Differentiable operad, Kuranishi correspondence, and Foundation of topological field theories

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Summary. In this article the author describes a general framework to establish foundation of various topological field theories. By taking the case of Lagrangian Floer theory as an example, we explain it in a way so that it is applicable to many similar situations including, for example, the case of 'symplectic field theory'. The results of this article is not really new in the sense that its proof was already written in [33], in detail. However several statements are formulated here, for the first time. Especially the relation to the theory of operad is clarified.

1 Introduction

The purpose of this article is to describe a general framework to construct topological field theory by using smooth correspondence (by various moduli spaces typically). We explain our general construction by taking the case of Lagrangian Floer theory [32, 33, 34] as an example. However we explain it in the way so that it is applicable to many similar situations, including, for example, the case of 'symplectic field theory' [18] (that is Gromov-Witten theory of symplectic manifold with cylindrical ends). This article is also useful for the readers who are interested in the general procedure which was established in [33] but are not familiar with the theory of pseudo-holomorphic curves (especially with its analytic detail).

In this article, we extract from [33] various results and techniques and formulate them in such a way so that its generalizations to other similar situations are apparent. We do so by clarifying its relation to the theory of operads. In this way, we may regard the analytic parts of the story as a 'black box', and separate geometric (topological) and algebraic constructions from analytic part of the story. The geometric and algebraic constructions include in particular the transversality and orientation issue, which are the heart of the rigorous construction of topological field theories of various kinds. The analytic part of the story, such as Fredholm theory, gluing, compactness and etc., are to be worked out for each individual cases. In many (but not all) of the cases which are important for applications to topological field theory, the analytic part of the story, by now, is well-established or understood by experts in principle. For example, in the case of pseudo-holomorphic discs with boundary condition given by a Lagrangian submanifold, we carried it out in [33] especially in its §29. (This part of [33] is not discussed in this article, except its conclusion.) The results of this article (and its cousin for other operads or props) clarify the output of the analytic part of the story which is required for the foundation of topological field theory, in a way so that one can state it without looking the proof of analytic part. This seems to be useful for various researchers, since building foundation of topological field theory now is becoming rather massive work to carry out which requires many different kinds of mathematics and is becoming harder to be worked out by a single researcher.

It is possible to formulate the axioms under which the framework of [33] and of this article is applicable. Those axioms are to be formulated in terms of a 'differential topology analogue' of operads (or props) (See for example [1, 49] for a review of operads, props etc. and Definition 2 for its 'differential topology analogue') and correspondence by spaces with Kuranishi structure (see [30]) parametrized by such an operad (or prop). In other words, output of the analytic part is to be formulated as the existence of spaces with Kuranishi structure with appropriate compatibility conditions. To formulate the compatibility conditions in a precise way is the main part of this article. In this article we give a precise formulation in the case of A_∞ operad. The author is planning to discuss it in more general situation elsewhere.

The main theorem of this article is as follows. We define the notion of G-gapped filtered Kuranishi A_∞ correspondence in §10. There we define the notion of morphism between them and also homotopy between morphisms. Thus we have a homotopy category of G-gapped filtered Kuranishi A_∞ correspondences, which we denote by $\mathfrak{HAKCorr}_G$. We also have a homotopy category of G-gapped filtered A_∞ algebras (with $\mathbb{Q}$ coefficient). This notion is defined in [33] Chapter 4. See also §7 and §9 of this article. We denote this category by $\mathfrak{HAlg}_G^{\mathbb{Q}}$.

Theorem 12. *There exists a functor* $\mathfrak{HAKCorr}_G \to \mathfrak{HAlg}_G^{\mathbb{Q}}$.

This theorem is in §10. Roughly speaking, Theorem 12 says that we can associate an A_∞ algebra in a canonical way to Kuranishi correspondence. Thus it reduces the construction of A_∞ algebra to the construction of Kuranishi correspondence.

Actually once the statement is given, we can extract the proof of Theorem 12 from [33]. So the main new point of this article is the statement itself. In other words, it is the idea to formulate the construction using the operad and correspondence by Kuranishi structure.

The contents of each sections are in order. §2 is a review of the general idea of topological field theory and smooth correspondence. We emphasize the important role of chain level intersection theory in it. In §3, we exhibits our construction in the simplest case, that is the Bott-Morse theory on finite dimensional manifold. §4 and §5 are reviews of A_∞ space and A_∞ algebra, respectively. Thus, up to §5, this article is a review and there is nothing new there. We start discussing our main theorem from §6. In §6, we study the case of a correspondence by a manifold which is parametrized by an A_∞ operad. In §7 we study morphism between such correspondences and in §8 we study homotopy between morphisms. We generalize it to its filtered version in §9. Such a generalization is essential to apply it to various topological field theories. Then in §10 we introduce the notion of Kuranishi correspondence and Theorem 12. §11 is again a review and explains how Theorem 12 is used in Lagrangian Floer theory. As we mentioned already the heart of the proof of Theorem 12 is the study of transversality and orientation. They are discussed in detail in [33] §30 and Chapter 9 respectively. The argument there can be directly applied to prove Theorem 12. In §12 we give the transversality part of the proof of Theorem 12 over $\mathbb{R}$ coefficient, in a way different from [33]. See the beginning of §12, where we discuss various known techniques to handle transversality. In §13 we discuss orientation. There we explain the way how to translate the argument on orientation in [33] Chapter 9 to our abstract situation.

A part of this article is a survey. But the main result Theorem 12 is new and its proof is completed in this paper (using the results quoted from [33]).

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2 Smooth correspondence and chain level intersection theory

Let us begin with explaining the notion of smooth correspondence.

A typical example of smooth correspondence is given by the following diagram :

$$M \xleftarrow{\pi_1} \mathfrak{M} \xrightarrow{\pi_2} N \tag{1}$$

of oriented closed manifolds of dimension $m, \mathfrak{r}, n$, respectively. It induces a homomorphism

$$\mathrm{Corr}_{\mathfrak{M}} : H_d(M) \to H_{d+\mathfrak{r}-m}(N) \tag{2}$$

by

$$\mathrm{Corr}_{\mathfrak{M}}([c]) = (\pi_{2*} \circ PD \circ \pi_1^* \circ PD)([c]) \tag{3}$$

where PD is the Poincaré duality. More explicitly we can define this homomorphism by using singular homology as follows. Let

$$c = \sum \sigma_i, \quad \sigma_i : \Delta^d \to M$$

be a singular chain representing the homology class $[c]$. We assume that σ_i are smooth and transversal to π_1. Then we take a simplicial decomposition

$$\Delta^d{}_{\sigma_i} \times_{\pi_1} \mathfrak{M} = \sum_j \Delta_{i,j}^{d+\mathfrak{r}-m} \tag{4}$$

of the fiber product. The map π_2 induces $\sigma_{ij} : \Delta_{i,j}^{d+\mathfrak{r}-m} \to N$. We thus obtain a singular chain on N by

$$\mathrm{Corr}_{\mathfrak{M}}(c) = \sum_{i,j} (\Delta_{i,j}^{d+\mathfrak{r}-m}, \sigma_{ij}). \tag{5}$$

An immediate generalization of it is

$$\underbrace{M \times \cdots \times M}_{k} \xleftarrow{\pi_1} \mathfrak{M} \xrightarrow{\pi_2} N \tag{6}$$

which defines a multi-linear map

$$\mathrm{Corr}_{\mathfrak{M}} : (H(M)^{\otimes k})_d \to H_{d+\mathfrak{r}-m}(N), \tag{7}$$

or, in other words, a family of operations on homology group.

An example is given by the diagram

$$M \times M \xleftarrow{\pi_1} M \xrightarrow{\pi_2} M \tag{8}$$

where

$$\pi_1(p) = (p,p), \qquad \pi_2(p) = p.$$

The homomorphism (7) in this case is nothing but the intersection pairing. We can apply a similar idea to the case when $\mathfrak{M}$ is a moduli space of various kinds.

Correspondence is extensively used in algebraic geometry. (In a situation closely related to topological field theory, correspondence was used by H. Nakajima [53] to construct various algebraic structures. His concept of 'generating space' ([54]) is somewhat similar to the notion of Kuranishi correspondence.) In complex algebraic geometry, it is, in principle, possible to include the case when M, $\mathfrak{M}$, N are singular spaces, since the singularity occurs in real codimension two.

However in case of real C^∞ manifold, if we include the manifold $\mathfrak{M}$ which is not necessary closed (that is a manifold which may have boundary and/or corner), we will be in a trouble. This is because Poincaré duality does not hold in the way appearing in Formula (3). We can still define operations *in the chain level* by the formula (5). However the operation, then, does not induce a map between homology groups, directly.

This problem can also be reformulated as follows. Let $f_i : P_i \to M$ and $f : Q \to N$ be maps from smooth oriented manifolds (without boundary), which represent cycles on M or N, respectively. Under appropriate transversality conditions, we count (with sign) the order of the set

$$\{(x, p_1, \cdots, p_k, q) \in \mathfrak{M} \times M^k \times N \mid \pi_1(x) = f_i(p_i), \pi_2(x) = f(q)\}, \tag{9}$$

in case when its (virtual) dimension is zero. The order (counted with sign) of (9) is an 'invariant' of various kinds in case $\mathfrak{M}$ is a moduli space. For example Donaldson invariant of a 4 manifold and Gromov-Witten invariant of a symplectic manifold both can be regarded as invariants of this kind[1]. When the boundary of $\mathfrak{M}$ is not empty, the order (counted with sign) of (9) is not an invariant of the homology classes but depends on the chains P_i, Q which represent the homology classes. So we need to perform our construction in the chain level. The first example where one needs such a chain level construction, is the theory of Floer homology. In that case the 'invariant' obtained by counting (with sign) of something similar to (9) depends on various choices involved. What is invariant in Floer's case is the homology group of the chain complex, of which the matrix coefficient of the boundary operator is obtained by such counting.

In various important cases, the boundary of the moduli space $\mathfrak{M}$, is described as a *fiber* product of other moduli spaces. To be slightly more precise, we consider the following situation. (See §9 and also [33] §30.2 for more detailed exposition.)

Let $(\mathcal{O}, \le)$ be a partially ordered set. We suppose that, for each $\alpha \in \mathcal{O}$, we have a space $\mathfrak{M}_\alpha$ together with a diagram

$$\underbrace{M \times \cdots \times M}_{k_\alpha} \xleftarrow{\pi_1} \mathfrak{M}_\alpha \xrightarrow{\pi_2} M \tag{10}$$

Here $\mathfrak{M}_\alpha$ is a manifold with boundary. (In more general case, $\mathfrak{M}_\alpha$ is a space with Kuranishi structure with boundary (See [30] and §10).) Moreover, the boundary of $\mathfrak{M}_\alpha$ is described as a *fiber product* of various $\mathfrak{M}_\beta$ with $\beta \in \mathcal{O}$, $\beta < \alpha$.

Such a situation occurs in case when moduli space has a Uhlenbeck type bubbles and noncompactness of the moduli space occurs only by this bubbling phenomenon. In those cases, the partially ordered set A encodes the energy of elements of our moduli space together with a data describing complexity of the combinatorial structure of the singular objects appearing in the compactification.

[1] We need to include the case when M or N are of infinite dimension in the case of Donaldson invariant. Namely they are the spaces of connections in that case.

In such situation, we are going to define some algebraic system on a set of chains on M so that the numbers defined by counting the order of a set like (9) are its structure constants. The structure constant itself is *not* well-defined. Namely it depends on various choices.

One of such choices is the choice of perturbation (or multisection of our Kuranishi structure) required to achieve relevant transversality condition.

Another choice is an extra geometric structure on our manifolds which we need to determine a partial differential equation defining the moduli space $\mathfrak{M}_\alpha$. In the case of self-dual Yang-Mills gauge theory, it is a conformal structure of 4-manifolds. In the case of Gromov-Witten theory, it is an almost complex structure which is compatible with a given symplectic structure.

This extra structure we put later on, plays a very different role from the structure we start with. In Gromov-Witten theory, for example, we start with a symplectic structure and add an almost complex structure later on. The invariant we finally obtain is independent of the almost complex structure but may depend on symplectic structure. We remark that symplectic structure is of 'topological' nature. Namely its moduli space is of finite dimension by Moser's theorem. On the other hand, the moduli space of almost complex structures is of infinite dimension. The word 'topological' in 'topological field theory' in the title of this article, means that it depends only on 'topological' structure such as symplectic structure but is independent of the 'geometric' structure such as an almost complex structure. In this sense the word 'topological field theory' in the title of this article is slightly different from those axiomatized by Atiyah [2]. Our terminology coincides with Witten [73].

In order to establish the independence of our 'topological field theory' of perturbation and of 'geometric' structures, it is important to define an appropriate notion of *homotopy equivalence* among algebraic systems which appear. We then prove that the algebraic system we obtain by smooth correspondence is independent of the choices *up to homotopy equivalence*. We remark that establishing appropriate notion of equivalence is the most important part of the application of homological algebra to our story, since the main role of homological algebra is to overcome the difficulty of dependence of the order of the set (9) on various choices.

This story we outlined above is initiated by Donaldson and Floer in 1980's, first in gauge theory. Based on it, Witten [73] introduced the notion of topological field theory. Around the same time, Gromov and Floer studied the case of pseudo-holomorphic curve in symplectic geometry. In those days, the relevant algebraic system are chain complex mainly. In early 1990's, more advanced homological algebra is introduced and several researchers started to use it more systematically. It appears in Mathematics (for example [66, 20, 21, 43, 40]) and in Physics (for example [72, 74]) independently.

For the development of the mathematical side of the story, one of the main obstacle to build topological filed theory, in the level where advanced homological algebra is included, was the fact that we did not have enough general framework for the transversality issue, at that time. The virtual fundamental cycle technique which was introduced by [30, 47, 59, 62] at the year 1996, resolved this problem in sufficiently general situation. Then this obstacle was removed, in the case when the group of automorphisms of objects involved is of finite order. Actually this was the main motivation for the author and K. Ono to introduce this technique and to write [30] Chapter 1 in a way so that it can be *directly* applied to other situations than those we need in [30]. Applying virtual fundamental chain/cycle technique in the chain level, sometimes requires more careful discussion, which was completed soon after and was written in detail in [33].

As we already mentioned, we need homological (or homotopical) algebra of various kinds, to develop topological field theory in our sense. The relevant algebraic structure sometimes had been known before. Especially, the notion of A_∞ algebra and L_∞ algebra were already known much earlier in algebraic topology. The importance of such structures in topological field theory was realized more and more by various researchers during 1990's. At the same time, homological algebra to handle those structures itself has been developed. Since the main motivation to use homological algebra in topological field theory is slightly different from those in homotopy theory, one needs to clean it up in a slightly different way. We need also to introduce several new algebraic structures in order to study various problems in topological field theory. Study of such homological algebra is still on the way of rapid progress by various researchers.

The general strategy we mentioned above (together with the basic general technology to realize it) was well established, as a principle, was known to experts around the end of 1990's, and was written in several articles. (See for example [21, 24, 57].) The main focus of the development then turned to rigorously establishing it in various important cases. Another main topic of the subject is a calculation and application of the structure obtained. Around the same time, the number of researchers working on topological field theory (in the sense we use in this article) increased much. Working out the above mentioned strategy in a considerable level of generality, is a heavy work. So its completion took lots of time after the establishment of the general strategy. In [33] we completed the case of Lagrangian Floer theory. Several other projects are now in progress by various authors in various situations.

In this article the author comes back to general frame work, and axiomatize it in a package, so that one can safely use it without repeating the proof.

3 Bott-Morse theory : a baby example

In this section, we discuss Bott-Morse theory as a simplest example of our story. Let X be a compact smooth manifold of finite dimension and $f : X \to \mathbb{R}$ be a smooth function on X. We put

$$\mathrm{Crit}(f) = \{x \in X \mid df(x) = 0\}. \tag{11}$$

Definition 1. *f is said to be a Bott-Morse function if each connected component R_i of $\mathrm{Crit}(f)$ is a smooth submanifold and the restriction of Hessian $\mathrm{Hess}_x f$ to the normal bundle $N_{R_i}X$ is non-degenerate.*

We define the Morse index of f at R_i to be the sum of the multiplicities of the negative eigenvalues of $\mathrm{Hess}_x f$ on $N_{R_i}X$. (Here $x \in R_i$.) We denote it by $\mu(R_i)$

An example of Bott-Morse function is as in the Figure 1 below. In this example

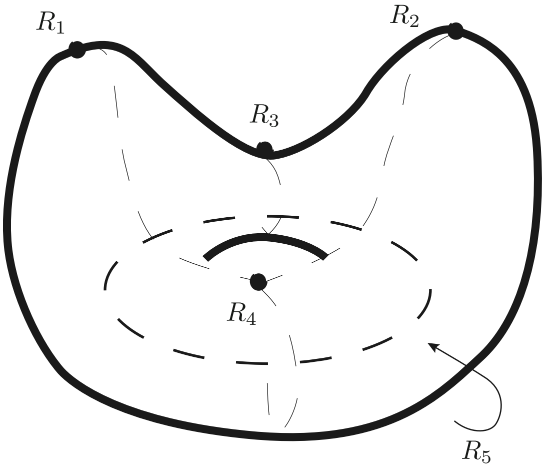

Fig. 1.

the critical point set is a union of 4 points R_1, R_2, R_3, R_4 and a circle R_5. The Morse indices of them are 2,2,1,1,0 respectively.

The main result of Bott-Morse theory is the following result due to Bott. We enumerate critical submanifolds R_i such that $f(R_i) \ge f(R_j)$ for $i < j$.

Let $N^-_{R_i}X$ be the subbundle of the normal bundle generated by the negative eigenspaces of $\mathrm{Hess}_x f$. Let $\Theta^-_{R_i}$ be the local system associated with the determinant real line bundle of $N^-_{R_i}X$. (It corresponds to a homomorphism $\pi_1(R_i) \to \{\pm 1\} = \mathrm{Aut}\mathbb{Z}$.)

Theorem 1 (Bott [8]). *There exists a spectral sequence E^*_{**} such that*

$$E^2_{i,j} = \bigoplus_i H_{j-\mu(R_i)}(R_i; \Theta^-_{R_i}) \tag{12}$$

and such that it converges to $H(X;\mathbb{Z})$.

Classical proof is based on the stratification of the space X to the union of stable manifolds of critical submanifolds. This approach is not suitable for its generalization to some of its infinite dimensional version, especially to the situation where Morse index is infinite. (This is the situation of Floer homology.) We need to use Floer's approach [19] to Morse theory in such cases. We explain Bott-Morse version of Floer's approach here following [24]. (See [4] for related results. The restriction on Bott-Morse function which was assumed in [4] by now can be removed as we explained in [33] §30.2 Remark 30.20.)

We take a Riemannian metric g on X. We then have a gradient vector field grad$f$ of f. Let $\tilde{\mathcal{M}}(R_i, R_j)$ be the set of all maps $\ell : \mathbb{R} \to X$ such that

$$\frac{d\ell}{dt}(t) = -\mathrm{grad}_{\ell(t)} f \tag{13}$$

$$\lim_{t\to-\infty} \ell(t) \in R_i, \quad \lim_{t\to+\infty} \ell(t) \in R_j. \tag{14}$$

The group $\mathbb{R}$ acts on $\tilde{\mathcal{M}}(R_i, R_j)$ by $(s \cdot \ell) = \ell(t+s)$. Let $\mathcal{M}(R_i, R_j)$ be the quotient space. We define the maps π_i by

$$\pi_1(\ell) = \lim_{t\to-\infty} \ell(t), \qquad \pi_2(\ell) = \lim_{t\to+\infty} \ell(t). \tag{15}$$

They define a diagram

$$R_i \xleftarrow{\ \pi_1\ } \mathcal{M}(R_i, R_j) \xrightarrow{\ \pi_2\ } R_j. \tag{16}$$

Now we have

Lemma 1. *By perturbing f on a set away from* Crit$f$*, we may choose f so that $\mathcal{M}(R_i, R_j)$ is a smooth manifold with boundary and corners. Moreover we have*

$$\partial \mathcal{M}(R_i, R_j) = \bigcup_{i<k<j} \mathcal{M}(R_i, R_k) \times_{R_k} \mathcal{M}(R_k, R_j). \tag{17}$$

Let us exhibit the lemma in case of the example of the Morse function in Figure 1. In this case we have the following :

$$\begin{aligned} \mathcal{M}(R_1, R_3) = \mathcal{M}(R_2, R_3) = \mathcal{M}(R_1, R_4) = \mathcal{M}(R_2, R_4) &= \text{one point}. \\ \mathcal{M}(R_3, R_5) = \mathcal{M}(R_4, R_5) &= \text{two points}. \\ \mathcal{M}(R_1, R_5) = \mathcal{M}(R_2, R_5) &= \text{union of two arcs}. \end{aligned}$$

We then have

$$\begin{aligned} \partial(\mathcal{M}(R_1, R_5)) &= (\mathcal{M}(R_1, R_3) \times \mathcal{M}(R_3, R_5)) \cup (\mathcal{M}(R_1, R_4) \times \mathcal{M}(R_4, R_5)) \\ &= 4 \text{ points}. \end{aligned}$$

Now we have the following :

Lemma 2. *There exists a subcomplex $C(R_i;\Theta^-_{R_i}) \subset S(R_i;\Theta^-_{R_i})$ of the singular chain complex $S(R_i;\Theta^-_{R_i})$ of R_i with $\Theta^-_{R_i}$ coefficient, such that the inclusion induces an isomorphism of homologies and that the following holds.*

If $c \in C(R_i;\Theta^-_{R_i})$ then $\mathrm{Corr}_{\mathcal{M}(R_i,R_j)}(c)$ is well-defined by (5) and is in $C(R_j;\Theta^-_{R_j})$. Moreover we have

$$[\partial, \mathrm{Corr}_{\mathcal{M}}] + \mathrm{Corr}_{\mathcal{M}} \circ \mathrm{Corr}_{\mathcal{M}} = 0. \tag{18}$$

Here in (18) we write $\mathrm{Corr}_{\mathcal{M}}$ in place of $\mathrm{Corr}_{\mathcal{M}(R_i,R_j)}$ for various i, j.

Lemmas 1 and 2 are in [22]. (See also [33] §30.2.)

We remark that (18) is a version of Maurer-Cartan equation. We define

$$\partial_{\mathcal{M}} = \partial + \mathrm{Corr}_{\mathcal{M}} : \bigoplus_i C(R_i;\Theta^-_{R_i}) \to \bigoplus_i C(R_i;\Theta^-_{R_i}). \tag{19}$$

(18) implies

$$\partial_{\mathcal{M}} \circ \partial_{\mathcal{M}} = 0. \tag{20}$$

Lemma 3. $\mathrm{Ker}\partial_{\mathcal{M}}/\mathrm{Im}\partial_{\mathcal{M}}$ *is isomorphic to the homology group of M.*

We can prove Lemma 3 as follows. First we prove that $\mathrm{Ker}\partial_{\mathcal{M}}/\mathrm{Im}\partial_{\mathcal{M}}$ is independent of the choice of the Bott-Morse function f. (An infinite dimensional version of this fact (whose proof is similar to and is harder than Lemma 3) is proved in [22]). Moreover in the case when $f \equiv 0$ the lemma is obvious.

By construction

$$F_j = \bigoplus_{i \ge j} C(R_i;\Theta^-_{R_i})$$

is a filtration of $(\bigoplus_i C(R_i;\Theta^-_{R_i}), \partial_{\mathcal{M}})$. The spectral sequence associated to this filtration is one required in Theorem 1.

We remark that in case $j > i+1$ the correspondence $\mathrm{Corr}_{\mathcal{M}(R_i,R_j)}$ may not define a homomorphism $H(R_i;\Theta^-_{R_i}) \to H(R_j;\Theta^-_{R_j})$ in the homology level, because $\partial\mathcal{M}(R_i,R_j) \neq \emptyset$ in general. This point is taken care of by the spectral sequence. Namely the third differential d_3 of the spectral sequence is defined only partially and has ambiguity, which is controlled by the second differential d_2. This is a prototype of the phenomenon which appears in the situation where we need more advanced homological algebra. (For example massey product has a similar property.) In the situation of Theorem 1, we study chain complex without any additional structure on it. In the situation we will discuss later, we consider chain complex together with various multiplicative structures. We remark including multiplicative structure (the cup and massey products) in Theorem 1 is rather a delicate issue, if we prove it in a way described above. See [33] §30.2 on this point. Actually there are many errors and confusions in various references on this point. If we prove it by stratifying the space X, it is easy to prove that the spectral sequence in Theorem 1 is multiplicative.

4 A_∞ space.

In the rest of this article we describe the construction outlined in §1 in more detail. To be specific, we concentrate to the case of A_∞ structure. We first recall the notion of A_∞ structure introduced by Stasheff [65]. (See also Sugawara [68].) (There are excellent books [1, 7, 49] etc., on related topics.)

The original motivation for Stasheff to introduce A_∞ space is to study loop space. Let us recall it briefly. Let (X, p) be a topological space with base point. We put

$$\Omega(X,p) = \{\ell : [0,1] \to X \mid \ell(0) = \ell(1) = p\}. \tag{21}$$

We define $\mathsf{m}_2 : \Omega(X,p) \times \Omega(X,p) \to \Omega(X,p)$ by

$$\mathsf{m}_2(\ell_1,\ell_2)(t) = \begin{cases} \ell_1(2t) & \text{if } t \le 1/2 \\ \ell_2(2t-1) & \text{if } t \ge 1/2 \end{cases} \tag{22}$$

$\mathfrak{m}_2$ is associative only modulo parametrization. In fact, for $t \le 1/4$, we have

$$(\mathsf{m}_2(\ell_1,\mathsf{m}_2(\ell_2,\ell_3)))(t) = \ell_1(2t) \ne$$
$$(\mathsf{m}_2(\mathsf{m}_2(\ell_1,\ell_2),\ell_3))(t) = \ell_1(4t).$$

On the other hand, there is a canonical homotopy between $\mathsf{m}_2(\ell_1,\mathsf{m}_2(\ell_2,\ell_3))$ and $\mathsf{m}_2(\mathsf{m}_2(\ell_1,\ell_2),\ell_3)$. Actually some stronger statements than the existence of homotopy hold. The A_∞ structure is the way to state it precisely.

To define A_∞ structure we need a series of contractible spaces $\mathcal{M}_{k+1}$ together with continuous maps

$$\circ_i : \mathcal{M}_{k+1} \times \mathcal{M}_{l+1} \to \mathcal{M}_{k+l}, \tag{23}$$

for $i = 1, \cdots, l$, such that the following holds for $a \in \mathcal{M}_{k+1}$, $b \in \mathcal{M}_{l+1}$, $c \in \mathcal{M}_{m+1}$.

$$(a \circ_j b) \circ_i c = a \circ_j (b \circ_{i-j+1} c), \tag{24}$$

for $i \ge j$ (see Figure 2) and

$$(a \circ_j b) \circ_i c = (a \circ_i c) \circ_{j+k_2-1} b \tag{25}$$

for $i < j$, $a \in \mathcal{M}_{k_1+1}$, $b \in \mathcal{M}_{k_2+1}$, $c \in \mathcal{M}_{k_3+1}$ (see Figure 3).

A topological space X is an A_∞ space if there is a sequence of continuous maps

$$\mathcal{M}_{k+1} \times X^k \to X; (a; x_1, \cdots, x_k) \mapsto \mathsf{m}(a; x_1, \cdots, x_k) \tag{26}$$

such that

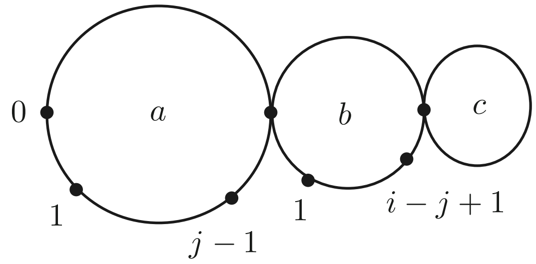

Fig. 2.

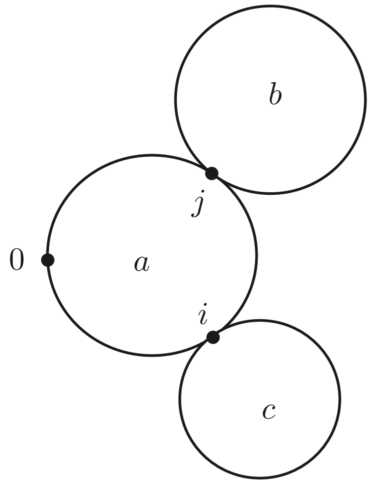

Fig. 3.

$$\begin{aligned}&\mathsf{m}(a \circ_i b; x_1, \cdots, x_{k+\ell-1})\\&= \mathsf{m}(a; x_1, \cdots, x_{i-1}, \mathsf{m}(b; x_i, \cdots, x_{i+\ell-1}), \cdots, x_{k+\ell-1}).\end{aligned} \tag{27}$$

We remark that (24), (25) are compatible with (27).

