.8 is

Corollary 1.9.

$$(\partial \eta_k)([x_1], \cdots, [x_k]) = \sum_{i<j} \epsilon \cdot \eta_{k-j+i+1}([x_1], \cdots, \eta_{j-i}([x_{i+1}], \cdots, [x_j]), \cdots, [x_k]).$$

Here $\epsilon = (-1)^{(j-i)(deg x_1 + \cdots + deg x_i)}$. Upto sign Corollary 1.9 is immediate from Theorem 1.8. The verificatin of the sign is omitted.

Let us consider the case when $k = 2$. Corollary 1.9 implies that η_2 is a chain map. Hence by Theorem 1.4 it induces a map

$$H^i(M; \mathbf{Z}) \otimes H^j(M; \mathbf{Z}) \to H^{i+j}(M; \mathbf{Z}) \tag{1.10}$$

Proposition 1.11. *The map (1.10) is the cup product.*

Sketch of the proof. Let $\sum a_m[x_m] \in C^i(M, f_2 - f_1)$, $\sum b_m[y_m] \in C^j(M, f_3 - f_2)$ be cocycles. The Poincaré dual of the corresponding cohomology class is represented by $\sum a_m X_m \in C^i(M, f_2 - f_1)$, and $\sum b_m Y_m \in C^j(M, f_3 - f_2)$ where X_m is the closure of the unstable manifold of the critical point x_m and Y_m is the closure of the unstable manifold of the critical point y_m. Hence the Poincaré dual of the cup product of $[\sum a_m[x_m]]$ and $[\sum b_m[y_m]]$ is represented by $\sum a_m b_{m'}[X_m \cap Y_{m'}]$.

On the other hand by definition we have, for $z \in Cr(f_3 - f_1)$,

$$\mathcal{M}(x_m, y_{m'}, z) = X_m \cap Y_{M'} \cap Z.$$

Where Z is the stable manifold of the critical point z of $grad(f_3 - f_1)$. The proposition now follows from the definition.

We next consider the case when $k = 3$. We have

$$\begin{aligned} 0 = {} & d(\eta_3(x_1, x_2, x_3)) - \eta_3(dx_1, x_2, x_3) \\ & - (-1)^{degx_1}\eta_3(x_1, dx_2, x_3) - (-1)^{degx_2+degx_3}\eta_3(x_1, x_2, dx_3) \\ & - \eta_2(\eta_2(x_1, x_2), x_3) - \eta_2(x_1, \eta_2(x_2, x_3)). \end{aligned} \tag{1.12}$$

There is two consequences of this formula. One is that the cup product is associative. In fact our map η_3 gives a chain homotopy for the cup product to be associative.

The other is that we can define Massey product using it. Let $x_i \in C(M, f_i - f_{i-1})$ be cocycles such that $[x_1] \cup [x_2] = [x_2] \cup [x_3] = 0$. We put $\eta_2(x_1, x_2) = dy_1$, $\eta_2(x_2, x_3) = dy_2$. Then we find

$$d(\eta_2(y_1, x_3) + (-1)^{degy_1}\eta_2(x_1, y_2) + \eta_3(x_1, x_2, x_3)) = 0.$$

Proposition 1.13. *The cocycle $\eta_2(y_1, x_3) + (-1)^{degx_1}\eta_2(x_1, y_2) + \eta_3(x_1, x_2, x_3)$ represents the Massey product of three elements $[x_1]$,$[x_2]$,$[x_3]$.*

The proof is similar to one of Proposition 1.11 and is omitted. In this way one can find the higher Massey product using our maps η_k. One can summarize these propositions by using the notion of (topological) A^∞-category.

Definition 1.14. A topological A^∞*-category* is a collection of the topological space, the set of object $\mathfrak{Ob}$, a cochain complex $C(a, b)$ for Baire subset of $(a, b) \in \mathfrak{Ob}^2$, and maps $\eta_k : C(a_0, a_1) \otimes \cdots \otimes C(a_{k-1}, a_k) \to C(a_0, a_k)$ of degree $k - 3$ for Baire subset of $(a_0, \cdots, a_k) \in \mathfrak{Ob}^{k+1}$, such that Formula (1.9) holds in a Baire subset.

Now it is easy to see that our discussion so far proves the following :

Theorem 1.15. *For each oriented manifold M, there exists a (topological) A^∞-Category whose object is a smooth function on it, whose morphism is an element of Witten complex, and whose (higher) composition is η_k.*

Problem 1.16. Can one construct Sullivan's minimal model [Su], using our A^∞-category ?

Remark 1.17. We remark that our operator η_k is defined everywhere and gives (higher) Massey product after appropriate modification. The idea that one can construct everywhere defined operator from secondary operator (which is defined only in an appropriate subset and is well defined modulo some indeterminacy) using something like A^∞-category is found independently by V.Smirnov [Sm].

§2 Quantization.

In this section we join the discussion of §1 with topological σ-model or Gromov-Ruan invariant [Ru]. It seems to the author that one can define quantum rational homotopy type of Calabi-Yau manifold in this way.

In physical literature, there is a construction know as quantum ring or A-model. ([W3].) This is a quantization of cup product. In this section we also quantize (higher) Massey product and Steenrod square. We first consider the moduli space $\widehat{\mathcal{T}}_{0,k}$ which modify a bit the moduli space $\widetilde{\mathcal{T}}_{0,k}$ we used in §1. For a positive integer m we put

$$\mathcal{T}_{0,\partial,m} = \left\{ (D^2; x_1, \cdots, x_k) \,\middle|\, \begin{array}{l} x_i \in \partial D^2,\ x_1, \cdots, x_k \text{ are disjoint to each other,} \\ \text{cyclic ordering of them is respected} \end{array} \right\} / \sim .$$

Here $\sim$ denotes the biholomorphic equivalence. We can compactify the moduli space $\mathcal{T}_{0,\partial,m}$ by adding the configuration where $x_i = x_{i+1}$ etc., and denote it by $\mathcal{CT}_{0,\partial,m}$. We may and will regard the element of $\mathcal{T}_{0,\partial,m}$ as Rieman sphere with m point on it which is contained in one circle, (the set conformal to equator.) We put

$$\widehat{\mathcal{T}}_{0,k} = \left\{ (\Gamma; D_1, \cdots, D_a) \,\middle|\, \begin{array}{l} \Gamma \in \widetilde{\mathcal{T}}_{0,k} \text{ with } a \text{ inner vertices} \\ D_b \in \mathcal{CT}_{0,\partial,m} \text{ where } b\text{-th vertex of } \Gamma \text{ has } m\text{-edges.} \end{array} \right\}$$

