Lagrangian Floer theory of arbitrary genus and Gromov-Witten invariant

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$$\mathbf{L} = \{(L_\kappa, b_\kappa)\}$$ A finite set of pairs

L_κ (relatively spin) Lagrangian submanifolds

$b_\kappa \in H^{\mathrm{odd}}(L;\Lambda_0)$ weak bounding cochains

$\mathscr{L}$ A cyclic unital filtered A inifinity category

$\mathbf{L}$ set of objects

$$\mathscr{L}\big((L,b),(L',b')\big) = H(L \cap L';\Lambda_0)$$

set of morphisms

F, Oh, Ohta, Ono (FOOO) (+ Abouzaid FOOO (AFOOO))

$HH^*(\mathscr{L},\mathscr{L})$ Hochshild cohomology

$HH_*(\mathscr{L},\mathscr{L})$ Hochshild homology

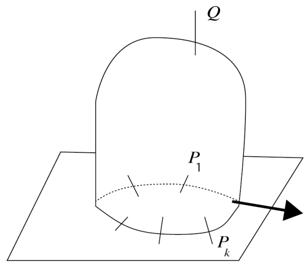
$HC^*(\mathscr{L})$ Cyclic cohomology

$HC_*(\mathscr{L})$ Cyclic homology

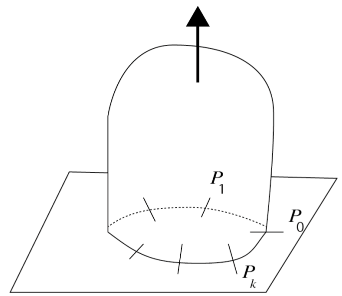
$$\mathsf{q} : H(X) \to HH^*(\mathscr{L},\mathscr{L})$$

$$\mathsf{p} : HH_*(\mathscr{L},\mathscr{L}) \to H(X)$$

$$\mathsf{p}_c : HC_*(\mathscr{L}) \to H(X)$$



q



p

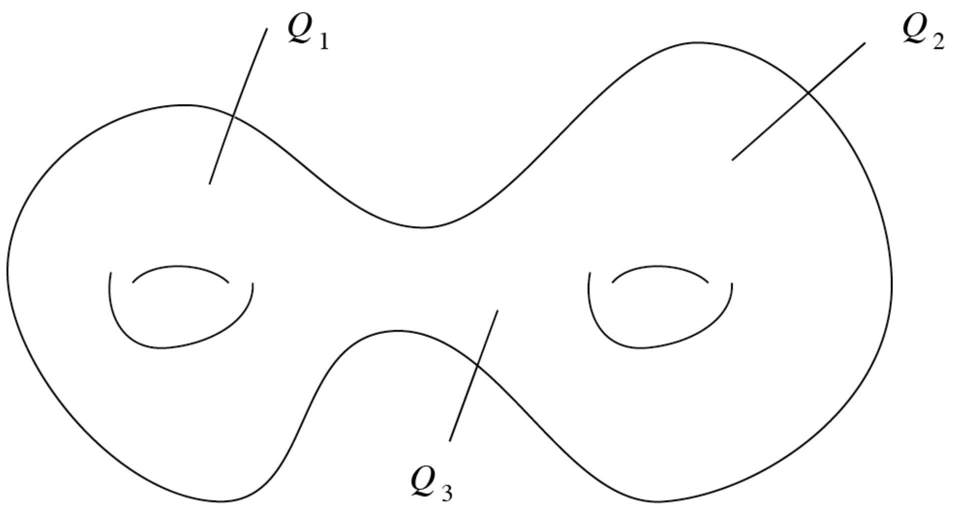
Open closed maps FOOO , AFOOO

Gromov-Witten invariant

$$GW_{g,\ell} : H(X;\Lambda)^{\otimes \ell} \to \Lambda$$

Counting genus g pseudo-holomorphic maps intersecting with cycles in X

$$\varphi : \Sigma \to X; \quad \Sigma : \text{Riemann surface}, \quad \varphi : \text{holomorphic}$$



Q_i : cycles on X

Problem to study

$$\mathbf{x}_i \in HH_*(\mathscr{L}, \mathscr{L}) \qquad \text{or} \qquad \mathbf{x}_i \in HC_*(\mathscr{L})$$

Compute

$$GW_{g,\ell}\left(\mathfrak{p}(\mathbf{x}_1), \ldots, \mathfrak{p}(\mathbf{x}_\ell)\right)$$

in terms of the structures of $\mathscr{L}$

A_∞ structure.

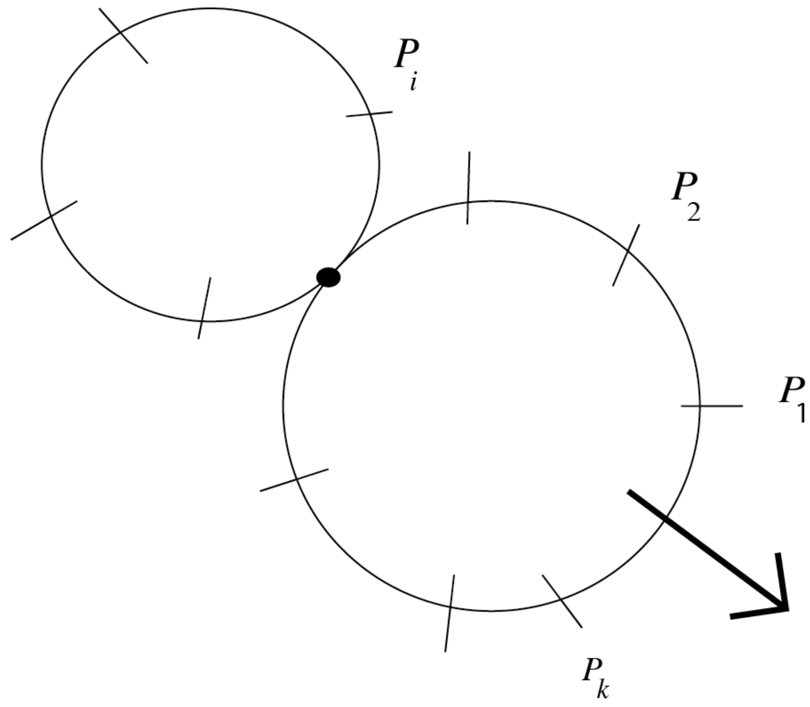
$$B_k \mathscr{L}\big((L,b),(L',b')\big)$$

$$= \bigoplus_{\substack{(L_0,b_0)=(L,b)\\(L_k,b_k)=(L',b')\\(L_i,b_i);\, i=1,\dots,k-1}} \Big(\mathscr{L}\big((L_0,b_0),(L_1,b_1)\big)\otimes\cdots\otimes\mathscr{L}\big((L_{k-1},b_{k-1}),(L_k,b_k)\big)\Big)$$

$$m_k : B_k \mathscr{L}\big((L,b),(L',b')\big) \to \mathscr{L}\big((L,b),(L',b')\big)$$