Suppose that X is an A_∞ space. Then, by taking an (arbitrary but fixed) element $a_0 \in \mathcal{M}_{2+1}$ we define

$$\mathsf{m}_2 = \mathsf{m}(a_0; \cdot, \cdot) : M^2 \to M.$$

Since $\mathcal{M}_{3+1}$ is contractible there exists a path joing $a_0 \circ_1 a_0$ to $a_0 \circ_2 a_0$. Using it we have a homotopy between

$$\mathsf{m}_2(\mathsf{m}_2(x, y), z) = \mathsf{m}(a_0; \mathsf{m}_2(a_0; x, y), z) = \mathsf{m}(a_0 \circ_1 a_0; x, y, z)$$

and

$$\mathsf{m}_2(x, \mathsf{m}_2(y, z)) = \mathsf{m}(a_0; x, \mathsf{m}_2(a_0; y, z)) = \mathsf{m}(a_0 \circ_2 a_0; x, y, z).$$

Namely $\mathfrak{m}_2$ is homotopy associative. The condition (27) is more involved than homotopy associativity.

An important result by Stasheff is that an H-space is an A_∞ space if and only if it is homotopy equivalent to a loop space $\Omega(X,p)$ for some (X,p). More precisely loop space corresponds to an A_∞ space with unit for this theorem. We do not discuss unit here. (See §14.1.)

In the rest of this section, we give two examples of $\mathcal{M}_{k+1}$ which satisfy (24). We call such system of $\mathcal{M}_{k+1}$ an A_∞ *operad.*

Remark 1. We remark that the notion of operad which was introduced by May [50] is similar to but is slightly different from above. It is a family of spaces $\mathcal{P}(k)$ together with operations

$$\mathcal{P}(l) \times (\mathcal{P}(k_1) \times \cdots \times \mathcal{P}(k_l)) \to \mathcal{P}(k_1 + \cdots + k_l). \tag{28}$$

Its axiom contains an associativity of the operation (28) and also symmetry for exchanging $\mathcal{P}_{k_i}$'s. In our case, the structure map (23) is slightly different from (28) and is closer to something called non Σ-operad. One important difference is that we do not require any kinds of commutativity to our operations $\mathsf{m}(a; x_1, \cdots, x_k)$.

There are several other variants of operad or prop. (The difference between operad and prop is as follows. An operad has several inputs but has only one output. A prop has several inputs and several outputs.) See [49] for those variants and history of its development. We can discuss correspondence parametrized by them in a way similar to the case of A_∞ operad which we are discussing in this paper.

The first example of A_∞ operad is classical and due to Boardman-Vogt [7] Definition 1.19. (See [33] §9.) Let us consider the planer tree $|T|$ (that is a tree embedded in $\mathbb{R}^2$). We divide the set $C^0(|T|)$ of vertices of $|T|$ into a disjoint union

$$C^0(|T|) = C^0_{\rm int}(|T|) \cup C^0_{\rm ext}(|T|)$$

where every vertex in $C^0_{\rm int}(|T|)$ has at least 3 edges and all the vertices in $C^0_{\rm ext}(|T|)$ have exactly one edge. We assume that there is no vertex with two edges. Elements of $C^0_{\rm int}(|T|)$, $C^0_{\rm ext}(|T|)$ are said to be interior edges and exterior edges, respectively. $C^1(|T|)$ denotes the set of all edges. We say an edge to be exterior if it contains an exterior vertex. Otherwise the edge is said to be interior.

We consider such $|T|$ together with a function $l : C^1(|T|) \to (0,\infty]$ which assigns the length $l(e)$ to each edge e. We assume $l(e) = \infty$ if e is an exterior edge.

We consider $(|T|, l)$ as above such that the tree has exactly $k+1$ exterior vertices. We fix one exterior vertex v_0 and consider the set of all the isomorphism classes of such $(|T|, l, v_0)$. We denote it by Gr_{k+1} and call its element *a rooted metric ribbon tree with* $k+1$ *exterior vertices.* We enumerate the exterior vertices as $\{v_0, v_1, \cdots, v_k\}$ so that it respects the counter-clockwise orientation of $\mathbb{R}^2$.

We can prove (see [31] for example) that Gr_{k+1} is homeomorphic to D^{k-2} and hence is contractible.

We define

$$\circ_i : Gr_{k+1} \times Gr_{l+1} \to Gr_{k+l}$$

as follows. Let $T = (|T|, l, v_0) \in Gr_{k+1}$, $T' = (|T'|, l', v'_0) \in Gr_{l+1}$. Let $v'_0, \cdots, v'_l$ be the exterior vertices of Gr_{l+1} enumerated according to the counter-clockwise orientation. We identify $v_i \in |T|$ and $v'_0 \in |T'|$ to obtain $|T| \circ_i |T'|$. The length of the edges of it is the same as one for $|T|$ or $|T'|$ except the new edge, whose length is defined to be infinity. We thus obtain an element $T \circ_i T'$ of Gr_{k+l}. (24), (25) can be checked easily. Thus, by putting

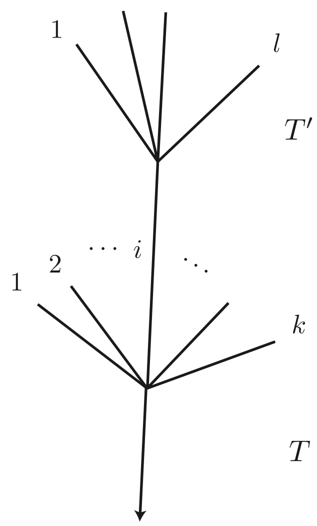

Fig. 4.

$\mathcal{M}_{k+1} = Gr_{k+1}$, we obtain an example of A_∞-operad.

We next discuss another example of A_∞-operad, which is closely related to Lagrangian Floer theory.

We consider $(D^2; z_0, \cdots, z_k)$ where D^2 is the unit disc in $\mathbb{C}$ centered at origin. z_i, $i = 0, \cdots, k$ are pair-wise distinct points of ∂D^2. We assume that $z_0, \cdots, z_k$ respects counter-clockwise cyclic order of ∂D^2. Let $PSL(2; \mathbb{R})$ be the group of all biholomorphic maps $D^2 \to D^2$. For $u \in PSL(2; \mathbb{R})$ we put

$$u \cdot (D^2; z_0, \cdots, z_k) = (D^2; u(z_0), \cdots, u(z_k)) \tag{29}$$

We denote by $\overset{\circ}{\mathcal{M}}_{k+1}$ the set of all the equivalence classes of such $(D^2; z_0, \cdots, z_k)$ with respect to the relation (29).

It is easy to see that $\overset{\circ}{\mathcal{M}}_{k+1}$ is diffeomorphic to $\mathbb{R}^{k-2}$. We can compactify $\overset{\circ}{\mathcal{M}}_{k+1}$ to obtain $\mathcal{M}_{k+1}$. An idea to do so is to take double and use Deligne-Mumford compactification of the moduli space of Riemann surface. (See [32] §3, where its generalization to higher genus is also discussed.)

An element of $\mathcal{M}_{k+1}$ is regarded as $(\Sigma; z_0, \cdots, z_k)$ which satisfies the following conditions. (See Figure 5.) We consider a Hausdorff topological space

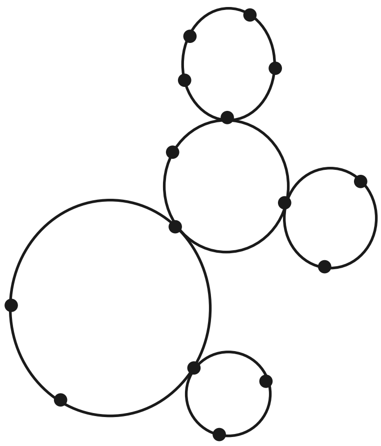

Fig. 5.

Σ which is a union of discs $D^2_1, \cdots, D^2_k$. We call D^2_i a *components* of Σ. We assume that, for each $i \neq j$, the intersection $D^2_i \cap D^2_j$ is either empty or consists of one point which lies on the boundaries of D^2_i and of D^2_j. We assume moreover that intersection of three components are empty. Furthermore Σ is assumed to be connected and simply connected.

The set of all points on Σ which belongs to more than 2 components are called *singular*.

We put

$$\partial\Sigma = \bigcup \partial D^2_i$$

We assume that $z_i \in \partial\Sigma$. We also assume that z_i is not singular. We embed Σ to $\mathbb{C}$ so that it is biholomorphic on each D^2_i. We require (the image of) $z_0, \cdots, z_k$ respects the counter-clockwise cyclic orientation induced by the orientation of $\mathbb{C}$.

Finally we assume the following stability condition. We say

$$\varphi : \Sigma \to \Sigma'$$

is biholomorphic if it is a homeomorphism and if its restriction to D^2_i induces a biholomorphic maps $D^2_i \to D^2_j$ for some j. Here D^2_i and D^2_j are components of Σ and Σ', respectively. Let $\mathrm{Aut}(\Sigma; z_0, \cdots, z_k)$ be the group of all biholomorphic maps $\varphi : \Sigma \to \Sigma$ such that $\varphi(z_i) = z_i$. We say that $(\Sigma; z_0, \cdots, z_k)$ is *stable* if $\mathrm{Aut}(\Sigma; z_0, \cdots, z_k)$ is of finite order.

Two such $(\Sigma; z_0, \cdots, z_k)$, $(\Sigma'; z'_0, \cdots, z'_k)$ are said to be *biholomorphic* to each other if there exists a biholomorphic map $\varphi : \Sigma \to \Sigma'$ such that $\varphi(z_i) = z'_i$.

$\mathcal{M}_{k+1}$ is the set of all the biholomorphic equivalence classes of $(\Sigma; z_0, \cdots, z_k)$ which is stable.

We can show that $(\Sigma; z_0, \cdots, z_k)$ is stable if and only if each components contain at least 3 marked or singular points. We remark that in our case of genus zero, $\mathrm{Aut}(\Sigma; z_0, \cdots, z_k)$ is trivial if $(\Sigma; z_0, \cdots, z_k)$ is stable.

We define

$$\circ_i : \mathcal{M}_{k+1} \times \mathcal{M}_{l+1} \to \mathcal{M}_{k+l}$$

as follows. Let $(\Sigma; z_0, \cdots, z_k) \in \mathcal{M}_{k+1}$ and $(\Sigma'; z'_0, \cdots, z'_l) \in \mathcal{M}_{l+1}$. We identify $z_i \in \Sigma$ and $z'_0 \in \Sigma'$ to obtain Σ''. We put

$$(z''_0, \cdots, z''_{k+l-1}) = (z_0, \cdots, z_{i-1}, z'_1, \cdots, z'_l, z_{i+1}, \cdots, z_k).$$

We now define

$$(\Sigma; z_0, \cdots, z_k) \circ_i (\Sigma'; z'_0, \cdots, z'_l) = (\Sigma''; z''_0, \cdots, z''_{k+l-1})$$

which represents an element of $\mathcal{M}_{k+\ell}$. We can easily check (24), (25).

Actually the two A_∞-operads we described above are isomorphic to each other. In fact the following theorem is proved in [31]. (This is a result along the line of theory of the quadratic differential by Strebel [67] etc.)

Theorem 2. *There exists a homeomorphism $Gr_{k+1} \cong \mathcal{M}_{k+1}$ which is compatible with $\circ_i$.*

We remark that $\mathcal{M}_{k+1} \cong D^{k-2}$. Moreover we have

$$\partial \mathcal{M}_{k+1} = \sum_{1 \le i < j \le k} \mathcal{M}_{j-i+1} \circ_i \mathcal{M}_{k-j+i} \tag{30}$$

where the images of the right hand sides intersect each other only at their boundaries. We remark that (30) can be regarded as a Maurer-Cartan equation.

Using (30) inductively we obtain a cell decomposition of the cell D^{k-2}. A famous example is the case $k = 5$. In that case we obtain a cell decomposition of D^3 which is called Stasheff cell.

It is classical that the existence of A_∞ structure on X is independent of the choice of the A_∞ operad $\mathcal{M}_{k+1}$. So in this article we always use the A_∞ operad we constructed above.

For the purpose of this paper, the structure of differentiable manifold on $\mathcal{M}_{k+1}$ is important. So we make the following definition.

Definition 2. *A differentiable A_∞ operad is an A_∞ operad $\mathcal{M}_{k+1}$ such that $\mathcal{M}_{k+1}$ is a compact and oriented smooth manifold (with boundary or corner) and the structure map* (23) *is an smooth embedding. Moreover we assume* (30).

It is straightforward to extend this definition to the case of variants of operad or prop.

5 A_∞ algebra.

A_∞ algebra is an algebraic analogue of A_∞ space and is a generalization of differential graded algebra. It is defined as follows.

Hereafter R is a commutative ring with unit. Let C be a graded R module. We assume it is free as R module. We define its suspension $C[1]$ (the degree shift) by $C[1]^k = C^{k+1}$. Hereafter we denote by $\deg'$ the degree after shifted and by deg the degree before shifted.

We define the *Bar complex* $BC[1]$ by

$$B_kC[1] = \underbrace{C[1] \otimes \cdots \otimes C[1]}_{k \text{ times}}, \qquad BC[1] = \bigoplus_{k=0}^{\infty} B_kC[1].$$

We define a coalgebra structure $\varDelta$ on $BC[1]$ by

$$\varDelta(x_1 \otimes \cdots \otimes x_k) = \sum_{i=0}^{k} (x_1 \otimes \cdots \otimes x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_k)$$

here in case $i = 0$, for example, the right hand side is $1 \otimes (x_1 \otimes \cdots \otimes x_k)$.

We consider a sequence of operations

$$\mathfrak{m}_k : B_kC[1] \to C[1] \tag{31}$$

of degree 1 for each $k \geq 1$. It is extended uniquely to a coderivation

$$\hat{\mathfrak{m}}_k : BC[1] \to BC[1]$$

whose $Hom(B_kC[1], B_1C[1]) = Hom(B_kC[1], C[1])$ component is $\mathfrak{m}_k$.

We recall $\varphi : BC[1] \to BC[1]$ is said to be a *coderivation* if and only if

$$\varDelta \circ \varphi = (\varphi \otimes 1 + 1 \otimes \varphi) \circ \varDelta. \tag{32}$$

(Note that the right hand side is defined by $(1 \otimes \varphi)(x \otimes y) = (-1)^{\deg' x \deg \varphi} x \otimes \varphi(y)$.)

We put

$$\hat{d} = \sum_k \hat{\mathfrak{m}}_k : BC[1] \to BC[1].$$

Definition 3. $(C, \mathfrak{m}_*)$ *is an* A_∞ *algebra if* $\hat{d} \circ \hat{d} = 0$.

We can rewrite the condition $\hat{d} \circ \hat{d} = 0$ to the following relation, which is called the A_∞ relation.

$$\sum_{1 \le i < j \le k} (-1)^* \mathfrak{m}_{k-j+i}(x_1, \cdots, \mathfrak{m}_{j-i+1}(x_i, \cdots, x_j), \cdots, x_k) = 0 \tag{33}$$

where

$$* = \deg' x_1 + \cdots \deg' x_{i-1} = \deg x_1 + \cdots + \deg x_{i-1} + i - 1.$$

Our sign convention is slightly different from Stasheff's [65].

We remark that (33) implies $\mathfrak{m}_1 \circ \mathfrak{m}_1 = 0$. Namely $(C, \mathfrak{m}_1)$ is a chain complex.

Example 1. If $(C, d, \wedge)$ is a differential graded algebra, we may regard it as an A_∞ algebra by putting

$$\mathfrak{m}_1(x) = (-1)^{\deg x} dx, \qquad \mathfrak{m}_2(x, y) = (-1)^{\deg x (\deg y + 1)} x \wedge y. \tag{34}$$

An alternative choice of sign $\mathfrak{m}_1(x) = dx$, $\mathfrak{m}_2(x, y) = (-1)^{\deg x} x \wedge y$, also works. Here we follow the convention of [33].

The following result is classical and is certainly known to Stasheff.

Theorem 3. *A structure of* A_∞ *space on* X *induces a structure of* A_∞ *algebra on its singular chain complex.*

Sketch of the proof : By using (30), we may take a simplicial decomposition of $\mathcal{M}_{k+1}$ so that $\circ_i$ are all simplicial embeddings.

We use the 'cohomological' notation for the degree of singular chain complex $S(X)$. Namely we put $S^{-d}(X) = S_d(X)$. (We remark that we do *not* assume X is a manifold. So we can not use Poincaré duality to identify chain with cochain.)

Let $\sigma_i : \Delta^{d_i} \to X$, $i = 1, \cdots, k$ be singular chains. We take the standard simplicial decomposition

$$\mathcal{M}_{k+1} \times \Delta^{d_1} \times \cdots \times \Delta^{d_k} = \sum_j \Delta^d_j$$

induced by the simplicial decomposition of $\mathcal{M}_{k+1}$, where $d = \sum d_i + k - 2$. We have

$$\mathfrak{m}_k(\sigma_1, \cdots, \sigma_k) = \sum \pm (\Delta^d_j; \sigma)$$

where

$$\sigma : \mathcal{M}_{k+1} \times \Delta^{d_1} \times \cdots \times \Delta^{d_k} \to X$$

is defined by

$$\sigma(a; p_1, \cdots, p_k) = \mathfrak{m}_k(a; \sigma_1(p_1), \cdots, \sigma_k(p_k)).$$

Since $-d - 1 = \sum(-d_i - 1) + 1$ the degree is as required. We do not discuss sign here. (33) can be checked by using (30). $\square$

6 A_∞ correspondence.

Theorem 3 we discussed at the end of the last section is a result on algebraic topology where we never use manifold structure etc.

Contrary to Theorem 3 we use manifold structure and Poincaré duality in the next theorem. To state it we need some notation.

We consider the following diagram of smooth maps :

$$\begin{array}{ccccc} & & \mathcal{M}_{k+1} & & \\ & & \uparrow{\scriptstyle \pi_0} & & \\ M^k & \xleftarrow[\pi_2]{} & \mathfrak{M}_{k+1} & \xrightarrow[\pi_1]{} & M \end{array} \tag{35}$$

where M is a closed oriented manifold $\mathfrak{M}_{k+1}$ is a compact oriented manifold which may have boundary or corner. We assume

$$\dim \mathfrak{M}_{k+1} = \dim M + \dim \mathcal{M}_{k+1} = \dim M + k - 2. \tag{36}$$

We define $ev_i : \mathfrak{M}_{k+1} \to M$ by :

$$\pi_2 = (ev_1, \cdots, ev_k), \qquad \pi_1 = ev_0.$$

Definition 4. *We say that $\mathfrak{M}_{k+1}$, $k = 1, 2, \cdots$ is an A_∞ correspondence on M, if there exists a family of smooth maps*

$$\circ_{\mathfrak{m},i} : \mathfrak{M}_{k+1} \ {}_{ev_i}\times_{ev_0} \mathfrak{M}_{l+1} \to \mathfrak{M}_{k+l} \tag{37}$$

for $i = 1, \cdots, l$ with the following properties.

(1) (operad axiom) *The following diagram commutes.*

$$\begin{array}{ccc} \mathcal{M}_{k+1} \times \mathcal{M}_{l+1} & \xrightarrow[\circ_i]{} & \mathcal{M}_{k+l} \\ {\scriptstyle \pi_0\times\pi_0}\uparrow & & {\scriptstyle \pi_0}\uparrow \\ \mathfrak{M}_{k+1} \ {}_{ev_i}\times_{ev_0} \mathfrak{M}_{l+1} & \xrightarrow[\circ_{\mathfrak{m},i}]{} & \mathfrak{M}_{k+l} \end{array} \tag{38}$$

(2) (cartesian axiom) $\mathfrak{M}_{k+1} \ {}_{ev_i}\times_{ev_0} \mathfrak{M}_{l+1}$ *coincides with the fiber product*

$$(\mathcal{M}_{k+1} \times \mathcal{M}_{l+1}) \times_{\mathcal{M}_{k+l}} \mathfrak{M}_{k+l}$$

obtained by Diagram (38).

(3) (associativity axiom) *We have*

$$(\mathbf{a} \circ_{\mathfrak{m},j} \mathbf{b}) \circ_{\mathfrak{m},i} \mathbf{c} = \mathbf{a} \circ_{\mathfrak{m},j} (\mathbf{b} \circ_{\mathfrak{m},i-j+1} \mathbf{c}), \tag{39}$$

for $i < j$ and

$$(\mathbf{a} \circ_{\mathfrak{m},j} \mathbf{b}) \circ_{\mathfrak{m},i} \mathbf{c} = (\mathbf{a} \circ_{\mathfrak{m},i} \mathbf{c}) \circ_{\mathfrak{m},j+k_2-1} \mathbf{b} \tag{40}$$

for $i < j$, $\mathbf{c} \in \mathfrak{M}_{k_2+1}$.

(4) (evaluation map axiom) *The following diagram commutes.*

$$\begin{array}{ccccc} M^{k+l} & \xleftarrow{\pi_2} & \mathfrak{M}_{k+l} & \xrightarrow{\pi_1} & M \\ \| & & \uparrow \circ_{\mathfrak{m},i} & & \| \\ M^{k+l} & \longleftarrow & \mathfrak{M}_{k+1} \ {}_{ev_i}\times_{ev_0} \mathfrak{M}_{l+1} & \xrightarrow[\pi_1 \circ \mathrm{pr}_1]{} & M \end{array} \tag{41}$$

Here the first arrow in the second line is

$$(ev_1 \circ \mathrm{pr}_1, \cdots, ev_{i-1} \circ \mathrm{pr}_1, ev_1 \circ \mathrm{pr}_2, \cdots, ev_l \circ \mathrm{pr}_2, ev_{i+1} \circ \mathrm{pr}_1, \cdots, ev_l \circ \mathrm{pr}_1)$$

where $\mathrm{pr}_1 : \mathfrak{M}_{k+1} \ {}_{ev_i}\times_{ev_0} \mathfrak{M}_{l+1} \to \mathfrak{M}_{k+1}$ *is the projection to the first factor and* $\mathrm{pr}_2 : \mathfrak{M}_{k+1} \ {}_{ev_i}\times_{ev_0} \mathfrak{M}_{l+1} \to \mathfrak{M}_{l+1}$ *is the projection to the second factor.*

(5) (Maurer-Cartan axiom) *We have*

$$\partial \mathfrak{M}_{n+1} = \bigcup_{\substack{k+l=n+1 \\ 1 \le i \le k}} \circ_{\mathfrak{m},i}(\mathfrak{M}_{k+1} \ {}_{ev_i}\times_{ev_0} \mathfrak{M}_{l+1}). \tag{42}$$

The images of the fiber product of the right hand sides, intersect to each other only at their boundaries.

(6) (orientation axiom) *The isomorphism* (42) *preserves orientation.*

We need a sign for (6) which will be discussed as Definition 27. The property (3) is regarded as a Maurer-Cartan equation. We recall that the diagram (38) is said to be *cartesian* when the condition (2) above holds.

A typical example of A_∞ correspondence is as follows.

Example 2. $\mathfrak{M}_{k+1} = \mathcal{M}_k \times M^{k+1}$ and π_0, π_1, π_2 are obvious projections.

Example 3. Let M be a manifold which is an A_∞ space such that the structure map $\mathsf{m} : \mathcal{M}_{k+1} \times M^k \to M$ is smooth. We put $\mathfrak{M}_{k+1} = \mathcal{M}_{k+1} \times M^k$, $\pi_1 = \mathsf{m}$, (π_0, π_1) =identity. This gives another example of A_∞ correspondence.

Now the next results can be proved in the same way as [33] §30.

Theorem 4. *If there is an* A_∞ *correspondence on* M *then there exists a cochain complex* $C(M;\mathbb{Z})$ *whose homology group is* $H^*(M;\mathbb{Z})$ *and such that* $C(M;\mathbb{Z})$ *has a structure of* A_∞ *algebra.*

Remark 2. In our situation, we can prove Theorem 4 over $\mathbb{Z}_2$ coefficient without assuming none of the conditions on orientations in Definition 4.

Applying Theorem 4 to Example 2 we obtain the following :

Corollary 1. *For any oriented closed manifold* M*, there exists an* A_∞ *algebra whose cohomology group is* $H^*(M;\mathbb{Z})$*.*

Corollary 1 is [33] Theorem 9.8. It is also proved by McClure [51] and by Wilson [71].

Remark 3. Actually the statement of Corollary 1 itself is a consequence of classical fact. In fact the singular cochain complex has a cup product which is associative in the chain level. What is important here is that the A_∞ operations are realized by the *chain level intersection theory* and by identifying chain with cochain by *Poincaré duality*. We emphasize that the chain level Poincaré duality is still a mysterious subject. We also emphasize that using chain (instead cochain) is more natural in our situation since the moduli space can be naturally regarded as a chain but can be regarded as a cochain only via Poincaré duality.

Sketch of the proof of Theorem 4 : Let $f_i : P_i \to M$ be 'chains' of dimension $\dim M - d_i$. (Actually the precise choice of the chain complex to work with is the important part of the proof. See [33] Remark 1.34 and the beginning of §12.) We consider the fiber product

$$\mathfrak{M}_{k+1} \ {}_{\pi_1}\times_{f_1,\cdots,f_k} (P_1 \times \cdots \times P_k) \tag{43}$$

over M^k. Assuming the transversality, (43) is a smooth manifold with boundary or corner. $\pi_2 = ev_0 : \mathfrak{M}_{k+1} \to M$ induces a smooth map ev_0 from the manifold (43). We now put

$$\mathfrak{m}_k(P_1, \cdots, P_k) = (ev_0)_* \left(\mathfrak{M}_{k+1} \ {}_{\pi_1}\times_{f_1,\cdots,f_k} (P_1 \times \cdots \times P_k)\right). \tag{44}$$

(44) is a chain of dimension

$$\dim M + k - 2 + \sum(\dim M - d_i) - k \dim M = \dim M - \left(\sum d_i - (k-2)\right).$$

Using Poincaré duality, we identify chain of dimension $\dim M - d$ on M with cochain of degree d. Then (44) induces a map of degree $k-2$ on cochains. This is (after degree shift) a map with required degree.

(42) implies (33) modulo transversality and sign. □

As we mentioned already, the main difficulty to prove Theorem 4 is the study of orientation and transversality. Transversality is discussed in detail in [33] §30 and orientation is discussed in detail in [33] Chapter 9. These points will be discussed also in §12 and §13 of this paper.

We remark that if we replace 'manifold' by 'space with Kuranishi structure', we can still prove Theorem 4 over $\mathbb{Q}$ coefficient. See §10.

Remark 4. In this article we discuss A_∞ structure since it is the only case which is worked out and written in detail, at the time of writing this article. However the argument of [33] can be generalized to show analogy of Theorem 4 for various other (differentiable) operads or props. Especially we can generalize it to the case of L_∞ structure, which appears in the loop space formulation of Floer theory ([29]) and involutive-bi-Lie infinity (or BV infinity) structure, which appears when we study symplectic manifolds with cylindrical end [16, 18] and also in the higher genus generalization of Lagrangian Floer theory. It appears also in string topology [12].

7 A_∞ homomorphism.

As we mentioned before, the main motivation for the author to study homotopy theory of A_∞ algebra etc. is to find a correct way to state the well-definedness of the algebraic system induced by the smooth correspondence by moduli space. Actually those algebraic systems are well-defined up to homotopy equivalence. To prove it is our main purpose. For this purpose, it is very important to define the notion of homotopy equivalence and derive its basic properties. In this section we define A_∞ homomorphism and describe a way to obtain it from smooth correspondence.

Let $(C, \mathfrak{m})$ and $(C', \mathfrak{m}')$ be A_∞ algebras. We consider a series of homomorphisms

$$\mathfrak{f}_k : B_kC[1] \to C'[1], \tag{45}$$

$k = 1, 2, \cdots$ of degree 0. We can extend it uniquely to a coalgebra homomorphism

$$\hat{\mathfrak{f}} : BC[1] \to BC[1]$$

whose $Hom(B_kC[1], C'[1])$ component is φ_k. Here $\hat{\varphi}$ is said to be a *coalgebra homomorphism* if $(\hat{\mathfrak{f}} \otimes \hat{\mathfrak{f}}) \circ \Delta = \Delta \circ \hat{\mathfrak{f}}$.