(See Figure 4.) We define a topology as in Figure 5. Now the following can be proved in a similar way as Theorem 1.5.

$$Figure 4 + 5$$

Theorem 2.1. *$\widehat{\mathcal{T}}_{0,k}$ is diffeomorphic to the $k-3$-dimensional open disk.*

See Figure 6 for the case when $k = 5$. Compactification of this moduli space is obtained in a similar way.

$$Figure 6$$

We regard an element of $\widehat{\mathcal{T}}_{0,k}$ as 2 dimensional space as in Figure 4. We now define a moduli space $\mathcal{M}^m(M, 0, k; f_1, \cdots, f_k; x_1, \cdots, x_k)$ as follows. Let M be a symplectic manifold. We choose a compatible almost complex structure on M. We assume that first Chern class of this almost complex structure vanishes. (In fact one can discuss in a similar way the case when the first Chern form is a multiplication of symplectic form by a positive number . See [Fu3] Chapter 4.) Let f_i be functions on it. x_i is a critical point of $f_i - f_{i-1}$. Let $P \in \widehat{\mathcal{T}}_{0,k}$. Then we consider a continuous map $I : P \to M$ such that they satisfy (1.6.1),(1.6.2),(1.6.3) on graph part of P, and

(2.2.1) I is pseudo-holomorphic on each sphere.

(2.2.2)

$$\int_P I^*\omega = m.$$

Here ω denotes the symplectic form. We let $\mathcal{M}^m(M,0,k;f_1,\cdots,f_k;x_1,\cdots,x_k)$ be the space of all such objects. One can define a topology on it in a similar way. There is again a map $\pi:\mathcal{M}^m(M,0,k;f_1,\cdots,f_k;x_1,\cdots,x_k)\to\widehat{\mathcal{T}}_{0,k}$.

To be more precise, we have to modify (2.3) and consider the equation

$$\overline{\partial} I = g$$

in place of Caucy-Rieman equation. (See [Ru].) But we do not discuss this point since the argument one needs for it is the same as Ruan's one. Now we have :

Theorem 2.3. *For generic f_i, almost complex structure, and perturbation of Caucy Rieman equation, the space $\mathcal{M}^m(M,0,k;f_1,\cdots,f_k;x_1,\cdots,x_k)$ is a manifold of dimension $n-\sum\mu(x_i)+k-3$.*

Here n is the real dimension of our manifold. The proof is a minor modification of one by Ruan [Ru] and is omitted.

(We remark that our assumption that the first Chern class vanishes implies that the dimension is independent of m. In case when $c^1=N\omega$, the dimension will be $n-\sum\mu(x_i)+k-3+nmN$.)

We can also prove the following analogy of Theorem 1.8.

Theorem 2.4. *One can find a compactification $\mathcal{CM}^m(M;f_1,\cdots,f_k;x_1,\cdots,x_k)$ of $\mathcal{M}^m(M;f_1,\cdots,f_k;x_1,\cdots,x_k)$ which is a manifold with boundary and corner. The codimensions one stratum (the boundary) of this compactification is one of the following spaces.*

(2.4.1) $\mathcal{M}^m(M;f_1,\cdots,f_k;x_1,\cdots,x'_i,\cdots,x_k)\times\mathcal{M}(x_i,x'_i)$. *Here* $x'_i\in Cr(f_i-f_{i-1})$.

(2.4.2) $\mathcal{M}^{m'}(M;f_1,\cdots,f_i,f_j,\cdots,f_k;x_1,\cdots,x_i,x,x_{j+1},\cdots,x_k)\times\mathcal{M}^{m-m'}(M;f_i,\cdots,f_j;x,x_{i+1},\cdots,x_j)$

We now define

$$\eta^m_{k-1}:C^{a_1}(M;f_2-f_1)\otimes\cdots\otimes C^{a_{k-1}}(M;f_k-f_{k-1})\to C^{\sum a_i+k-3}(M;f_k-f_1)$$

by the same formula as η_{k-1}. Namely

$$\eta^m_{k-1}([x_1]\otimes\cdots\otimes[x_{k-1}])=\sum_{x_k}\sharp\mathcal{M}^m(M;f_1,\cdots,f_k;x_1,\cdots,x_k)[x_k].$$

We put $\widehat{C}^a(M;f)=C^a(M;f)\otimes\mathbf{Z}[[T]]$. (The ring $\mathbf{Z}[[T]]$ arises here by the same reason as Novikov ring is used in symplectic Floer theory ([Si] [HS] [LO] [On] [Fu3]). Then we define

$$\widehat{\eta}_{k-1}:\widehat{C}^{a_1}(M;f_2-f_1)\otimes\cdots\otimes\widehat{C}^{a_{k-1}}(M;f_k-f_{k-1})\to\widehat{C}^{\sum a_i+k-3}(M;f_k-f_1)$$

by

$$\widehat{\eta}_{k-1}=\sum T^m\eta^m_{k-1}.$$

Hence using Theorem 2.4 we obtain :

Corollary 2.5.

$$(\partial\widehat{\eta}_k)([x_1],\cdots,[x_k]) = \sum_{i<j}\epsilon\cdot\widehat{\eta}_{k-j+i}([x_1],\cdots,\widehat{\eta}_{j-i}([x_{i+1}],\cdots,[x_j]),\cdots,[x_k]).$$

Here ϵ is as in Corollary 1.9. It is a direct consequence of the definition that η_2^m coincides to Gromov-Ruan invariant [Ru]. Thus $\widehat{\eta}_2$ is the same as quantum ring in physical literature. This coincidence is pointed out by M.Kontsevitch to the author in the case of open string quantization. Kontsevitch proposed the following :

Conjecture 2.6. $\sum_m \sharp\mathcal{M}^m(M;f_1,\cdots,f_k;x_1,\cdots,x_k)\,t^m$ converges for small t.

If Conjecture 2.6 is true, then one can consider $\widehat{\eta}_k$ over real coefficient in place of formal powerseries ring.