A_∞ relation

$$0 = \sum_{k_1+k_2=k+1} \sum_{i=1}^{k_1} \pm m_{k_1}(x_1,\dots,m_{k_2}(x_i,\dots,x_{i+k_2-1}),\dots,x_k)$$



$$0 = \sum_{k_1+k_2=k+1} \sum_{i=1}^{k_1} \pm m_{k_1}(x_1,\ldots,m_{k_2}(x_i,\ldots,x_{i+k_2-1}),\ldots,x_k)$$

Inner product and cyclicity

$$\mathscr{L}\big((L,b),(L',b')\big) = H(L \cap L';\Lambda_0)$$

$$\langle\ \ \rangle : \mathscr{L}\big((L,b),(L',b')\big) \otimes \mathscr{L}\big((L',b'),(L,b)\big) \to \Lambda$$

is (up to sign) a Poincare duality on $H(L \cap L';\Lambda_0)$

$$\big\langle m_k(x_1,\ldots,x_k),x_0 \big\rangle = \pm \big\langle m_k(x_0,x_1,\ldots,x_{k-1}),x_k \big\rangle$$

cyclicity

Problem to study

$$\mathbf{x}_i \in HH_*(\mathscr{L},\mathscr{L})$$ or $$\mathbf{x}_i \in HC_*(\mathscr{L})$$

Compute

$$GW_{g,\ell}\left(\mathsf{p}(\mathbf{x}_1),\ldots,\mathsf{p}(\mathbf{x}_\ell)\right)$$

in terms of the structures of $\mathscr{L}$

$$m_k : B_k\mathscr{L}\left((L,b),(L',b')\right) \to \mathscr{L}\left((L,b),(L',b')\right)$$

$$\langle\ \rangle : \mathscr{L}\left((L,b),(L',b')\right) \otimes \mathscr{L}\left((L',b'),(L,b)\right) \to \Lambda$$

Answer NO we can't !

$$GW_{g,\ell}\left(p(\mathbf{x}_1),\ldots,p(\mathbf{x}_\ell)\right)$$

is determined by the structures of $\mathscr{L}$

in case $g = 0, \quad \ell = 3$

In general we need extra information.

I will explain those extra information below.
It is Lagrangian Floer theory of higher genus (loop).

dIBL structure (differential involutive bi-Lie structure) on **B**

3 kinds of operations

$d : \mathbf{B} \to \mathbf{B}$ differential $dd = 0$

$\{\quad\} : \mathbf{B} \otimes \mathbf{B} \to \mathbf{B}$ Lie bracket Jacobi

$\}\{ : \mathbf{B} \to \mathbf{B} \otimes \mathbf{B}$ co Lie Bracket co Jacobi

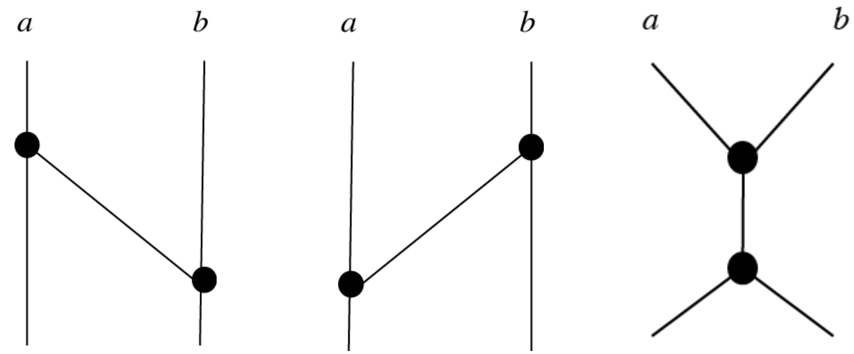
$\{\quad\}$ is a derivation

$\}\{$ is a coderivation

with respect to d

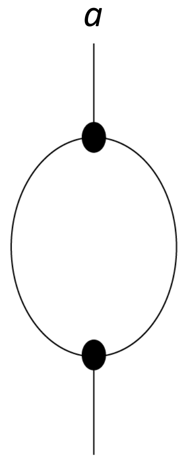
- { } is compatible with }{ $\}a\{=\sum a_{1,c}a_{2,c}$

$$\sum \pm a_{1,c}\{a_{2,c},b\}\pm a_{2,c}\{a_{1,c},b\}$$
$$+\sum \pm b_{1,c}\{b_{2,c},a\}\pm b_{2,c}\{b_{1,c},a\}$$
$$+\}\{a,b\}\{=0$$



- Involutive

$$\sum \{a_{1,c}a_{2,c}\}=0$$



IBL infinity structure = its homotopy everything analogue

operations :

$$\mathscr{P}_{n,m} : E_n\mathbf{B} \to E_m\mathbf{B} \qquad E_n\mathbf{B} = \underbrace{\mathbf{B} \otimes \cdots \otimes \mathbf{B}}_{n} / S_n$$

Homotopy theory of IBL infinity structure is built (Cielibak-Fukaya-Latschev)

