§35. Normally polynomial perturbations.

The main purpose of this section is to prove Theorem 34.11. For this purpose, we need to first recall some notations and lemmas from [FuOn01] whose proofs we duplicate here for readers' convenience.

35.1. Fiberwise polynomial sections.

Let M be a manifold and Γ a finite group. Let $\mathcal{E}_1$, $\mathcal{E}_2$ be real vector bundles over M on which Γ acts. The action of Γ on M is trivial. *We assume that the action of* Γ *on* $\mathcal{E}_1$ *is effective.* We pull back $\mathcal{E}_2$ to the total space of $\mathcal{E}_1$ by the projection $\pi : \mathcal{E}_1 \to M$ and denote it by $\tilde{\mathcal{E}}_2$. Let D be a sufficiently large integer.

Definition 35.1. A section s of $\tilde{\mathcal{E}}_2$ is said to be a *fiberwise polynomial section* of degree $\le D$, if its restriction to each fiber $\pi^{-1}(x) \subset \mathcal{E}_1$ is a polynomial of degree $\le D$.

We put $I_v = \{\gamma \in \Gamma \,|\, \gamma v = v\}$. For each subgroup $\Gamma' \subseteq \Gamma$, we put

$$\mathcal{E}_1^{=}(\Gamma') = \{v \in \mathcal{E}_1 \,|\, I_v = \Gamma'\}.$$

It follows that $\mathcal{E}_1^{=}(\Gamma')$ is a smooth submanifold of $\mathcal{E}_1$. Let $x \in M$ denote $V_1 = \mathcal{E}_{1x}$, $V_2 = \mathcal{E}_{2x}$ for the fibers at x of $\mathcal{E}_{1x}$ and $\mathcal{E}_{2x}$ respectively. Let $V_i^{\Gamma'}$ be the Γ' invariant part of V_i. We define

$$i(\Gamma') = \dim V_1^{\Gamma'} - \dim V_2^{\Gamma'}. \tag{35.2}$$

The main technical result we use in this section is the following :

Proposition 35.3. *Let D be sufficiently large. Then for any generic fiberwise polynomial section s of degree $\le D$, we have*

$$\dim(s^{-1}(0) \cap \mathcal{E}_1^{=}(\Gamma')) = \dim M + i(\Gamma'), \tag{35.4}$$

and that $s^{-1}(0) \cap \mathcal{E}_1^{=}(\Gamma')$ is a smooth submanifold of $\mathcal{E}_1^{=}(\Gamma')$.

Moreover $s^{-1}(0)/\Gamma$ has a triangulation compatible with smooth structures on $s^{-1}(0) \cap \mathcal{E}_1^{=}(\Gamma')$.

Proof. Let $\mathrm{Poly}_\Gamma^D(V_1, V_2)$ be the set of all Γ-equivariant polynomial maps $P : V_1 \to V_2$ of degree $\le D$. There is an evaluation map

$$ev : \mathrm{Poly}_\Gamma^D(V_1, V_2) \times V_1 \to V_2.$$

We first consider the case $\Gamma' = \{1\}$. We put $V_1^{=}(1) = \{v \in V_1 | I_v = \{1\}\}$. We note that $V_1^{=}(1)$ is an *open* subset of V_1.

Lemma 35.5. *Suppose the action of* Γ *on* V_1 *is effective. Then* ev *is a submersion on* $\mathrm{Poly}^D_\Gamma(V_1, V_2) \times V_1^{=}(1)$ *for sufficiently large* D*. In particular, the space* $ev^{-1}(0) \cap (\mathrm{Poly}^D_\Gamma(V_1, V_2) \times V_1^{=}(1))$ *is a smooth manifold of dimension*

$$\dim(ev^{-1}(0) \cap (\mathrm{Poly}^D_\Gamma(V_1, V_2) \times V_1^{=}(1))) = \dim V_1 + \dim \mathrm{Poly}^D_\Gamma(V_1, V_2) - \dim V_2.$$

Lemma 35.5 is an easy consequence of the following.