We can define the quantum (higher) Massey product by exactly the same formula as in §1. Namely

$$< x_1,x_2,x_3 >= \widehat{\eta}_2(y_1,x_3) + (-1)^{deg x_1}\widehat{\eta}_2(x_1,y_2) + \widehat{\eta}_3(x_1,x_2,x_3).$$

Next we discuss the quantum Steenrod square based on Betz-Cohen's result. For this purpose we first recall their idea to relate Steenrod square to Morse theory. Our treatment is basically the same as Betz-Cohen [BC]. However we modify and organize the construction so that it is easy to see the way of quantization. Let M be an oriented manifold. (We do not need to assume that it is symplectic at this point.) First we need to modify a bit our moduli space. $\mathcal{M}(M;f_1,\cdots,f_k;x_1,\cdots,x_k)$ and define $\mathcal{M}(M;f_1,\cdots,f_k)$. We recall that in the definition of $\mathcal{M}(M;f_1,\cdots,f_k;x_1,\cdots,x_k)$ we regard the length of exterior edges to be infinity. Here we take an arbitrary but a fixed finite number (we choose 1) and suppose the length of exterior edges to be that number. More precisely we define $\mathcal{M}(M;f_1,\cdots,f_k)$ to be the set of all maps $I:\Gamma\to M$ such that it satisfies (1.6.2) and

(2.7) Let e be the i-th exterior edge. We identify $e=[0,1]$. Then $I|_e$ is a gradient line of $f_i - f_{i-1}$.

(Here we do not need to take the critical points x_i.) There is a map :

$$\pi : \mathcal{M}(M;f_1,\cdots,f_k) \to \widetilde{\mathcal{T}}_{0,k}$$

and

$$\pi_2 : \mathcal{M}(M;f_1,\cdots,f_k) \to M^k.$$

Here π_2 is defined by $\pi_2(\Gamma,I) = (I(v_1),\cdots,I(v_k))$. We consider the union of $\mathcal{M}(M;f_1,\cdots,f_k)$ for all f_i and denote it by $\mathcal{M}_{0,k}$. Then we have

$$\pi_1 : \mathcal{M}_{0,k} \to \widetilde{\mathcal{T}}_{0,k}\times(C^\infty(M))^k. \tag{2.8}$$

Here we remark that the cyclic group $\mathbf{Z}_k$ of order k acts as a symmetry group of this map. We choose an appropriate subspace $\overline{(C^\infty(M))^k}$ of $(C^\infty(M))^k$ such that

$\mathbf{Z}_k$ act freely there and that $(C^\infty(M))^k - \overline{(C^\infty(M))^k}$ is of infinite codimension. We obtain

$$\pi_1 : \overline{\mathcal{M}}_{0,k} \to \frac{\widetilde{\mathcal{T}}_{0,k} \times \overline{(C^\infty(M))^k}}{\mathbf{Z}_k}.$$

By dividing the map (2.8). We also obtain a map

$$\pi_2 : \overline{\mathcal{M}}_{0,k} \to M^k/\mathbf{Z}_k.$$

Now we can prove by a standard transversality argument that

Theorem 2.9.

$$\pi_1 : \overline{\mathcal{M}}_{0,k} \to \frac{\widetilde{\mathcal{T}}_{0,k} \times \overline{(C^\infty M)^k}}{\mathbf{Z}_k}.$$

is a Fredholm map of index n*.*

We need one more map. We remark that there is a universal family

$$Univ \to \frac{\widetilde{\mathcal{T}}_{0,k} \times \overline{(C^\infty M)^k}}{\mathbf{Z}_k},$$

whose fiber is a tree. Since a tree is contractible, we can choose a section s to this map. Then we get a map

$$\pi_3 : \overline{\mathcal{M}}_{0,k} \to M$$

by

$$\pi_3([\Gamma, I]) = I(s([\Gamma, I])).$$

Now let p be an arbitrary prime. We remark that $\frac{\widetilde{\mathcal{T}}_{0,p} \times \overline{(C^\infty M)^p}}{\mathbf{Z}_p}$ is a $K(\mathbf{Z}_p, 1)$ space. Hence its homology over $\mathbf{Z}_p$ has a canonical generator $[X_i]$ in each degree i. We choose a cycle X_i which represents this class. Next let U be a cycle of degree $n - j$ which represents a cohomology class $H^j(M;\mathbf{Z}_p)$. (To be precise one needs to use the notion of geometric cycle of degree i. See [Fu1].) Then in case we choose everything transversal we get a cycle

$$\pi_1^{-1} X_i \cap \pi_2^{-1}(U^p/\mathbf{Z}_p).$$

This is a cycle of dimension $n - pj + i$. Hence by using π_3 it gives an element of

$$H_{n-pj+i}(M;\mathbf{Z}_p) = H^{pj-i}(M;\mathbf{Z}_p).$$

Theorem 2.10. $\pi_3(\pi_1^{-1} X_i \cap \pi_2^{-1}(U^p/\mathbf{Z}_p))$ *represents the element* $D_i([U])$*. Where* D_i *is as in [SE]*

The proof is rather immediate from the constructions and hence omitted. We recall, for example, $Sq^k = D_{j-k}$ for an element of degree j.

Now it is quite obvious how to quantize the construction. We can construct the moduli space $\mathcal{M}^m(M; f_1, \cdots, f_k)$ etc. Hence we obtain a map

$$D^m_{i,(p)} : C^j(M;\mathbf{Z}_p) \to C^{pj-i}(M;\mathbf{Z}_p),$$

in case M is a symplectic manifold with trivial first Chern class. We define

$$\widehat{D}_{i,(p)} : C^j(M; \mathbf{Z}_p[[T]]) \to C^{pj-i}(M; \mathbf{Z}_p[[T]])$$

by

$$\widehat{D}_{i,(p)} = \sum_m T^m D^m_{i,(p)}.$$

Thus we get quantum Steenrod square and higher reduced power.

We close this section by proposing some open questions.

Problem 2.11. Does Adem and Cartan relations hold for quantum Steenrod square and higher reduced power ? If not what is a quantized version of it ?

Problem 2.12. Can one define secondary operations by combining ideas of this and last sections ?

We remark that the idea of §1 is to use Teichmüller parameter (moduli parameter of graphs) to define a secondary operator (Massey product). We remark that in the definition of quantum Steenrod square we do not in fact need a moduli parameter since there is an element of $\widetilde{T}_{0,p}$ with is fixed by $\mathbf{Z}_p$. (But this does not the case for higher genus.)