$(C, \langle\ \rangle, d)$ chain complex with inner product

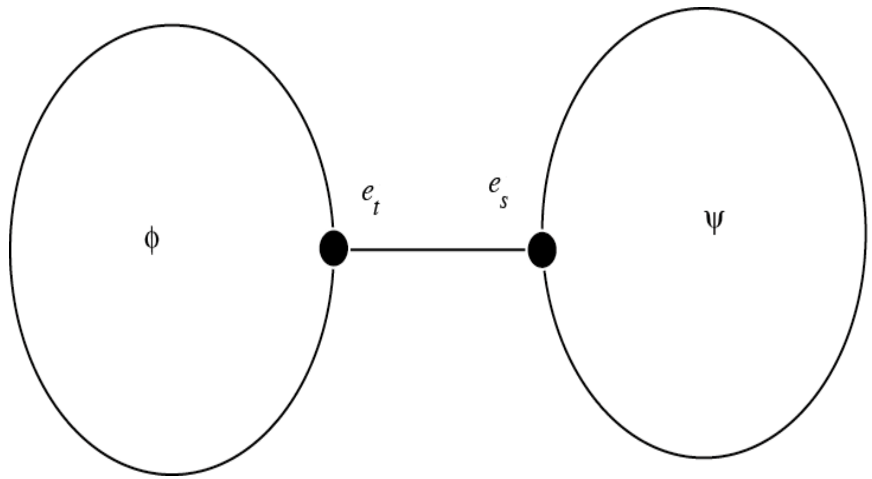
$\mathbf{B} = \left(B^{cyc}C\right)^*$ (dual cyclic bar complex) has a structure of dIBL algebra (cf. Cielibak-F-Latschev)

$$\varphi \in \left(B^{cyc}C\right)^*, \quad \varphi^{i_1 \cdots i_k} = \varphi(e^{i_1}, \cdots, e^{i_k})$$

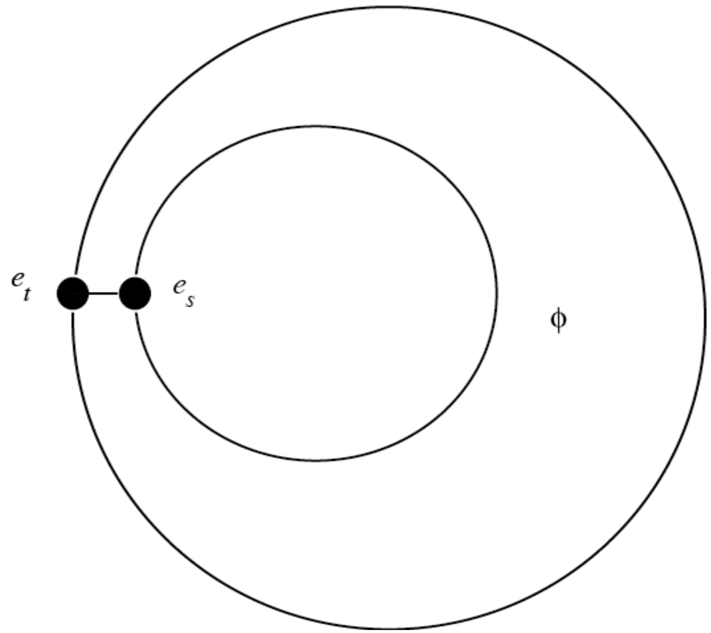
e_i basis of C

$$g^{st} = \left\langle e^s, e^t \right\rangle \qquad (g_{st}) = (g^{st})^{-1}$$

- $\{\varphi,\psi\}^{l_1\cdots l_{k+k'-2}} = \sum \pm g_{st}\varphi^{l_1\cdots l_{a-1} s l_{a+b+1}\cdots i_c}\psi^{l_a\cdots l_{a+b} t\, l_c\cdots l_{k+k'-2}}$



- $\}\varphi\{^{i_1\cdots i_k ; j_1\cdots j_l} = \sum \pm g_{st}\varphi^{i_a\cdots i_{a-1} s j_b\cdots j_{b-1} t}$



There is a category version.

Dual cyclic bar complex $\mathbf{B} = \left(B^{cyc}\mathscr{L}\right)^{*}$ has dIBL structure

Remark: This structure does NOT (yet) use A_∞ operations m_k
except the classical part of m_1, that is the usual boundary operator.

Cyclic A_∞ stucture (operations m_k) on $\mathscr{L}$

⇕

$\mathscr{M}_{1,0} \in \mathbf{B} = \left(B^{cyc}\mathscr{L} \right)^*$ satisfying Maurer-Cartan equation

$$d\mathscr{M}_{1,0} + \frac{1}{2}\{\mathscr{M}_{1,0}, \mathscr{M}_{1,0}\} = 0$$

This is induced by a holomorphic DISK

$\mathscr{M}_{1,0} \in \mathbf{B} = \left(B^{cyc}\mathscr{L} \right)^{*}$ is given by

$$\mathscr{M}_{1,0}(x_1, \ldots, x_k) = \left\langle m_{k-1}(x_1, \ldots, x_{k-1}), x_k \right\rangle$$

$$d\mathscr{M}_{1,0} + \frac{1}{2}\{\mathscr{M}_{1,0}, \mathscr{M}_{1,0}\} = 0$$

$\Updownarrow$

A_∞ relation among m_k

Theorem (Lagrangian Floer theory of arbitrary genus) (to be written up)

There exists $\mathscr{M}_{\ell,g} \in E_\ell \mathbf{B}$ such that BV master equation $\mathbf{B} = \left(B^{cyc}\mathscr{Q}\right)^*$

$$d\mathscr{M}_{\ell,g} + \frac{1}{2}\sum_{\ell_1+\ell_2=\ell+1}\sum_{g_1+g_2=g}\{\mathscr{M}_{\ell_1,g_1},\mathscr{M}_{\ell_2,g_2}\}_{\text{out}} + \}\mathscr{M}_{\ell-1,g}\{ + \{\mathscr{M}_{\ell+1,g-1}\}_{\text{int}} = 0$$

is satisfied.

The gauge equivalence class of $\{\mathscr{M}_{\ell,g}\}$ is well-defined.

Remark added after talks

After this talk M. Kontsevitch commented that we may not need all of

$$\mathscr{M}_{\ell,g} \in E_\ell \mathbf{B} \qquad \text{but only} \qquad \mathscr{M}_{1,0} \quad \mathscr{M}_{2,0}$$

that is enough to determine the way how `Hodge to de Rham degeneration' occurs.

This point is related to K. Costello's paper
The partition function of a topological field theory, J. Topol. 2 (2009), no. 4, 779--822.

I talked with Kontsevitch, Soibelman, Katzarkov, Costello, Abousaid during Miami conference 2012 January.

They mostly persuaded me.
However it is still necessary to study more to claim that stronger statement as a Theorem.
Anyway the theorem in the last page should be written up before really established.