Sublemma 35.6. *Let* $v \in V_1$ *and* $w \in V_2$*. We assume* $I_v = \{1\}$*. Then there exists a* Γ*-equivariant polynomial map* $P : V_1 \to V_2$ *such that* $P(v) = w$*.*

Proof. Without loss of generalities, we may assume that V_2 is an irreducible (over $\mathbb{R}$) Γ module. We put

$$W = \bigoplus_{\gamma \in \Gamma} \mathbb{R}[\gamma]$$

and define a Γ action on it by

$$g\left(\sum c_\gamma[\gamma]\right) = \sum c_\gamma[\gamma g^{-1}].$$

Since W is a regular representation of Γ, there exists a surjective Γ linear map $\Psi : W \to V_2$. We choose $w_\gamma \in \mathbb{R}$ such that :

$$\Psi\left(\sum w_\gamma[\gamma]\right) = w.$$

Since $I_v = \{1\}$, there exists an ($\mathbb{R}$ valued) polynomial f on V_1 such

$$f(\gamma v) = w_\gamma$$

for each $\gamma \in \Gamma$. We put

$$P(x) = \Psi\left(\sum_\gamma f(\gamma x)[\gamma]\right)$$

for $x \in V_1$. It is straightforward to see that P has the required property. $\square$

Two bundles $\mathcal{E}_1, \mathcal{E}_2 \to M$ induce a bundle $\mathrm{Poly}^D_\Gamma(\mathcal{E}_1, \mathcal{E}_2) \to M$ whose fiber is $\mathrm{Poly}^D_\Gamma(V_1, V_2)$. We may identify a fiberwise polynomial section of degree $\le D$ with a section of the bundle $\mathrm{Poly}^D_\Gamma(\mathcal{E}_1, \mathcal{E}_2)$. We consider the evaluation maps

$$C^\infty(M; \mathrm{Poly}^D_\Gamma(\mathcal{E}_1, \mathcal{E}_2)) \times \mathcal{E}_1^{=}(1) \xrightarrow{ev_1} \mathrm{Poly}^D_\Gamma(\mathcal{E}_1, \mathcal{E}_2) \times \mathcal{E}_1^{=}(1) \xrightarrow{ev_2} \mathcal{E}_2.$$

Here $ev_1(\zeta, v) = (\zeta(\pi(v)), v)$. By Lemma 35.5, the set $ev_2^{-1}(0)$ is a smooth submanifold of $\mathrm{Poly}^D_\Gamma(\mathcal{E}_1, \mathcal{E}_2) \times \mathcal{E}_1^{=}(1)$ of codimension $\dim V_2$. Since ev_1 is a submersion we find that $ev_1^{-1}(ev_2^{-1}(0))$ is a smooth submanifold of codimension $\dim V_2$.

We next consider the map

$$\pi : ev_1^{-1}(ev_2^{-1}(0)) \to C^\infty(M; \mathrm{Poly}^D_\Gamma(\mathcal{E}_1, \mathcal{E}_2))$$

induced by the projection. Then this map is a Fredholm map of index

$$\dim \mathcal{E}_1 - \dim V_2.$$

By the Sard-Smale theorem, the set of regular values is residual and so dense. We take a regular value of π. It corresponds to a normally polynomial section s of $\tilde{\mathcal{E}}_2$. Since we have the identity

$$s^{-1}(0) \cap \mathcal{E}_1^{=}(1) = ev_1^{-1}(ev_2^{-1}(s)) \cap \pi^{-1}(0),$$

$s^{-1}(0) \cap \mathcal{E}_1^{=}(1)$ becomes a smooth manifold with its dimension given by

$$\dim \mathcal{E}_1^{=}(1) - \mathrm{rank}\, \mathcal{E}_2 = \dim M + \mathrm{rank}\, \mathcal{E}_1 - \mathrm{rank}\, \mathcal{E}_2 = \dim M + i(\Gamma').$$

Here we use the fact that $\mathcal{E}_1^{=}(1)$ is an open subset of $\mathcal{E}_1$ for the first equality. This proves that for the case of $\Gamma' = \{1\}$, $s^{-1}(0) \cap \mathcal{E}_1^{=}(\Gamma')$ is smooth with dimension given by (35.4).