Definition 5. $\mathfrak{f}_k$ $(k = 1, 2, \cdots)$ *is said to be an A_∞ homomorphism if $\hat{d}' \circ \hat{\mathfrak{f}} = \hat{\mathfrak{f}} \circ \hat{d}$. An A_∞ homomorphism is said to be linear if $\mathfrak{f}_k = 0$ for $k \neq 1$.*

We can rewrite the condition $\hat{d}' \circ \hat{\mathfrak{f}} = \hat{\mathfrak{f}} \circ \hat{d}$ as follows.

$$\begin{aligned}\sum_l \sum_{k_1+\cdots+k_l=k} &\mathfrak{m}'_l(\mathfrak{f}_{k_1}(x_1, \cdots, x_{k_1}), \cdots, \mathfrak{f}_{k_l}(x_{k-k_l+1}, \cdots, x_k)) \\ &= \sum_{1\le i<j\le k} (-1)^* \mathfrak{f}_{k-j+i}(x_1, \cdots, \mathfrak{m}_{j-i+1}(x_i, \cdots, x_j), \cdots, x_k)\end{aligned} \tag{46}$$

where

$$* = \deg' x_1 + \cdots + \deg' x_{i-1}.$$

We remark that (46) implies that $\mathfrak{f}_1 : (C, \mathfrak{m}_1) \to (C', \mathfrak{m}_1)$ is a chain map.

Definition 6. *Let $\mathfrak{f} : C^{(1)} \to C^{(2)}$ and $\mathfrak{g} : C^{(2)} \to C^{(3)}$ be A_∞ homomorphisms. The composition $\mathfrak{f} \circ \mathfrak{g}$ of them is defined by $\widehat{\mathfrak{f} \circ \mathfrak{g}} = \widehat{\mathfrak{f}} \circ \widehat{\mathfrak{g}}$.*

We next are going to discuss how we obtain A_∞ homomorphism by smooth correspondence. For this purpose we need to find spaces $\mathcal{F}_{k+1}$ whose relation to A_∞ homomorphism is the same as the relation of the spaces $\mathcal{M}_{k+1}$ to A_∞ operations. We use the space $\mathcal{F}_{k+1}$ also to define the notion of A_∞ maps between two A_∞ spaces.

The following result which is [33] Theorem 29.51 gives such spaces $\mathcal{F}_{k+1}$. (Note in [33] we wrote $\mathcal{N}_{k+1}$ in place of $\mathcal{F}_{k+1}$.)

Theorem 5 (FOOO). *There exists a cell decomposition $\mathcal{F}_{k+1}$ of D^{k-1} such that the boundary $\partial\mathcal{F}_{k+1}$ is a union of the following two types of spaces, which intersect only at their boundaries.*

(1) *The spaces $\mathcal{M}_{l+1} \times \mathcal{F}_{k_1+1} \times \cdots \mathcal{F}_{k_l+1}$ where $\sum_{i=1}^{l} k_i = k$.*
(2) *The spaces $\mathcal{F}_{(k-j+i)+1} \times \mathcal{M}_{(j-i+1)+1}$ where $1 \le i < j \le k$.*

Sketch of the proof : We define $\mathcal{F}_{k+1}$ by modifying $\mathcal{M}_{k+1}$. Let $(\Sigma; z_0, \cdots, z_k)$ be an element of $\mathcal{M}_{k+1}$ and $\Sigma = \cup_{a\in A} D^2_a$ be the decomposition to the components. We assume that $z_0 \in D_{a_0}$. We say $a \le b$ if any path connecting D_a to D_{a_0} intersects with D_b. The relation $\le$ defines a partial order on A.

We say a map $\rho : A \to [0,1]$ to be a *time allocation* if $\rho(a) \le \rho(b)$ for all $a, b \in A$ with $a \le b$. Let $\mathcal{F}_{k+1}$ be the set of all isomorphism classes of $(\Sigma; z_0, \cdots, z_k; \rho)$ where $(\Sigma; z_0, \cdots, z_k) \in \mathcal{M}_{k+1}$ and ρ is a time allocation. We define a topology on it in an obvious way.

Lemma 4. *$\mathcal{F}_{k+1}$ is homeomorphic to D^{k-1}.*

We omit the proof. See [33] §29.5.

Let us consider the boundary of $\mathcal{F}_{k+1}$. Let $(\Sigma^{(i)}; z^{(i)}_0, \cdots, z^{(i)}_k; \rho^{(i)})$ be a sequence of elements of $\mathcal{F}_{k+1}$. There are several cases where it converges to a potential boundary point. It can be classified as follows.

(I) A component $D^{(i)}_a$ splits into two components in the limit $i \to \infty$.
(II) There exists a, b such that $D^{(i)}_a \cap D^{(i)}_b \ne \emptyset$ and such that $\lim_{i\to\infty} \rho^{(i)}(a) = \lim_{i\to\infty} \rho^{(i)}(b)$.
(III) $\lim_{i\to\infty} \rho^{(i)}(a_0) = 1$.
(IV) $\lim_{i\to\infty} \rho^{(i)}(a) = 0$, for some a.

We observe that (I) cancels with (II). Namely they do not correspond to a boundary point of $\mathcal{F}_{k+1}$.

On the other hand, we can check that (III) corresponds to (1) of Theorem 5 and (IV) corresponds to (2) of Theorem 5. This implies the theorem. □

Let us explain how we use $\mathcal{F}_{k+1}$ to define an A_∞ map between two A_∞ spaces. We denote by

$$\circ_{\rm mf} : \mathcal{M}_{l+1} \times \mathcal{F}_{k_1+1} \times \cdots \mathcal{F}_{k_l+1} \to \mathcal{F}_{k_1+\cdots+k_l+1} \tag{47}$$

and

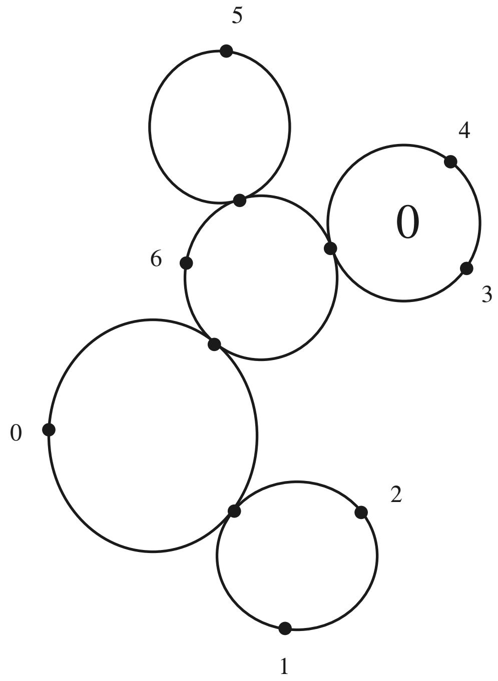

Fig. 6. $\mathfrak{f}(x_1, x_2, \mathfrak{m}(x_3, x_4), x_5, x_6)$

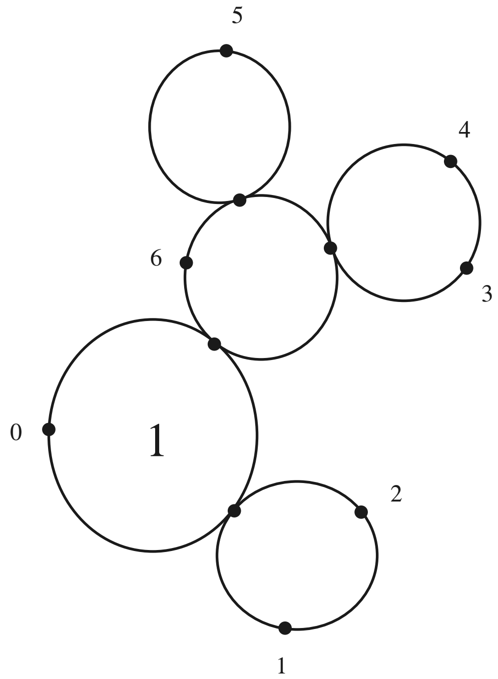

Fig. 7. $\mathfrak{m}(\mathfrak{f}(x_1, x_2), \mathfrak{f}(x_3, x_4, x_5, x_6))$

$$\circ_{\mathrm{fm},i} : \mathcal{F}_{(k-j+i)+1} \times \mathcal{M}_{(j-i+1)+1} \to \mathcal{F}_{k+1} \tag{48}$$

the inclusions obtained by Theorem 5 (1) and (2), respectively. They satisfy the following compatibility conditions (49), (50), (51), (52).

$$\begin{aligned} &(a \circ_i b) \circ_{\mathrm{mf}} (x_1, \cdots, x_l) \\ &= a \circ_{\mathrm{mf}} (x_1, \cdots, x_{i-1}, b \circ_{\mathrm{mf}} (x_i, \cdots, x_{i+l_2-1}), \cdots, x_l) \end{aligned} \tag{49}$$

where $a \in \mathcal{M}_{l_1+1}$, $b \in \mathcal{M}_{l_2+1}$, $x_n \in \mathcal{F}_{k_n+1}$. See Figure 8.

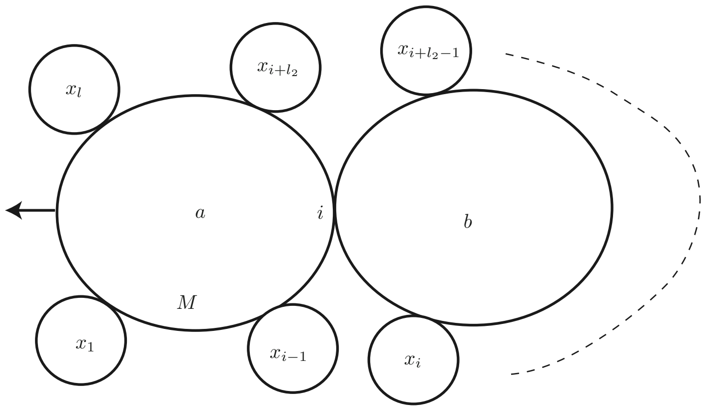

Fig. 8.

$$\begin{aligned} &a \circ_{\mathrm{mf}} (x_1, \cdots, x_{i-1}, x_i \circ_{\mathrm{fm},j} b, x_{i+1}, \cdots, x_l) \\ &= (a \circ_{\mathrm{mf}} (x_1, \cdots, x_l)) \circ_{\mathrm{fm},j'} b, \end{aligned} \tag{50}$$

where $x_n \in \mathcal{F}_{k_n+1}$, $j' = j + k_1 + \cdots + k_{i-1}$. See Figure 9.

$$(x \circ_{\mathrm{fm},i} a) \circ_{\mathrm{fm},j+l_1-1} b = (x \circ_{\mathrm{fm},j} b) \circ_{\mathrm{fm},i} a \tag{51}$$

where $x \in \mathcal{F}_{k+1}$, $a \in \mathcal{M}_{l_1+1}$, $b \in \mathcal{M}_{l_2+1}$, $i < j$. See Figure 10.

$$(x \circ_{\mathrm{fm},i} a) \circ_{\mathrm{fm},i+j-1} b = x \circ_{\mathrm{fm},i} (a \circ_j b). \tag{52}$$

See Figure 11.

Let X, X' be A_∞ spaces with structure maps

$$\mathsf{m}_k : \mathcal{M}_{k+1} \times X^k \to X, \qquad \mathsf{m}'_k : \mathcal{M}_{k+1} \times X'^k \to X'$$

respectively.

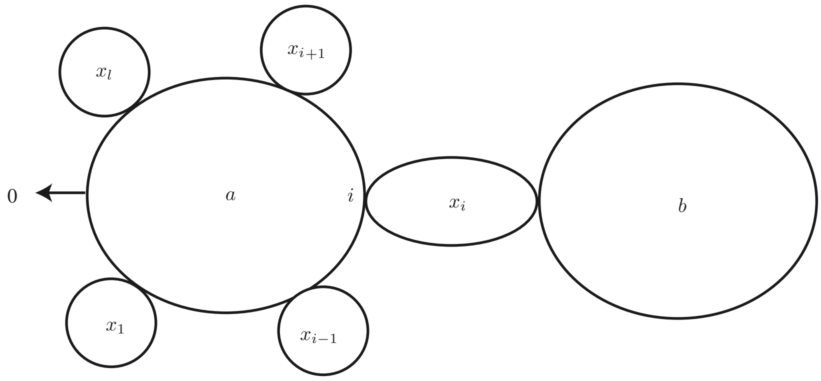

Fig. 9.

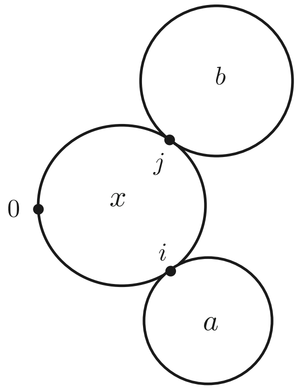

Fig. 10.

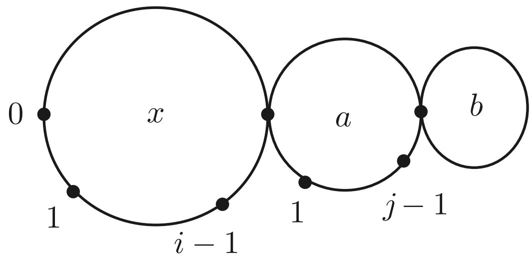

Fig. 11.

Definition 7. *An A_∞ map from X to X' is a sequence of maps*

$$\mathsf{f}_k : \mathcal{F}_{k+1} \times X^k \to X'$$

with the following properties.

(1) *Let $k = k_1 + \cdots + k_l$, $x_{i,j} \in X$ for $1 \le i \le l, 1 \le j \le k_i$, and $a \in \mathcal{M}_{l+1}$, $c_i \in \mathcal{F}_{k_i+1}$. Then, we have*

$$\begin{aligned} &\mathsf{f}_k((a \circ_{\rm mf} (c_1, \cdots, c_l)), (x_{1,1}, \cdots, x_{l,k_l})) \\ &\quad = \mathsf{m}_l(a; \mathsf{f}_{k_1}(c_1; x_{1,1}, \cdots, x_{1,k_1}), \cdots, \mathsf{f}_{k_l}(c_l; x_{l,1}, \cdots, x_{l,k_l})) \end{aligned} \tag{53}$$

(2) *Let $0 \le i \le k$ and $a \in \mathcal{M}_{(j-i+1)+1}$, $c \in \mathcal{F}_{(k-j+i)+1}$, $x_1, \cdots, x_k \in X$. Then we have*

$$\begin{aligned} &\mathsf{f}_k(c \circ_{{\rm mf},i} a; x_1, \cdots, x_k) \\ &\quad = \mathsf{f}_{k-j+i}(c; x_1, \cdots, \mathsf{m}_{j-i+1}(a; x_i, \cdots, x_j), \cdots, x_k). \end{aligned} \tag{54}$$

We remark that in Stasheff [65] and in [49], the notion of A_∞ map is defined in the case X' is a monoid. (See [7].) Stasheff told the author that A_∞ map between A_∞ spaces is defined in his thesis. A construction of $\mathcal{F}_{k+1}$ can be found in [41]. It does not seem to be easy to see that the space in [41] is a cell. For the purpose of homotopy theory, this point is not important at all. However to apply it to the study of smooth correspondence, the fact that $\mathcal{F}_{k+1}$ is a smooth manifold is useful.

We mention the following which the author believes to be a classical result in homotopy theory.

Proposition 1. *An A_∞ map $X \to X'$ between two A_∞ spaces induces an A_∞ homomorphisms between A_∞ algebras in Theorem 3.*

The proof is similar to the proof of Theorem 3 and is omitted.

Now we use the spaces $\mathcal{F}_{k+1}$ to define the notion of a morphism between A_∞ correspondences. Let

$$\begin{array}{ccccc} & & \mathcal{M}_{k+1} & & \\ & & \uparrow{\scriptstyle \pi_0} & & \\ M^k & \xleftarrow[\pi_2]{} & \mathfrak{M}_{k+1} & \xrightarrow[\pi_1]{} & M \end{array} \tag{55}$$

and

$$\begin{array}{ccccc} & & \mathcal{M}_{k+1} & & \\ & & \uparrow{\scriptstyle \pi_0} & & \\ M'^k & \xleftarrow[\pi_2]{} & \mathfrak{M}'_{k+1} & \xrightarrow[\pi_1]{} & M' \end{array} \tag{56}$$

be A_∞ correspondences.

Definition 8. *A morphism between two A_∞ correspondences is the following diagram of smooth manifolds (with boundary or corners)*

$$
\begin{array}{ccccc}
 & & \mathcal{F}_{k+1} & & \\
 & & \uparrow{\scriptstyle \pi_0} & & \\
M^k & \xleftarrow[\pi_2]{} & \mathfrak{F}_{k+1} & \xrightarrow[\pi_1]{} & M'
\end{array}
\tag{57}
$$

together with smooth maps

$$
\circ_{\mathfrak{mf}} : \mathfrak{M}_{l+1} \ {}_{\pi_2}\times_{\pi_1,\cdots,\pi_1} (\mathfrak{F}_{k_1+1} \times \cdots \times \mathfrak{F}_{k_l+1}) \to \mathfrak{F}_{k_1+\cdots+k_l+1} \tag{58}
$$

$$
\circ_{\mathfrak{fm},i} : \mathfrak{F}_{(k-j+i)+1} \ {}_{ev_i}\times_{ev_0} \mathfrak{M}_{(j-i+1)+1} \to \mathfrak{F}_{k+1} \tag{59}
$$

with the following properties.

(1) *The following diagrams commute.*

$$
\begin{array}{ccc}
\mathcal{M}_{l+1} \times \mathcal{F}_{k_1+1} \times \cdots \times F_{k_l+1} & \xrightarrow[\circ_{\rm mf}]{} & \mathcal{F}_{k_1+\cdots+k_l+1} \\
{\scriptstyle \pi_0\times\cdots\times\pi_0}\uparrow & & {\scriptstyle \pi_0}\uparrow \\
\mathfrak{M}_{l+1} \ {}_{\pi_2}\times_{\pi_1,\cdots,\pi_1} (\mathfrak{F}_{k_1+1} \times \cdots \times \mathfrak{F}_{k_l+1}) & \xrightarrow[\circ_{\mathfrak{mf}}]{} & \mathfrak{F}_{k_1+\cdots+k_l+1}
\end{array}
\tag{60}
$$

$$
\begin{array}{ccc}
\mathcal{F}_{(k-j+i)+1} \times \mathcal{M}_{(j-i+1)+1} & \xrightarrow[\circ_{{\rm fm},i}]{} & \mathcal{F}_{k+1} \\
{\scriptstyle \pi_0\times\pi_0}\uparrow & & {\scriptstyle \pi_0}\uparrow \\
\mathfrak{F}_{(k-j+i)+1} \ {}_{ev_i}\times_{ev_0} \mathfrak{M}_{(j-i+1)+1} & \xrightarrow[\circ_{\mathfrak{fm},i}]{} & \mathfrak{F}_{k+1}
\end{array}
\tag{61}
$$

(2) *Diagrams* (60), (61) *are cartesian.*
(3) *Formulae* $(49), (50), (51), (52)$ *hold after replacing* $\circ_{{\rm fm},i}$, $\circ_{\rm mf}$ *by* $\circ_{\mathfrak{fm},i}$, $\circ_{\mathfrak{mf}}$.
(4) *The following diagrams commute. We put* $k = k_1 + \cdots + k_l$

$$
\begin{array}{ccccc}
M^k & \xleftarrow[\pi_2]{} & \mathfrak{F}_{k+1} & \xrightarrow[\pi_1]{} & M' \\
\| & & \uparrow{\scriptstyle \circ_{\mathfrak{mf}}} & & \| \\
M^k & \longleftarrow & \mathfrak{M}'_{l+1} \ {}_{\pi_2}\times_{\pi_1,\cdots,\pi_1} (\mathfrak{F}_{k_1+1} \times \cdots \times \mathfrak{F}_{k_l+1}) & \xrightarrow[\pi_1\circ{\rm pr}_1]{} & M'
\end{array}
\tag{62}
$$

Here the first arrow in the second line is $(\pi_2 \cdots \pi_2)$ *on the factor* $\mathfrak{F}_{k_1+1} \times \cdots \times \mathfrak{F}_{k_l+1}$.

$$
\begin{array}{ccccc}
M^k & \xleftarrow[\pi_2]{} & \mathfrak{F}_{k+1} & \xrightarrow[\pi_1]{} & M' \\
\| & & \uparrow{\scriptstyle \circ_{\mathfrak{fm},i}} & & \| \\
M^k & \longleftarrow & \mathfrak{F}_{(k-j+i)+1} \ {}_{ev_i}\times_{ev_0} \mathfrak{M}_{(j-i+1)+1} & \xrightarrow[\pi_1\circ{\rm pr}_1]{} & M'
\end{array}
\tag{63}
$$

Here the first arrow in the second line is

$(ev_1 \circ \mathrm{pr}_1, \cdots, ev_{i-1}\circ \mathrm{pr}_1, ev_1\circ \mathrm{pr}_2, \cdots, ev_{j-i+1}\circ \mathrm{pr}_2, ev_{i+1}\circ \mathrm{pr}_1, \cdots, ev_{k-j+i}\circ \mathrm{pr}_1)$.

(5) *The union of the images of* $\circ_{\mathfrak{mf}}$ *and of* $\circ_{\mathfrak{fm},i}$ *is the boundary of* $\mathfrak{F}_{k+1}$. *Those images intersect only at their boundaries.*
(6) *The identification of* (5) *preserves orientation.*

Now we have :

Theorem 6. *A morphism between two* A_∞ *correspondences induces an* A_∞ *homomorphism between* A_∞ *algebras in Theorem 4.*

The proof is similar to the proof of Theorem 4 and can be extracted from [33] §30.

Before closing this section, we remark that there is a map :

$$\mathrm{Comp}_{\mathbf{k},\mathbf{k}'} : \mathcal{F}_{k'+l+1} \times (\mathcal{F}_{k_1+1} \times \cdots \mathcal{F}_{k_l+1}) \to \mathcal{F}_{k+k'+1} \tag{64}$$

where $\mathbf{k} = (k_1, \cdots, k_l)$, $\mathbf{k}' = (k'_0, \cdots, k'_l)$, $k_1 + \cdots + k_l = k$ and $k'_0 + \cdots + k'_l = k'$. The map (64) describes the way how the A_∞ maps and A_∞ correspondences are composed. The map (64) is defined as follows.

Let $\mathcal{S} = (\Sigma; z_0, \cdots, z_{k'+l}, \rho) \in \mathcal{F}_{k'+l}$ and $\mathcal{S}_i = (\Sigma_i; z_0^{(i)}, \cdots, z_{k_i}^{(i)}, \rho_i) \in \mathcal{F}_{k_i+1}$ $(i = 1, \cdots, l)$.

We identify $z_{k'_0+\cdots+k'_{i-1}+i} \in \partial\Sigma$ with $z_0^{(i)} \in \partial\Sigma_i$ for each i and obtain Σ'.

Time allocations ρ and ρ_i induce a time allocation ρ' on Σ' as follows. When D_a is a component of Σ, we regard it as a component of Σ' and put $\rho'(a) = (1 + \rho(a))/2$. When D_a is a component of Σ_i, we regard it as a component of Σ' and put $\rho'(a) = \rho_i(a)/2$.

We define $(z'_0, z'_1, \cdots, z'_{k+k'})$ as

$$\begin{aligned}(z_0, &z_1, \cdots, z_{k'_0}, z_1^{(1)}, \cdots, z_{k_1}^{(1)}, z_{k'_0+2}, \cdots \\ &\cdots, z_{k'_1+\cdots+k'_{l-1}+l-1}, z_1^{(l)}, \cdots, z_{k_l}^{(l)}, z_{k'_1+\cdots+k'_{l-1}+l+1}, \cdots, z_{k+k'}).\end{aligned} \tag{65}$$

(See Figure 12.) Then, we put

$$\mathrm{Comp}_{\mathbf{k},\mathbf{k}'}(\mathcal{S}, \mathcal{S}_1, \cdots, \mathcal{S}_l) = (\Sigma'; z'_0, z'_1, \cdots, z'_{k+k'}; \rho'). \tag{66}$$

Lemma 5. $\mathcal{F}_{n+1}$ *is a union of the images of* $\mathrm{Comp}_{\mathbf{k},\mathbf{k}'}$ *for various* $\mathbf{k}, \mathbf{k}'$ *with* $n = k_1 + \cdots + k_l + k'_0 + \cdots + k'_l$. *Those images intersect only at their boundaries.*

The proof is easy from the combinatorial description of the elements of $\mathcal{F}_{k+1}$ and is omitted. (See also [33] §19.3.)

The map (64) is compatible with (47) and (48). We omit the precise description of the compatibility condition and leave it to the interested readers.

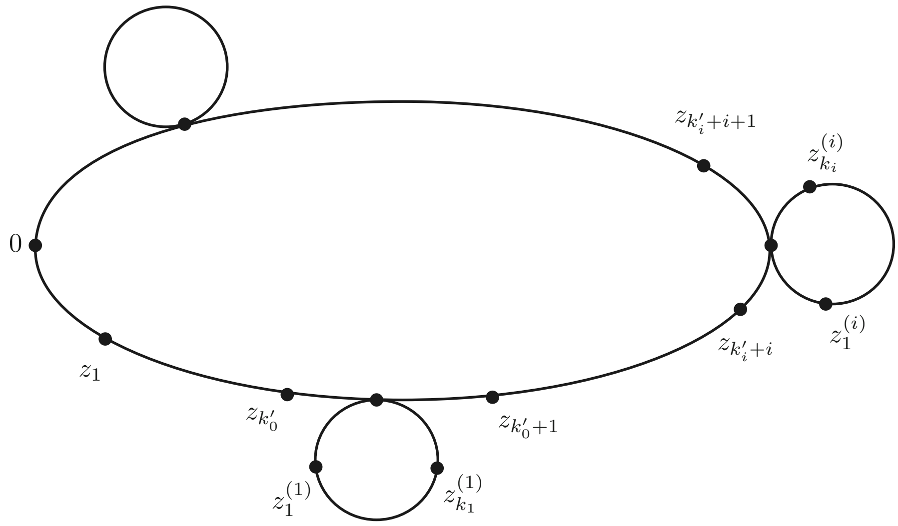

Fig. 12.

By using (64) and Lemma 5 we can define a composition of two A_∞ maps. We omit it since we do not use it. The map (64) is related to the composition of A_∞ correspondences. Since composition of A_∞ correspondences is more naturally defined in case we include correspondence by spaces with Kuranishi structure, we will introduce it in §10.

8 A_∞ homotopy.

We begin this section with algebraic side of the story. In this section we define the notion of homotopy between two A_∞ homomorphisms. As far as the author knows, there are two definitions of homotopy between two A_∞ homomorphisms in the literature. One of then can be found, for example, in [64, 46]. (In the case of graded commutative differential graded algebra, a similar formulation is due to Sullivan [69].) Another is an A_∞ version of the definition of homotopy which is written in [35] in the case of differential graded algebra. (See [28] for A_∞ version of this second definition.) When we were writing [33], we were trying to find a relation between these two definitions but were not able to find it in the literature. (The author believes that this equivalence was known to experts long ago.) So in [33] Chapter 4 we took an axiomatic approach and gave a definition which is equivalent to both of them (and hence proved the equivalence of those two definitions as a consequence). We will discuss this approach here.

Let $(C, \mathfrak{m})$ be an A_∞ algebra.

Definition 9. *An A_∞ algebra $(\mathfrak{C}, \mathfrak{m})$ together with the following diagram is said to be a model of $[0,1] \times C$ if the conditions (1) - (4) below hold.*

$$\begin{array}{ccccc} & & C & & \\ & & \big\downarrow{\scriptstyle \mathrm{Incl}} & & \\ C & \xleftarrow[\mathrm{Eval}_0]{} & \mathfrak{C} & \xrightarrow[\mathrm{Eval}_1]{} & C \end{array} \tag{67}$$

(1) Incl, Eval_0, Eval_1 *are linear* A_∞ *homomorphisms.*
(2) $\mathrm{Eval}_0 \circ \mathrm{Incl} = \mathrm{Eval}_1 \circ \mathrm{Incl} = \mathrm{identity}$.
(3) $\mathrm{Eval}_0 \oplus \mathrm{Eval}_1 : \mathfrak{C} \to C \oplus C$ *is surjective.*
(4) $\mathrm{Incl} : C \to \mathfrak{C}$ *is a chain homotopy equivalence.*

Example 4. Let M be a manifold and C be its de Rham complex regarded as an A_∞ algebra. Let $\mathfrak{C}$ be the de Rham complex of $\mathbb{R} \times M$. Let Incl, Eval_0, and Eval_1 be the linear maps induced by the projection $\mathbb{R} \times M \to M$, the inclusion $M \to \{0\} \times M \subset \mathbb{R} \times M$, and the inclusion $M \to \{1\} \times M \subset \mathbb{R} \times M$, respectively. It is easy to see that $\mathfrak{C}$ is a model of $[0,1] \times C$.

Proposition 2. *For any* $(C, \mathfrak{m})$ *there exists a model of* $[0,1] \times C$.