Note: $\{\quad\}: \mathbf{B} \otimes \mathbf{B} \to \mathbf{B}$ induces

$\{\quad\}_{\text{out}} : E_n\mathbf{B} \otimes E_m\mathbf{B} \to E_{n+m-1}\mathbf{B}$ and

$\{\quad\}_{\text{int}} : E_n\mathbf{B} \to E_{n-1}\mathbf{B}$

$$\{x_1 \cdots x_n, y_1 \cdots y_m\}_{\text{out}} = \sum_{i,j} \pm \{x_i, y_j\} x_1 \cdots \hat{x}_i \cdots x_n y_1 \cdots \hat{y}_j \cdots y_m$$

$$\{x_1 \cdots x_n\}_{\text{int}} = \sum_{i,j} \pm \{x_i, x_j\} x_1 \cdots \hat{x}_i \cdots \hat{x}_j \cdots y_n$$

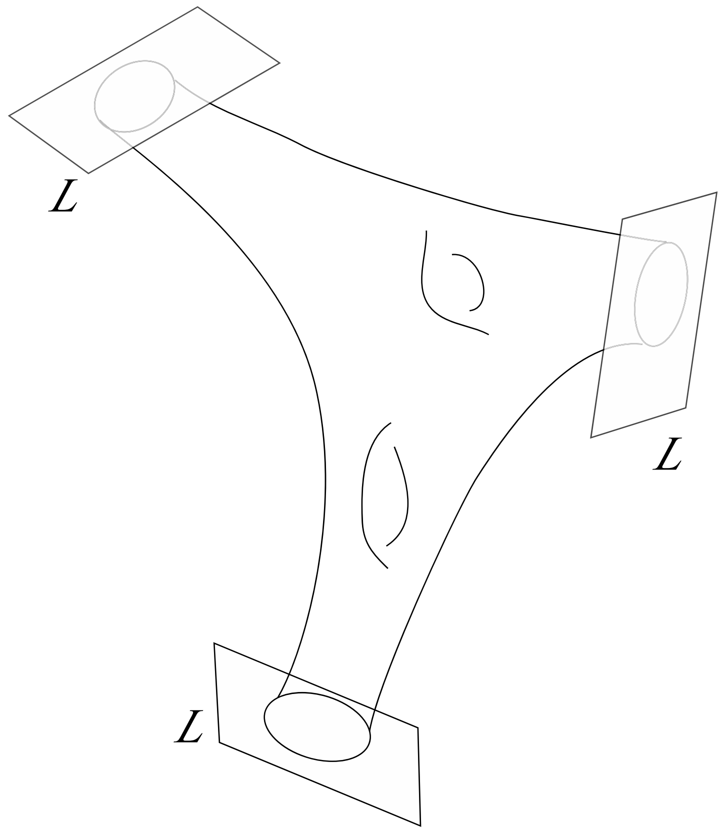
$\}\{:\mathbf{B}\to\mathbf{B}\otimes\mathbf{B}$ induces

$$\}\{:E_{n-1}\mathbf{B}\to E_n\mathbf{B}$$

$$\}x_1\cdots x_n\{=\sum_i \pm x_1\cdots\}x_i\{\cdots\hat{x}_j\cdots y_n$$

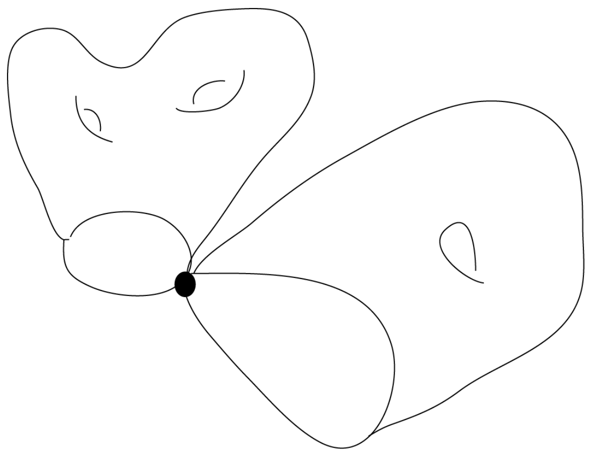
$$\mathscr{M}_{\ell,g} \in \left(\boldsymbol{E}_\ell \mathbf{B}\right)^*$$

is obtained from moduli space of genus g bordered Riemann surface with ℓ boundary components

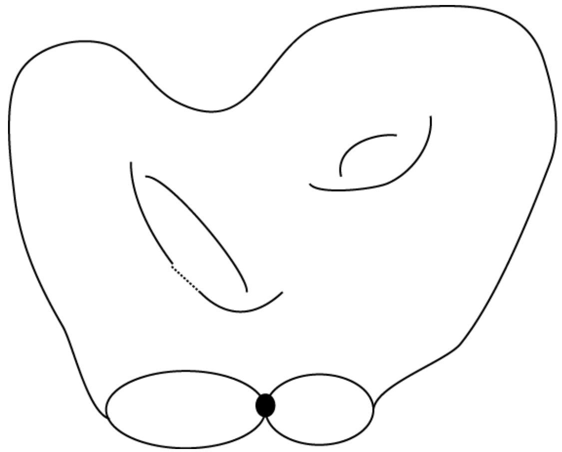


$$g = 2, \quad \ell = 3$$

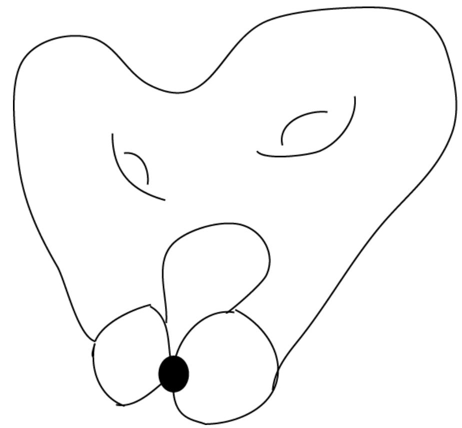
$$d\mathscr{M}_{\ell,g} + \frac{1}{2}\sum_{\ell_1+\ell_2=\ell+1}\sum_{g_1+g_2=g}\{\mathscr{M}_{\ell_1,g_1},\mathscr{M}_{\ell_2,g_2}\}_{\mathrm{out}} + \}\mathscr{M}_{\ell-1,g}\{ + \{\mathscr{M}_{\ell+1,g-1}\}_{\mathrm{int}} = 0$$



$\{\mathscr{M}_{\ell_1,g_1},\mathscr{M}_{\ell_2,g_2}\}_{\mathrm{out}}$



$\}\mathscr{M}_{\ell-1,g}\{$



$\{\mathscr{M}_{\ell+1,g-1}\}_{\mathrm{int}}$

Remark:

(1): In case the target space M is a point, a kind of this theorem appeared in papers by various people including Baranikov, Costello Voronov, etc. (In Physics there is much older work by Zwieback.)