To prove the latter for general Γ' we need to use somewhat more complicated argument. We define

$$\mathcal{E}_2^{\supseteq}(\Gamma') = \{v \in \mathcal{E}_2 | I_v \supseteq \Gamma'\}.$$

We pull it back to $\mathcal{E}_1^{=}(\Gamma')$ and denote it by $\tilde{\mathcal{E}}_2(\Gamma')$. For given $x \in M$, we consider the fiber

$$\mathrm{Poly}^D_\Gamma(\mathcal{E}_1, \mathcal{E}_2)|_x = \mathrm{Poly}^D_\Gamma(V_1, V_2)$$

and define a linear subspace $V_2^{\supseteq}(\Gamma')$ of V_2 by

$$V_2^{\supseteq}(\Gamma') = \{v \in V_2 \,|\, I_v \supseteq \Gamma'\}.$$

We obtain an evaluation map

$$ev : \mathrm{Poly}^D_\Gamma(V_1, V_2) \times V_1^{=}(\Gamma') \to V_2^{\supseteq}(\Gamma') \tag{35.7}$$

by the restriction. Recall that

$$V_1^{=}(\Gamma') = \{v \in V_1 | I_v = \Gamma'\} \quad (\neq V_1^{\Gamma'}).$$

Lemma 35.8. *The map* (35.7) *is a submersion.*

Proof. Let $v \in V_1^{=}(\Gamma')$. It suffices to show that the linear map : $\mathrm{Poly}_\Gamma^D(V_1, V_2) \to V_2^{\supseteq}(\Gamma')$, $P \mapsto P(v)$ is surjective. Let $N(\Gamma')$ be the normalizer of Γ' in Γ. We put $H(\Gamma') = N(\Gamma')/\Gamma'$. Then $H(\Gamma')$ acts freely on $V_1^{=}(\Gamma')$. Also $H(\Gamma')$ acts on $V_2^{\supseteq}(\Gamma')$.

Let $w \in V_2^{\supseteq}(\Gamma')$. By Sublemma 35.6, we find an $H(\Gamma')$ invariant polynomial map $P_1 : V_1^{=}(\Gamma') \to V_2^{\supseteq}(\Gamma')$ such that $P_1(v) = w$. P_1 is also $N(\Gamma')$-invariant.

Since $I_v = \Gamma'$, it follows that $\gamma(v) \in V_1^{=}(\Gamma')$ if and only if $\gamma \in N(\Gamma')$. Therefore there exists a polynomial map $P_2 : V_1 \to V_2^{\supseteq}(\Gamma')$ such that

$$P_2(\gamma v) = \begin{cases} P_1(\gamma v) = \gamma P_1(v) & \text{if } \gamma \in N(\Gamma'), \\ 0 & \text{if } \gamma \notin N(\Gamma'). \end{cases} \tag{35.9}$$

Note that P_2 is not Γ invariant. If we define $P \in \mathrm{Poly}_\Gamma^D(V_1, V_2)$ by

$$P(x) = \frac{1}{\# N(\Gamma')} \sum_{\gamma \in \Gamma} \gamma^{-1} P_2(\gamma x),$$

then P is obviously Γ-invariant. The $N(\Gamma')$-invariance of P_1 and (35.9) imply that $P(v) = w$. The proof of Lemma 35.8 is complete. □

Using Lemma 35.8, we derive that $s^{-1}(0) \cap \mathcal{E}_1^{=}(\Gamma')$ is a smooth submanifold of $\mathcal{E}_1^{=}(\Gamma')$ for a generic Γ'-invariant fiberwise polynomial sections s of degree less than equal to D. And it follows that (35.4) also holds similarly as in the case $\Gamma' = \{1\}$ that we have already discussed.