Problem 2.13. Can one quantize Adams spectral sequence ?

Solving these problems should be the way to find "quantum homotopy type" of, say, Calabi-Yau manifold.

§3 Remarks.

First we describe another way to quantize Morse homotopy. Let X be a symplectic manifold. For simplicity we again assume that the first Chern class of X vanishes. We assume also that the cohomology class of the symplectic form ω is integral. We choose and fix a complex line bundle L with connection ∇ on it such that its Chern form is equal to symplectic form (not only as cohomology class but also as forms.) We say a Lagrangian $\Lambda \subset X$ to be a Borh-Sommerfert orbit (BS-orbit) if the restriction of (L, ∇) to it is trivial. For two oriented BS-orbits Λ_1,Λ_2 we define its (Lagrangian intersection) Floer homology (roughly) as follows. (See [Fu3] for detail.) Let $p, q \in \Lambda_1 \cap \Lambda_2$. We consider the moduli space

$$\mathcal{M}(p,q) = \left\{ h : D^2 \to X \,\middle|\, \begin{array}{ll} h(-1) = p, h(1) = q, & \\ h(e^{\sqrt{-1}\theta}) \in \Lambda_1, & \text{if } \theta \in [0,\pi], \\ h(e^{\sqrt{-1}\theta}) \in \Lambda_2, & \text{if } \theta \in [\pi, 2\pi]. \end{array} \right\}$$

We splits the space using symplectic area. Namely we put

$$\mathcal{M}_m(p,q) = \left\{ h \in \mathcal{M}(p,q) \,\middle|\, \int_{D^2} h^*\omega = m + c(p,q) \right\}.$$

Here $c(p,q)$ is a number depending only on p and q. (See [Fu3] Chapter 4.) Because of the presence of fundamental group of Λ_i and of Maslov index, the dimension of this space depends on the components. We can split $\mathcal{M}_m(p,q)$, as

$$\mathcal{M}_m(p,q) = \bigcup_{\ell > \ell_o} \mathcal{M}_{m,\ell}(p,q)$$

and that $M_{m,\ell}(p,q)$ is a manifold of dimension $\mu(p) - \mu(q) + 2\ell$. Here $\mu(p)$ is a integer depending only on p. We divide the moduli space by $\mathbf{R} = Aut(D^2; -1, +1)$ and let $\overline{\mathcal{M}}_{m,\ell}(p,q)$ be the quotient space. Now we put

$$\widehat{C}(\Lambda_1, \Lambda_2) = \bigoplus_{x \in \Lambda_1 \cap \Lambda_2} \mathbf{Z}[[T]][T^{-1}][x].$$

Here $[x]$ is regarded to have degree $\mu(x)$ and T is regarded to be degree 2. Boundary operator is defined by

$$\partial[x] = \sum_y \sharp \overline{\mathcal{M}}_{m,\ell}(x,y) T^m [y].$$

Here we choose ℓ such that $dim \overline{\mathcal{M}}_{m,\ell}(x,y) = 0$. Thus we obtain a chain complex.

From now on we again use cohomology notation. Hence we put

$$\widehat{C}^k(\Lambda_1, \Lambda_2) = \widehat{C}_{n-k}(\Lambda_1, \Lambda_2).$$

Here $2n = dim_{\mathbf{R}} X$.

Let $\Lambda_1, \cdots, \Lambda_k$ be BS-orbits and $p_i \in \Lambda_i \cap \Lambda_{i-1}$. ($\Lambda_0 = \Lambda_k$.) We put

$$\mathcal{M}(\Lambda_1, \cdots, \Lambda_k; p_1, \cdots, p_k) = \mathcal{M}(p_1, \cdots, p_k)$$
$$= \left\{ (h; x_1, \cdots, x_k) \,\middle|\, \begin{array}{l} [D^2; x_1, \cdots, x_k] \in \mathcal{T}_{0,\partial,k} \\ h : D^2 \to X \text{ is pseudo holomorphic,} \\ h(x_i) = p_i, \quad h(\overline{x_i x_{i+1}}) \subset \Lambda_i \end{array} \right\} / \sim$$

Here $(h; x_1, \cdots, x_k) \sim (h\varphi; \varphi^{-1}(x_1), \cdots, \varphi^{-1}(x_k))$ for $\varphi \in Aut(D^2)$ and $\overline{x_i x_{i-1}}$ denotes the subset of ∂D^2 which bounds $\{x_i, x_{i-1}\}$. We let $\mathcal{M}_k(p_1, \cdots, p_k)$ be the subset of $\mathcal{M}(p_1, \cdots, p_k)$ such that $\int h^*\omega = m + c(p_1, \cdots, p_k)$. (See [Fu3] Chapter 4.)

We again split $\mathcal{M}_k(p_1, \cdots, p_k)$ into $\mathcal{M}_{k,\ell}(p_1, \cdots, p_k)$ such that

$$dim \mathcal{M}_{k,\ell}(p_1, \cdots, p_k) = n - \sum \mu(p_i) + (k-3) + 2\ell.$$

We count its number and obtain

$$\eta_k^\ell([p_1], \cdots, [p_{k-1}]) = \sharp \mathcal{M}_{k,\ell}(p_1, \cdots, p_k) \cdot [p_k].$$

We put

$$\widehat{\eta}_k = \sum T^\ell \eta_k^\ell.$$

Theorem 3.1. ([Fu2],[Fu3])

$$(\partial \widehat{\eta}_k)([x_1], \cdots, [x_k]) = \sum_{i<j} \epsilon \cdot \widehat{\eta}_{k-j+i}([x_1], \cdots, \widehat{\eta}_{j-i}([x_{i+1}], \cdots, [x_j]), \cdots, [x_k]).$$

Here ϵ is as in Corollary 1.9. This is a consequence of a result similar to Theorem 2.3.

Let us mention here relation of this construction to one in §2.

First we mention one difference between them. In the construction of §2, we do *not* consider the quantum correction, (which is defined by using the moduli space of pseudo holomorphic maps with nonzero symplectic area), in the definition of boundary operator. (But it appeared in cup and Massey product.)