(2): Theorem itself is also expected to hold by various people including F for a long time.

(3): The most difficult part of the proof is transversality. It becomes possible by recent progress on the understanding of transversality issues. It works so far only over $\mathbb{R}$. It also requires machinery from homological algebra of IBL infinity structure to work out the problem related to take projective limit, in the same way as A infinity case of [FOOO]. This homological algebra is provided by Cielibak-F-Latshev.

(4): Because of all these, the novel part of the proof of this theorem is extremely technical. So I understand that it should be written up carefully before being really established.

(5): In that sense the novel point of this talk is the next theorem (in slide 33) which contains novel point in the statement also.

Relation to `A model Hodge structure'

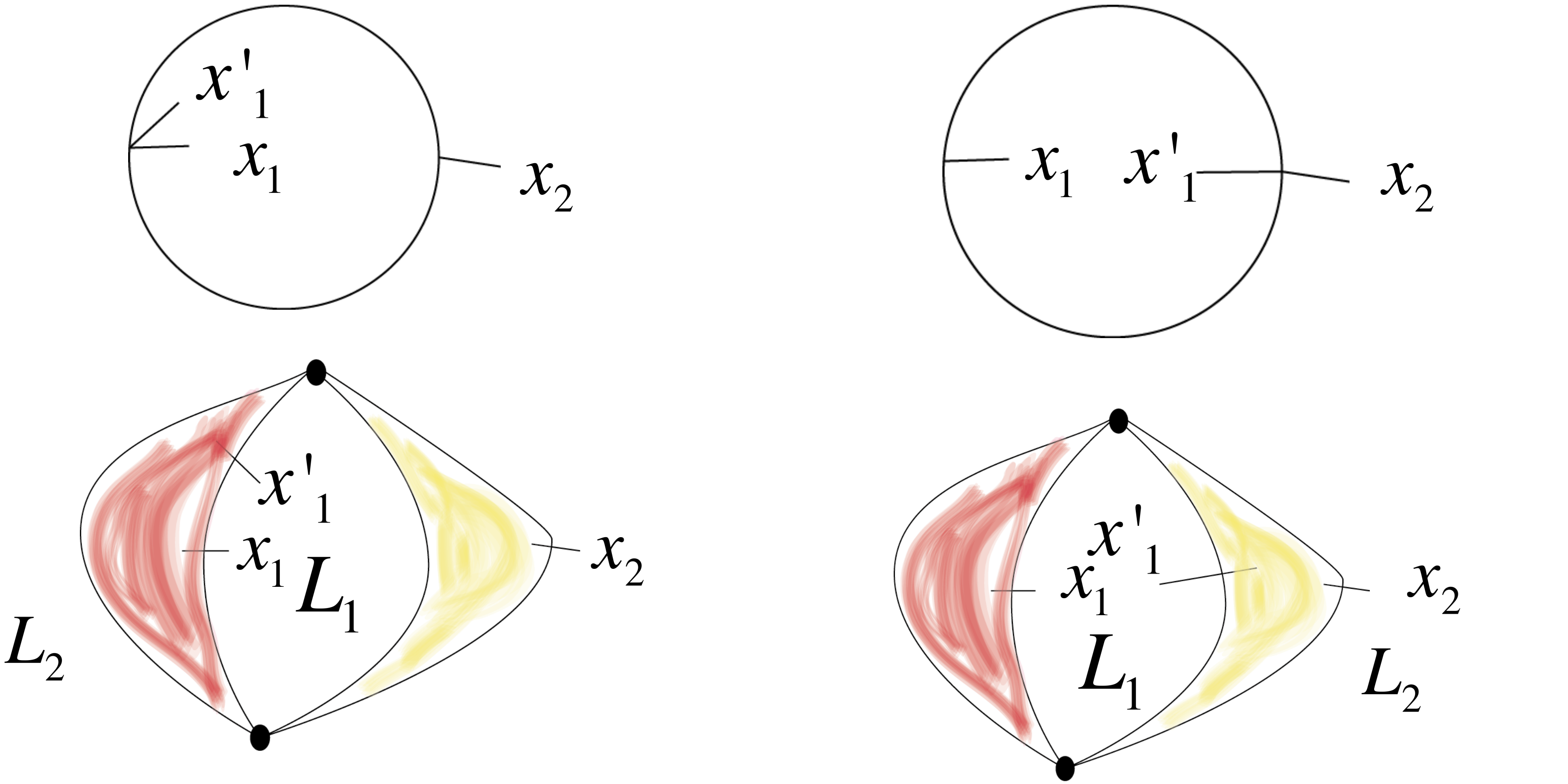
We need a digression first.

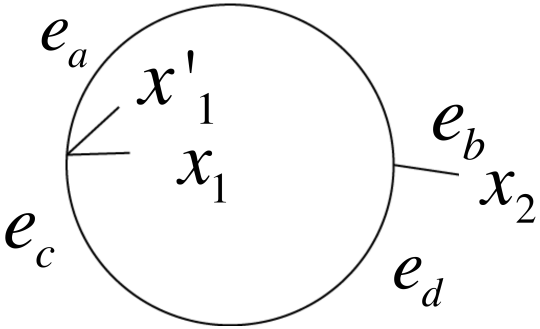
Let $\mathbf{x}_i \in HH_*(\mathscr{L},\mathscr{L})$ Put $Z(\mathbf{x}_1,\mathbf{x}_2) = \langle p(\mathbf{x}_1), p(\mathbf{x}_2) \rangle$

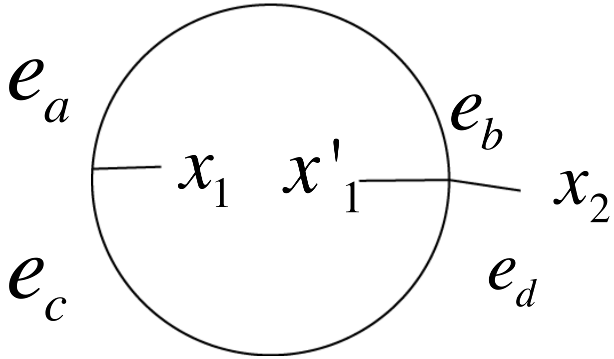
We (AFOOO) have an explicit formula to calculate it based on Cardy relation.