We finally prove that $s^{-1}(0)/\Gamma$ has a triangulation compatible to the smooth structures on each stratum. We again consider the evaluation map

$$ev : \mathrm{Poly}_\Gamma^D(V_1, V_2) \times V_1 \longrightarrow V_2.$$

This map is real analytic : In fact, this is the reason why we consider the notion of normally polynomial mappings here in this chapter. Hence $Z = ev^{-1}(0)$ and Z/Γ are real analytic sets. Therefore Z/Γ has a Whitney stratification. We have a fiber bundle over M whose fiber is Z. We denote it by $\mathcal{Z} \to M$. The Whitney stratification of Z/Γ induces a Γ invariant Whitney stratification of $\mathcal{Z}$. Therefore $\mathcal{Z}$ has a Γ invariant simplicial decomposition such that each stratum is a sub-complex. (See [BCR98], for example.) On the other hand, since the evaluation map

$$C^\infty(M; \mathrm{Poly}_\Gamma^D(\mathcal{E}_1, \mathcal{E}_2)) \times \mathcal{E}_1 \longrightarrow \mathrm{Poly}_\Gamma^D(\mathcal{E}_1, \mathcal{E}_2) \times \mathcal{E}_1$$

is a submersion, the map

$$(x, v) \mapsto (s(x), v)\,;\, \mathcal{E}(1) = M \times \mathcal{E}(1) \to \mathrm{Poly}_\Gamma^D(\mathcal{E}_1, \mathcal{E}_2) \times \mathcal{E}_1$$

is transverse to each simplex of $\mathcal{Z} \subset \mathrm{Poly}^D_\Gamma(\mathcal{E}_1, \mathcal{E}_2) \times \mathcal{E}_1$ for generic

$$s \in C^\infty(M; \mathrm{Poly}^D_\Gamma(\mathcal{E}_1, \mathcal{E}_2))$$

by the Thom transversality theorem. For such s, the set $s^{-1}(0)/\Gamma$ has a compatible triangulation. The proof of Proposition 35.3 is now completed. □

35.2. Normal bundle and normally polynomial sections.

Let X be an orbifold and E be an orbi-bundle on it. (See the end of Definition A1.33 for the definition of orbi-bundle.) Let Γ be an abstract group. We put

$$X^{\cong}(\Gamma) = \{x \in X \mid I_x \cong \Gamma\}.$$

Here I_x is defined as follows: We identify a neighborhood of x locally as V_x/Γ_x: Let $\bar{x}$ be the element of V_x corresponding to x: Then I_x is the group of elements of Γ_x fixing the point $\bar{x}$. We remark that $X^{\cong}(\Gamma)$ is a smooth manifold.

In §A1.6, we define an element of $Sh(X^{\cong}(\Gamma), \underline{\Gamma})$ on $X^{\cong}(\Gamma)$, which we call standard stack structure on $X^{\cong}(\Gamma)$. (See Example-Definition A1.80.) We also define a normal bundle $N_{X^{\cong}(\Gamma)}X$ (Definition A1.91).

For each orbi-bundle E on X, we can define a restriction $E|_{X^{\cong}(\Gamma)}$ to $X^{\cong}(\Gamma)$. (See Definition A1.95.)

We remark that the normal bundle $N_{X^{\cong}(\Gamma)}X$ and the restriction $E|_{X^{\cong}(\Gamma)}$ are vector bundles on the standard stack structure on $X^{\cong}(\Gamma)$ in the sense of Definition A1.86 and is *not* necessary a vector bundle in the usual sense on the topological space $X^{\cong}(\Gamma)$. (An example where $N_{X^{\cong}(\Gamma)}X$ is not a vector bundle in the usual sense is in Example A1.65.)

By Lemma A1.96 we have an orbifold $(N_{X^{\cong}(\Gamma)}X)/\Gamma$ and by Lemma A1.97 the orbifold $(N_{X^{\cong}(\Gamma)}X)/\Gamma$ is diffeomorphic to a neighborhood of $N_{X^{\cong}(\Gamma)}X$ in X as an orbifold. We alert that the 'total space' of the vector bundle $N_{X^{\cong}(\Gamma)}X$ over stack is *not* a well-defined notion. On the other hand, the quotient $(N_{X^{\cong}(\Gamma)}X)/\Gamma$ is well-defined as a point set also.