The reason for it is as follows. (This obserbation is due to Floer [Fl].) The quantum correction of boundary operator, if exists, should be defined by counting the order of the moduli space of the map from the following figure :

$$Figure 7$$

The point is that the 2-sphere with two points is *not* a stable curve. That is it has an automorphism group S^1. So if such a moduli space is nonempty then its (virtual) dimension is at lease one. Since we count the moduli space of dimension zero, the nontrivial contribution comes only from the degenerate case, which corresponds to the usual boundary operator.

On the other hand, in our situation of Lagrangian intersection, we are considering a pseudo holomorphic map from D^2 with two points on its boudary. Its automorphism group is $\mathbf{R}$, which corresponds, in Morse theory, the reparametrization of the gradient line. Hence there is no reason that the quantum correction is 0 in this situation, even for boundary operator. However the author do not have an explicit example that it is really nonzero.

We next discuss how the version of this section is related to one in §2. Let M be a symplectic manifold of dimension n. Suppose that its first Chern class vanishes. We embed M to $X = M \times M$ as diagonal. Then M is a BS-orbit. One can prove that a neighborhood of M in X is symplectic diffeomorphic to a neighborhood of zero section in T^*M, the cotangent bundle. Let $f \in C^\infty(M)$. We consider a graph of ϵdf in T^*M. Then for ϵ small, we may regard it as a submanifold in X. We denote it by Λ_f. We remark that $\Lambda_f \cap \Lambda_g = Cr(f-g)$. Let $p_i \in \Lambda_{f_i} \cap \Lambda_{f_{i-1}} = Cr(f_i - f_{i-1})$. We may choose $c(p_1, \cdots, p_k) \sim \epsilon$. Namely for

$$h \in \mathcal{M}_{m,\ell}(\Lambda_{f_1}, \cdots, \Lambda_{f_k}; p_1, \cdots, p_k)$$

we have

$$\int h^*\omega = m + O(\epsilon).$$

Then $\mathcal{M}_{m,\ell}(\Lambda_{f_1}, \cdots, \Lambda_{f_k}; p_1, \cdots, p_k)$ is nonempty only if $m \geq 0$. Hence we can take $\mathbf{Z}[[T]]$ as coefficient ring rather than $\mathbf{Z}[[T]][T^{-1}]$. Moreover we can verify that the Maslov index $\pi_1(M) \to \mathbf{Z}$ defined in [Fu3] Chapter 4 is trivial in this case where M is diagonal. As a consequence we do not need to split using ℓ.

Conjecture 3.2. *For small ϵ the space $\mathcal{M}_m(\Lambda_{f_1}, \cdots, \Lambda_{f_k}; p_1, \cdots, p_k)$ is diffeomorphic to $\mathcal{M}^m(M; f_1, \cdots, f_k; p_1, \cdots, p_k)$.*

Here the first moduli space is introduced in this section and the second one is introduced in §2.

Idea of the proof. We write $\mathcal{M}^\epsilon_m(\Lambda_{f_1}, \cdots, \Lambda_{f_k}; p_1, \cdots, p_k)$ in place of $\mathcal{M}_m(\Lambda_{f_1}, \cdots, \Lambda_{f_k}; p_1, \cdots, p_k)$. We consider elements $h_\epsilon \in \mathcal{M}_m(\Lambda_{f_1}, \cdots, \Lambda_{f_k}; p_1, \cdots, p_k)$ and study what happens when ϵ goes to zero. We want to describe the moduli space which we expect to be the set of limit of the images $h_\epsilon(D^2)$, (say, in Haudsorff topology.)

Let $(\Gamma; D_1, \cdots, D_a) \in \widehat{T}_{0,\partial,k}$. In place of regarding it as a two dimensional space like Figure 4. we regard it as

$$Figure 8$$

Then we consider the maps $h : P \to X$ such that on the graph part it satisfies (1.6.1), (1.6.2),(1.6.3), on disks it satisfies (3.3.1),(3.3.2) and boundary condition (3.3.3).

(3.3.1) P is pseudo holomorphic on each disk,
(3.3.2) $\int_P h^*\omega = m$.
(3.3.3) $h(\partial P) \subset M =$ diagonal.

We write moduli space of such maps $\mathcal{M}^0_m(\Lambda_{f_1}, \cdots, \Lambda_{f_k}; p_1, \cdots, p_k)$

Lemma 3.4. *$\mathcal{M}^0_m(\Lambda_{f_1},\cdots,\Lambda_{f_k};p_1,\cdots,p_k)$ is diffeomorphic to $\mathcal{M}^m(M;f_1,\cdots,f_k;p_1,\cdots,p_k)$.*

This is an immediate consequence of reflection principle. Thus to show Conjecture 3.2 we need only to prove

$$\lim_{\epsilon\to 0}\mathcal{M}^\epsilon_m(\Lambda_{f_1},\cdots,\Lambda_{f_k};p_1,\cdots,p_k)=\mathcal{M}^0_m(\Lambda_{f_1},\cdots,\Lambda_{f_k};p_1,\cdots,p_k) \tag{3.5}$$

The author do not know the proof of it. (See also Hofer-Salamon [HS].)

In fact we can prove that the left hand side is contained in right hand side. But the opposite is more delicate. (It is an analogy of Taubes' construction in Gauge theory.) To use Conjecture 3.2 to find relations between two versions we need probably Ono's idea in [On].

Finally we give a remark about our assumption that $x_1,\cdots,x_k$ is on one circle, in the definition of $\mathcal{T}_{0,\partial,m}$. From the point of view of moduli space of Rieman surface with points, this assumption looks strange and it looks natural to remove it. (Of course one needs it for Conjecture 3.2 to be true.) So it seems not appropriate to regard the construction of §2 as closed string theory.

However there is another reason why we put that assumption. Under this assumption the moduli space $\mathcal{T}_{0,\partial,m}$ has codimension one boundary which we use to construct our moduli space $\widehat{\mathcal{T}}_{0,k}$ by patching moduli spaces of configurations of different combinatorial types. If we remove that assumption in the definition of $\mathcal{T}_{0,\partial,m}$, it will have only codimension 2 boundary. This is inconvenient to patch it to other spaces. The author do not understand well have to handle various components in the case of closed string.

§4 Higher genus Morse homotopy.

In this section we generalize the construction of §1 by using graphs (more precisely Ribbon graphs) which is not necessary a tree.

Let $g \ge 0, k \ge 2$ be integers. We define a space $\widetilde{\mathcal{T}}_{g,k}$ as follows.