Formula for $Z(\mathbf{x}_1,\mathbf{x}_2)=\left\langle p\left(\mathbf{x}_1\right),p\left(\mathbf{x}_2\right)\right\rangle$

$x_1 \otimes x'_1 = \mathbf{x}_1 \qquad x_2 = \mathbf{x}_2 \qquad x_1, x'_1 \in H(L_1) \quad x_2 \in H(L_2)$







$$\sum_{a,b,c,d} g^{ab} g^{cd} \langle m_3(x_1, x'_1, e_a), e_c \rangle \langle m_2(x_2, e_b), e_d \rangle \quad + \sum_{a,b,c,d} g^{ab} g^{cd} \langle m_2(x_1, e_a), e_c \rangle \langle m_3(x_2, e_b, x'_1), e_d \rangle$$

Theorem (AFOOO, FOOO ....)

$$Z(\mathbf{x}_1, \mathbf{x}_2) = \langle \mathsf{p}(\mathbf{x}_1), \mathsf{p}(\mathbf{x}_2) \rangle$$

e_a a basis of $HF(L_1, L_2)$ $\qquad g_{ab} = \langle e_a, e_b \rangle$

Hochshild complex

Let $B; CH_*(\mathscr{L},\mathscr{L}) \to CH_{*+1}(\mathscr{L},\mathscr{L})$

be the operator obtained by `circle' action.

Hochshild homology $HH_*(\Omega(L),\Omega(L))$

is homology of the free loop space of L.

$$B; H_*(\mathcal{L}(L)) \to H_{*+1}(\mathcal{L}(L))$$

is obtained from the S^1 action on the free loop space.

Hochshild complex

Let $$B; CH_*(\mathscr{L}, \mathscr{L}) \to CH_{*+1}(\mathscr{L}, \mathscr{L})$$

be the operator obtained by `circle' action.

Proposition $$Z(B\mathbf{x}_1, \mathbf{x}_2) = \left(\}\mathscr{M}_{1,0}\{\right)(\mathbf{x}_1, \mathbf{x}_2)$$

it implies that there exists $$K; CH_*(\mathscr{L}, \mathscr{L}) \to CH_{*+2}(\mathscr{L}, \mathscr{L})$$

such that $\delta K + K\delta = B \qquad BK = 0$ if Z is non-degenerate

because $$\pm\}\mathscr{M}_{1,0}\{ \quad = d\mathscr{M}_{2,0} + \{\mathscr{M}_{2,0}, \mathscr{M}_{1,0}\}_{\text{out}}$$

2nd of BV master equation

Corollary (Hodge – de Rham degeneration) (Conjectured by Kontsevitch-Soibelman)

If Z is non-degenerate then

$$HC(\mathscr{Q},\mathscr{Q};\Lambda) \cong HH(\mathscr{Q},\mathscr{Q};\Lambda) \otimes \Lambda[[S]]$$

Remark: This uses only $\mathscr{M}_{2,0}$: moduli space of annulus.

To recover $GW_{g,\ell}(\mathsf{p}(\mathbf{x}_1),\ldots,\mathsf{p}(\mathbf{x}_\ell))$

we must to use all the informations $\mathscr{M}_{\ell,g}$

Remark: Why this is called `Hodge – de Rham degeneration' ?

Hodge structure uses $\partial, \quad \bar{\partial}$ with $\partial\bar{\partial} + \bar{\partial}\partial = \partial\partial = \bar{\partial}\bar{\partial} = 0$

One main result of Hodge theory is $\dfrac{\mathrm{Ker}\,\partial}{\mathrm{Im}\,\partial} \cong \dfrac{\mathrm{Ker}\,\bar{\partial}}{\mathrm{Im}\,\bar{\partial}} \cong \dfrac{\mathrm{Ker}\,d}{\mathrm{Im}\,d}$ $\quad d = \partial + \bar{\partial}$

we may rewrite this to $d_u = \partial + u\bar{\partial}$ $\quad H(d_u) \cong \dfrac{\mathrm{Ker}\,d_u}{\mathrm{Im}\,d_u}$ is independent of u

We have $\delta B + B\delta = BB = \delta\delta = 0$

Put $d_u = \delta + uB$

$\delta K + K\delta = B \qquad BK = 0$ ⇒ $H(d_u) \cong \dfrac{\mathrm{Ker}\,d_u}{\mathrm{Im}\,d_u}$ is independent of u

construction will be generalized and rewritten in this categorical language.

objects in this paper	generalization	
$(f, \mathbb{C}^{n+1})$	$\mathcal{A}$	A_∞-category (compact, smooth, Calabi-Yau)
J_f	$HH^\bullet(\mathcal{A}, \mathcal{A})$	Hochschild cohomology
$(\Omega^\bullet_{\mathbb{C}^{n+1}}, df\wedge)$	$(C_\bullet(\mathcal{A}, \mathcal{A}), b)$	Hochschild chain complex
Ω_f	$HH_\bullet(\mathcal{A}, \mathcal{A})$	Hochschild homology
d	B	Conne's differential
δ_w^{-1}	u	parameter of degree 2
$\mathcal{H}_f$	$HP_\bullet(\mathcal{A})$	periodic cyclic homology
$\mathcal{H}_f^{(0)}$	$HC_\bullet^-(\mathcal{A})$	negative cyclic homology
$\Omega_f[[\delta_w^{-1}]] \Rightarrow \mathcal{H}_f^{(0)}$	$HH_\bullet(\mathcal{A}, \mathcal{A})[[u]] \Rightarrow HC_\bullet^-(\mathcal{A})$	Hodge to de Rham spectral sequence
$\frac{d}{d\delta_w}$	$-u^2\frac{d}{du}$	flat connection on $HC_\bullet^-(\mathcal{A})$ with poles of order 2 at $u=0$
K_f	?	higher residue pairings
$J_f \overset{\zeta}{\simeq} \Omega_f$	$HH^\bullet(\mathcal{A}, \mathcal{A}) \overset{\zeta}{\simeq} HH_\bullet(\mathcal{A}, \mathcal{A})$	primitive form (at the origin $0 \in S$)