Actually in §A1.6 we give a detail of the construction of the normal bundle $N_{X^{\cong}(\Gamma)}X$ and of the restriction $E|_{X^{\cong}(\Gamma)}$ only in case X is a good orbifold, that is a global quotient of a manifold by a group action. So in this book we use only this case. (We can construct $N_{X^{\cong}(\Gamma)}X$ and $E|_{X^{\cong}(\Gamma)}$ in general however.)

In order to define and study normally polynomial sections of orbifold and more generally of Kuranishi structure, we first generalize Proposition 35.3 to the case of vector bundle in the sense of Definition A1.86. We consider $(M, [\{h_{ij}\}, \{\gamma_{ijk}\}])$ where $[\{h_{ij}\}, \{\gamma_{ijk}\}] \in Sh(M; \underline{G})$. Let $\mathcal{F} = (\{F_i\}, \{g^F_{ij}\})$ and $\mathcal{E} = (\{E_i\}, \{g^E_{ij}\})$ be vector bundles on $[\{h_{ij}\}, \{\gamma_{ijk}\}] \in Sh((M, \mathcal{U}); \underline{G})$ where $\mathcal{U} = \{U_i\}$ is an open covering of M. (In later application $\mathcal{F}$ will be the normal bundle and $\mathcal{E}$ be the obstruction bundle.)

We assume that the G action on the fiber of F_i is effective. Then $\mathcal{F}/G$ is an orbifold by Lemma A1.96. We pull back $\mathcal{E}$ to $\mathcal{F}/G$. It is easy to see that the pull back $\pi^*\mathcal{E}$ is an orbi-bundle on $\mathcal{F}/G$.

By Lemma A1.98 and a discussion right before that, the set of G equivariant smooth maps of degree $\le D$, $C^\infty(M; \mathrm{Poly}^D(\mathcal{F},\mathcal{E})^G)$, is well-defined. Let $s \in C^\infty(M; \mathrm{Poly}^D(\mathcal{F},\mathcal{E})^G)$. Then it is easy to see that s induces a (single valued) section of the pull back orbi-bundle $\pi^*\mathcal{E}$ on $\mathcal{F}/G$. Hereafter we regard an element of $C^\infty(M; \mathrm{Poly}^D(\mathcal{F},\mathcal{E})^G)$ as a section of $\pi^*\mathcal{E}$.

Let Γ be an abstract group. Let $(\mathcal{F}/G)^{\cong}(\Gamma)$ be as before. (Note $\mathcal{F}/G$ is an orbifold.) We consider a connected component $(\mathcal{F}/G)^{\cong}(\Gamma)_k$ of it. We take $[p,v] \in \mathcal{F}/G$ where $p \in U_i \subset M$, $v \in F_i$. We remark that $I_v \cong \Gamma$.

We put

$$i(\Gamma;k) = \dim(F_i)_p^{I_v} - \dim(E_i)_p^{I_v}. \tag{35.10}$$

Here $(F_i)_p$ is the fiber of the vector bundle F_i at p and

$$(F_i)_p^{I_v} = \{w \in (F_i)_p \mid \forall g \in I_v \ g\cdot p = p\}.$$

Proposition 35.3bis. *For each sufficiently large D and generic smooth section s of $C^\infty(M; Poly^D(\mathcal{F},\mathcal{E})^G)$ the set $s^{-1}(0) \subset \mathcal{F}/G$ has the following properties.*

For each abstract group Γ, the intersection $(\mathcal{F}/G)^{\cong}(\Gamma)_k \cap s^{-1}(0)$ is a smooth manifold of dimension $\dim M + i(\Gamma;k)$.

Moreover $s^{-1}(0)$ has a triangulation which is compatible with the smooth structures of $(\mathcal{F}/G)^{\cong}(\Gamma)_k \cap s^{-1}(0)$.

If there is a compact subset K_0 of M and a section of s_0 satisfying the conclusion above on a neighborhood of K_0, we may perturb it to s so that s satisfies the conclusion everywhere and s coincides with s_0 on K_0.

The proof is the same as the proof of Proposition 35.3.