A ribbon graph Γ is a one dimensional simplicial complex together with cyclic order of the set of the edges containing a vertex. Any ribbon graph is embedded in a unique way to a Riemann surface Σ such that each connected component of $\Sigma - \Gamma$ is a cell and that the cyclic order is respected. The genus of Γ is by definition the genus of Σ.

A k-marked ribbon graph is a ribbon graph Γ and a vertex v_0 on it with exactly k-edges containing v_0 together with an order of the set of the edges containing v_0 compatible with the cyclic order. (Here we only consider the case with one marked point. One can generalize it to the case when there are several marked points. But we do not try to do it here.) In case when genus is zero and the graph minus v_0 is a tree, we go back to the situation in §1 by removing the vertex v_0 and taking its completion.

For a k-marked ribbon graph, we call an edge to be an interior edge if it is disjoint to v_0. Otherwise the edge is called an exterior edge. A length function ℓ is a function from the set of interior edges to the set of positive numbers. We assume that the number of edges containing each vertex is not smaller than 3. A circle of a k-marked ribbon graph Γ is by definition a subcomplex $\subset \Gamma$ which is a boundary of a connected component of $\Sigma - i(\Gamma)$. A circle is called an interior circle if it is disjoint from v_0. Otherwise it is called an exterior circle. By our assumption there are exactly k-exterior circles. In other words each exterior circle is embedded to Γ in a neighborhood of v_0.

We put

$$\widetilde{\mathcal{T}}_{g,k} = \left\{ (\Gamma, v_0), \ell, [i]) \;\middle|\; \begin{array}{l} (\Gamma, v_0) \text{ is a } k\text{-marked ribbon graph,} \\ \ell \text{ is a length function,} \\ [i] \text{ is an isotopy class of an embedding of } \Gamma \text{ to a Rieman} \\ \text{surface together with an order of the set of interior circles.} \end{array} \right\}$$

Let $\widetilde{\mathcal{T}}_{g,m,k}$ be the set of all elements of $\widetilde{\mathcal{T}}_{g,k}$ with m interior circles and k exterior circles.

We remark that we can define a topology of these spaces $\widetilde{\mathcal{T}}_{g,k}$, in an obvious way.

The space $\widetilde{\mathcal{T}}_{g,k,m}$ is almost the same as the triangulation of Teichmüller space based on Strebel's quadratic differentials. This fact is important for quantization of the construction of this section. However in this section we use this space in a different way. We remark that one can prove that $\widetilde{\mathcal{T}}_{g,k}$ is a smooth manifold in exactly the same way as the case of the triangulation of Teichmüller space. (The argument by Stasheff quoted in §1 can be applied to this case.) But contrary to the case of §1 this space is not contractible. For example in case of $g = 0$ $m \ne 0$ the fundamental group of it is the braid group.

Now let us take an oriented manifold M (of finite dimension). We choose an ordered set $(f_1, \cdots, f_n)$ of smooth functions on M. We assume that $f_i - f_j$ is a Morse function for each $i \neq j$.

We take the point $x_1, \cdots, x_k$ of M such that x_i is a critical point of $f_i - f_{i-1}$.

We now generalize the construction of §1 and define a moduli space

$$\mathcal{M}(M, g, m, k; f_1, \cdots, f_k, f_1', \cdots, f_m', x_1, \cdots, x_k),$$

together with a map from there to $\widetilde{\mathcal{T}}_{g,m,k}$ as follows. (We remark that

$$\mathcal{M}(M, 0, 0, k; f_1, \cdots, f_k; x_1, \cdots, x_k) = \mathcal{M}(M; f_1, \cdots, f_k; x_1, \cdots, x_k)$$

where the right hand side is the space defined in §1.)

Let $(\Gamma, (v_1, \cdots, v_n), \ell, [i]) \in \widetilde{\mathcal{T}}_{g,m,k}$. Let $\mathcal{V}$ be the set of all vertices of Γ. We consider the set of all maps $I : \Gamma - \{v_0\} \to M$ such that the following is satisfied for each edge e. We put $\partial e = \{v_1, v_2\}$ and the circle containing e is i-th and j-th one such that the orientation is as in Figure 3.

(4.1.1) $\partial e \cap \{v_0\} = \emptyset$.

We assume $I|_e$ (together with its parameterization) is a gradient line of $f_j - f_i$. (We identify the edge e with $[0, \ell(e)]$.) (It can happen that $i = j$. But it does not matter.)

(4.1.2) $v_1 = v_0$.

Then e is an i-th exterior edge for $i \in \{1, \cdots, k\}$. We identify $e - v_1$ by $(-\infty, 0]$. We assume $I|_{e-v_1}$ is a gradient line of $f_i - f_{i-1}$. Such that $\lim_{t \to -\infty} I(t) = x_1$. (We remark that x_i is a critical point of $f_i - f_{i-1}$.)

We define

$$\pi^{-1}((\Gamma, (v_1, \cdots, v_n), \ell, [i])) \subset \mathcal{M}(M, g, m, k; f_1, \cdots, f_k, f_1', \cdots, f_m', x_1, \cdots, x_k),$$

to be the set of all maps $I : \Gamma \to M$ satisfying the above condition for each edges.

Taking the union of all $\pi^{-1}((\Gamma, (v_1, \cdots, v_n), \ell, [i]))$ we obtain the moduli space

$$\mathcal{M}(M, g, m, k; f_1, \cdots, f_k, f_1', \cdots, f_m', x_1, \cdots, x_k)$$

and the map

$$\mathcal{M}(M, g, m, k; f_1, \cdots, f_k, f_1', \cdots, f_m', x_1, \cdots, x_k) \to \widetilde{\mathcal{T}}_{g,m,k}.$$

Theorem 4.2. *For generic choice of $f_1, \cdots, f_m'$ the moduli space*

$$\mathcal{M}(M, g, m, k; f_1, \cdots, f_k, f_1', \cdots, f_m', x_1, \cdots, x_k)$$

is a manifold of dimension

$$n - \sum \mu(x_i) + k - 3 + (m + 2g)(3 - n).$$

Proof. The fact that our moduli space is a smooth manifold is a consequence of standard transversality argument. To count its dimension, we first remark that we can forget ribbon structure to consider the dimension of the fibre. (The ribbon structure come to play when we patch the moduli spaces which belong to the graphs of different combinatorial types.) To count the dimension we need only the case when each interior vertex has three edges. In that case to add one edge decreases the dimension of the fibre by n.