If one takes $\mathcal{A}$ in the list as the dg enhancement of the category of singularity, then one

Floer's boundary operator

$f = \mathrm{PO}$ Landau-Gizburg potential

Table from Saito-Takahashi's paper

FROM PRIMITIVE FORMS TO FROBENIUS MANIFOLDS

(Similar table is also in a paper by Katzarkov-Kontsevich-Pantev)

Z plus paring between HH_* and HH^*

MainTheorem (work in progress)

The gauge equivalence class of $\{\mathscr{M}_{\ell,g}\}$ determines Gromov-Witten invariants

$$GW_{g,\ell}\left(p(\mathbf{x}_1),\ldots,p(\mathbf{x}_\ell)\right)$$

for $\mathbf{x}_i \in HH_*(\mathscr{L},\mathscr{L})$

if Z is non-degenerate.

The story of $\{\mathscr{M}_{\ell,g}\}$ fits naturally to the on going project

to prove homological Mirror symmetry by family Floer homologies.

So we can expect that it can be used to enhance

homological Mirror symmetry and classical Mirror symmetry

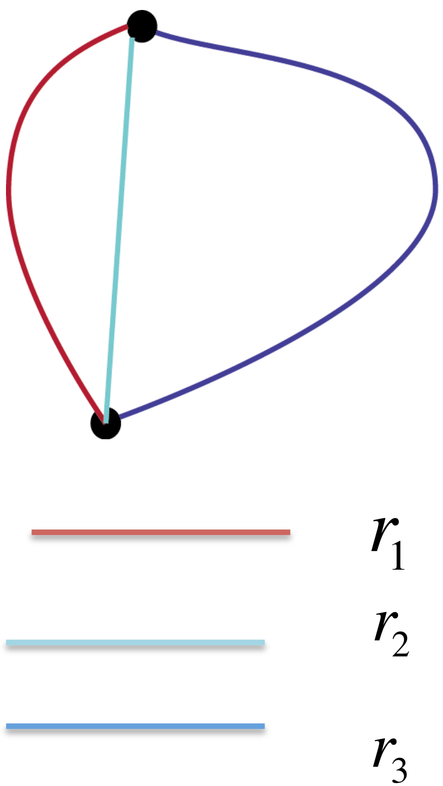
to include arbitrary genus.

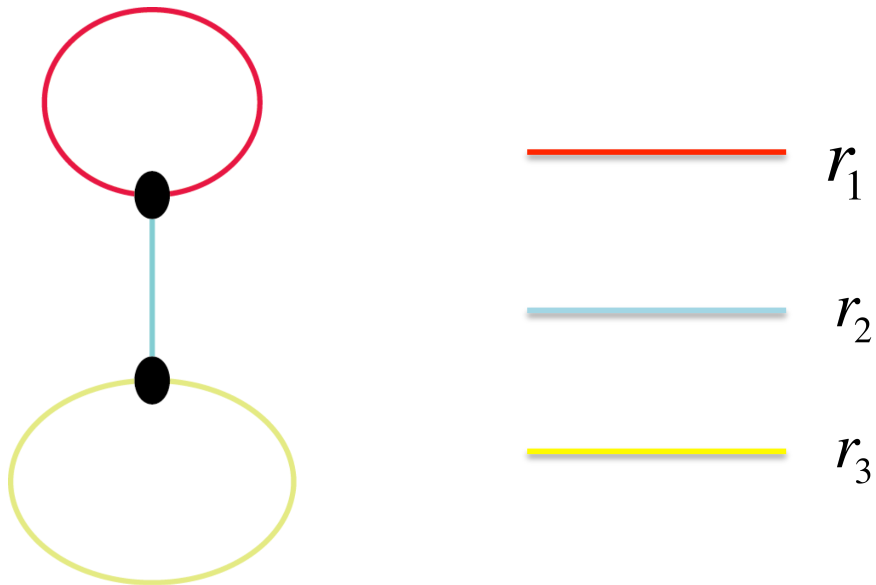
(B sides should be quantum Kodaira-Spencer theory. (BCOV).)