We can prove it either by an explicit construction or by using obstruction theory to show the existence. We omit the proof and refer [33] §15.1.

The following is a kind of uniqueness theorem of model of $[0,1] \times C$.

Theorem 7. *Let* $(C, \mathfrak{m})$, $(C', \mathfrak{m}')$ *be* A_∞ *algebras and* $\mathfrak{f} : C \to C'$ *be an* A_∞ *homomorphism. Let* $\mathfrak{C}$, $\mathfrak{C}'$ *be models of* $[0,1] \times C$, $[0,1] \times C'$ *respectively. Then there exists an* A_∞ *homomorphism* $\mathfrak{F} : \mathfrak{C} \to \mathfrak{C}'$ *such that*

$$\mathfrak{F} \circ \mathrm{Incl} = \mathrm{Incl} \circ \mathfrak{f}, \qquad \mathrm{Eval}_{s_0} \circ \mathfrak{F} = \mathfrak{f} \circ \mathrm{Eval}_{s_0}$$

where $s_0 = 0$ *or* 1.

This is [33] Theorem 15.34.

Definition 10. *Let* $(C, \mathfrak{m})$, $(C', \mathfrak{m}')$ *be* A_∞ *algebras and* $\mathfrak{f} : (C, \mathfrak{m}) \to (C', \mathfrak{m}')$ *and* $\mathfrak{g} : (C, \mathfrak{m}) \to (C', \mathfrak{m}')$ *be* A_∞ *homomorphisms. Let* $\mathfrak{C}'$ *be a model of* $[0,1] \times C'$.

We say an A_∞ *homomorphism* $\mathfrak{H} : C \to \mathfrak{C}'$ *to be a homotopy from* $\mathfrak{f}$ *and* $\mathfrak{g}$ *if*

$$\mathrm{Eval}_0 \circ \mathfrak{H} = \mathfrak{f}, \quad \mathrm{Eval}_1 \circ \mathfrak{H} = \mathfrak{g}.$$

We write $\mathfrak{f} \sim_{\mathfrak{C}'} \mathfrak{g}$ *if there exists a homotopy* $\mathfrak{H} : C \to \mathfrak{C}'$ *between them.*

Using Theorem 7, we can prove the following.

Proposition 3. (1) $\sim_{\mathfrak{C}'}$ *is independent of the choice of* $\mathfrak{C}'$. *(We write* $\sim$ *in place of* $\sim_{\mathfrak{C}'}$ *hereafter.)*
(2) $\sim$ *is an equivalence relation.*
(3) $\sim$ *is compatible with composition of* A_∞ *homomorphisms. Namely if* $\mathfrak{f} \sim \mathfrak{g}$ *then*

$$\mathfrak{f} \circ \mathfrak{h} \sim \mathfrak{g} \circ \mathfrak{h}, \qquad \mathfrak{h}' \circ \mathfrak{f} \sim \mathfrak{h}' \circ \mathfrak{g}.$$

Sketch of the proof : We assume $\mathfrak{f} \sim_{\mathfrak{C}'} g$. Let $\mathfrak{H} : C \to \mathfrak{C}'$ be the homotopy between them. Let $\mathfrak{C}''$ be another model of $[0,1] \times C'$. We apply Theorem 7 to the identity from C' to C' and obtain $\mathfrak{ID} : \mathfrak{C}' \to \mathfrak{C}''$. It is easy to see that $\mathfrak{ID} \circ \mathfrak{H}$ is a homotopy from $\mathfrak{f}$ to $\mathfrak{g}$. We have $\mathfrak{f} \sim_{\mathfrak{C}''} g$. (1) follows.

Let $\mathfrak{f} \sim g$ and $\mathfrak{g} \sim h$. Let $\mathfrak{H}^{(1)} : C \to \mathfrak{C}_1$ and $\mathfrak{H}^{(2)} : C \to \mathfrak{C}_2$ be homotopies from $\mathfrak{f}$ to $\mathfrak{g}$ and from $\mathfrak{g}$ to $\mathfrak{h}$, respectively. We put

$$\mathfrak{C}' = \{(y_1, y_2) \in \mathfrak{C}_1 \oplus \mathfrak{C}_2 \mid \mathrm{Eval}_1 y_1 = \mathrm{Eval}_0 y_2\}.$$

It is easy to see that $\mathfrak{C}'$ is a model of $[0,1] \times C'$. We define $\mathfrak{H} : C \to \mathfrak{C}'$ by

$$\mathfrak{H}_k(x_1, \cdots, x_k) = (\mathfrak{H}_k^{(1)}(x_1, \cdots, x_k), \mathfrak{H}_k^{(2)}(x_1, \cdots, x_k)).$$

It is easy to see that $\mathfrak{H}$ is a homotopy from $\mathfrak{f}$ to $\mathfrak{h}$. We thus proved that $\sim$ is transitive. Other part of the proof of Proposition 3 is similar and is omitted. (See [33] §15.2.) □

Using Proposition 3 we can define the notion of two A_∞ algebras to be homotopy equivalent to each other and also the notion of A_∞ homomorphism to be a homotopy equivalence, in an obvious way.

We can then prove the following two basic results of homotopy theory of A_∞ algebras.

Theorem 8. *If $\mathfrak{f} : (C, \mathfrak{m}) \to (C', \mathfrak{m})$ is an A_∞ homomorphism such that $\mathfrak{f}_1 : (C, \mathfrak{m}_1) \to (C', \mathfrak{m}_1)$ is a chain homotopy equivalence. Then $\mathfrak{f}$ is a homotopy equivalence. Namely there exists $\mathfrak{g} : (C', \mathfrak{m}) \to (C, \mathfrak{m})$ such that $\mathfrak{g} \circ \mathfrak{f}$ and $\mathfrak{f} \circ \mathfrak{g}$ are homotopic to identity.*

This theorem seems to be known to experts. The proof based on our definition of homotopy is in [33] §18.

Theorem 9. *Let $(C, \mathfrak{m})$ be an A_∞ algebra. Let $C' \subset C$ be a subchain complex of $(C, \mathfrak{m}_1)$ such that the inclusion $(C', \mathfrak{m}_1) \to (C, \mathfrak{m}_1)$ is a chain homotopy equivalence.*

Then there exists a sequence of operators $\mathfrak{m}'_k$ for $k \geq 2$ such that $\mathfrak{m}'_k$ and $\mathfrak{m}'_1 = \mathfrak{m}_1$ define a structure of A_∞ algebra on C'.

Moreover there exists $\mathfrak{f}_k : B_k C'[1] \to C[1]$ for $k \geq 2$ such that $\mathfrak{f}_k$ together with $\mathfrak{f}_1 =$ inclusion define a homotopy equivalence $C' \to C$.

We put $H(C) = \mathrm{Ker}\mathfrak{m}_1 / \mathrm{Im}\mathfrak{m}_1$.

Corollary 2. *If R is a field (or $H(C)$ is a free R module) then there exists a structure of A_∞ algebra on $H(C)$ for which $\mathfrak{m}_1 = 0$ and which is homotopy equivalent to $(C, \mathfrak{m})$*

Theorem 9 and Corollary 2 have a long history which starts with [42]. Theorem 9 and Corollary 2 are proved in [33] §23.4. (The proof in [33] is similar to one in [45].)

We also refer [61] and references therein for more results on homological algebra of A_∞ structures.

We now discuss relations of the algebraic machinery described above to geometry. We first remark that we can define the notion of homotopy for two A_∞ maps between A_∞ spaces. Moreover we can prove that, if two A_∞ maps are homotopic to each other, then induced A_∞ homomorphisms are also homotopic. We omit the proof of this since we do not use it.

Let us consider the case of A_∞ correspondence. We consider two A_∞ correspondences (55), (56) and two morphisms

$$\begin{array}{ccccc} & & \mathcal{F}_{k+1} & & \\ & & \uparrow{\scriptstyle \pi_0} & & \\ M^k & \xleftarrow[\pi_2]{} & \mathfrak{F}_{k+1} & \xrightarrow[\pi_1]{} & M' \end{array} \tag{68}$$

$$\begin{array}{ccccc} & & \mathcal{F}_{k+1} & & \\ & & \uparrow{\scriptstyle \pi_0} & & \\ M^k & \xleftarrow[\pi_2]{} & \mathfrak{F}'_{k+1} & \xrightarrow[\pi_1]{} & M' \end{array} \tag{69}$$

between them.

Definition 11. *A homotopy from* (68) *to* (69) *is a sequence of diagrams*

$$\begin{array}{ccccc} & & \mathcal{F}_{k+1} & & \\ & & \uparrow{\scriptstyle \pi_0} & & \\ M^k & \xleftarrow[\pi_2]{} & \mathfrak{H}_{k+1} & \xrightarrow[\pi_1]{} & M' \times [0,1] \end{array} \tag{70}$$

together with the smooth maps

$$\begin{aligned} \circ_{\mathfrak{mh}} : \mathfrak{M}_{l+1} \,{}_{\pi_2}\times_{\pi_1,\cdots,\pi_1} (\mathfrak{H}_{k_1+1} \times_{[0,1]} \cdots \\ \cdots \times_{[0,1]} \mathfrak{H}_{k_l+1}) \to \mathfrak{H}_{k_1+\cdots+k_l+1} \end{aligned} \tag{71}$$

$$\circ_{\mathfrak{hm},i} : \mathfrak{H}_{(k-j+i)+1} \,{}_{ev_i}\times_{ev_0} \mathfrak{M}_{(j-i+1)+1} \to \mathfrak{H}_{k+1} \tag{72}$$

with the following properties. (We remark that in (71) *we take fiber product over* $[0,1]$ *using the* $[0,1]$ *factor of* π_1*.)*

(1) $\pi_1^{-1}(M \times \{0\}) = \mathfrak{F}_{k+1}$, $\pi_1^{-1}(M \times \{1\}) = \mathfrak{F}'_{k+1}$. *The restriction of the maps* (71), (72) *are the maps* (58), (59), *respectively.*
(2) *The following diagrams commute and are cartesian.*

$$
\begin{array}{ccc}
\mathcal{M}_{l+1} \times \mathcal{F}_{k_1+1} \times \cdots \mathcal{F}_{k_l+1} & \xrightarrow[\circ_{\mathfrak{m}\mathfrak{f}}]{} & \mathcal{F}_{k_1+\cdots+k_l+1} \\
{\scriptstyle \pi_0\times\cdots\times\pi_0}\big\uparrow & & {\scriptstyle \pi_0}\big\uparrow \\
\mathfrak{M}_{l+1\ \pi_2} \times (\mathfrak{H}_{k_1+1} \times_{[0,1]} \cdots \times_{[0,1]} \mathfrak{H}_{k_l+1}) & \xrightarrow[\circ_{\mathfrak{m}\mathfrak{h}}]{} & \mathfrak{H}_{k_1+\cdots+k_l+1}
\end{array}
\tag{73}
$$

$$
\begin{array}{ccc}
\mathcal{F}_{(k-j+i)+1} \times \mathcal{M}_{(j-i+1)+1} & \xrightarrow[\circ_{\mathfrak{f}\mathfrak{m},i}]{} & \mathcal{F}_{k+1} \\
{\scriptstyle \pi_0\times\pi_0}\big\uparrow & & {\scriptstyle \pi_0}\big\uparrow \\
\mathfrak{H}_{(k-j+i)+1\ ev_i} \times_{ev_0} \mathfrak{M}_{(j-i+1)+1} & \xrightarrow[\circ_{\mathfrak{h}\mathfrak{m},i}]{} & \mathfrak{H}_{k+1}
\end{array}
\tag{74}
$$

(3) *Formulae* (49), (50), (51), (52) *holds after replacing* $\circ_{\mathrm{fm},i}$, $\circ_{\mathrm{mf}}$ *by* $\circ_{\mathfrak{hm},i}$, $\circ_{\mathfrak{mh}}$.
(4) *The following diagrams commute. We put* $k = k_1 + \cdots + k_l$

$$
\begin{array}{ccccc}
M^k & \xleftarrow[\pi_2]{} & \mathfrak{H}_{k+1} & \xrightarrow[\pi_1]{} & M' \times [0,1] \\
\big\| & & \big\uparrow {\scriptstyle \circ_{\mathfrak{m}\mathfrak{f}}} & & \big\| \\
M^k & \longleftarrow & \mathfrak{M}_{l+1\ \pi_2} \times (\mathfrak{H}_{k_1+1} \times_{[0,1]} \cdots \times_{[0,1]} \mathfrak{H}_{k_l+1}) & \longrightarrow & M' \times [0,1]
\end{array}
\tag{75}
$$

$$
\begin{array}{ccccc}
M^k & \xleftarrow[\pi_2]{} & \mathfrak{H}_{k+1} & \xrightarrow[\pi_1]{} & M' \times [0,1] \\
\big\| & & \big\uparrow {\scriptstyle \circ_{\mathfrak{h}\mathfrak{m},i}} & & \big\| \\
M^k & \longleftarrow & \mathfrak{H}_{(k-j+i)+1\ ev_i} \times_{ev_0} \mathfrak{M}_{(j-i+1)+1} & \xrightarrow[\pi_1 \circ \mathrm{pr}_1]{} & M' \times [0,1]
\end{array}
\tag{76}
$$

(5) *The union of the images of* $\circ_{\mathfrak{mh}}$ *and of* $\circ_{\mathfrak{hm},i}$ *is the boundary of* $\mathfrak{H}_{k+1}$. *Those images intersect only at their boundaries.*
(6) *The identification* (5) *preserves orientations.*

Now we have

Theorem 10. *A homotopy* $\mathfrak{H}$ *between two* A_∞ *correspondences* $\mathfrak{F}$ *and* $\mathfrak{F}'$ *induces a homotopy between the two* A_∞ *homomorphisms induced by Theorem 6.*

The proof can be extracted from [33] §30.10 - 13.

9 Filtered A_∞ algebra and Filtered A_∞ correspondence.

So far we developed a machinery which works at least to construct A_∞ algebra defined by the intersection theory on a manifold. (See Example 2.) To apply the machinery to the case when we use moduli spaces (of pseudo-holomorphic curves for example), we need some generalizations. One generalization we need is related to the fact that the structure constants of the algebraic system we will construct is not a number but a kind of formal power series. The reason why we need to consider formal power series lies on the following fact. In case of the Gromov-Witten theory, for example, the moduli space $\mathfrak{M}_{k+1}$ which appears in the definition of correspondence, is not compact and does not have good compactification either. We need to put some energy bound to prove the compactness of the moduli space of pseudo-holomorphic curves. Filtered A_∞ algebra (and its cousins) will be used in order to take care of this problem.

Definition 12. *A proper submonoid G is a submonoid of $\mathbb{R}_{\ge 0} \times 2\mathbb{Z}$ with the following properties.*

(1) *If $(0,\mu) \in G$ then $\mu = 0$.*
(2) *For each $E_0 \in \mathbb{R}_{\ge 0}$ the set $\{(E,\mu) \in G \mid E \le E_0\}$ is finite.*

This definition is closely related to the Gromov compactness in the theory of pseudo-holomorphic curve.

Let $E : G \to \mathbb{R}_{\ge 0}$, $\mu : G \to \mathbb{Z}$ be the projections to the first and second factors. We define the *Novikov ring* Λ_G^R ([52]) associated with G as the set of all such (formal) series

$$\sum_{\beta \in G} a_i T^{E(\beta)} e^{\mu(\beta)/2} \tag{77}$$

where T and e are formal generators of degree 0 and 2 respectively, and $a_i \in R$ (a commutative ring with unit, which we fixed at the beginning). It is easy to see that, by Definition 12 (2), we can define sum and product between two elements of the form (77), and Λ_G^R becomes a ring.

The ring Λ_G^R is contained in the *universal Novikov ring* $\Lambda_{0,nov}^R$ which is the set of all the (formal) series

$$\sum_{\beta \in G} a_i T^{E_i} e^{\mu_i} \tag{78}$$

where $E_i \in \mathbb{R}_{\ge 0}$ and $\mu_i \in \mathbb{Z}$ are sequences such that $\lim_{i\to\infty} E_i = \infty$. (The fact that Novikov ring is a natural coefficient ring of Floer homology was first observed by Floer. It was used by [38] and [58].)

Remark 5. We consider a monoid G together with a partial order $\le$ such that the following holds.

(1) $g \le g'$, $h \le h'$ implies $g \cdot h \le g' \cdot h'$.
(2) For any g_0 there exists only a finite number of $g \in G$ with $g \le g_0$.
(3) We have $\beta_0 \le g$ for any $g \in G$. Here $\beta_0 = (0,0)$.

We then take the completion of its group ring

$$\hat{R}(G) = \left\{ \sum_{g\in G} a_g[g] \;\middle|\; a_g \in R, \quad \text{infinite sum} \right\}.$$

By (2) we can define a products of two elements of $\hat{R}(G)$ in an obvious way. $\hat{R}(G)$ then is a ring.

In case of our $G \subset \mathbb{R} \times 2\mathbb{Z}$, we have $\hat{R}(G) = \Lambda_G^R$.

In the case when noncommutative G appears (such as the case we consider Lagrangian submanifolds with noncommutative fundamental group in Floer theory) to use appropriate $\hat{R}(G)$ with noncommutative G may give more information.

The reason why we use universal Novikov ring $\Lambda_{0,nov}^R$ here is that, in our application, the monoid G depends on various choices (such as almost complex structure in the case Gromov-Witten or Floer theory). So to state the independence of the structure of the choices, it is more convenient to use $\Lambda_{0,nov}^R$ which contains all of Λ_G^R.

Let $\overline{C}$ be a free graded R module. We put $C = \overline{C}\widehat{\otimes}_R \Lambda_{0,nov}^R$. Here $\widehat{\otimes}_R$ is the completion of the algebraic tensor product $\otimes$ with respect to the non-Archimedean norm defined by the ideal generated by T.

Definition 13. *A structure of G-gapped filtered A_∞ algebra on C is defined by a family of the operations*

$$\mathfrak{m}_{k,\beta} : B_k\overline{C}[1] \to \overline{C}[1]$$

of degree $1 - \mu(\beta)$, for $\beta \in G$ and $k = 0, 1, \cdots$, satisfying the following conditions.

(1) $\mathfrak{m}_{k,\beta_0} = 0$ *if* $\beta_0 = (0, 0)$ *and* $k = 0$.

(2) *We define*

$$\mathfrak{m}_k = \sum_{\beta\in G} T^{E(\beta)} e^{\mu(\beta)/2} \mathfrak{m}_{k,\beta} : B_kC[1] \to C[1].$$

Then it satisfies (33)*.*

Definition 14. *Let $(C, \mathfrak{m})$, $(C', \mathfrak{m}')$ be G-gapped filtered A_∞ algebras. A G-gapped filtered A_∞ homomorphism $\mathfrak{f} : C \to C'$ is defined by a family of R module homomorphisms*

$$\mathfrak{f}_{k,\beta} : B_k\overline{C}[1] \to \overline{C}'[1]$$

of degree $-\mu(\beta)$, for $\beta \in G$ and $k = 0, 1, \cdots$, satisfying the following conditions.

(1) $\mathfrak{m}_{k,\beta_0} = 0$ *if* $\beta_0 = (0,0)$ *and* $k = 0$.
(2) *We define*

$$\mathfrak{f}_k = \sum_\beta T^{E(\beta)} e^{\mu(\beta)/2} \mathfrak{f}_{k,\beta} : B_k C[1] \to C'[1].$$

Then it satisfies (46)*.*

Remark 6. We remark that in the case of *filtered* A_∞ algebra, the maps $\mathfrak{m}_0$ or $\mathfrak{f}_0$ may be nonzero. Filtered A_∞ algebra (resp. homomorphism) is said to be strict if $\mathfrak{m}_0 = 0$ (resp. $\mathfrak{f}_0 = 0$).

We can develop homotopy theory of filtered A_∞ algebra in the same way as that of A_∞ algebra. Namely Propositions 2, 3 and Theorems 7, 8, 9 hold without change. (See [33] for their proofs.) (We can do it for each fixed G.)

We next explain how to obtain a filtered A_∞ algebra and filtered A_∞ homomorphisms by smooth correspondence. We first define $\mathcal{M}_{k+1,\beta}$ for each (k,β) such that $k \ge 2$ or $\beta \ne \beta_0$. We consider $\Sigma = \bigcup_{a\in A} D^2_a$ and $z_i \in \partial\Sigma$ satisfying the same condition as the definition of $\mathcal{M}_{k+1}$ *except the definition of stability.* Let $\beta(\cdot) : A \to G$ be a map with $\beta = \sum \beta(a)$. We define the following stability condition for $(\Sigma; z_1, \cdots, z_k; \beta(\cdot))$.

Definition 15. *For each component* D_a *either one of the following holds.*
(1) D_a *contains at least three marked or singular points.* (2) $\beta(a) \ne (0,0)$*.*

$\mathcal{M}_{k+1,\beta}$ is the set of all the isomorphism classes of such $(\Sigma; z_0, \cdots, z_k; \beta(\cdot))$. This definition is closely related to the notion of stable map [44].

We can define a topology on $\mathcal{M}_{k+1,\beta}$ in an obvious way. Namely, in a neighborhood of $(\Sigma; z_0, \cdots, z_k; \beta(\cdot))$ there are $(\Sigma'; z'_0, \cdots, z'_k; \beta'(\cdot))$ where Σ' is obtained by resolving a singularity $p \in \Sigma$. If a component D_a of Σ' is obtained by gluing two components D_{a_1} and D_{a_2} of Σ at p, then we put $\beta'(a) = \beta(a_1) + \beta(a_2)$.

By a similar gluing as §4, we obtain a continuous map :

$$\circ_i : \mathcal{M}_{k+1,\beta_1} \times \mathcal{M}_{l+1,\beta_2} \to \mathcal{M}_{k+l,\beta_1+\beta_2}.$$

We remark here that the topology on $\mathcal{M}_{k+1,\beta}$ is rather pathological. Namely it is *not* Hausdorff. Let us exhibit it by an example. We consider $\overset{\circ}{\mathcal{M}}_{2+1,\beta}$ (which is the set of elements of $\mathcal{M}_{2+1,\beta}$ with no singularity), for $\beta \ne (0,0)$. Actually $\overset{\circ}{\mathcal{M}}_{2+1,\beta} \cong \overset{\circ}{\mathcal{M}}_{2+1}$ is a point. On the other hand

$$\mathcal{M}_{0+1,\beta} \circ_3 \mathcal{M}_{3+1,\beta_0} \subset \mathcal{M}_{2+1,\beta}.$$

(Here $\beta_0 = (0,0)$.) The left hand side is diffeomorphic to $\mathcal{M}_{3+1}$ and is an interval $[0,1]$.

Thus *any* neighborhood of *any* point of $[0,1] \cong \mathcal{M}_{0+1,\beta} \circ_3 \mathcal{M}_{4+1,\beta_0}$ contains the point $\overset{\circ}{\mathcal{M}}_{2+1,\beta}$. Thus $\mathcal{M}_{2+1,\beta}$ is not Hausdorff.

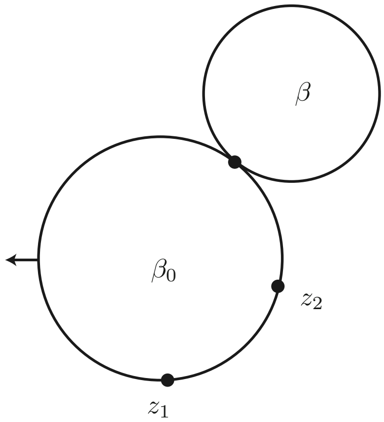

Fig. 13.

Remark 7. An appropriate language to describe this situation is (Artin) stack. We do not use the notion of stack later in this article. So the reader can skip this remark safely if he wants.

Let us consider the example we discussed above. We consider an element $(\Sigma; z_0, z_1, z_2, z_3; \beta(\cdot)) \in \mathcal{M}_{0,\beta} \circ_3 \mathcal{M}_{3+1,\beta_0}$. The group of its automorphisms is the group $\mathrm{Aut}(D^2, \{1\})$ which consists of the biholomorphic maps $u : D^2 \to D^2$ with $u(1) = 1 \in \partial D^2$. This group is isomorphic to

$$\left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \;\middle|\; a \in \mathbb{R}_+,\ b \in \mathbb{R} \right\} \subset PSL(2;\mathbb{R}) \cong \mathrm{Aut}(D^2).$$

The (infinitesimal) neighborhood of the element $(\Sigma; z_0, z_1, z_2, z_3; \beta(\cdot))$ in $\mathcal{M}_{2+1,\beta}$ is, by definition, a quotient of

$$\mathcal{M}_{0+1,\beta} \circ_3 \mathcal{M}_{3+1,\beta_0} \;\times\; [0,\epsilon)$$

by the action of the group $\mathrm{Aut}(D^2, \{1\})$.

Note the parameter a above acts on $[0,\epsilon)$ factor by $t \mapsto at$. Also the element $\partial/\partial b$ in the Lie algebra of $\mathrm{Aut}(D^2, \{1\})$ moves the position of third marked point z_3 of the $\mathcal{M}_{3+1,\beta_0}$ factor, when $[0,\epsilon)$ factor is positive. Thus we have

$$\frac{\mathcal{M}_{0+1,\beta} \circ_3 \mathcal{M}_{3+1,\beta_0} \;\times\; [0,\epsilon)}{\mathrm{Aut}(D^2, \{1\})} = (\mathcal{M}_{0+1,\beta} \circ_3 \mathcal{M}_{3+1,\beta_0} \times \{0\}) \cup \{\text{one point}\}.$$

Here $\{$one point$\}$ in the right hand side is the quotient of $\mathcal{M}_{0+1,\beta} \circ_3 \mathcal{M}_{3+1,\beta_0} \times (0,\epsilon)$ by $\mathrm{Aut}(D^2, \{1\})$ action and is identified with $\overset{\circ}{\mathcal{M}}_{2+1,\beta}$.

Thus the neighborhood of $(\Sigma; z_0, z_1, z_2, z_3; \beta(\cdot))$ in $\mathcal{M}_{3+1,\beta_0}$ is as we mentioned above.

The fact that $\mathcal{M}_{3+1,\beta_0}$ is not Hausdorff is a consequence of the noncompactness of the group $\mathrm{Aut}(D^2, \{1\})$.

Now we define the notion of G-gapped filtered A_∞ correspondence. Actually the definition is almost the same as Definition 4. Suppose we have a family of the following commutative diagrams.

$$\begin{array}{ccccc} & & \mathcal{M}_{k+1,\beta} & & \\ & & \uparrow{\scriptstyle \pi_0} & & \\ M^k & \xleftarrow[\pi_2]{} & \mathfrak{M}_{k+1}(\beta) & \xrightarrow[\pi_1]{} & M \end{array} \tag{79}$$

such that $\mathfrak{M}_{k+1}(\beta)$ is a smooth (Hausdorff) manifold with boundary or corners and that

$$\dim \mathfrak{M}_{k+1}(\beta) = \dim M + \mu(\beta) + k - 2. \tag{80}$$

We assume also that we have a family of smooth maps

$$\circ_{\mathfrak{m},i} : \mathfrak{M}_{k+1}(\beta_1)\ {}_{ev_i}\times_{ev_0} \mathfrak{M}_{l+1}(\beta_2) \to \mathfrak{M}_{k+l}(\beta_1+\beta_2). \tag{81}$$

To state one of the conditions (cartesian axiom (2) below) we need some notations. Let us consider the space $\mathcal{M}_{k+l,\beta}$. It is not Hausdorff as mentioned before. It is decomposed into union of *smooth manifolds* according to its combinatorial structure. Namely if we collect all of the elements $(\Sigma; z_0, \cdots, z_k; \beta(\cdot))$ which is homeomorphic to a given element of $\mathcal{M}_{k+l,\beta}$ then it is a smooth manifold. (This manifold is actually a ball.) We write this stratum $\mathcal{M}_{k+l,\beta}(\mathbf{S})$, where $\mathbf{S}$ stands for a homeomorphism type of $(\Sigma; z_0, \cdots, z_k; \beta(\cdot))$. (We remark that $\mathcal{M}_{k+l,\beta}(\mathbf{S})$ is Hausdorff.)