We next count the dimension of the Teichmüller parameter. We consider the dimension of the moduli space of ribbon graphs of each combinatorial type. Among them the stratum of the largest dimension is one when each interior vertex has three edges. Since we fix combinatorial type we can forget again the ribbon structure. Then we can find easily from Euler's formula that the number of internal edges is $k-3+3(m+2g)$. The theorem follows.

We can also consider the case when we take the length of exterior edges to be one. (Namely we assume (2.7) in place of (4.1.2).) (We do not need to take the critical point in this case by the same reason as in §2.) By moving functions $f_1, \cdots, f'_m$, we obtain $\mathcal{M}(M,g,m,k)$ and a map $\pi : \mathcal{M}(M,g,m,k) \to (C^\infty(M))^{k+m}$. Also there is a map $\pi_2 : \mathcal{M}(M,g,m,k) \to \widetilde{\mathcal{T}}_{g,m,k}$ and a map $\pi : \mathcal{M}(M,g,m,k) \to (C^\infty(M))^{k+m} \times \widetilde{\mathcal{T}}_{g,m,k}$.

Theorem 4.3. *$\pi : \mathcal{M}(M,g,m,k) \to (C^\infty(M))^{k+m}$ is a Fredholm map of index $n+k-3+(m+2g)(3-n)$.*

We omit the proof. One may be able to find an "invariant" of manifold by counting the number of the moduli space when it is 0-dimensional. However one finds from the formula that the dimension decreases quite rapidly as m or g grows, unless $n=3$. So one might worry that there is only a few invariant obtained in that way. Of course by taking k large there is always something. But the author do not know how much nontrivial it is.

In case dimension is 3 there is infinitely many invariant obtained in this way.

In higher dimensional case, there are two ways to get something of positive dimension. (We can do them at the same time also.) One is to use symmetry (extended mapping class group), as we will discuss soon. The other is to try to construct an invariant of M fibre bundle. Namely we consider the following situation. Let $p : E \to B$ be a smooth oriented fibre bundle with M as a fibre. We consider the mapping $F : E \to \mathbf{R}$ and regard it as a family of smooth function on M parameterized by B . By applying the construction fibrewise we obtain $\pi : \mathcal{M}(E,B,g,m,k) \to (C^\infty(E))^{k+m}$, $\pi_2 : \mathcal{M}(E,B,g,m,k) \to E^k$, $\pi_1 : \mathcal{M}(E,B,g,m,k) \to (C^\infty(E))^{k+m} \times \widetilde{\mathcal{T}}_{g,m,k}$. The map $\pi : \mathcal{M}(E,B,g,m,k) \to (C^\infty(E))^{k+m}$ is Fredholm map of index $n+k-3+(m+2g)(3-n)+\dim N$. In this way we can get extra dimension and may find a nontrivial invariant.

This construction and the critical dimension 3 is similar to Kontsevitch's construction in [Ko2]. Namely in his case, one obtains an invariant in 3 dimensional case, and an invariant of fibre bundle in higher dimension. (However Kontsevitch used in that case graph in place of ribbon graph.) Kontsevitch's construction is a generalization of one obtained for Chern-Simons perturbation theory by Axelrod-Singer [AS], Kontsevitch, and is based on De-Rham theory. There might be a relation of it to this construction (together with mapping class group we will soon discuss). Witten [W4] might be related to it.

We now consider the action of mapping class group. We first take a neighborhood U of p_0 which are disjoint to each other and take a differomorphism $\varphi : U \to D^2$. We put

$$K = \varphi^{-1}\left\{re^{2\pi\ell\sqrt{-1}/k} | \ell \in \mathbf{Z}, r \in \mathbf{R}_+\right\}.$$

For an element (Γ, i) of $\widetilde{\mathcal{T}}_{g,m,k}$ we may assume that $i(\Gamma) \cap U = K$. (Since we fixed the order of the edges around marked point the isotopy class satisfying, this condition is same as one without assuming it.)

$$\mathfrak{M}_{g,k} = \{\Psi : \Sigma_g \to \Sigma_g | \Psi(p_0) = p_0, \Psi(K_i) = K_i, \Psi \text{ is a diffeomorphism.}\}$$

This group acts on our space $\widetilde{\mathcal{T}}_{g,m,k}$. There is a group homomorphism

$$\sigma : \mathfrak{M}_{g,k} \to \mathbf{Z}_k$$

(Namely an element of $\mathfrak{M}_{g,k}$ acts as a cyclic permutation of exterior edges around the marked point.) The kernel of this homomorphism is isomorphic to the usual mapping class group $\mathfrak{M}_g$. (The group of isotopy classes of homeomorphisms of surface of genus g which fix the based point.)

One the other hand, the symmetric group $\mathfrak{S}_m$ of order $m!$ acts on $\widetilde{\mathcal{T}}_{g,m,k}$ by changing the order of interior circles.

We put

$$\begin{aligned}
\mathcal{T}_{g,k,m} &= \widetilde{\mathcal{T}}_{g,m,k}/(\mathfrak{M}_g \times \mathfrak{S}_m),\\
\overline{\mathcal{T}}_{g,k,m} &= \widetilde{\mathcal{T}}_{g,m,k}/(\mathfrak{M}_{g,k} \times \mathfrak{S}_m),
\end{aligned}$$

It seems to the author that the fundamental group of $T_{g,k,m}$ is $\pi_0(Diff(\Sigma_g; p_1, \cdots, p_{n+1}))$, and this space is a $K(\pi, 1)$ space. Now we can divide the moduli space and maps in Theorem 4.2 and Theorem 4.3, by these groups. In the case of Theorem 4.3, we can simply divide everything by the group $\mathfrak{M}_{g,k} \times \mathfrak{S}_m$ and obtain the spaces $\overline{\mathcal{M}}(M, g, m, k) = \mathcal{M}(M, g, m, k)/(\mathfrak{M}_{g,k} \times \mathfrak{S}_m)$, and maps

$$\begin{aligned}
\pi_1 &: \overline{\mathcal{M}}(M, g, m, k) \to ((C^\infty(M))^{k+m} \times M^k)/(\mathbf{Z}_k \times \mathfrak{S}_m)\\
\pi_2 &: \overline{\mathcal{M}}(M, g, m, k) \to \overline{\mathcal{T}}_{g,k,m}.
\end{aligned}$$