We next proceed to the study of normally polynomial section of orbi-bundle on an orbifold X. Let E be an orbi-bundle on X. For each abstract group Γ and a connected component $X^{\cong}(\Gamma)_k$ of $X^{\cong}(\Gamma)$ we define $d(\Gamma;k)$ as follows. Let $p \in X^{\cong}(\Gamma)_k$ and V_p/Γ_p be its neighborhood. Here $\Gamma_p \cong \Gamma$. The orbibundle E defines a Γ_p equivariant vector bundle on V_p.

$$d(\Gamma;k) = \dim X^{\cong}(\Gamma)_k - \dim E_p^{\Gamma}. \tag{35.11}$$

It is independent of p and depends only on Γ and k since the right hand side is integer valued and continuous on $X^{\cong}(\Gamma)$.

We remark that

$$d(\Gamma;k) = \dim M + i(\Gamma;k)$$

for $X = \mathcal{F}/G$ and in the situation of (35.10).

We next define the notion of normally polynomial section of s of E on X.

Definition 35.12. A single valued section s of an orbi-bundle E on an orbifold X is said to be a *normally polynomial section* if the following holds for each abstract group Γ.

We identify a tubular neighborhood $\mathrm{Tube}(X^{\cong}(\Gamma))$ of $X^{\cong}(\Gamma)$ with $(N_{X^{\cong}(\Gamma)}X)/\Gamma$. We put

$$K^{\cong}(X;\Gamma) = X^{\cong}(\Gamma) \setminus \bigcup_{\Gamma' \supset \Gamma} \mathrm{Tube}(X^{\cong}(\Gamma')).$$

Then there exists a section $s^{\Gamma} \in \mathrm{Poly}^D(N_{X^{\cong}(\Gamma)}X, E|_{X^{\cong}(\Gamma)})^{\Gamma}$ such that s^{Γ} induces s on a neighborhood of $K^{\cong}(X;\Gamma)$ in $\mathrm{Tube}(X^{\cong}(\Gamma))$.

Proposition 35.13. *For any normally polynomial section s of E there exists a sequence of normally polynomial sections s_{ϵ} converging to s in C^0 topology, such that the following holds.*

For each Γ the intersection $X^{\cong}(\Gamma) \cap s_{\epsilon}^{-1}(0)$ is a smooth manifold of dimension $d(\Gamma;k)$. Moreover $s_{\epsilon}^{-1}(0)$ has a triangulation which is compatible with the smooth structures of $X^{\cong}(\Gamma) \cap s_{\epsilon}^{-1}(0)$.

If there is a compact subset K_0 of X such that s has the properties above on a neighborhood of K_0, we may choose s_{ϵ} so that it coincides with s on K_0.

Note to state Proposition 35.13, we use the fact that $N_{X^{\cong}(\Gamma)}X$ exists as a bundle over stack, which we prove in detail only in case X is a good orbifold. So we will use Proposition 35.13 only in that case.

Proof. We are going to construct s_{ϵ} on

$$W_m = U(K_0) \cup \bigcup_{\#\Gamma' \geq m} \mathrm{Tube}(X^{\cong}(\Gamma'))$$

by downward induction on m. Here $U(K_0)$ is a sufficiently small neighborhood of K_0. We remark we will prove by induction that the section s_{ϵ} on W_m satisfies the conclusion for arbitrary finite group Γ and not only for finite group Γ' with $\#\Gamma' \geq m$. (Note $\mathrm{Tube}(X^{\cong}(\Gamma')) \cap X^{\cong}(\Gamma)$ can be nonempty for $\#\Gamma' \geq m$ and $\#\Gamma < m$.)

We assume X is compact for simplicity. (We can prove noncompact case by an obvious modification of the argument.) Then for sufficiently large m the set W_m is $U(K_0)$ and we can put $s_{\epsilon} = s$.

We assume that we have constructed s_{ϵ} on W_{m+1}. Let Γ' be a group of order m. We have already s_{ϵ} on $\bigcup_{\Gamma'' \supset \Gamma'} \mathrm{Tube}(X^{\cong}(\Gamma''))$. Hence we can extend it to a neighborhood of $\mathrm{Tube}(X^{\cong}(\Gamma'))$ by Proposition 35.3bis. Thus the induction works. The proof of Proposition 35.13 is complete. □

Now we are going to use Proposition 35.13 to study $X^{\cong}(\Gamma)$, where X is a space with Kuranishi structure. We first define the notion of normally polynomial section of the obstruction bundle of Kuranishi structure. Before doing so, we need a digression.