Definition 16. *A system of objects as in* (79), (80), (81) *is said to be a G-gapped filtered A_∞ correspondence if the following holds.*

(1) *The following diagram commutes.*

$$\begin{array}{ccc} \mathcal{M}_{k+1,\beta_1} \times \mathcal{M}_{l+1,\beta_2} & \xrightarrow[\circ_i]{} & \mathcal{M}_{k+l,\beta_1+\beta_2} \\ {\scriptstyle \pi_0\times\pi_0}\uparrow & & {\scriptstyle \pi_0}\uparrow \\ \mathfrak{M}_{k+1}(\beta_1)\ {}_{ev_0}\times_{ev_i} \mathfrak{M}_{l+1}(\beta_2) & \xrightarrow[\circ_{\mathfrak{m},i}]{} & \mathfrak{M}_{k+l}(\beta_1+\beta_2) \end{array} \tag{82}$$

(2) *The inverse image*

$$\pi_0^{-1}(\mathcal{M}_{k+l,\beta}(\mathbf{S})) \subset \mathfrak{M}_{k+l}(\beta) \tag{83}$$

of each such stratum $\mathcal{M}_{k+l,\beta}(\mathbf{S})$ of $\mathcal{M}_{k+l,\beta}$ is a smooth submanifold of $\mathfrak{M}_{k+l}(\beta)$. Its codimension is the number of singular points of $\mathbf{S}$. We denote (83) *by $\mathfrak{M}_{k+l}(\beta;\mathbf{S})$. Restriction of ev_0 to each such stratum $\mathfrak{M}_{k+l}(\beta;\mathbf{S})$ is a smooth map $\mathfrak{M}_{k+l}(\beta;\mathbf{S}) \to \mathcal{M}_{k+l,\beta}(\mathbf{S})$. Diagram* (82) *is a cartesian diagram as a diagram of sets.*

(3) *Formulae* (39) *and* (40) *hold.*

(4) *The following diagram commutes.*

$$
\begin{array}{ccccc}
M^{k+l} & \xleftarrow[\pi_2]{} & \mathfrak{M}_{k+l}(\beta_1+\beta_2) & \xrightarrow[\pi_1]{} & M \\
\| & & \uparrow{\scriptstyle \circ_{\mathfrak{m},i}} & & \| \\
M^{k+l} & \longleftarrow & \mathfrak{M}_{k+1}(\beta_1)\ {}_{ev_i}\times_{ev_0} \mathfrak{M}_{l+1}(\beta_2) & \xrightarrow[\pi_1\circ \mathrm{pr}_2]{} & M
\end{array}
\tag{84}
$$

(5) *For each n the boundary of $\mathfrak{M}_{n+1}(\beta)$ is a union of*

$$\circ_{\mathfrak{m},i}(\mathfrak{M}_{k+1}(\beta_1)\ {}_{ev_i}\times_{ev_0} \mathfrak{M}_{l+1}(\beta_2))$$

for various $k, l, i, \beta_1, \beta_2$ with $k+l = n$, $i = 1, \cdots, l$, $\beta_1+\beta_2 = \beta$. They intersect each other only at their boundaries.
(6) *The identification in* (5) *preserves orientation.*

Note the axiom (2) above is more complicated than the corresponding axiom in Definition 4. This is because $\mathcal{M}_{k+1,\beta}$ is not Hausdorff and hence we can not say the Diagram (82) being cartesian in the category of smooth manifolds. One might say that the diagram (82) is cartesian in the sense of stacks. (The author wants to avoid using the notion of stack here since he is not familiar with it.)

We can define the notion of (G-gapped filtered) morphism between two G-gapped filtered A_∞ correspondences in the same way as Definition 8. The homotopy between two morphisms are defined in the same way as Definition 11. We then have :

Theorem 11. *G-gapped filtered A_∞ correspondence on M induces a structure of G-gapped filtered A_∞ algebra on a cochain complex representing the cohomology group of M.*

A morphism between G-gapped filtered A_∞ correspondences induces a G-gapped filtered A_∞ homomorphism. A homotopy between two morphisms induces a homotopy between G-gapped filtered A_∞ homomorphisms.

The proof of this theorem can be extracted from [33] §30.

10 Kuranishi correspondence

To study correspondence, manifold is too much restrictive category of spaces, since we can not take fiber product in general, for example. As a consequence, composition of correspondences is defined only under some transversality assumptions. We can use the notion of Kuranishi structure to resolve this problem. Moreover Kuranishi structure is a general frame work to handle various transversality problem and to study moduli spaces arising in differential geometry, in a uniform way. In this section we define the notion of A_∞ Kuranishi correspondence and use it to generalize Theorem 11 furthermore.

We first review briefly the notion of Kuranishi structure. (See [30] Chapter 1 or [33] Appendix 1 for more detail.) The notion of Kuranishi structure is simple and elementary. The author believe that the main obstacle to understand it is rather psychological. Actually the definition of Kuranishi structure is very much similar to the definition of manifold.

We consider a space Z which is Hausdorff and compact.

Definition 17. *A Kuranishi chart is a quintet* (V, E, Γ, s, ψ) *such that*

(1) $V \subset \mathbb{R}^n$ *is an open set and* $E = V \times \mathbb{R}^m$*. Here* n, m *are nonnegative integers which may depend on the chart.*
(2) Γ *is a finite group acting effectively on* V *and has a linear action on the fiber* $\mathbb{R}^m$ *of* E*.*
(3) $s : V \to \mathbb{R}^m$ *is a* Γ *equivalent map.*
(4) ψ *is a homeomorphism from* $s^{-1}(0)/\Gamma$ *to an open subset of* Z*.*

If $p \in \psi(s^{-1}(0))$ we call (V, E, Γ, s, ψ) a *Kuranishi neighborhood* of p. Sometimes we call V a Kuranishi neighborhood, by abuse of notation. We call E the *obstruction bundle.* s is called the *Kuranishi map.* We remark that in case Z is a moduli space s is actually a Kuranishi map in the usual sense.

A Kuranishi chart is said to be it oriented if there is an orientation of $\Lambda^{\rm top} TV \otimes \Lambda^{\rm top} E$ which is preserved by the Γ-action.

Remark 8. Roughly speaking Kuranishi neighborhood of p gives a way to describe a neighborhood of p in Z as a solution of an equation $s(x) = (s_1(x), \cdots, s_m(x)) = 0$. In case s_i is a polynomial, it defines a structure of scheme as follows. (More precisely since the finite group action is involved it gives a structure of Deligne-Mumford stack.) Let us consider the quotient ring R_s of the polynomials ring $R[X_1, \cdots, X_n]$ by the ideal which is generated by the polynomials $s_i(X_1, \cdots, X_n)$ $(i = 1, \cdots, m)$. Then we obtain a ringed space $\mathrm{Spec}(R_s)$, that is the affine scheme defined by the ring R_s. By gluing them we obtain a scheme.

If we try to apply this construction of algebraic geometry to differential geometry (that is our situation), then we will be in the trouble. We can indeed construct a sheaf of rings (of smooth functions modulo the components of s) on Z and then Z becomes a ringed space (that is a space together with a sheaf of local ring). However the structure of ringed space does not seem to hold enough information we need. For example, since the Krull dimension of the ring of the germs of smooth functions is infinite, it follows that the dimension (that is $n - m$ in case of (1) of Definition 17) does not seem to be determined from the structure of ringed space. So it seems difficult to obtain the notion of (virtual) fundamental chains using the structure of ringed space.

Therefore, in place of using the structure of ringed space, we 'remember' the equation $s = 0$ itself as a part of the structure and glue the Kuranishi chart in that sense as follows.

Let $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ and $(V_j, E_j, \Gamma_j, s_j, \psi_j)$ be two Kuranishi charts.

Definition 18. *A coordinate change from* $(V_i, E_i, \Gamma_p, s_i, \psi_i)$ *to* $(V_j, E_j, \Gamma_j, s_j, \psi_j)$ *consists of a* Γ_i*-invariant open subset* $V_{ji} \subset V_i$*, maps* $\phi_{ji} : V_{ji} \to V_j$*,* $\hat\phi_{ji} : E_i|_{V_{ji}} \to E_j$ *and a homomorphism* $h_{ji} : \Gamma_i \to \Gamma_j$ *with the following properties.*

(1) ϕ_{ji} *is an* h_{ji}*-equivariant smooth embedding.*
(2) $\hat\phi_{ji}$ *is an* h_{ji}*-equivariant embedding of the vector bundles over* ϕ_{ji}*.*
(3) h_{ji} *is defined and is injective if* $V_{ji} \neq \emptyset$*.*
(4) $s_j \circ \phi_{ji} = \hat\phi_{ji} \circ s_i$*.*
(5) $\psi_j \circ \phi_{ji} = \psi_i$ *on* $(s_i^{-1}(0) \cap V_{ji})/\Gamma_j$*.*
(6) $\psi_i((s_i^{-1}(0) \cap V_{ji})/\Gamma_i)$ *contains a neighborhood of* $\psi_i(s_i^{-1}(0)/\Gamma_i) \cap \psi_j(s_j^{-1}(0)/\Gamma_j)$ *in* Z*.*
(7) *If* $\gamma(\phi_{ji}(V_{ji})) \cap \phi_{ji}(V_{ji}) \neq \emptyset$ *and* $\gamma \in \Gamma_j$*, then* $\gamma \in h_{ji}(\Gamma_i)$*.*

Remark 9. We can state (7) also as $h_{ij}((\Gamma_j)_p) = (\Gamma_i)_{\phi_{ij}(p)}$. (Here $(\Gamma_j)_p$ etc. is the isotropy group.) This condition is assumed in [30] as a part of the assumption that $V_{ij}/\Gamma_j \to V_i/\Gamma_i$ is an embedding of orbifold. Therefore Definition 18 (including (7)) is equivalent to [30] Definition 5.3.

We say that coordinate change is compatible with orientation if there exists a Γ_i equivariant bundle isomorphism

$$\frac{\phi_{ji}^* TV_j}{TV_{ji}} \cong \frac{\phi_{ji}^* E_j}{E_i|_{ji}} \tag{85}$$

which is compatible with orientations of $\Lambda^{\mathrm{top}} TV_i \otimes \Lambda^{\mathrm{top}} E_i$ and of $\Lambda^{\mathrm{top}} TV_j \otimes \Lambda^{\mathrm{top}} E_j$.

Definition 19. *A Kuranishi structure on a compact metrizable space* Z *is* $((V_i, E_i, \Gamma_i, s_i, \psi_i); i \in I)$ *with the following properties. Here* $(I, \preceq)$ *is a partially ordered set.*

(1) *If and* $i \preceq j$*, then we have a coordinate change* $(\phi_{ji}, \hat\phi_{ji}, h_{ji})$*.*
(2) *If* $\psi_i(s_i^{-1}(0)/\Gamma_i) \cap \psi_j(s_j^{-1}(0)/\Gamma_j) \neq \emptyset$ *then either* $i \preceq j$ *or* $j \preceq i$ *holds.*
(3) *If* $i \preceq j \preceq k$ *and* $V_{ki} \cap \phi_{ji}^{-1}(V_{kj}) \neq \emptyset$ *then*

$$\phi_{kj} \circ \phi_{ji} = \phi_{ki}, \quad \hat\phi_{kj} \circ \hat\phi_{ji} = \hat\phi_{ki}, \quad h_{kj} \circ h_{ji} = h_{ki}$$

on $V_{ki} \cap \phi_{ji}^{-1}(V_{kj})$*.*
(4) $\psi_i(s_i^{-1}(0)/\Gamma_i)$ $(i = 1, \cdots, I)$ *is an open covering of* Z*.*

We call $((V_i, E_i, \Gamma_i, s_i, \psi_i); i \in I)$ *a Kuranishi atlas.*

Note Kuranishi chart Definition 19 is called a good coordinate system in [30] Definition 6.1. Hence by [30] Lemma 6.3 the above definition of Kuranishi structure is equivalent to one in [30].

Kuranishi structure is said to have tangent bundle and is oriented if the all the coordinate changes preserve orientation and if we have a commutative diagram :

$$\begin{array}{ccccc} \dfrac{\phi_{ji}^* TV_j}{TV_{ji}} & \longrightarrow & \dfrac{\phi_{ki}^* TV_k}{TV_i} & \longrightarrow & \dfrac{\phi_{kj}^* TV_k}{TV_j} \\ \cong\Big\downarrow & & \cong\Big\downarrow & & \cong\Big\downarrow \\ \dfrac{E_j}{\hat\phi_{ji}(E_i)} & \longrightarrow & \dfrac{E_k}{\hat\phi_{ki}(E_i)} & \longrightarrow & \dfrac{E_k}{\hat\phi_{kj}(E_j)} \end{array} \tag{86}$$

for i where the vertical arrows are as in (85) and horizontal lines are obvious exact sequences.

We thus defined a 'space with Kuranishi structure which has a tangent bundle and is oriented'. Since the notation in the quote is rather lengthy we call it oriented *K-space* from now on. More precisely when we say K-space, we *fix* isomorphism (85) such that Diagram (86) commutes, *as a part of the structure.*

We remark that by (85)

$$\dim V_i - \operatorname{rank} E_i \tag{87}$$

is independent of i (if Z is connected). We call it the *dimension* of the Kuranishi structure or K-space.

We next define a map from K-space to a manifold.

Definition 20. *Let M be a manifold and Z be a K-space. A system $f = (f_i)$ of maps $f_i : V_i \to N$ is said to be a strongly smooth map if each of f_i is smooth and $f_j \circ \phi_{ji} = f_i$.*

$f = (f_i)$ is said to be weakly submersive if each of $f_i : V_i \to N$ is a submersion.

We remark that strongly smooth map induces a continuous map $Z \to N$ in an obvious way.

We also define K-space with boundary and corners as follows. If we replace the condition 'V is an open subsets in $\mathbb{R}^n$' in (1) of Definition 17, by 'V is an n dimensional submanifold with corners in $\mathbb{R}^n$' it will be the definition of Kuranishi neighborhood with corners. We then proceed in the same way as Definitions 18, 19 we define (oriented) K-space with corners.

A point x of K-space is said to be in the codimension k corner if $x = \psi_i(y)$ with y in the codimension k corner of V_i/Γ_i. We can easily show that the set of all codimension k corner of a given K-space Z has a structure of K-space with corners.

For our purpose to study correspondence, the notion of fiber product of K-space is important. Let Z and Z' be K-spaces with their Kuranishi atlas $((V_i, E_i, \Gamma_i', s_i, \psi_i); i \in I)$, $((V_{i'}', E_{i'}', \Gamma_{i'}, s_{i'}', \psi_{i'}'); i' \in I')$, respectively.

Let (f_i) and $(f_{i'}')$ be weakly submersive strongly smooth maps from Z to N and Z' to N, respectively. Here N is a smooth manifold. We first take a fiber product $Z \times_N Z'$ in the category of topological space. The next lemma is in [33] §A1.2.

Lemma 6. *$Z \times_N Z'$ has a structure of K-space.*

Proof : We consider the fiber products

$$V_{(i,i')} = V_i \ {}_{f_i}\times_{f'_{i'}} V'_{i'}.$$

By the assumption that (f_i) and $(f'_{i'})$ are weakly submersive, the above fiber product is well-defined in the category of smooth manifold. We define $E_{(i,i')}$ as the pull back of the exterior product of E_i and $E'_{i'}$. The group $\Gamma_{(i,i')} = \Gamma_i \times \Gamma'_{i'}$ acts on $V_{(i,i')}$ and $E_{(i,i')}$ as the restriction of the direct product action. Using weakly submersivity of f_i, $f'_{i'}$ we can prove that this action is effective. We put $s_{(i,i')}(x,y) = (s_i(x), s'_{i'}(y))$. It is easy to see that

$$s_{(i,i')}^{-1}(0) = s_i^{-1}(0) \ {}_{f_i}\times_{f'_{i'}} s'^{-1}_{i'}(0).$$

Hence we obtain $\phi_{(i,i')}$. Thus we have a Kuranishi chart

$$(V_{(i,i')}, E_{(i,i')}, \Gamma_{(i,i')}, s_{(i,i')}, \phi_{(i,i')}).$$

It is easy to see that we can glue coordinate transformation and construct a K-space. □

Remark 10. In the above construction, it may happen that $i_1 \prec i_2$, $i'_1 \prec i'_2$, $\psi_{(i_1,i'_2)}(s^{-1}_{(i_1,i'_2)}(0)) \cap \psi_{(i'_1,i_2)}(s^{-1}_{(i'_1,i_2)}(0)) \neq \emptyset$, but neither $(i_1,i'_2) \prec (i'_1,i_2)$ nor $(i'_1,i_2) \prec (i_1,i'_2)$. In this case Definition 19 (2) is not satisfied. However we can shrink $V_{i,i'}$ in the way as in Figure 14 below so that Definition 19 (2) is satisfied.

In the situation of Lemma 6, we assume that we also have a strongly smooth map $g : Z' \to N'$ such that $f' \times g : Z' \to N \times N'$ are weakly submersive. Then, it is easy to see that g induces a strongly smooth map $g : Z \times_N Z' \to N'$ which is weakly submersive.

A few more notations are in order.

Let Z be a K-space with corner and $p, q \in \partial Z$. We say that they are in the same *component* of ∂Z and write $p \sim q$ if there exists a sequence of Kuranishi charts $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ $(i = 0, \cdots, l)$ such that

(1) $p = \psi_1(x_1) \in \psi_1(s_1^{-1}(0)/\Gamma_1)$ and $q = \psi_l(x'_l) \in \psi_l(s_l^{-1}(0)/\Gamma_l)$.
(2) Either $x_i = \phi_{i(i+1)}(x'_i)$ or $x'_i = \phi_{(i+1)i}(x_i)$. Here in the first case $i+1 \prec i$ and $x'_i \in \partial V_{i(i+1)}$. In the second case $i \prec i+1$ and $x_i \in \partial V_{(i+1)i}$.
(3) x'_i and x_{i+1} can be joined by a path which is contained in ∂V_{i+1}.

A component of ∂Z is a closure of $\sim$ equivalence class. It has a structure of K-space.

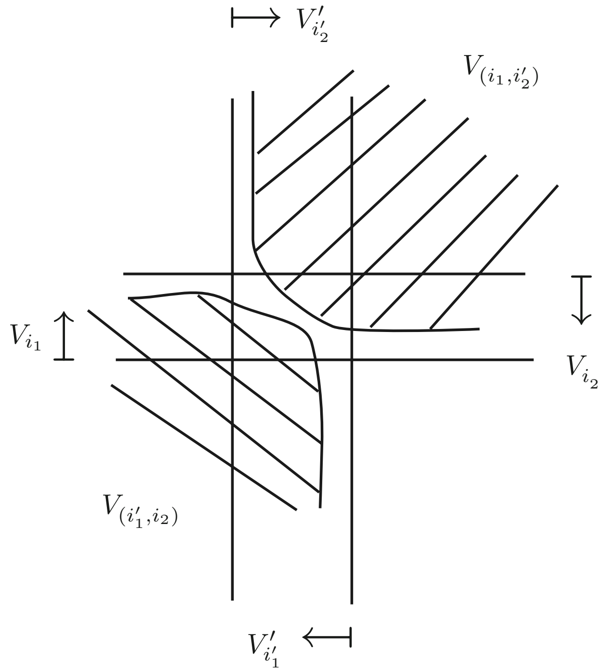

Fig. 14.

We say that 'p and q are connected by a path contained in the set of boundary points in the Kuranishi neighborhood' if above condition is satisfied.

Let $p, q \in \partial Z \setminus$ corner. We say that they are in the same *stratum* of ∂Z and write $p \sim' q$, if p and q are connected by a path which is contained in the set of boundary points in the Kuranishi neighborhood and does not intersect with corner points. A stratum is a closure of $\sim'$ equivalence class. It has a structure of K-space.

We can define stratum of codmension d corner of Z in the same way. It has a structure of K-space also.

Let M be a closed and oriented manifold. We assume that we have a diagram

$$\begin{array}{ccccc} & & \mathcal{M}_{k+1,\beta} & & \\ & & \uparrow{\scriptstyle \pi_0} & & \\ M^k & \xleftarrow[\pi_2]{} & \mathfrak{M}_{k+1}(\beta) & \xrightarrow[\pi_1]{} & M. \end{array} \tag{88}$$

Here $\mathfrak{M}_{k+1}(\beta)$ is a K-space and

$$(\pi_0, \pi_1, \pi_2) : \mathfrak{M}_{k+1}(\beta) \to \mathcal{M}_{k+1,\beta} \times M^{k+1}$$

is assumed to be strongly smooth and is *weakly submersive*. We assume

$$\dim \mathfrak{M}_{k+1}(\beta) = \dim M + \mu(\beta) + k - 2. \tag{89}$$

Definition 21. (88) *is said to be a G-gapped filtered Kuranish A_∞ correspondence, if there exists a map*

$$\circ_{\mathfrak{m},i} : \mathfrak{M}_{k+1}(\beta_1)\ {}_{ev_i}\times_{ev_0} \mathfrak{M}_{l+1}(\beta_2) \to \mathfrak{M}_{k+l}(\beta_1+\beta_2). \tag{90}$$

which identifies the left hand side with a union of finitely many stratum of the boundary of the right hand side as K-spaces, such that the following holds.

(1) *The following diagram commutes.*

$$\begin{array}{ccc} \mathcal{M}_{k+1,\beta_1} \times \mathcal{M}_{l+1,\beta_2} & \xrightarrow[\circ_i]{} & \mathcal{M}_{k+l,\beta_1+\beta_2} \\ {\scriptstyle \pi_0\times\pi_0}\big\uparrow & & {\scriptstyle \pi_0}\big\uparrow \\ \mathfrak{M}_{k+1}(\beta_1)\ {}_{ev_i}\times_{ev_0} \mathfrak{M}_{l+1}(\beta_2) & \xrightarrow[\circ_{\mathfrak{m},i}]{} & \mathfrak{M}_{k+l}(\beta_1+\beta_2) \end{array} \tag{91}$$

(2) *The inverse image*

$$\pi_0^{-1}(\mathcal{M}_{k+l,\beta}(\mathbf{S})) \subset \mathfrak{M}_{k+l}(\beta) \tag{92}$$

of each such stratum $\mathcal{M}_{k+l,\beta}(\mathbf{S})$ of $\mathcal{M}_{k+l,\beta}$ is a union of strata of codimension d corner of $\mathfrak{M}_{k+l}(\beta)$. Here d is the number of singular points of $\mathbf{S}$. We denote (92) *by $\mathfrak{M}_{k+l}(\beta;\mathbf{S})$. Restriction of ev_0 to each such stratum $\mathfrak{M}_{k+l}(\beta;\mathbf{S})$ is a strongly smooth weakly submersive map $\mathfrak{M}_{k+l}(\beta;\mathbf{S}) \to \mathcal{M}_{k+l,\beta}(\mathbf{S})$. Diagram* (91) *is a cartesian diagram as a diagram of sets.*
(3) *The Formulae* (24)*,* (25) *holds after replacing $\circ_i$ by $\circ_{\mathfrak{m},i}$*
(4) *The following diagram commutes.*

$$\begin{array}{ccccc} M^{k+l} & \xleftarrow[\pi_2]{} & \mathfrak{M}_{k+l}(\beta_1+\beta_2) & \xrightarrow[\pi_1]{} & M \\ \Big\Vert & & \big\uparrow{\scriptstyle \circ_{\mathfrak{m},i}} & & \Big\Vert \\ M^{k+l} & \longleftarrow & \mathfrak{M}_{k+1}(\beta_1)\ {}_{ev_i}\times_{ev_0} \mathfrak{M}_{l+1}(\beta_2) & \xrightarrow[\pi_1\circ \mathrm{pr}_1]{} & M \end{array} \tag{93}$$

(5) *For each n the boundary of $\mathfrak{M}_{n+1}(\beta)$ is a union of*

$$\circ_{\mathfrak{m},i}(\mathfrak{M}_{k+1}(\beta_1)\ {}_{ev_i}\times_{ev_0} \mathfrak{M}_{l+1}(\beta_2))$$

for various k,l,i,β_1,β_2 with $k+l=n$, $i=1,\cdots,l$, $\beta_1+\beta_2=\beta$. They intersect each other only at their boundaries.
(6) *The identification* (5) *preserves orientations, with signs which will be described by Definition* 27.

In the rest of this article we say Kuranishi correspondence sometimes in place of G-gapped filtered Kuranish A_∞ correspondence for simplicity.

Now, in a similar way as the Definitions 8 and 11, we can define morphism between Kuranishi correspondences, and homotopy between morphisms. To rewrite Definitions 8 and 11 to our situation is straightforward and hence we omit them here.

As we mentioned before we can define composition of morphisms between Kuranishi correspondences, as follows. Let

$$\begin{array}{ccccc} & & \mathcal{M}_{k+1,\beta} & & \\ & & \uparrow{\scriptstyle \pi_0} & & \\ M_i^k & \xleftarrow[\pi_2]{} & \mathfrak{M}^{(i)}_{k+1}(\beta) & \xrightarrow[\pi_1]{} & M_i. \end{array} \tag{94}$$

be Kuranishi correspondences for $i = 1, 2, 3$. Let

$$\begin{array}{ccccc} & & \mathcal{F}_{k+1,\beta} & & \\ & & \uparrow{\scriptstyle \pi_0} & & \\ M_i^k & \xleftarrow[\pi_2]{} & \mathfrak{F}^{(ij)}_{k+1}(\beta) & \xrightarrow[\pi_1]{} & M_j \end{array} \tag{95}$$

be morphisms between Kuranishi correspondences for $(ij) = (12), (23)$. We will define a morphism of Kuranishi correspondence $(\mathfrak{F}^{(13)}_{k+1}(\beta))$ which is a composition of $(\mathfrak{F}^{(12)}_{k+1}(\beta))$ and $(\mathfrak{F}^{(23)}_{k+1}(\beta))$ as follows.

In the same way as (64) we can define

$$\begin{aligned} \mathrm{Comp}^{\beta_0,\beta_1,\cdots,\beta_l}_{k_1,\cdots,k_l;k'_0,\cdots;k'_l} : &\mathcal{F}_{l+k'+1,\beta_0} \\ &\times (\mathcal{F}_{k_1+1,\beta_1} \times \cdots \mathcal{F}_{k_l+1,\beta_l}) \to \mathcal{F}_{k+k'+1,\beta_0+\cdots+\beta_l} \end{aligned} \tag{96}$$

where $k' = k'_0 + \cdots + k'_l$ and $k_1 + \cdots + k_l = k$. By a filtered analogue of Lemma 5, the images of (96) (for various l, k_i, k'_i, β_i with $k + k' = n$, $\sum \beta_i = \beta$) decompose $\mathcal{F}_{n+1,\beta}$.

Now we consider the fiber product

$$\mathfrak{F}^{\beta_0,\beta_1,\cdots,\beta_l}_{k_1,\cdots,k_l;k'_0,\cdots;k'_l} = \mathfrak{F}^{(12)}_{l+k'+1}(\beta_0) \times_{M_2^l} (\mathfrak{F}^{(23)}_{k_1+1}(\beta_1) \times \cdots \times \mathfrak{F}^{(23)}_{k_l+1}(\beta_l)). \tag{97}$$

Here the map

$$\mathfrak{F}_{k_1+1}(\beta_1) \times \cdots \times \mathfrak{F}_{k_l+1}(\beta_l) \to M_2^l$$

is $(ev_0, \cdots, ev_0)$ and

$$\mathfrak{F}_{l+k'+1}(\beta_0) \to M_2^l$$

is $(ev_{k'_0+1}, ev_{k'_1+2}, \cdots, ev_{k'_{l-1}+l})$. (See (65).) Now by a filtered analogue of Lemma 5, we can glue spaces $\mathfrak{F}^{\beta_0,\beta_1,\cdots,\beta_l}_{k_1,\cdots,k_l;k'_0,\cdots;k'_l}$ for various l, k_i, k'_i, β_i with

$k + k' = n$, $\sum \beta_i = \beta$ along their boundaries to obtain a K-space $\mathfrak{F}^{(13)}_{n+1}(\beta)$. Moreover we can define

$$\pi_0 : \mathfrak{F}^{(13)}_{n+1}(\beta) \to \mathcal{F}^{(13)}_{n+1,\beta}$$

such that (96), (97) commute with π_0. Furthermore we can define π_1 and π_2 such that

$$\begin{array}{ccccc} & & \mathcal{F}_{k+1,\beta} & & \\ & & \uparrow{\scriptstyle \pi_0} & & \\ M_1^k & \xleftarrow[\pi_2]{} & \mathfrak{F}^{(13)}_{k+1}(\beta) & \xrightarrow[\pi_1]{} & M_3 \end{array} \tag{98}$$

is a morphism between Kuranishi correspondences.

Definition 22. (98) *is a composition of* $(\mathfrak{F}^{(12)}_{k+1}(\beta))$ *and* $(\mathfrak{F}^{(23)}_{k+1}(\beta))$. *We write it as :*

$$(\mathfrak{F}^{(13)}_{k+1}(\beta)) = (\mathfrak{F}^{(23)}_{k+1}(\beta)) \circ (\mathfrak{F}^{(12)}_{k+1}(\beta)).$$

It is easy to see that the notion of composition between morphisms are compatible with the notion of homotopy of morphisms.