In the case of Theorem 4.2, situation is a bit more complicated. First we fix $f_1, \cdots, f_k$, functions corresponding to exterior circles, and $x_1, \cdots, x_k$, the critical points. Then we take the union of moduli spaces

$$\mathcal{M}(M, g, m, k; f_1, \cdots, f_k, f'_1, \cdots, f'_m, x_1, \cdots, x_k)$$

moving $f'_1, \cdots, f'_m$. Let

$$\mathcal{M}(M, g, m, k; f_1, \cdots, f_k, x_1, \cdots, x_k)$$

be the space we obtained. We can divide this space by $\mathfrak{M}_g \times \mathfrak{S}_m$, to obtain

$$\overline{\mathcal{M}}(M, g, m, k; f_1, \cdots, f_k, x_1, \cdots, x_k).$$

There are maps

$$\pi_1 : \overline{\mathcal{M}}(M, g, m, k; f_1, \cdots, f_k, x_1, \cdots, x_k) \to (C^\infty(M))^m/\mathfrak{S}_m,$$
$$\pi_2 : \overline{\mathcal{M}}(M, g, m, k) \to \mathcal{T}_{g,k,m}.$$

One can use these spaces and maps (and the homology classes of $\overline{\mathcal{T}}_{g,k,m}$ and $\mathcal{T}_{g,k,m}$) to find an "invariant" of M, as we did in §1,2. We remark that $K(\mathbf{Z}_k \times \mathfrak{S}_m, 1)$ has nonzero torsion homology class in arbitrary high degree but has finite $\mathbf{Q}$-homology dimension. As we mentioned in introduction, it is obscure for the author what we get in that way, since we do not have algebraic machinery to describe it. We do not have an explicit calculation of these invariants yet so there is still a gloomy possibility that everything is trivial, though it is quite unlikely.

§5 Preliminary ideas for quantization of higher genus Morse homotopy.

Now we try to "quantize" the construction of §4. Let $g \geq 0$,$k \geq 2$,$m > 0$ be integers. We consider a compact Rieman surface Σ of genus g and $m+1$ boundaries. Let $(p_1, \cdots, p_k)$ be points in first component of the boundary. We assume that the cyclic order $(p_1, \cdots, p_k)$ is respected in $S^1 \subset \partial\Sigma$. Let $\widetilde{T}'_{g,m,k}$ be the set of all such objects modulo biholomorphic isomorphisms isotopic to identity. In a way similar to [Mu], [Sr], [Ha], [Pe], we can prove that $\widetilde{T}'_{g,m,k}$ is diffeomorphic to $\widetilde{T}_{g,m,k}$. Where the later is defined in §4.

Let X be a symplectic manifold with trivial first Chern class and $\Lambda_1, \cdots, \Lambda_k$,$\Lambda'_1, \cdots, \Lambda'_m$ be BS-orbits in it. Let $x_i \in \Lambda_i \cap \Lambda_{i-1}$. We define the space

$$\mathcal{M}_{g,m,k}(X; \Lambda_1, \cdots, \Lambda_k, \Lambda'_1, \cdots, \Lambda'_m; x_1, \cdots, x_k).$$

Let $(\Sigma, p_1, \cdots, p_m) \in \widetilde{T}'_{g,m,k}$. We consider the map $h : \Sigma \to X$ such that

(5.1.1) h is pseudo holomorphic.
(5.1.2) $h(p_i) = x_i$.
(5.1.3) $h(\partial_i\Sigma) \subset \Lambda'_i$, for $i \geq 2$.
(5.1.4) $h(\overline{x_i x_{i-1}}) \subset \Lambda_i$.

Here $\partial_i\Sigma$ denote the i-th component of $\partial\Sigma$. Let $\mathcal{M}_{g,m,k}(X; \Lambda_1, \cdots, \Lambda_k, \Lambda'_1, \cdots, \Lambda'_m; x_1, \cdots, x_k)$ be the set of all such objects. We can split it as in §2 using symplectic area a and Maslov index ℓ . Then a similar dimension formula can be proved. Namely

$$\begin{aligned} &dim \mathcal{M}_{g,m,k;a,\ell}(X; \Lambda_1, \cdots, \Lambda_k, \Lambda'_1, \cdots, \Lambda'_m; x_1, \cdots, x_k) \\ &\qquad = n - \sum \mu(x_i) + (k-3) + (m+2g)(3-n) + 2\ell. \end{aligned}$$

(Where $2n = dim_{\mathbf{R}} X$.)

Conjecture 5.2.

$$\begin{aligned} &\lim_{\epsilon \to 0} \mathcal{M}^{\epsilon}_{g,m,k;a=0}(X;\Lambda_{f_1}, \cdots, \Lambda_{f_k}, \Lambda_{f'_k}, \cdots, \Lambda_{f'_m}; x_1, \cdots, x_k) \\ &\qquad = \mathcal{M}(M, g, k, m; f_1, \cdots, f_k, f'_1, \cdots, f'_m; x_1, \cdots, x_k). \end{aligned}$$

Here Λ_{f_i} is the graph of ϵdf_i. And in the left hand side we can omit ℓ by the same reason as §3. The right hand side is the moduli space we introduced in §4.

We remark that the compactification of $\mathcal{M}_{g,m,k}(X; \Lambda_{f_1}, \cdots, \Lambda_{f_k}, \Lambda_{f'_k}, \cdots, \Lambda_{f'_m}; x_1, \cdots, x_k)$ has various troubles.

Finally we remark a bit the quantization similar to §2. Let us consider the ribbon graph

*Figure*9

Naturally to quantize it in a way similar to §2, one needs to consider the configuration like

*Figure*10

Deforming it continuously we get

$$Figure 11$$

and finally

$$Figure 12$$

Thus we need to study pseudo holomorphic map from Rieman surface of higher genus.

In this say we can guess what is the limit

$$\lim_{\epsilon \to 0} \mathcal{M}^{\epsilon}_{g,m,k}(X; \Lambda_{f_1}, \cdots, \Lambda_{f_k}; \Lambda_{f'_1}, \cdots, \Lambda_{f'_m}; x_1, \cdots, x_k).$$

There are a lot of difficulties to prove it.

As we remarked at the end of §3, Rieman surface we can use in this way is restricted. Namely it should have an anti holomorphic involution.

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