We recall that in the proof of Lemma A1.97 (tubular neighborhood theorem), we use Riemannian metric and exponential map on our orbifold. If we change the Riemannian metric we use, then the diffeomorphism $(N_{X^{\cong}(\Gamma)}X)/\Gamma \cong \mathrm{Tube}(X^{\cong}(\Gamma))$ changes. Of course, the diffeomorphism does not change up to isotopy. However the notion of section being normally polynomial is *not* invariant of smooth isotopy. So we need to fix the Riemannian metric in order to define it. We also need to fix an isomorphism between pullback of the restriction $\pi^*(E|_{X^{\cong}(\Gamma)})$ with the restriction of E to $\mathrm{Tube}(X^{\cong}(\Gamma))$. We specify the choice of them below.

Let X be a space with Kuranishi structure with tangent bundle. We fix a good coordinate system as in Lemma A1.11 and use the notation there. We take a Riemannian metric g_p on V_p for each $p \in P$ such that g_p is Γ_p invariant. We moreover assume that $\phi_{pq} : (V_{pq}, g_q) \to (V_p, g_p)$ is a totally geodesic and isometric embedding of Riemannian manifold. We can easily construct such g_p by induction on the order $<$ on P.

For each $p \in P$ we consider the (good) orbifold V_p/Γ_p and an orbibundle E_p/Γ_p on it. Let Γ be any abstract group (of finite order). We then obtain a standard stack structure on $N_{(V_p/\Gamma_p)^{\cong}(\Gamma)}(V_p/\Gamma_p)$ by applying Definition-Example A1.80 to $\tilde X = V_p$, $G = \Gamma_p$, $\Gamma = \Gamma$ etc. The choice of our metric g_p determines a diffeomorphism from $(N_{((V_p/\Gamma_p)^{\cong}(\Gamma))}(V_p/\Gamma_p))/\Gamma$ to a neighborhood $\mathrm{Tube}((V_p/\Gamma_p)^{\cong}(\Gamma))$ of $(V_p/\Gamma_p)^{\cong}(\Gamma)$ in V_p/Γ_p by Lemma A1.97.

By Definition A1.95 we have a vector bundle $E_p|_{(V_p/\Gamma_p)^{\cong}(\Gamma)}$ on $(V_p/\Gamma_p)^{\cong}(\Gamma)$, on which we equip with the standard stack structure.

We next fix a family of connections ∇^p on E_p for each $p \in P$. We assume that the bundle embedding $\hat\phi_{pq} : (E_q, \nabla^q)|_{V_{pq}} \to (E_p, \nabla^p)$ is totally geodesic. Namely we assume that the image of $\hat\phi_{pq}$ is preserved by ∇^p parallel transport and the restriction of ∇^p to the image of $\hat\phi_{pq}$ coincides with ∇^q. Such family of connections can be constructed also by induction on $p \in P$ with respect to the order $<$.

Note $\pi^*(E|_{(V_p/\Gamma_p)^{\cong}(\Gamma)})$ is an orbi-bundle on the orbifold $(N_{(V_p/\Gamma_p)^{\cong}(\Gamma)}(V_p/\Gamma_p))/\Gamma$. Then we can use the parallel transport of ∇^p along the g_p minimal geodesic to obtain an isomorphism between $\pi^*(E|_{(V_p/\Gamma_p)^{\cong}(\Gamma)})$ and the restriction of E_p to the tubular neighborhood $\mathrm{Tube}((V_p/\Gamma_p)^{\cong}(\Gamma))$.

We thus fixed two identifications. We use them to define the notion of section s_p of E_p/Γ_p on V_p/Γ_p being normally polynomial. (Definition 35.12).