Lemma 7. *Composition of morphisms are homotopy associative. Namely, the morphisms of Kuranishi correspondence*

$$(\mathfrak{F}^{(34)}_{k+1}(\beta)) \circ \Big((\mathfrak{F}^{(23)}_{k+1}(\beta)) \circ (\mathfrak{F}^{(12)}_{k+1}(\beta))\Big)$$

and

$$\Big(\mathfrak{F}^{(34)}_{k+1}(\beta) \circ (\mathfrak{F}^{(23)}_{k+1}(\beta))\Big) \circ (\mathfrak{F}^{(12)}_{k+1}(\beta))$$

are homotopic to each other.

Proof : We remark that in the definition of (66), the time allocation of the component of the stable curve Σ' which comes from the the first factor Σ lies in $[0,1/2]$. For the component which comes from the second factors Σ_i, its time allocation is in $[1/2,1]$. Hence by the same reason as the nonassociativity of the product in loop space (see §4), our composition is not strictly associative. However by the same reason as the product in loop space is homotopy associative, we can easily construct the required homotopy by using the homotopy between parametrizations. □

Definition 23. *We define a category* $\mathfrak{HAKCorr}_G$ *as follows. Its object is a manifold* M *together with a* G*-filtered Kruranishi correspondence on it. A morphisms between them is a homotopy class of the morphisms of Kuranishi correspondence. The composition is defined by Definition 22. It is well-defined by Lemma 7. We call it the homotopy category of* G*-filtered Kuranishi correspondence.*

We define a category $\mathfrak{HAlg}_G^{\mathbb{Q}}$ as follows. Its object is a G-filtered A_∞ algebra with $\mathbb{Q}$ coefficient. The morphism is a homotopy class of G-filtered A_∞ homomorphisms. We call it the homotopy category of G-filtered A_∞ algebras.

Now the main theorem of this paper is as follows.

Theorem 12. *There exists a functor $\mathfrak{HAKCorr}_G \to \mathfrak{HAlg}_G^{\mathbb{Q}}$.*

The proof can be extracted from [33]. It is proved also in §12, §13 over $\mathbb{R}$ coefficient.

Remark 11. In Definition 23, we take *homotopy class* of the morphism of Kuranishi correspondence as a morphism of our categories. One of the reasons we did so is the fact that the associativity holds only up to homotopy. On the other hand, as we explained in the proof of Lemma 7, the way how the associativity breaks down is the same as the way how the associativity of the product in loop space breaks down. Therefore, it is very likely that we can define an appropriate notion of 'A_∞ category' (or ∞ category) in place of taking the quotient by the homotopy. Note A_∞ category as defined in [20] is a category where the set of morphisms has a structure of chain complex. The 'A_∞ category' above is different from that. Namely the set of morphisms do not have a structure of chain complex. Its relation to the A_∞ category in [20] is similar to the relation of A_∞ space to A_∞ algebra.

On the other hand, there is a notion of 2-category of A_∞ category. (See [48] §7.) In particular, there is a 2-category of A_∞ algebras. Since, in the world of A_∞ structure, we can define 'homotopies of homotopies of ... of homotopies of ... ' in a natural way (see [33] §30.12), it is also very likely that we can define ∞ category (or 'A_∞ category') whose object is an A_∞ category or an A_∞ algebra and whose morphisms are A_∞ functor or A_∞ homomorphism.

Then it seems very likely that we can generalize Theorem 12 to the existence of an 'A_∞ functor' (or ∞ functor) in an appropriate sense.

11 Floer theory of Lagrangian submanifolds.

In this section we explain briefly how the general construction of the earlier sections were used in [33] to study Floer homology of Lagrangian submanifolds.

Let (M,ω) be a compact symplectic manifold and L be its Lagrangian submanifold. We assume that L is oriented and is relatively spin. Here L is said to be relatively spin if its second Stiefel-Whiteney class lifts to a cohomology class in $H^2(M;\mathbb{Z}_2)$. Moreover we fix relative spin structure. (See [33] §44.1 for its definition.) For example, if L is spin, the choice of spin structure determines a choice of its relative spin structure.

We denote by $\mu : \pi_2(M;L) \to \mathbb{Z}$ the Maslov index. (See [3] or [33] §2.1 for its definition.) Since L is oriented its image is contained in $2\mathbb{Z}$. We next define $E : \pi_2(M,L) \to \mathbb{R}$ by

$$E(\beta) = \int_\beta \omega.$$

This is well-defined since L is a Lagrangian submanifold. We put

$$G_+(L) = \mathrm{Im}(E,\mu) \subset \mathbb{R} \times 2\mathbb{Z}. \tag{99}$$

Note this does not satisfy the conditions of Definition 12.

We next take and fix a compatible almost complex structure J on M. Let Σ be a Riemann surface (which may have a boundary). We say that $u : \Sigma \to M$ is J-holomorphic if

$$J \circ du = du \circ j_\Sigma$$

where j_Σ is the complex structure of Σ. Now we define

$$G_0(J) = \left\{ (E(\beta),\mu(\beta)) \,\middle|\, \begin{matrix} \text{There exists a } J\text{-holomorphic map} \\ u : (D^2,\partial D^2) \to (M,L) \text{ of homotopy class } \beta \end{matrix} \right\}.$$

By Gromov compactness [36], the monoid $G(J)$ generated by $G_0(J)$ satisfies the conditions of Definition 12.

Now for $\beta \in G(J)$ we define a moduli space $\mathfrak{M}_{k+1}(\beta)$ as follows. Let us consider an element $(\Sigma; z_0, \cdots, z_k; \beta(\cdot)) \in \mathcal{M}_{k+1,\beta}$. Let $\Sigma = \bigcup_{a\in A} D^2_a$ be its decomposition. (See §9.) We consider a continuous map $u : (\Sigma, \partial\Sigma) \to (M,L)$ such that

(1) $u : D^2_a \to M$ is J-holomorphic.
(2) $\big(E([u|_{D^2_a}]), \mu([u|_{D^2_a}])\big) = \beta(a)$.

Let $\overset{\circ}{\mathfrak{M}}_{k+1}(\beta)$ be the isomorphism classes of all such $(\Sigma; z_0, \cdots, z_k; \beta(\cdot); u)$. We can compactify it by including the stable maps with sphere bubble. Let $\mathfrak{M}_{k+1}(\beta)$ be the compactification. (See [33] §2.2.)

We define

$$(ev_0, \cdots, ev_k) : \mathfrak{M}_{k+1}(\beta; J) \to L^{k+1}$$

by putting

$$ev_i(\Sigma; z_0, \cdots, z_k; \beta(\cdot); u) = u(z_i)$$

and extending it to the compactification. We also define

$$\pi_0 : \mathfrak{M}_{k+1}(\beta; J) \to \mathcal{M}_{k+1,\beta}$$

by putting

$$\pi_0(\Sigma; z_0, \cdots, z_k; \beta(\cdot); u) = (\Sigma; z_0, \cdots, z_k; \beta(\cdot))$$

and extending it to the compactification. We next define

$$\circ_{\mathfrak{m},i} : \mathfrak{M}_{k+1}(\beta_1; J) \ {}_{ev_i}\times_{ev_0} \mathfrak{M}_{l+1}(\beta_2; J) \to \mathfrak{M}_{k+l}(\beta_1 + \beta_2; J) \tag{100}$$

as follows. Let

$$\begin{aligned} \mathcal{S} &= (\Sigma; z_0, \cdots, z_k; \beta(\cdot); u) \in \mathfrak{M}_{k+1}(\beta_1; J) \\ \mathcal{S}' &= (\Sigma'; z'_0, \cdots, 'z_l; \beta'(\cdot); u') \in \mathfrak{M}_{l+1}(\beta_2; J) \end{aligned}$$

with

$$ev_0(\mathcal{S}') = u'(z_0) = u(z_i) = ev_i(\mathcal{S}). \tag{101}$$

We identify $z_i \in \Sigma$ and $z'_0 \in \Sigma'$ to obtain Σ''. By (101) we obtain a J-holomorphic map $u'' : (\Sigma'', \partial\Sigma'') \to (M, L)$ by putting $u'' = u$ on Σ and $u'' = u'$ on Σ'. We set.

$$(z''_0, \cdots, z''_{k+l-1}) = (z_0, \cdots, z_{i-1}, z'_1, \cdots, z'_l, z_{i+1}, \cdots, z_k).$$

We now define :

$$(\Sigma''; z''_0, \cdots, z''_{k+l-1}; \beta''(\cdot); u'') = \mathcal{S} \circ_{\mathfrak{m};i} \mathcal{S}' \in \mathfrak{M}_{k+l}(\beta_1 + \beta_2; J).$$

We thus defined (100).

Proposition 4. *$\mathfrak{M}_{k+1}(\beta; J)$ is a $G(J)$-gapped filtered A_∞ Kuranishi correspondence.*

This is [33] Propositions 29.1 and 29.2. (We remark that the moduli space $\mathfrak{M}_{k+1}(\beta; J)$ here is denoted by $\mathcal{M}^{\rm main}_{k+1}(\beta; J)$ in [33].)

Theorem 12 and Proposition 4 (together with filtered analogue of Corollary 2) imply that $H(L; \Lambda^{\mathbb{Q}}_{0,nov})$ has a structure of $G(J)$-gapped filtered A_∞ algebra, which we write $\mathfrak{m}^J$.

We next explain its independence of almost complex structure J. (We remark that $\mathfrak{m}^J$ may also depend on the various choices (other than J) which we make during the constructions. However Theorem 12 implies that it is independent of such choices up to homotopy equivalence.) Let J_0, J_1 be two compatible almost complex structures. Since the set of all compatible almost structures are contractible, it follows that there exists a path $t \mapsto J_t$ of almost complex structures joining J_0 to J_1. We denote this path by $\mathcal{J}$. We are going to associate a morphisms of Kuranishi correspondence to $\mathcal{J}$. We define a set

$$G_0(\mathcal{J}) = \bigcup_{t\in[0,1]} G(J_t).$$

Let $G(\mathcal{J})$ be the monoid generated by it. Again by Gromov compactness $G(\mathcal{J})$ satisfies the conditions of Definition 12.

Now we consider $(\Sigma; z_0, \cdots, z_k; \beta(\cdot); \rho(\cdot)) \in \mathcal{F}_{k+1,\beta}$. Here we remark $(\Sigma; z_0, \cdots, z_k; \beta(\cdot)) \in \mathcal{M}_{k+1,\beta}$ and $\rho : A \to [0, 1]$ is a time allocation. We decompose Σ as $\Sigma = \bigcup_{a\in A} D^2_a$. We consider a continuous map $u : (\Sigma, \partial\Sigma) \to (M, L)$ such that :

(1) $u : D^2_a \to M$ is $J_{\rho(a)}$-holomorphic.
(2) $(E([u|_{D^2_a}]), \mu([u|_{D^2_a}])) = \beta(a)$.

Let $\overset{\circ}{\mathfrak{F}}_{k+1}(\beta; \mathcal{J})$ be the set of all isomorphism classes of such objects $(\Sigma; z_0, \cdots, z_k; \beta(\cdot); \rho(\cdot); u)$. By adding the stable map with sphere bubble we can compactify it and obtain $\mathfrak{F}_{k+1}(\beta; \mathcal{J})$. (See [33] §19.1.) (We remark that $\mathfrak{F}_{k+1}(\beta; \mathcal{J})$ is denoted by in $\mathcal{N}_{k+1}(\beta; \mathcal{J})$ in [33].)

Proposition 5. *$\mathfrak{F}_{k+1}(\beta; \mathcal{J})$ is a morphism between two $G(\mathcal{J})$-gapped filtered A_∞ Kuranishi correspondences $\mathfrak{M}_{k+1}(\beta; J_0)$ and $\mathfrak{M}_{k+1}(\beta; J_1)$.*

We remark that $G(J_i) \subset G(\mathcal{J})$. Hence every $G(J_i)$-gapped filtered A_∞ Kuranishi correspondence may be regarded as a $G(\mathcal{J})$-gapped filtered A_∞ Kuranishi correspondence.

Thus, using Theorem 12, we obtain a filtered A_∞ homomorphism

$$\mathfrak{f}^{\mathcal{J}} : (H(L; \Lambda^{\mathbb{Q}}_{0,nov}), \mathfrak{m}^{J_0}) \to (H(L; \Lambda^{\mathbb{Q}}_{0,nov}), \mathfrak{m}^{J_1}). \tag{102}$$

We remark that if $\beta = \beta_0 = (0, 0)$ then

$$\mathfrak{F}_{k+1}(\beta_0; \mathcal{J}) = \mathcal{F}_{k+1} \times L, \tag{103}$$

since every J-holomorphic map u with $\int u^*\omega = 0$ is necessary constant. Using this fact and filtered version of Theorem 8 (that is [33] Theorem 15.45) we can prove that $\mathfrak{f}^{\mathcal{J}}$ is a homotopy equivalence.

We next assume that there are two paths $\mathcal{J}$ and $\mathcal{J}'$ of almost complex structures joining J_0 to J_1. Again since the set of compatible almost complex structures is contractible, it follows that there is a two parameter family $\widehat{\mathcal{J}}$ of almost complex structures interpolating $\mathcal{J}$ and $\mathcal{J}'$. Using it we can prove that the following.

Proposition 6. *There exists $G(\widehat{\mathcal{J}}) \supseteq G(\mathcal{J}) \cup G(\mathcal{J}')$ and a homotopy $\mathfrak{H}_{k+1}(\beta; \widehat{\mathcal{J}})$ of the morphisms of filtered $G(\widehat{\mathcal{J}})$-gapped Kuranishi correspondences between $\mathfrak{F}_{k+1}(\beta; \mathcal{J})$ and $\mathfrak{F}_{k+1}(\beta; \mathcal{J}')$.*

See [33] §19.2.

Thus, by Theorem 12 and filtered version of Corollary 2, we have the following :

Theorem 13. *For each relatively spin Lagrangian submanifold L of a compact symplectic manifold M we can associate a structure of filtered A_∞ algebra on $H(L; \Lambda^{\mathbb{Q}}_{0,nov})$.*

It is independent of the choices up to the homotopy equivalence. The homotopy class of the homotopy equivalences is also independent of the choices.

This is Theorem A of [33]. We remark that since $\mathfrak{m}_{1,\beta_0} = 0$ on $H(L;\Lambda_{0,nov}^{\mathbb{Q}})$, it follows that any homotopy equivalence between two filtered A_∞ structures on it is an isomorphism, that is filtered A_∞ homomorphism which has an inverse. In [33] it is also proved that the filtered A_∞ algebra we obtain is unital.

The structure obtained in Theorem 13 is highly nontrivial. We gave various calculations in [34] §37 and §55. See [14, 15] for some other calculations. We gave also various applications to symplectic topology in [33] Chapter 6, [34] Chapter 8 etc.. Since, in this article, we concentrate on foundations, we do not explain calculations or applications here.

12 Transversality.

In §12 and §13, we will prove Theorem 12. Once stated appropriately, this theorem can be proved by the argument we wrote in §6 as a proof of Theorem 4 *except transversality and orientation.* So in §12 and §13 we focus these two points. In this section we discuss transversality.

We first remark that, after 1996, the transversality problem (in the theory of pseudo-holomorphic curve, for example) becomes a problem of finite dimensional topology rather than one on (linear or nonlinear) analysis. In early days of gauge theory or pseudo-holomorphic curve theory, various kinds of perturbations were introduced and used by various authors for various purposes. In those days, the heart of the study of the transversality problem was to find an appropriate geometric parameter, by which we have enough room to perturb the partial differential equations so that relevant transversality is achieved. Therefore the transversality problem was closely tied to the analysis of the particular nonlinear differential equation we use. This situation changed since the virtual fundamental chain technique was introduced. We now can reduce the problem to one of a finite dimensional topology in quite general situation, including *all* the cases in pseudo-holomorphic curve theory. So the main point to work out is finite dimensional problem. One of the main outcome of the discussion of the preceding sections is a formulation of this finite dimensional problem in a precise and rigorous way. (Of course finding explicit geometric parameter for perturbation can be interesting since it may give additional information on the algebraic system we obtain and may have geometric applications.)

When our situation is 'Morse type' and not 'Bott-Morse' type, the transversality can be achieved in general by taking abstract multivalued perturbation, that is by applying [30] Theorem 3.11 and Lemma 3.14, *directly.* Here 'Morse type' in our situation means that the correspondence we study is a correspondence between 0 dimensional spaces (that are discrete sets). Thus in this case the transversality problem had been solved by the method of [30].

In the 'Bott-Morse' case, the problem is more involved. Namely in case we study correspondence between manifolds of positive dimension, we need to perform the construction of virtual fundamental chains more carefully. This is the point we focus in this section. We refer [33] §30.2 (especially right after Situation 30.7) for the explanation of the reason why Bott-Morse case is harder to study.

As far as the author knows at the time of writing this article, there are two methods to deal with Bott-Morse case, both of which works in all the situations that are important for the applications to pseudo-holomorphic curve theory. One uses a kind of singular (co)homology and the other uses de Rham cohomology.

The first method was worked out in detail in [33] §30. As far as the author knows, this is the only way which works over $\mathbb{Q}$ (or $\mathbb{Z}$ sometimes) coefficient, in the general situation. The other advantage of this method is that singular homology is more flexible and so is useful for explicit calculations. (See [34] §56, §57 for some examples of calculations using singular homology.) The disadvantage of this method is that it destroys various symmetry of the problem.

The first method is summarized as follows. We first choose a countable set of smooth singular chains on our manifold M (in the case of Theorem 4 for example) and perturb the moduli space $\mathfrak{M}_{k+1}$ etc. so that the fiber product (43) is transversal for each $(P_1, \cdots, P_k)$ with P_i in the set we choosed above. We then define the operations by Formula (43). The trouble is that the chain which is an output of the operation, may not be in the chain complex generated by the chains we start with. So we increase our chain complex by adding those outputs. We next perturb again the moduli space $\mathfrak{M}_{k+1}$ to achieve transversality with those newly added chains. One important point is that we need to perturb $\mathfrak{M}_{k+1}$ in a way depending on the chains P_i on M. We continue this process countably many times and obtain a structure we want. One needs to work out rather delicate argument to organize the induction so that we can take such perturbations in a way so that they are all compatible to each other. We omit the detail and refer [33].

The second approach, using de Rham theory, works only over $\mathbb{R}$ coefficient. It is however somewhat simpler than the first one. In fact, for example, to prove Corollary 1 over $\mathbb{R}$ coefficient using de Rham theory, there is nothing to do in geometric side. Namely de Rham complex has a ring structure which is associative in the chain level. Therefore, by applying Corollary 2, we immediately obtain Corollary 1 over $\mathbb{R}$ coefficient. The case of Kuranishi correspondence is not such easy but is somewhat simpler than working with singular homology. The method using de Rham cohomology is somewhat similar to the discussion by Ruan in [59] and also to the argument of [27] §16. It was used systematically in [33] §33 and in [29]. Another advantage of this method is that it is easier to keep symmetry of the problem. For example we can prove the cyclic symmetry of the A_∞ algebra in Theorem 13 in this way. We will explain this method more later in this section.

We remark that there is a third method which works under some restrictions. It is the method to use Morse homology [19] (see for example [60]) or Morse homotopy ([20, 6, 23]). Let us discuss this method briefly.

We first consider the case of Theorem 4. We take *several* functions f_i on M (in the case of Theorem 4 for example) so that $f_i - f_j$ for $i \neq j$ are Morse functions and the gradient flow of $f_i - f_j$ are Morse-Smale. We also assume that the stable and unstable manifolds of $f_i - f_j$ for various $i \neq j$ are transversal to each other. (Of course the stable manifold of $f_i - f_j$ is not transversal to itself. So we exclude this case.) We then regard the stable manifolds of $f_i - f_{i+1}$ as a chain P_i and consider (44). Then we obtain an operator

$$\mathfrak{m}_k : C^*(M; f_0 - f_1) \otimes \cdots \otimes C^*(M; f_{k-1} - f_k) \to C^*(M; f_0 - f_k). \quad (104)$$

Here $C^*(M; f_1 - f_2)$ is the Morse-Witten complex of $f_1 - f_2$. (It is the complex (19) for *Morse* function $f = f_1 - f_2$. See for example [60].) More precisely, since we need to squeeze our structure to the Morse-Witten complex which is much smaller than singular chain complex, we need to combine the construction above with the proof of Theorem 2 as is done in [45] §6.4. Then the structure constant of the operation (104) turns out to be obtained by counting the order of appropriate sets of maps from a metric rooted ribbon tree to M such that each edge will be mapped to a gradient line of $f_i - f_j$. (See [20] §3,4 and [31] §12, §13.) It satisfies the relation (33) and hence defines a (topological) A_∞ *category*. Since $f_i = f_j$ is excluded. It is difficult to define A_∞ *algebra* in this way, directly.

We can generalize this construction to the case of Theorem 13 under some restrictions. In the situation of Theorem 13, operators

$$\mathfrak{m}_{k,\beta} : C^*(L; f_0 - f_1) \otimes \cdots \otimes C^*(L; f_{k-1} - f_k) \to C^*(L; f_0 - f_k) \quad (105)$$

are defined by counting a map from the configuration as in Figure 15 below to L. Here we put functions $f_0, \ldots, f_k$ on $D^2 \setminus$ tree according to the counter clockwise order. The small circles in the figure are mapped to the boundary value of a pseudo-holomorphic discs which bounds L. We assume that the sum of the homology classes of those pseudo-holomorphic disc is β. If e is an edge of the tree, then we assume that e is mapped to a gradient line of $f_i - f_j$. Here e is between two domains on which f_i and f_j are put. (Figure 15.1 is copied from page 429 of [23]. In [23] the case when the Lagrangian submanifold is a diagonal of the direct product $M \times M$ of symplectic manifold M was discussed. Y.-G. Oh [56] page 260 generalized it to more general Lagrangian submanifold L in the case of $\mathfrak{m}_1$, and also pointed out in [56] page 264 that it can be generalized to higher $\mathfrak{m}_k$ in some case.)

An important point of this construction is a cancellation of the two potential boundary of the moduli space of such maps. One of them corresponds to the shrinking of the edge, and the other is a splitting of the pseudo-holomorphic disc into a union of two discs. (See Figure 16.) This point was used for example in [32] page 290 for this purpose. (It was written there as the cancellation of (A4.70.2) and (A4.70.4).) We need however to put a restriction on our Lagrangian submanifold L to make this argument rigorous. Namely for general Lagrangian submanifold L for which $\mathfrak{m}_0 \neq 0$, we need to

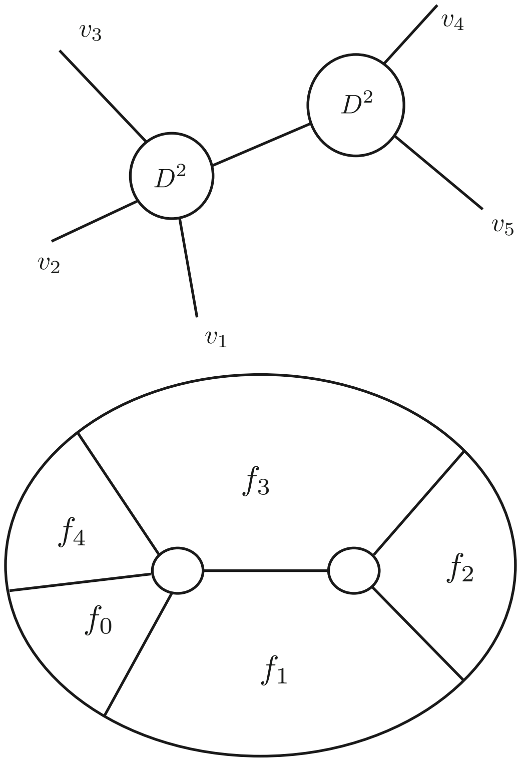

Fig. 15.

include tad pole (such as in the Figure 17 below). This causes some problem to rigorously define (104). In case when our Lagrangian submanifold is monotone with minimal maslov index ≥ 2, we can exclude such phenomenon. This fact was proved by Y.-G. Oh in [55] who established Floer homology of Lagrangian submanifold under this assumption. Under the same condition, Buhovsky [9] recently studied multiplicative structure of Floer homology using Morse homotopy.

Now we will discuss transversality problem in more detail using de Rham cohomology. We consider the situation of Kuranishi correspondence over M, that is the situation of Definition 21. Let $\Lambda^d(M)$ be the set of all smooth d forms on M. Using the correspondence (88) we want to construct a homomorphism $\mathfrak{m}_{k,\beta} : \Lambda^d(M^k) \to \Lambda^{d+1-\mu(\beta)}(M)$. Intuitively we might take

$$\mathfrak{m}_{k,\beta} \text{ ``}=\text{''} \ \pi_1! \circ \pi_2^*, \tag{106}$$

where π_2 is pull back of the differential form and $\pi_1!$ is integration along fiber. We remark however that integration along fiber is not well-defined as a smooth form unless π_1 is a submersion. Therefore we need to take appropriate

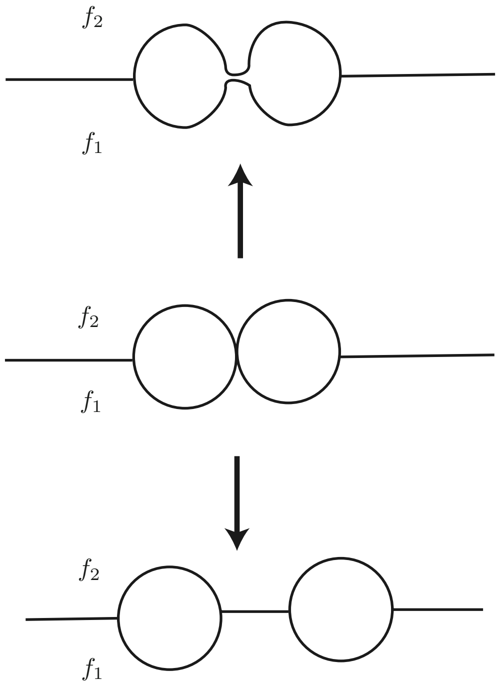

Fig. 16.

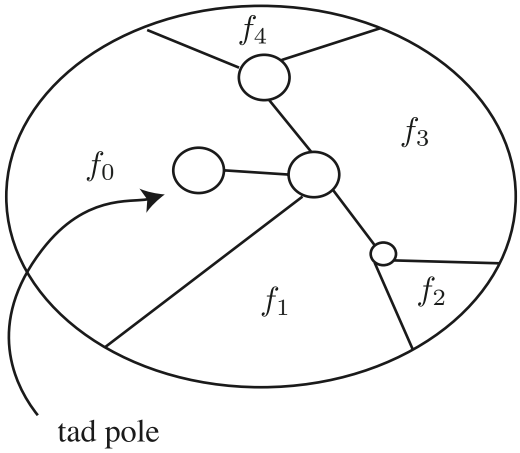

Fig. 17.

smoothing of the virtual fundamental chain to make sense of (106). We need to perform smoothing in a way compatible with the operadic structure of our K-spaces, which describes how they are glued. We explain this construction in three steps. At the first step, we work in one Kuranishi chart. At the second step, we work on each K-space $\mathfrak{M}_{k+1}(\beta)$. Finally we explain the way to make it compatible for various k, β.

Step 1 : Let $\mathcal{V} = (V, E, \Gamma, s, \psi)$ be a Kuranishi chart of $\mathfrak{M}_{k+1}(\beta)$. We first review the notion of multisection which was introduced [30] Definition 3.1, 3.2.

Let $E = V \times \mathbb{R}^{n_\mathcal{V}}$. We denote by meas$(\mathbb{R}^{n_\mathcal{V}})$ the space of all compactly supported Borel measures on $\mathbb{R}^{n_\mathcal{V}}$. Let $\mathfrak{k} \in \mathbb{Z}_{>0}$. We denote by $\text{meas}_{\mathfrak{k}}(\mathbb{R}^{n_\mathcal{V}})$ the set of measures of the form $\frac{1}{\mathfrak{k}}\sum_{c=1}^{\mathfrak{k}} \delta_{v_c}$ where δ_v is a delta measure on $\mathbb{R}^{n_\mathcal{V}}$ supported at v. The Γ action on $\mathbb{R}^{n_\mathcal{V}}$ induces a Γ action on $\text{meas}_{\mathfrak{k}}(\mathbb{R}^{n_\mathcal{V}})$.

Definition 24. *A $\mathfrak{k}$-multisection of E is by definition a Γ equivalent map $\mathfrak{s} : V \to \text{meas}_{\mathfrak{k}}(\mathbb{R}^{n_\mathcal{V}})$.*

It is said to be smooth if, for any sufficiently small $U \subset V$, we have smooth maps $s_c : U \to \mathbb{R}^{n_\mathcal{V}}$ $(c = 1, \cdots, k)$ and smooth functions $a_c : U \to \mathbb{R}$ such that

$$\mathfrak{s}(x) = \frac{1}{\mathfrak{k}} \sum_{c=1}^{\mathfrak{k}} \delta_{\mathfrak{s}_c(x)} \tag{107}$$

We say $\mathfrak{s}_c$ a branch of $\mathfrak{s}$.