We also use the Riemannian metric g_p and the exponential map to define a diffeomorphism between the normal bundle $N_{\phi_{pq}(V_{pq})}(V_p)$ and the tubular neighborhood $\mathrm{Tube}(\phi_{pq}(V_{pq}))$ of $\phi_{pq}(V_{pq})$ in V_p. We use ∇^p to define an isomorphism of the pull back bundle $\pi^*(E_p|_{\phi_{pq}(V_{pq})})$ on $N_{\phi_{pq}(V_{pq})}(V_p)$ and the restriction of E_p to the tubular neighborhood $\mathrm{Tube}(\phi_{pq}(V_{pq}))$. These identifications are used in Definition A1.21 to define compatibility of (multi)sections of E_q and of E_p.

Definition 35.14. Let $\mathfrak{s} = \{\mathfrak{s}_p\}$ be the system of single valued sections $\mathfrak{s}_p$ of E_p. Assume that they are compatible in the sense of Definition A1.21. We say that it is a *normally polynomial section* if $\hat{\phi}_{pq}\mathfrak{s}_q = \mathfrak{s}_p\phi_{pq}$ and if it is a normally polynomial (local) section for each $p \in P$. For each Γ and a connected component $V_p(\Gamma)_k$ of $V_p(\Gamma)$, we define $d(\Gamma; p; k)$ by the formula (35.11).

Proposition 35.15. *Let X be given a Kuranishi structure that has a tangent bundle in the sense of Definition* A1.14. *Let $\{(V_p, E_p, \Gamma_p, \psi_p, s_p)\}_{p\in P}$ be a good coordinate system. Then there exists a family of normally polynomial sections $\mathfrak{s}^\epsilon = \{\mathfrak{s}^\epsilon_p\}$ parameterized by ϵ so that $X^{\mathfrak{s}^\epsilon} = \bigcup_p (\mathfrak{s}^\epsilon_p)^{-1}(0)$ has the following properties for any Γ.*

(1) $X^{\mathfrak{s}^\epsilon}_{\cong}(\Gamma) := \bigcup_p (\mathfrak{s}^\epsilon_p)^{-1}(0) \cap V_p^{\cong}(\Gamma)$ *is a smooth manifold.*
(2) *The dimension of $X^{\mathfrak{s}^\epsilon}_{\cong}(\Gamma)$ is $d(\Gamma; p; k)$, which depends only on the connected component of $X^{\mathfrak{s}^\epsilon}_{\cong}(\Gamma)$.*
(3) $\bigcup_p (\mathfrak{s}^\epsilon_p)^{-1}(0)/\Gamma_p$ *has a triangulation compatible with the smooth structures of $X^{\mathfrak{s}^\epsilon}_{\cong}(\Gamma)$.*
(4) $\lim_{\epsilon\to 0} \mathfrak{s}^\epsilon = s$*, where s is the Kuranishi map of the given Kuranishi structure.*

Proof. The proof is by induction on $p \in P$ with respect to the order $<$. If p is minimal, we apply Proposition 35.13 to obtain $\mathfrak{s}^\epsilon_p$. Let us assume that we have $\mathfrak{s}^\epsilon_q$ for every $q < p$. We consider $\mathfrak{s}^\epsilon_q$ and the image $\phi_{pq}(V_{pq})$. We restrict $\mathfrak{s}^\epsilon_q$ on the image $\phi_{pq}(V_{pq})$ and use the embedding $\hat{\phi}_{pq}$ to obtain a section of $E_q|_{\phi_{pq}(V_{pq})} \to V_{pq}$. We can extend it to its neighborhood, so that the compatibility in the sense of Definition A1.21 is satisfied. We remark that this extended section is normally polynomial *since the part we added is linear.* In fact our choice of diffeomorphism of normal bundle with tubular neighborhood is designed so that it is compatible with the coordinate change. Also the connection ∇^p is chosen to be compatible with coordinate change.

Moreover the required properties (1) - (4) above are satisfied on the tubular neighborhood $N_{\phi_{pq}(V_{pq})}$ if it is satisfied by $\mathfrak{s}^\epsilon_q$.

Now we can use Propositions 35.13 to obtain the section $\mathfrak{s}^\epsilon_p$. The proof of Proposition 35.15 is complete. $\square$