We remark that we do not require $\mathfrak{s}_c$ to be Γ equivariant. Namely the Γ action may exchange them.

Remark 12. The multisection is introduced in [30] in a slightly different but equivalent way as above. The smoothness of multisection is a bit tricky thing to define. Here we assume that the branch $\mathfrak{s}_c$ exists locally. This is related to the notion liftability discussed in [30]. The liftable and smooth $\mathfrak{k}$-multisection in the sense of [30] is a smooth $\mathfrak{k}$-multisection in the sense above.

In case each branch is transversal to 0 the inverse image of 0 of multisection looks like the following Figure 18. (Figure 18 is a copy of [30] Figure 4.8.) Since we assumed that (π_0, π_1, π_2) is weakly submersive, it follows that

$$(\pi_1, \pi_2) : V \to M^{k+1}$$

is a submersion.

Let $W_\mathcal{V}$ be a manifold which is oriented and without boundary. We do not assume $W_\mathcal{V}$ is compact. (We choose that the dimension of $W_\mathcal{V}$ is huge.) We consider smooth $\mathfrak{k}$-multisection

$$\mathfrak{s}_\mathcal{V} : V \times W_\mathcal{V} \to \text{meas}_{\mathfrak{k}}(\mathbb{R}^{n_\mathcal{V}})$$

of the pull back of E to $V \times W_\mathcal{V}$. The action of Γ on $W_\mathcal{V}$ is the trivial action.

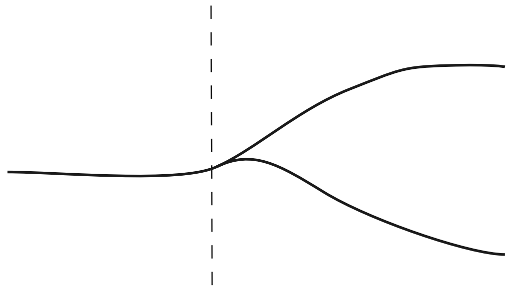

Fig. 18.

Definition 25. *We say that* $\mathfrak{s}_{\mathcal{V}}$ *is strongly submersive if the following holds.*
(1) *For each* $(x,w) \in V \times W_{\mathcal{V}}$ *we may choose its branches* $\mathfrak{s}_{\mathcal{V},c}$ $(c = 1,\cdots,\mathfrak{k})$ *on a neighborhood* U *of* (x,w)*, such that* 0 *is a regular vales of it.*
(2) *We put*

$$\mathfrak{s}_{\mathcal{V},c}^{-1}(0) \cap U = \{(x,w) \in U \mid \mathfrak{s}_{\mathcal{V},c}(x,w) = 0\},$$

which is a smooth manifold by (1)*. Then*

$$(\pi_1,\pi_2) : \mathfrak{s}_{\mathcal{V},c}^{-1}(0) \cap U \to M^{k+1}.$$

is a submersion.

Hereafter we say multisection in place of $\mathfrak{k}$-multisection in case we do not need to specify $\mathfrak{k}$.

Lemma 8. *We may choose* $W_{\mathcal{V}}$ *so that for any* ϵ *there exists a smooth multisection such that each branch of it is in the* ϵ *neighborhood of* s *point-wise.*

Proof : We first choose $W_{\mathcal{V}}$ huge and find a single valued section $\mathfrak{s}_{\epsilon} : V \times W_{\mathcal{V}} \to \mathbb{R}^{n_{\mathcal{V}}}$ which approximate s and that 0 is its regular value. We then put

$$\mathfrak{s}(x,w) = \frac{1}{\#\Gamma} \sum \delta_{\gamma \mathfrak{s}_{\epsilon}(\gamma^{-1}x,w)}. \tag{108}$$

It is straightforward to see that (108) has the required properties. □

We take a smooth multisection $\mathfrak{s}_{\mathcal{V}}$ which is strongly submersive. We next take a smooth provability measure $\omega_{\mathcal{V}}$ on $W_{\mathcal{V}}$ which is Γ invariant and is of compact supported. We put

$$\mathcal{W}_{\mathcal{V}} = (W_{\mathcal{V}}, \mathfrak{s}_{\mathcal{V}}, \omega_{\mathcal{V}}).$$

We next choose an open covering U_{α} of $V \times W_{\mathcal{V}}$ such that $\mathfrak{s}_{\mathcal{V}}$ has an expression

$$\mathfrak{s}_{\mathcal{V}} = \frac{1}{\mathfrak{k}_\alpha}\sum_{c=1}^{\mathfrak{k}_\alpha}\delta_{\mathfrak{s}_{\mathcal{V},\alpha,c}} \tag{109}$$

on U_α, and choose a partition of unity χ_α subordinate to U_α.

We regard $\omega_{\mathcal{V}}$ as a differential form of degree $\dim W_{\mathcal{V}}$ and pull it back to $\mathfrak{s}^{-1}_{\mathcal{V},\alpha,c}(0)$. We denote it by the same symbol. Now we define perturbed correspondence $\mathrm{Corr}^{\mathcal{W}_{\mathcal{V}}}_{\mathcal{V}} : \Lambda^d(M^k) \to \Lambda^{d+1-\mu(\beta)}(M)$ by

$$\mathrm{Corr}^{\mathcal{W}_{\mathcal{V}}}_{\mathcal{V}}(u) = \pm\sum_\alpha \frac{1}{\#\Gamma\cdot\mathfrak{k}_\alpha}\sum_i(\pi_{1+}!)\left((\chi_\alpha\pi^*_{2+}(u)\wedge\omega_{\mathcal{V}})|_{\mathfrak{s}^{-1}_{\mathcal{V},\alpha,c}(0)}\right). \tag{110}$$

Here $\pi_{1+} : \mathfrak{s}^{-1}_{\mathcal{V}}(0) \to M$ is a composition of projection with π_1, and the map π_{2+} is defined in the same way. The integration along fiber in (110) is well-defined since π_{1+} is a submersion and $\omega_{\mathcal{V}}$ is of compact support. We do not discuss sign in this section. See §13.

We remark that the right hand side of (110) is independent of the choices of the covering U_α, decomposition (109), and the partition of unity, but depend only on the data encoded in $\mathcal{W}_{\mathcal{V}}$. So hereafter we write the right hand side of (110) as

$$\pm\frac{1}{\#\Gamma}(\pi_{1+}!)\left((\pi^*_{2+}(u)\wedge\omega_{\mathcal{V}})|_{\mathfrak{s}^{-1}_{\mathcal{V}}(0)}\right), \tag{111}$$

for simplicity.

We remark that $\mathfrak{s}_{\mathcal{V}}$ may be regarded as a family of multisections $\mathfrak{s}_{\mathcal{V},w}(\cdot) = \mathfrak{s}_{\mathcal{V}}(\cdot, w)$ parametrized by $w \in W_{\mathcal{V}}$. The correspondence (111) is an average of the correspondences by $\mathfrak{s}^{-1}_{\mathcal{V},w}(0)$ by the smooth probability measure $\omega_{\mathcal{V}}$. The technique of multisection uses finitely many perturbations and its average. Here we take a family of uncountably many perturbations and use its average.

Step 2 : We next combine and glue the correspondence for the Kuranishi charts in a given Kuranishi atlas of $\mathfrak{M}_{k+1}(\beta)$. Let $\mathcal{V}_i = (V_i, E_i, \Gamma_i, s_i, \psi_i)$, $i = 1, \cdots, I$ be a Kuranishi atlas. We may enumerate them so that $i \prec j$ implies $i < j$.

We will construct $\mathcal{W}_{\mathcal{V}_i}$ by induction on i as follows. Suppose we constructed them for each j with $j < i$. We define $W_{0,i}$ and $W_{\mathcal{V}_i}$ inductively. We put

$$W_{\mathcal{V}_i} = \left(\prod_{j<i} W_{0,j}\right)\times W_{0,i}$$

By induction hypothesis, $W_{0,j}$ $(j < i)$ are defined. We will define $W_{0,i}$ later.

We can extend $\mathfrak{s}_{\mathcal{V}_j}$ uniquely to a section $\mathfrak{s}_{i,\mathcal{V}_j}$ of E_j on $\Gamma_i(\varphi_{ij}(V_{ij}))\times W_{\mathcal{V}_j}$ so that it is Γ_i invariant. Note that we use Condition (7) in Definition 18 here. We may regard it as a section on

$$\Gamma_i(\varphi_{ij}(V_{ij}))\times W_{\mathcal{V}_i} \tag{112}$$

by composing it with an obvious projection. We next extend it to a tubular neighborhood of (112) as follows. By (85) we can identify

$$\varphi_{ij}^* E_i \cong E_j \oplus N_{\varphi_{ij}(V_{ij})} V_i,$$

where $N_{\varphi_{ij}(V_{ij})}V_i$ is he normal bundle. A point in a tubular neighborhood of $\varphi_{ij}(V_{ij})$ can be written as $(\varphi_{ij}(x), v)$ here v is in the fiber of $N_{\varphi_{ij}(V_{ij})}V_j$. We now put

$$\mathfrak{s}_{i,\mathcal{V}_j}((\varphi_{ji}(x), v), (w_k)_{k\le i}) = \mathfrak{s}_{i,\mathcal{V}_j}(\varphi_{ji}(x), (w_k)_{k\le j}) \oplus v.$$

(More precisely we take branch of our multivalued section $\mathfrak{s}_{i,\mathcal{V}_j}$ and apply the above formula to each branch.)

We extend it by using Γ_i invariance. We denote it by the same symbol $\mathfrak{s}_{i,\mathcal{V}_j}$. By using induction hypothesis, the sections we constructed above for various j, coincide at the part where the tubular neighborhoods of $\varphi_{ij}(V_{ij})$ for different j intersect to each other. Thus we obtain a desired section on a neighborhood of the union of the images φ_{ij} for various j. Now we choose $W_{0,i}$ and extend this section so that it satisfies the conditions (1) (2) (3) of Step 1. Moreover we can choose

$$\omega_{\mathcal{V}_i} = \prod \omega_{0,j} \times \omega_{0,i},$$

where $\omega_{0,j}$ is a provability measure on $W_{0,j}$ chosen in earlier stage of induction. We thus have

$$\mathcal{W}_{\mathcal{V}_i} = (W_{\mathcal{V}_i}, s_{\mathcal{V}_i}, \omega_{\mathcal{V}_i}).$$

Now we define

$$\mathrm{Corr}^{\mathcal{W}_{\mathcal{V}_i}}_{\mathfrak{M}_{k+1}(\beta)}(u) = \sum_i \pm \frac{1}{\#\Gamma_i} (\pi_{1+}!) \left((\pi_{2+}^*(u) \wedge \omega_{\mathcal{V}_i})|_{\mathcal{V}_i^o \cap \mathfrak{s}_{\mathcal{V}_i}^{-1}(0)} \right). \tag{113}$$

Here

$$\mathcal{V}_i^o = \mathcal{V}_i \setminus \bigcup_{i \prec l} (V_{li} \times W_i).$$

We remark that precisely speaking the right hand side of (113) should be written in a way similar to (110) using partition of unity and branches.

By construction, we have the following equality.

$$\begin{aligned} &\frac{1}{\#\Gamma_j} (\pi_{1+}!) \left((\pi_{2+}^*(u) \wedge \omega_{\mathcal{V}_i})|_{(V_{ij} \times W_j) \cap \mathfrak{s}_{\mathcal{V}_i}^{-1}(0)} \right) \\ &= \frac{1}{\#\Gamma_i} (\pi_{1+}!) \left((\pi_{2+}^*(u) \wedge \omega_{\mathcal{V}_i})|_{(\mathrm{Tube}(\Gamma_i \cdot \varphi(V_{ij})) \times W_i) \cap \mathfrak{s}_{\mathcal{V}_i}^{-1}(0)} \right). \end{aligned} \tag{114}$$

Here $\mathrm{Tube}(\Gamma_i \cdot \varphi(V_{ij}))$ is the tubular neighborhood of $\Gamma_i \cdot \varphi(V_{ij})$. We can use this fact to prove (113) is smooth.

(114) also imply that when we apply Stokes' formula, the boundary term is the integral on the boundary of $\mathcal{M}_{k+1}(\beta)$.

Step 3 : Now we will explain the way how we perform the above construction for various K-spaces, $\mathfrak{M}_{k+1}(\beta)$ in a way so that they are compatible for various k, β.

We first explain the reason why, by the method of continuous family of perturbations, we can construct the compatible system of virtual fundamental cochains inductively, in the situation where fiber product appears.

Let $\mathcal{V} = (V, E, \Gamma, s, \psi)$, $\mathcal{V}' = (V', E', \Gamma', s', \psi')$ be Kuranishi neighborhoods of $p \in Z$ and $p' \in Z'$ respectively. We consider the diagram

$$M_1 \xleftarrow{\pi_1} V \xrightarrow{\pi_2} M \xleftarrow{\pi_1'} V' \xrightarrow{\pi_2'} M_2 \tag{115}$$

of smooth manifolds such that $(\pi_1, \pi_2) : V \to M_1 \times M$, $(\pi_1', \pi_2') : V' \to M \times M_2$ are submersions.

We take $\mathcal{W}_{\mathcal{V}} = (W_{\mathcal{V}}, \mathfrak{s}_{\mathcal{V}}, \omega_{\mathcal{V}})$, $\mathcal{W}_{\mathcal{V}'} = (W_{\mathcal{V}'}, \mathfrak{s}_{\mathcal{V}'}, \omega_{\mathcal{V}'})$ as in step one. Namely we assume that $(\pi_1, \pi_2) : \mathfrak{s}_{\mathcal{V}}^{-1}(0) \to M_1 \times M$ is a submersion and we put a similar assumption for (π_1', π_2'). Then, in the same way as (110), we obtain homomorphisms

$$\mathrm{Corr}_{\mathcal{V}}^{\mathcal{W}_{\mathcal{V}}} : \Lambda(M_1) \to \Lambda(M), \qquad \mathrm{Corr}_{\mathcal{V}'}^{\mathcal{W}_{\mathcal{V}'}} : \Lambda(M) \to \Lambda(M_2) \tag{116}$$

by correspondences. The composition of them is obtained by a correspondence which is a fiber product of $\mathcal{V}$ and $\mathcal{V}'$, as follows. We consider

$$\mathcal{V} \times_M \mathcal{V}' = (V \,{}_{\pi_1}\!\times_{\pi_2} V', E \times E', \Gamma \times \Gamma', s \times s')$$

and

$$(W_{\mathcal{V}} \times W_{\mathcal{V}'}, \mathfrak{s}_{\mathcal{V}} \times \mathfrak{s}_{\mathcal{V}'}, \omega_{\mathcal{V}} \times \omega_{\mathcal{V}'}). \tag{117}$$

We write (117) as $\mathcal{W}_{\mathcal{V}} \times \mathcal{W}_{\mathcal{V}'}$. We use them in the same way as Step 1 and obtain $\mathrm{Corr}_{\mathcal{V} \times_M \mathcal{V}'}^{\mathcal{W}_{\mathcal{V}} \times \mathcal{W}_{\mathcal{V}'}}$. It is easy to see that :

$$\mathrm{Corr}_{\mathcal{V} \times_M \mathcal{V}'}^{\mathcal{W}_{\mathcal{V}} \times \mathcal{W}_{\mathcal{V}'}} = \pm \mathrm{Corr}_{\mathcal{V}'}^{\mathcal{W}_{\mathcal{V}}} \circ \mathrm{Corr}_{\mathcal{V}}^{\mathcal{W}_{\mathcal{V}'}}. \tag{118}$$

(We do not discuss sign in this section. See the next section.) The formula (118) plays a crucial role to prove the A_∞ relation for the operations which we will define by the smooth correspondence as in (110). In order to use (118) for this purpose, we need to choose the continuous family of perturbations so that at the boundary of each of the spaces $\mathfrak{M}_{k+1}(\beta)$, the perturbation is obtained as the fiber product such as (117). Here we remark that by (5) of Definition 21, the boundary of $\mathfrak{M}_{k+1}(\beta)$ is decomposed to a union of fiber products of various $\mathfrak{M}_{k'+1}(\beta')$. We thus proceed inductively and construct continuous family of perturbations.

To carry out the idea described above, we start with defining the order by which we organize the induction. The following definition is taken from [33] Definitions 30.61 and 30.63. Let G be as in Definition 12.

Definition 26. *For $\beta \in G$, we put :*

$$\|\beta\| = \sup\{n \,|\exists \beta_i \in G \setminus \{(0,0)\} \;\; \beta_1 + \cdots + \beta_n = \beta\} + [E(\beta)] - 1.$$

Here $[x]$ is the largest integer $\le x$. We put $\|(0,0)\| = -1$.

We define a partial order $<$ on $G \times \mathbb{Z}_{\ge 0}$ as follows. Let $\beta_1, \beta_2 \in G$, $k_1, k_2 \in \mathbb{Z}_{\ge 0}$. We define $>$ so that $(\beta_1, k_1) > (\beta_2, k_2)$ if and only if one of the following holds.

(1) $\|\beta_1\| + k_1 > \|\beta_2\| + k_2$. (2) $\|\beta_1\| + k_1 = \|\beta_2\| + k_2$ *and* $\|\beta_1\| > \|\beta_2\|$.

We will define the continuous family of perturbations on $\mathfrak{M}_{k+1}(\beta)$ according to the (partial) order $<$ of (β, k).

We assume that we have a continuous family of perturbations for all $\mathfrak{M}_{k'+1}(\beta')$ with $(\beta', k') < (\beta, k)$ and construct a perturbation on $\partial\mathfrak{M}_{k+1}(\beta)$. Let $\mathbf{x} \in \partial\mathfrak{M}_{k+1}(\beta)$. We assume that $\mathbf{x}$ is contained in the codimension d corner of $\mathfrak{M}_{k+1}(\beta)$ (but is not in the codimension $d+1$ corner of it). (Here $d \ge 1$.) We put

$$\mathcal{S} = (\Sigma; z_0, \cdots, z_k; \beta(\cdot)) = \pi_0(\mathbf{x}) \in \mathcal{M}_{k+1,\beta}.$$

Σ has exactly d singular points. Let $\Sigma = \cup_{a \in A} D_a^2$ be the decomposition of Σ. (Here $\#A = d+1$.)

For each $a \in A$, we define

$$\mathcal{S}_a = (D_a^2; z_{a;0}, \cdots, z_{a;k_a}; \beta_a(\cdot)) \in \mathcal{M}_{k_a+1,\beta_a}$$

as follows. D_a^2 is the disc. The marked points of D_a^2 are singular or marked points of Σ which is on D_a^2. $\beta_a = \beta(a)$ and $\beta_a(\cdot)$ is the map which assigns β_a to the unique component of D_a^2. The 0-th marked point $z_{a;0}$ is defined as follows. If $z_0 \in D_a^2$ then $z_{a;0} = z_0$. If not there is unique D_b^2 such that $a < b$ and $D_b^2 \cap D_a^2 \ne \emptyset$. Here $<$ is the order on A which is defined during the proof of Theorem 5. (See Figure 19.) Then $z_{a;0}$ is the unique point in $D_a^2 \cap D_b^2$. We can use (1), (2) and (3) of Definition 21 repeatedly to find a unique element $\mathbf{x}_a \in \mathfrak{M}_{k_a+1}(\beta_a)$ such that $\pi_0(\mathbf{x}_a) = \mathcal{S}_a$ and that $\mathbf{x}_a$ is sent to $\mathbf{x}$ after applying $\circ_{\mathfrak{m},*}$ repeatedly. In fact $\mathcal{S}$ is obtained from $\mathcal{S}_a$ by applying $\circ_i$ several times. We apply $\circ_{\mathfrak{m},*}$ to $\mathbf{x}_a$ in the same way as $\circ_i$ is applied to $\mathcal{S}_a$. Then Definition 21 (3) implies that this composition is independent of the order to apply it.

We have

Lemma 9. $(k_a, \beta_a) < (k, \beta)$ *for each a.*

The proof is elementary and is omitted. (See [34] Lemma 30.65.)

Now we fix Kuranishi neighborhood $\mathcal{V}_a = (V_a, E_a, \Gamma_a, s_a, \psi_a)$ of $\mathbf{x}_a$ for each $a \in A$. Then by applying (3) and (4) of Definition 21 repeatedly, we find that a Kuranishi neighborhood of $\mathbf{x}$ is obtained as follows.

$$\mathcal{V} = \prod_{a, M^{d-1}} \mathcal{V}_a = \left(\prod_{a, M^{d-1}} V_a \times [0,\epsilon)^{d-1}, \prod_a E_a, \prod s_a, \prod_a \psi_a \right) \tag{119}$$

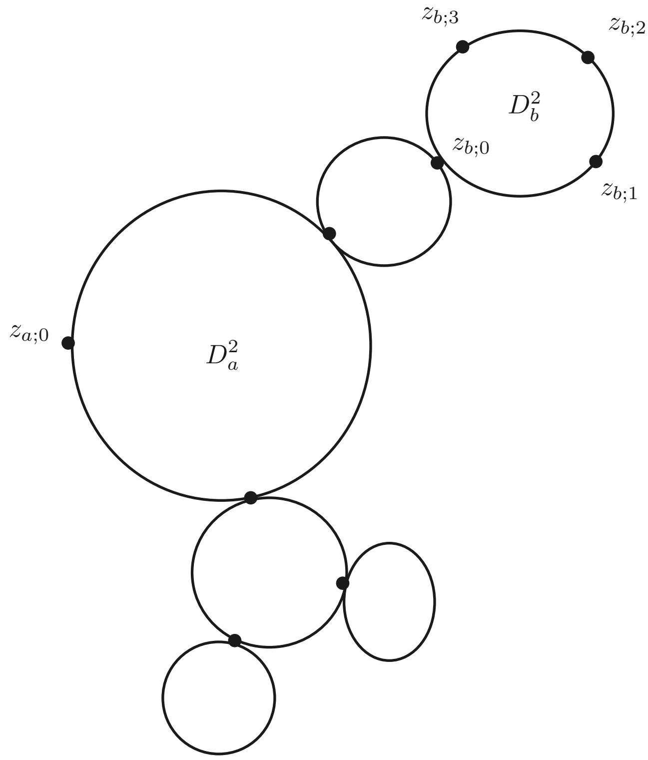

Fig. 19.

Here $\prod_{a,M^{d-1}} V_a$ is an appropriate fiber product of V_a over M^{d-1}. (See [33] §29.1 for the detailed description of this fiber product.) The factor $[0,\epsilon)^d$ appears since the point $\mathbf{x}$ lies on the codimension d corner.

By inductive hypothesis, we already defined a continuous family of perturbations $\mathcal{W}_{\mathcal{V}_a} = (W_{\mathcal{V}_a}, \mathfrak{s}_{\mathcal{V}_a}, \omega_{\mathcal{V}_a})$ for each a. We then define $W_{\mathcal{V}} = \prod_a W_{\mathcal{V}_a}$. (Note that some of the factors $W_{\mathcal{V}_a}$ coincides to each other. In that case we repeat the same factor as many times as $W_{\mathcal{V}_a}$ appeared in $W_{\mathcal{V}}$.)

We also define $\mathfrak{s}_{\mathcal{V}}$ etc. by restricting the direct product of $\mathfrak{s}_{\mathcal{V}_a}$ etc.. to $\prod_{a,M^{d-1}} V_a \times \{0\}^{d-1}$. We thus obtain the required continuous family of perturbations $\mathcal{W}_{\mathcal{V}}$ on the set of codimension d corners of the Kuranishi neighborhood of $\mathbf{x}$. We can extend its to its neighborhood by composing with the obvious projection of $[0,\epsilon)^{d-1}$ factor to $\{0\}$. We thus obtain a continuous family of perturbations in a Kuranishi neighborhood of $\mathbf{x}$.

By construction it is obvious that the system of family of perturbations above is compatible with the way we glued them at the earlier stage of induction using Step 2. So we obtain a continuous family of perturbations in a neighborhood of the boundary. Then we use Steps 1 and 2 to extend it to the whole $\mathfrak{M}_{k+1}(\beta)$.

Now since the continuous family of perturbations we constructed is consistent with the decomposition of the boundary given in Definition 21 (6), it follows from (118) and Stokes' formula that the operations $\mathfrak{m}_{k,\beta}$ obtained in this way satisfies the A_∞ formula (33). We thus obtain a filtered A_∞ structure on the de Rham complex.

This is the transversality part of the proof of Theorem 12. (We discussed only the construction of operations. The construction of morphisms and their homotopies are similar.)

Remark 13. Actually there is another serious trouble to be taken care of in order to rigorously establish filtered A_∞ structure. This trouble is pointed out in [33] §30.3, discussed in detail in [33] §30 and is summarized as follows. We took continuous families of perturbations inductively. The zero set $\mathfrak{s}_{\mathcal{V}}^{-1}(0)$ should be in a small neighborhood of the original moduli space $s^{-1}(0)$, since if it runs out of the Kuranishi neighborhood we will be unable to use Stokes' theorem to prove A_∞ formula. By taking the perturbation small, we can do it without difficulty as long as we have only finitely many steps to work out. However in an actual geometric situation, we will define an infinite number of operations $\mathfrak{m}_{k,\beta}$. This causes a trouble in the following way. Let us consider the argument of Step 3 above. We first choose ϵ enough small so that $\mathfrak{s}_{\mathcal{V}}^{-1}(0)$ which we construct at first stage of the induction lies in an ϵ neighborhood of the original moduli space. Then in the k-th step, the perturbation we find on the boundary or corner is a fiber product of k perturbations of the earlier steps. So the perturbation which is already defined is away from the original moduli space by a distance something like $k\epsilon$. We remark that, in the fiber product decomposition like (119), the same factor (which was already determined at the first stage of the induction, for example) may appear many times. And once we fixed the perturbation at some stage of the induction, we are not supposed to change it later. Thus the zero set of the perturbed section runs out of the Kuranishi neighborhood at some finite stage.

The idea to overcome this difficulty is as follows. For each fixed (n, K) we can choose our perturbations so that we can continue the construction for each (β, k) with $(\beta, k) \le (n, K)$. (Here $<$ is the order defined in Definition 26.) We next define an appropriate notion of $A_{n,K}$ structure. Then, in this way, we can construct an $A_{n,K}$ structure for any but fixed (n, K). We can also prove that the $A_{n,K}$ structure which is constructed above, is homotopy equivalent to $A_{n',K'}$ structure as an $A_{n,K}$ structure, for arbitrary n', K'. We finally use homological algebra to show that this implies that the $A_{n,K}$ structure can be extended to an A_∞ structure.

The argument outlined above is carried out in detail in [33] §30. The same trouble seems to occur frequently for the rigorous constructions of various topological field theories by Kuranishi correspondence. The method we explained above seems to work in all the cases. (At the time of writing this article, the author does not know any other way to resolve this trouble.) An earlier example where a similar trouble appeared is the study of Floer homology of periodic Hamiltonian system in monotone symplectic manifold. K. Ono [58] resolved this difficulty in that case in a similar but slightly different way, using projective limit. (This is the reason why the result of [58] is more general than that of [38].)

In order to prove Theorem 12, we need to work out the same argument for morphisms and homotopies. In order to carry it out in the case of homotopies, we need to show compatibility of $A_{n,K}$ homotopies between various different n, K. This requires us to study the notion of homotopy of homotopies. Two (rather heavy) subsections §30.12 and §30.13 of [33] are devoted to this point.

13 Orientation.

In this section we discuss sign or orientation. The problem of orientation and sign appears in two related but different ways, in the construction of topological field theory.

(1) To prove that the K-space of the appropriate moduli problem is orien*table*. To find and describe the geometric data which determines the orientation of the K-space.
(2) To organize the orientations of several fiber products appearing in the construction in a consistent way. To fix sign convention of the algebraic system involved. To prove that the system of orientations organized above is consistent with the sign convention of the algebraic systems.

The point (1) is a problem of family index theory. (This observation goes back to [17].) For example, in the case of the moduli space of pseudo-holomorphic discs which bounds a given Lagrangian submanifold, it is proved that the moduli space is orientable in case L is relatively spin, in [32] and [63]. We also remark that even in case we can prove that the moduli space involved is orientable (in a way consistent with the fiber products as in Definition 21 (5)), it is a different problem to specify the geometric data which determines the orientation. In other words, proving the *existence* of a coherent orientation is not enough to complete this step. Actually there can be several different choices of coherent orientations, in general. (See [13] for explicit example of this phenomenon in Lagrangian Floer theory.)

Such phenomenon already appears in the classical Morse theory as follows. Let M be a smooth manifold with $H^1(M;\mathbb{Z}_2) \neq 0$ and $f : M \to \mathbb{R}$ be a Morse function. To define Morse homology we need to specify orientation of the moduli space $\mathcal{M}(p,q)$ of the gradient lines joining two given critical points p, q. The system of orientations of them for various p, q are said to be coherent if it is consistent with fiber product structure in Formula (17). We can find such coherent orientation for each representation : $\pi_1(M) \to \mathbb{Z}_2$ and different choices induce different homology group. So we need to find some way to distinguish trivial representation from other ones, to find a system of orientations which gives the ordinary homology.

The point (2) is of different nature. At first sight, it might look rather a technical problem that could be resolved by 'patience' and 'carefulness'. (By this reason the importance of this point is frequently overlooked.) Indeed, in early days, when the structure involved was rather simple, one could fix sign convention by hand once the point (1) was understood. However as time going and as the structure to deal with becoming more advanced, it gets harder to find a correct sign convention. Especially the work to check whether it coincides with the sign or orientation of geometric origin becomes more and more cumbersome. (It seems that the amount of the works to study sign grows exponentially as the complexity of the structure we deal with grows.) Then, one arrived in the point where fixing sign and orientation only by patience and carefulness becomes impossible. We thus need some 'principle' to fix sign convention and to show that it coincides with one of geometric origin. In other words, studying sign is related to the procedure to ensure that the construction is enough canonical.

In this section, we do not discuss point (1) since it is related to the geometric origin of the moduli space (or K space) and so is not a part of the *general* theory we are building. Our focus in this section is point (2). The major part of [33] Chapter 9 is actually devoted to this point. There we still gave an explicit choice of signs and of the orientations of the moduli spaces and its fiber products. Though there are some 'principle' behind each of our choices, it is hard to state it in a mathematical and rigorous way, so it was rarely mentioned explicitly. And the proof in [33] Chapter 9 of the consistency of the orientation and sign was based on calculations.

The purpose of this section is to explain the way how we translate the discussion of [33] Chapter 9 to the more abstract situation of this paper. (In [33] Chapter 9 the situation of Lagrangian Floer theory is discussed.) On the way, we state precisely the compatibility condition of orientations among various spaces $\mathfrak{M}_{k+1}(\beta)$. This point was postponed in Definition 21.

We first introduce some notations. Let $\mathfrak{S}_k$ be the symmetric group of order $k!$. We put $\mathfrak{M}^+_{k+1}(\beta) = \mathfrak{S}_k \times \mathfrak{M}_{k+1}(\beta)$ on which $\mathfrak{S}_k$ acts by the left multiplication of the first factor. There is a one to one correspondences between the set of orientations on $\mathfrak{M}_{k+1}(\beta)$ and the set of orientations on $\mathfrak{M}^+_{k+1}(\beta)$ such that the action of σ is orientation preserving if and only if $\sigma \in \mathfrak{S}_k$ is an even permutation. We hereafter identify them.

Remark 14. In the case when $\mathfrak{M}_{k+1}(\beta)$ is the moduli space $\mathfrak{M}_{k+1}(\beta; J)$ of pseudo-holomorphic discs (which we introduced in §11), the space $\mathfrak{M}^+_{k+1}(\beta)$ is regarded as a compactification of the set of $(D^2; z_0, \cdots, z_k; u)$ such that $z_i \in \partial D^2$ and that u is J-holomorphic map with $(\int u^*(\omega), \eta([u])) = \beta$. Note we do *not* require the points $z_0, \cdots, z_k$ to respect the cyclic order. The $\mathfrak{S}_k$ action is defined by $(z_0, z_1, \cdots, z_k) \mapsto (z_0, z_{\sigma(1)}, \cdots, z_{\sigma(k)})$. The geometric meaning of the discussion below becomes clearer if the reader keeps this example in his mind.

The map

$$(\pi_1, \pi_2) : (ev_0, e_1, \cdots, ev_k) : \mathfrak{M}_{k+1}(\beta) \to M^{k+1}$$

is extended to $\mathfrak{M}^+_{k+1}(\beta)$ by

$$ev_i(\sigma, \mathbf{x}) = ev_{\sigma(i)}(\mathbf{x}) \quad (i \neq 0), \qquad ev_0(\sigma, \mathbf{x}) = ev_0(\mathbf{x}).$$

We extend $\circ_{\mathfrak{m},i}$ to

$$\circ_{\mathfrak{m},i} : \mathfrak{M}^+_{k+1}(\beta_i)_{ev_i} \times_{ev_0} \mathfrak{M}^+_{l+1}(\beta_2) \to \mathfrak{M}^+_{k+l}(\beta_1 + \beta_2)$$

by

$$(\sigma_1, \mathbf{x}_1) \circ_{\mathfrak{m},i} (\sigma_2, \mathbf{x}_2) = (\sigma, \mathrm{x}_1 \circ_{\mathfrak{m},\sigma_1(i)} \mathbf{x}_2)$$

where σ is defined by

$$\sigma(j) = \begin{cases} \sigma_1(j) & j < i, \quad \sigma_1(j) < \sigma_1(i) \\ \sigma_1(j) + l - 1 & j < i, \quad \sigma_1(j) > \sigma_1(i) \\ \sigma_2(j - i + 1) + \sigma_1(i) - 1 & i \le j < i + l - 1 \\ \sigma_1(j - l + 1) & j \ge i + l, \quad \sigma_1(j - l + 1) < \sigma_1(i) \\ \sigma_1(j - l + 1) + l - 1 & j \ge i + l, \quad \sigma_1(j - l + 1) > \sigma_1(i) \end{cases} \tag{120}$$

$\mathfrak{M}^+_{k+1}(\beta)$ is a K-space whose boundary stratum is a union of the images of $\circ_{\mathfrak{m},i}$. Namely

$$\circ_{\mathfrak{m},i} \left(\mathfrak{M}^+_{k+1}(\beta_i)\ _{ev_i} \times_{ev_0} \mathfrak{M}^+_{l+1}(\beta_2)\right) \subset \partial\mathfrak{M}^+_{k+l}(\beta_1 + \beta_2)$$

The compatibility condition of the orientations is defined as follows.

Definition 27. *We say the orientations of $\mathfrak{M}_{k+1}(\beta)$ for various k, β are compatible if the embedding*

$$\circ_{\mathfrak{m},i} : \mathfrak{M}^+_{k+1}(\beta_1)_{ev_1} \times_{ev_0} \mathfrak{M}^+_{l+1}(\beta_2) \subset (-1)^{(k-1)(l-1)+(\dim M + k)} \partial\mathfrak{M}^+_{k+l}(\beta_1 + \beta_2)$$

is orientation preserving for every k, l, β_1, β_2.

Actually this is a copy of the conclusion of [33] Proposition 46.2.

Remark 15. (1) We remark that, in Definition 27, the condition is put only on the fiber product by ev_1. Using the action of symmetric group, the compatibility of the orientations of other cases (namely the case when the fiber product is taken by ev_i) are induced automatically. This is the reason why we introduced $\mathfrak{M}^+_{k+1}(\beta_1)$.
(2) Actually we need to fix several conventions to discuss signs. Especially we need to specify the orientation of the fiber product. Here we omit them and refer [33] Chapter 9.

Now we consider chains (P_i, f_i) on M. Here P_i is a smooth manifold and f_i is a smooth map $P_i \to M$. Under the appropriate transversality condition we consider the fiber product of them with $\mathfrak{M}_{k+1}(\beta)$ and obtain a compact oriented and smooth manifold. We define an orientation of it by

$$(-1)^* \mathfrak{M}_{k+1}(\beta) \times_{M^k} (P_1 \times \cdots \times P_k) \tag{121}$$

with

$$* = (\dim M + 1) \sum_{j=1}^{k-1} \sum_{i=1}^{j} (\dim M - \dim P_i).$$

This definition is taken from [33] §48. Now we can translate the proof of [33] Proposition 48.1 *word to word* to our more abstract situation and show the A_∞ relation for the operations $\mathfrak{m}_{k,\beta}$ defined by (121), as far as the transversality condition is satisfied. We remark that the notations of this section and one in [33] Chapter 9 corresponds as follows :

$$\mathfrak{M}^+_{k+1}(\beta) \longleftrightarrow \mathcal{M}_{k+1}(\beta), \qquad \mathfrak{M}_{k+1}(\beta) \longleftrightarrow \mathcal{M}^{\rm main}_{k+1}(\beta).$$

In §12, to discuss transversality problem, we use de Rham complex. So the orientation problem which is required to work out the proof of §12 is fixing the sign of the operations defined by (111) etc. on differential forms on M. We can reduce this problem to the problem about sign on the operations among the chains (P_i, f_i) in M as follows. (This is explained in more detail in [33] §53.)

In §12 we constructed the operations by using continuous family of perturbations as follows. We took W a huge parameter space and consider correspondence

$$M^k \xleftarrow{\pi_2} V \times W \xrightarrow{\pi_1} M.$$

Here we have an obstruction bundle E over V and have a multisection $\mathfrak{s} : V \times W \to E \times W$. We fix a branch $\mathfrak{s}_c$ and took $\mathfrak{s}_c^{-1}(0) \subset V \times W$. We also use ω a top form on W. We pull it back to $\mathfrak{s}_c^{-1}(0)$ and the operation is defined by

$$u \mapsto \pm \pi_1 ! \left(\pi_2^*(u) \wedge \omega|_{\mathfrak{s}_c^{-1}(0)} \right), \tag{122}$$

where u is a differential form on M^k. The other part of the construction such as taking partition of unity etc. does not affect the problem of sign. Therefore, we only need to find a way to define the sign $\pm$ in (122) so that the resulting operation satisfies the A_∞ relation.

To reduce this problem to the orientation of fiber product as in (121), we proceed as follows. We can approximate our smooth form u by a current realized by the product of chains (P_i, f_i). So while discussing orientation problem we only need to consider the case $u = (u_1, \cdots, u_k)$ and u_i is realized by (P_i, f_i). We next take generic $w \in W$. Then the fiber product

$$(\mathfrak{s}_c^{-1}(0) \cap (V \times \{w\})) \times_{M^k} (P_1 \times \cdots \times P_k) \tag{123}$$

is well-defined. (Namely the transversality holds.) For such w we can define the sign by the same formula (121). In fact $\mathfrak{s}_c^{-1}(0) \cap (V \times \{w\})$ is (an open subset of) a perturbation of our moduli space $\mathfrak{M}_{k+1}(\beta)$.

We next regard ω as a smooth *measure*. We remark that we need to fix an orientation of W for this purpose. We did it already when we identify smooth measure on W with differential form of degree $\dim W$ in §12.

Using this smooth probability measure we average the current which is obtained by pushing out (123) to M by ev_0. It is easy to see that the average coincides with (122). Thus using (121) we can fix the sign in (122). A_∞ relation (with sign) of the operation given by (121) implies the A_∞ relation *with sign* of the operation defined by using (122).

This is the argument to reduce the problem of sign in Theorem 12 to the result of [33] Chapter 9. We discussed only the case of construction of filtered A_∞ algebra. The orientation problem in the construction of filtered A_∞ homomorphism and homotopy between them can be reduced to [33] Chapter 9 in the same way. The proof of Theorem 12 is now complete.

We finally go back to a point mentioned before, that is the data used to determine the sign of our A_∞ algebra. Using the consistency condition as in Definition 27, the orientation of the K-spaces $\mathfrak{M}_{k+1}(\beta)$ is determined by $\mathfrak{M}_{k'+1}(\beta')$ for other k', β' with $(\beta', k') < (\beta, k)$. Here the order $<$ is introduced in Definition 26. So the choice of orientation of $\mathfrak{M}_{k+1}(\beta)$ for which (β, k) is minimal determines the orientation of the other $\mathfrak{M}_{k+1}(\beta)$. (More precisely we can slightly modify $<$ to $<'$ so that $(\beta', k') < (\beta, k)$ in and only if $\mathfrak{M}_{k'+1}(\beta')$ appears in the boundary of $\mathfrak{M}_{k+1}(\beta)$. $<'$ above implies $<$ in Definition 26. But the converse may not be true.) The minimal (β, k) is $(\beta_0, 2)$ and $(\beta, 0)$ where $\beta_0 = (0, 0)$ and β is a primitive element of G.

In the situation of §11, $\mathfrak{M}_{2+1}(\beta_0)$ is L itself. We can fix the orientation of it so that $\mathfrak{m}_{2,\beta_0}$ is induced by usual cup product as in Example 1.

The orientation of $\mathfrak{M}_1(\beta)$ is more involved. It depends on the geometric data such as relative spin structure in the case of Lagrangian Floer theory. We remark that in general we can not choose orientations of various $\mathfrak{M}_1(\beta)$ with β primitive independently, because then the compatibility condition may not be satisfied. In fact if $\beta_1 + \beta_2 = \beta'_1 + \beta'_2 = \beta$ are decompositions of β to different sum of primitive elements, then by looking the consistency at $\mathfrak{M}_1(\beta)$, the choice of the orientations of three of $\mathfrak{M}_1(\beta_1)$, $\mathfrak{M}_1(\beta_2)$, $\mathfrak{M}_1(\beta'_1)$, $\mathfrak{M}_1(\beta'_2)$ determine the orientation of the fourth one automatically. This kind of phenomenon occurs since our monoid G may not be free.

If G is free (namely is isomorphic to $\mathbb{Z}^m_{\ge 0}$ for some m), then the choice of the orientation of $\mathfrak{M}_1(\beta)$ for the generators β of our monoid G corresponds one to one to the choice of system of orientations of all $\mathfrak{M}_{k+1}(\beta)$ satisfying the compatibility condition. In such a situation there is a simpler proof of the existence of consistent system of orientations and signs. See [27] §7 for such an argument.

We remark that we choose G satisfying Definition 12, because of Gromov compactness which is related to nonlinear analysis of pseudo-holomorphic curve theory. The problem of orientation is related to index theory and to linear analysis. Therefore during the discussion of the sign, we can replace G by a bigger monoid. Actually we can take $G = G_+(L)$ in (99). This can be a way to reduce the problem to the case when G is free.

14 Variations and generalizations.

There are many directions we can generalize the construction of this article. We mention some of them briefly below. Many of them are subjects of the future research and the argument of this section is rather brief. Proof of none of them are regarded to be completed except those which are proved in the reference rigorously.

14.1 Unitality

There is a unital version of the notion of A_∞ space and A_∞ algebra. Usually the unital version is called A_∞ space in the literature. So the version in §4 should be called non-unital A_∞ space. Let M be a space with a base point $*$. Then we require

$$\mathsf{m}_k(a; x_1, \cdots, x_{i-1}, *, x_{i+1}, \cdots, x_k) = \mathsf{m}_{k-1}(a; x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k)$$

in addition to define the notion of a unital A_∞ space. For A_∞ algebra, its unit $\mathbf{e}$ is an element of degree 0 (before shifted) such that :

$$\mathfrak{m}_k(x_1, \cdots, x_{i-1}, \mathbf{e}, x_{i+1}, \cdots, x_k) = 0$$

for $k \neq 2$ and

$$\mathfrak{m}_2(x, \mathbf{e}) = (-1)^{\deg x}\mathfrak{m}_2(\mathbf{e}, x) = x.$$

There is also the notion of homotopy unit. (See [33] §8. See also [61] §2 and reference therein for various versions of unit or homotopy unit and the relationships among them.)

The singular homology of unital A_∞ space is a unital A_∞ algebra with 0 chain $*$ as its unit. It can be proved in the same way as Theorem 3.

The situation is different for Kuranishi correspondence. The candidate of unit is the fundamental chain which is regarded as a degree 0 cochain by Poincaré duality. In the case of de Rham theory which we worked out in §12, it is a 0 form and is the constant function $\equiv 1$. This is actually the unit when there is a map

$$\mathfrak{forget}_i : \mathfrak{M}_{k+1}(\beta) \to \mathfrak{M}_k(\beta)$$

which is compatible with the map : $\mathcal{M}_{k+1} \to \mathcal{M}_k$ forgetting the i-th marked point and which is compatible with $ev_j : \mathfrak{M}_{k+1}(\beta) \to M$. We can prove it in the same way as [33] Lemma 31.2. In this way we can prove that our A_∞ algebra of Theorem 13 on de Rham complex of the Lagrangian submanifold has a strict unit.

14.2 Module and Category

There is a notion of A_∞ (bi)module and also A_∞ category. See [33] §12 and Chapter 15 for the bimodule and [26] etc. for the A_∞ category. We can modify Theorem 12 to include those cases, in a straightforward way.

14.3 Local coefficient

In the case of Bott-Morse theory, we constructed a structure (that is a higher boundary operator $\partial_{\mathcal{M}}$) on the direct sum $\bigoplus_a C(R_a; \Theta_a^-)$ of chain complex of singular chains on R_a *with local coefficient*. In order to include such a situation in our machinery, we consider correspondence such as

$$(M, \Theta_M)^k \xleftarrow{ev_1} \mathfrak{M}_{k+1} \xrightarrow{ev_2} (M, \Theta_M). \tag{124}$$

In this case, in place of assuming K-space $\mathfrak{M}_{k+1}$ to be oriented, we assume that it has relative orientation. Namely we assume that

$$ev_1^* (\Theta_M \otimes \cdots \otimes \Theta_M) \otimes ev_2^* \Theta_M \otimes \Lambda^{\rm top} TV \otimes \Lambda^{\rm top} E \tag{125}$$

has a trivialization. Here V is a Kuranishi neighborhood and E is an obstruction bundle. We also assume that the trivialization of (125) is compatible with the coordinate change. Namely we assume it is compatible with Diagram (86).

This situation appears when we study Bott-Morse version of Lagrangian Floer homology for a pair of Lagrangian submanifolds of clean intersection. See [33] §12.5 and §51, for detail. The argument there can be directly generalized to our abstract situation.

14.4 Family version

For a family of Lagrangian submanifolds L in M we can study family Floer homologies. (See [25, 39].) An abstract version of this construction can be formulated as follows.

We can generalize homotopy of Kuranishi correspondence (that is $[0,1]$ parametrized family of Kuranishi correspondences) to a family parametrized by an arbitrary manifold. Namely we can consider the following situation. Let $M \to X \xrightarrow{\pi} B$ be a family of manifolds M parametrized by a manifold B. We consider

$$\begin{array}{ccccc} & & \mathcal{M}_{k+1,\beta} & & \\ & & \uparrow{\scriptstyle \pi_0} & & \\ X^k & \xleftarrow[\pi_2]{} & \mathfrak{M}_{k+1}(\beta) & \xrightarrow[\pi_1]{} & X. \end{array} \tag{126}$$

such that

$$(\pi_1, \pi_2)(\mathfrak{M}_{k+1}(\beta)) \subseteq \{(x_0, \cdots, x_k) \in X^{k+1} \mid \pi(x_0) = \cdots = \pi(x_k)\} \subset X^{k+1}.$$

Then by replacing the fiber product $\mathfrak{M}_{k+1}(\beta_1)\ {}_{ev_i} \times_{ev_0} \mathfrak{M}_{l+1}(\beta_2)$ over M (in (1) Definition 21) by the fiber product over X we can generalize the Definition 21 to a B parametrized version. We call it the B parametrized family of Kuranishi correspondence.

Let us briefly describe the corresponding algebraic object. Let B be a simplicial complex. We define the notion of B parametrized family of A_∞ algebra as follows. For each simplex σ there is a (filtered) A_∞ algebra $C(\sigma)$ whose homology group is one of M. If σ_i is the i-th face of σ there is a linear A_∞ homomorphism

$$\mathrm{Eval}_{\partial_i} : C(\sigma) \to C(\sigma_i)$$

which is a homotopy equvalence. Let σ_{ij} be the set of all codimension 2 simplex of σ. Then we require the existence of the following exact sequence

$$C(\sigma) \to \bigoplus_i C(\sigma_i) \to \bigoplus_{ij} C(\sigma_{ij}).$$

See [33] Definition 30.68.5 for a similar exact sequence for rectangle.

We can associate B parametrized family of A_∞ algebra to B parametrized family of Kuranishi correspondence in a way similar to the proof of Theorem 12.

14.5 Group action and localization to fixed point set

In the study of Gromov-Witten invariant, localization to the fixed point set plays an important role. Gromov-Witten invariant of a manifold M is a family of numbers parametrized by homology classes of M and by homology classes of the Deligne-Mumford compactification of moduli space of Riemann surfaces. The localization formula gives a way to reduce its calculation to the study of neighborhood of the fixed point locus of the moduli space of pseudo-holomorphic curves, when a group act on it.

A problem to extend it to our story, for example to the study of Lagrangian Floer theory, lies in the fact that it is rather hard to find a correct statement of the (expected) result. This is because the structure constant of the algebraic system (which is the number obtained by counting the order of the moduli space in an appropriate sense) itself is not well-defined.

The result of this paper gives a way to formulate such a statement.

Let us exhibit a way to do so by considering the special case where the group is S^1 and the action of it on M is trivial. Let $\mathfrak{M}_{k+1}(\beta)$ be a Kuranishi correspondence on M. We assume S^1 acts on $\mathfrak{M}_{k+1}(\beta)$ such that the structure map and evaluation map is S^1 equivalent. (We put trivial action on M and on $\mathcal{M}_{k+1,\beta}$.) We put

$$(\mathfrak{M}_{k+1}(\beta))^{S^1} = \{x \in \mathfrak{M}_{k+1}(\beta) \mid \forall g \in S^1 \;\; gx = x\}. \tag{127}$$

We will define a K-space $(\mathfrak{M}_{k+1}(\beta))^{S^1+}$ as follows. Let $(V_i, E_i, \Gamma_i, \psi_i)$ be a Kuranishi chart. Let $V_i^{S^1}$ be the set of S^1 fixed point of V_i. We fix sufficiently small ϵ and take an ϵ neighborhood $B_\epsilon(V_i^{S^1})$ of $V_i^{S^1}$. Here we use S^1 invariant Riemannian metric on V_i. We identify $B_\epsilon(V_i^{S^1})$ with an open subset of the normal bundle. We consider the sphere bundle which is a boundary of $B_\epsilon(V_i^{S^1})$ and denote it by $S_\epsilon(V_i^{S^1})$. Our group S^1 acts on it so that the isotropy group is finite. We take a quotient of $S_\epsilon(V_i^{S^1})$ by the S^1 action and glue the quotient with $B_\epsilon(V_i^{S^1}) \setminus S_\epsilon(V_i^{S^1})$. We denote the resulting space by $\mathbb{P}V_i^{S^1}$. It is an orbifold. Our obstruction bundle E_i induces an orbibundle E_i' on it. The Kuranishi map s_i induces a section s_i' of E_i'. Let $Z_i = s_i'^{-1}(0)/\Gamma_i$. We can glue them using coordinate transformation to obtain a space Z. By covering Z_i with open subset of V_i and using the restriction of E_i', s_i' there, we obtain a Kuranishi chart for each points on Z_i. We can glue them in an obvious way to obtain a Kuranishi chart on Z. We thus obtain a Kuranishi structure on Z. We denote the space Z together with the above Kuranishi structure by $(\mathfrak{M}_{k+1}(\beta))^{S^1+}$.

Using the fact that the structure map and evaluation map of $\mathfrak{M}_{k+1}(\beta)$ S^1 equivalent, we can show that $(\mathfrak{M}_{k+1}(\beta))^{S^1+}$ is regarded as a Kuranishi correspondence on M.

In the next theorem we use $\mathbb{R}$ coefficient.

Theorem 14. *The filtered A_∞ structure associated to $(\mathfrak{M}_{k+1}(\beta))^{S^1+}$ by Theorem 12 is homotopy equivalent to one associated to $\mathfrak{M}_{k+1}(\beta)$.*

Sketch of the proof : Let δ be a positive number sufficiently smaller than ϵ. For each Kuranishi neighborhood V_i we consider $V_i \setminus B_\delta(V_i^{S^1})$. Since S^1 action is locally free there the quotient space $\frac{V_i \setminus B_\delta(V_i^{S^1})}{S^1}$ is an orbifold. The bundle E_i induces an orbibundle $\overline{E}_i$ on it. In the same way as §12, we can take a continuous family of multisection $\overline{E}_i$ of this bundle and lift it to $V_i \setminus B_\delta(V_i^{S^1})$. We thus have a continuous family of S^1 equivariant multisection on $V_i \setminus B_\delta(V_i^{S^1})$ which are transversal to zero. Obviously we can do it in a way compatible with the coordinate change. We can extend this multisection to $B_\delta(V_i^{S^1})$ so that it is transversal to zero but is not necessary S^1 invariant. We use this continuous family of multisections to define the operators $\mathfrak{m}_{k,\beta}$ as in §12.

Since evaluation map ev is S^1 equivariant, we can show that the contribution of the part to $V_i \setminus B_\delta(V_i^{S^1})$ to $\mathfrak{m}_{k,\beta}$ is 0. Hence the theorem. □

14.6 Other operad or prop

As we mentioned several times, the construction works for other operads or props than A_∞ operads. The argument of §12 can be generalized with littile change. For the part of the proof we gave in §13, we need certain modification.

The construction of appropriate *differentiable* operad or prop is also a nontrivial problem. It seems to the author that claims in the talks [5], [70] etc. can be reinterpreted as an existence of a differentiable prop associated to the moduli space of higher genus Riemann surface. Namely master-equation claimed in those talks are Maurer-Cartan equation (30), which is an important part of the axiom of differentiable prop. (See Definition 2.)

14.7 Gravitational descendant

In the usual theory of operads, spaces $\mathcal{P}(n)$ (or $\mathcal{M}_{k+1}$) are assumed to be contractible. However in the situation of several 'operads' or 'props' appearing in topological field theory there is a situation where they have a nontrivial homotopy type. In the case of A_∞ operad, $\mathcal{M}_{k+1}$ is contractible. The important case where nontrivial homotopy type appears is the case of higher genus Riemann surface and/or the case where interior marked point is included.

We can modify our construction of the structure to include the nontrivial homotopy type of operad or prop. Various related ideas are discussed by various people (See for example [11].) mainly from the algebraic side.

An example of such a construction is as follow. We consider the direct sum

$$\Lambda(\mathcal{M}) = \bigoplus_k \Lambda(\mathcal{M}_{k+1})$$

of the de Rham complexes of our differential operads. We use Maurer-Cartan axiom (30) to obtain a homomorphism

$$\Delta : \Lambda(\mathcal{M}) \to \Lambda(\partial\mathcal{M}) \to \Lambda(\mathcal{M}) \hat{\otimes} \Lambda(\mathcal{M}), \tag{128}$$

by restriction. Here $\hat{\otimes}$ is the tensor product in the sense of Fréchet-Schwartz. space.

We call a sequence $c_m \in \Lambda(\mathcal{M})$ a *multiplicative sequence* ([37] §1) if $c_0 = 1$ and

$$\Delta c_m = \sum_{i=0}^{m} c_i \otimes c_{m-i}, \qquad dc_i = 0. \tag{129}$$

When a multiplicative sequence c_m is given, we use it to replace (111) by

$$\sum_{m=0}^{\infty} \pm \frac{s^m}{\#\Gamma}(\pi_{1+}!)\left((\pi_{2+}^*(u) \wedge \omega_{\mathcal{V}} \wedge \pi_0^* c_m)|_{\mathfrak{s}_{\mathcal{V}}^{-1}(0)}\right), \tag{130}$$

where s is another formal parameter. We then obtain a formal deformation of our structure parametrized by s.

Unfortunately in case of A_∞ operad, the space $\mathcal{M}_{k+1}$ is contractible. Therefore there is no multiplicative sequence other than trivial one. However in case we include higher genus Riemann surface and interior marked points, non trivial example is obtained by using Mumford-Morita class. ([10].)

14.8 Infinite dimensional M

In the situation of String topology ([12]) and the loop space formulation of Lagrangian Floer theory ([29, 16]), the correspondence we use is slightly different from those discussed in this paper and can be described by a diagram :

$$(\Omega M)^k \xleftarrow{(ev_1,\cdots,ev_k)} \mathfrak{M}_{k+1}(\beta) \xrightarrow{ev_0} \Omega M. \tag{131}$$

Here ΩL is the free loop space and is of infinite dimension. The structure map is

$$\mathfrak{M}_{k+1}(\beta_1)\ {}_{ev_*\circ ev_i}\times_{ev_*\circ ev_0} \mathfrak{M}_{l+1}(\beta_2) \to \mathfrak{M}_{k+l+1}(\beta_1+\beta_2) \tag{132}$$

Here $ev_* : \Omega(M) \to M$ is the map $\ell \mapsto \ell(*)$ and $* \in S^1$ is the base point. The interesting new point (due to Chas and Sullivan) appearing here is that we take fiber product over M and not over $\Omega(M)$.

We need several modifications of the argument of this paper to include this case. We however remark that the method in [29] to realize transversality in the case of loop space is very similar to one in §12 of this paper.

We may also consider the case of gauge theory (of 4 manifolds, for example) where our M is an infinite dimensional space consisting of gauge equivalence classes of connections. There seems to be much more works to be done to extend the frame work of this paper to include gauge theory.

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