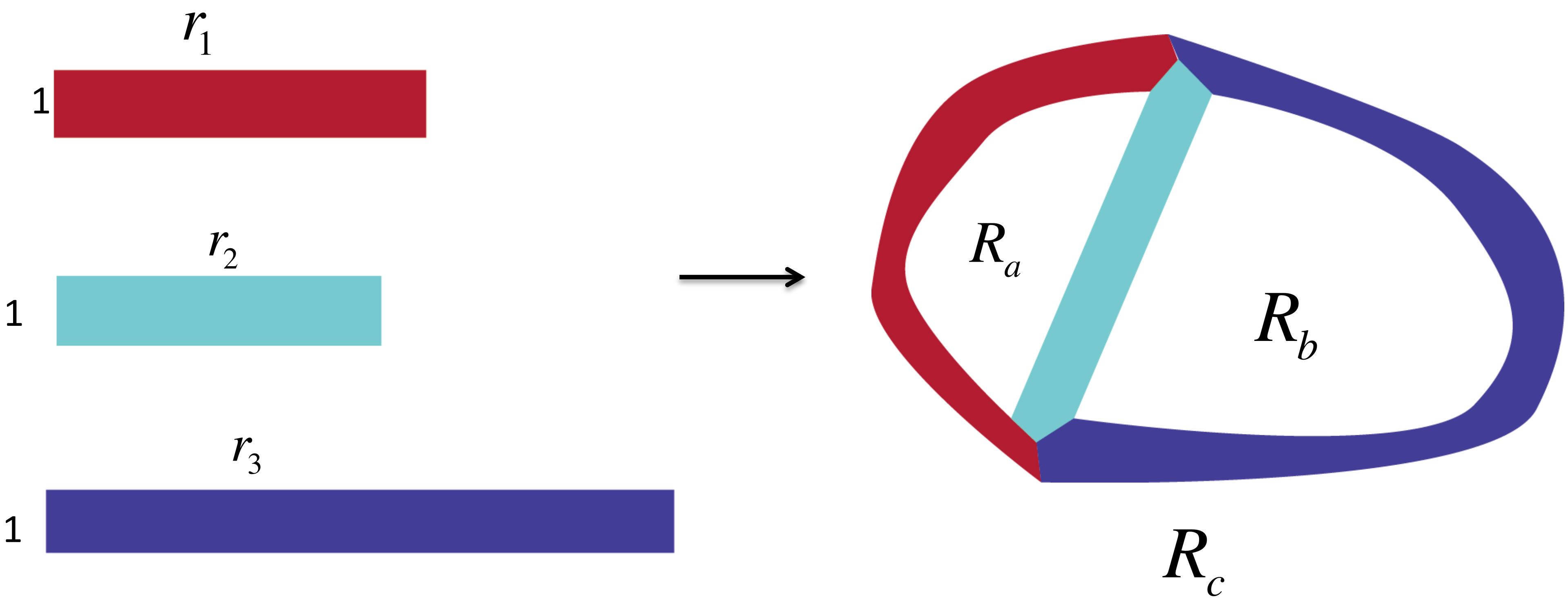
Idea of the proof of Main theorem

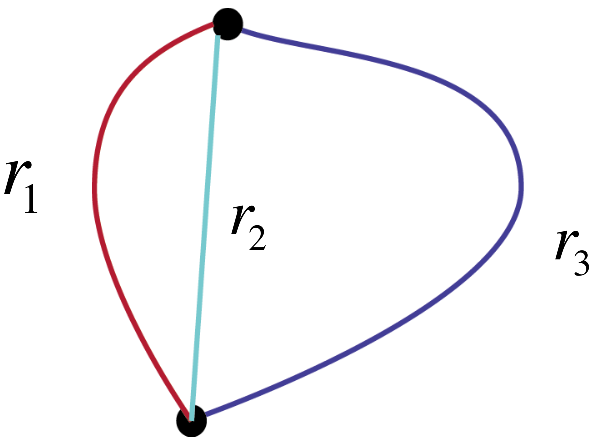
Metric Ribbon tree → Bordered Riemann surface

Example: 2 loop





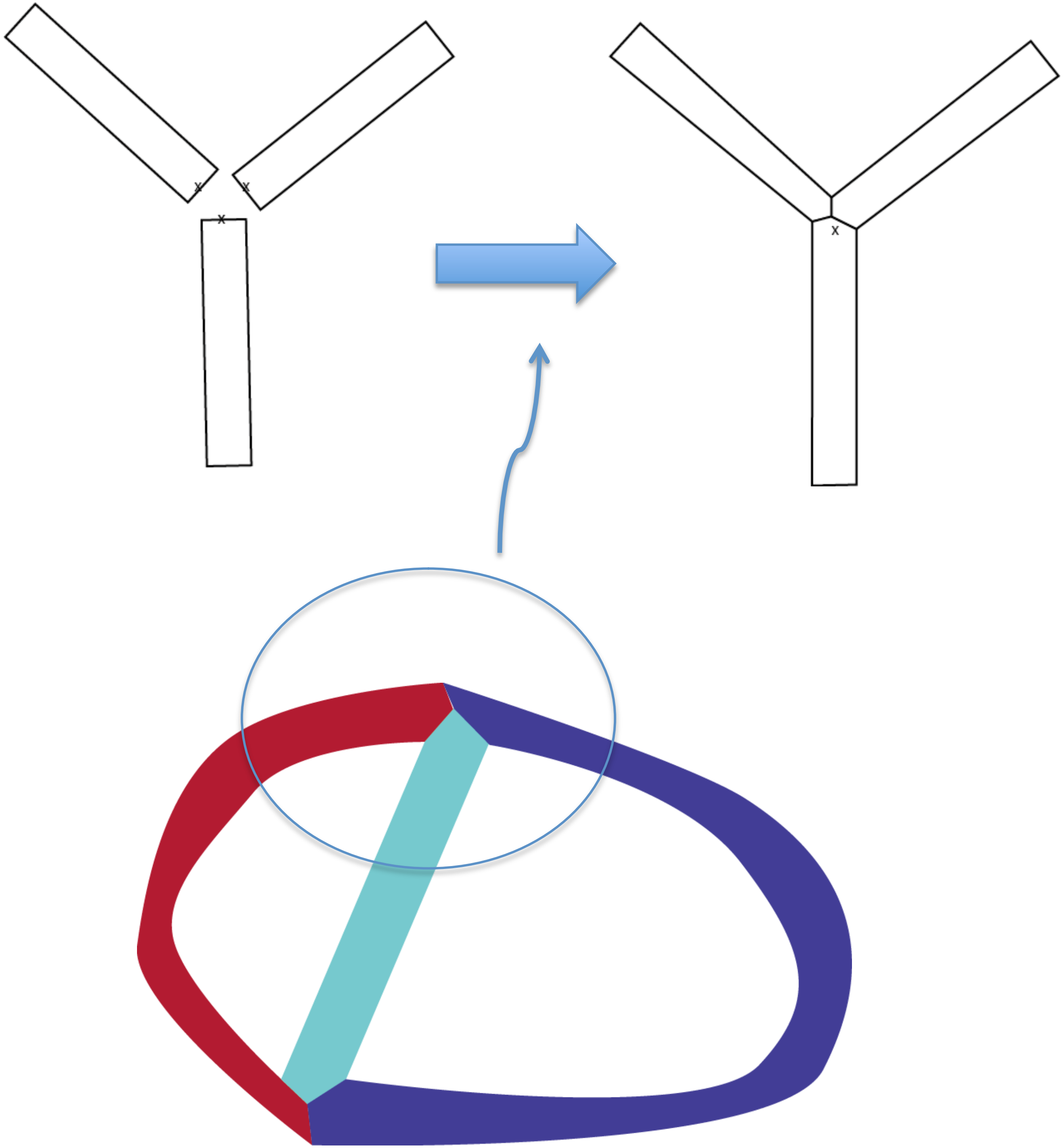




$$R_a = r_1 + r_2$$

$$R_b = r_2 + r_3$$

$$R_c = r_1 + r_3$$



Moduli of metric ribbon graph $=$ $\mathcal{M}_{l,g} \times \mathbb{R}_+^l$

R_a

R_b

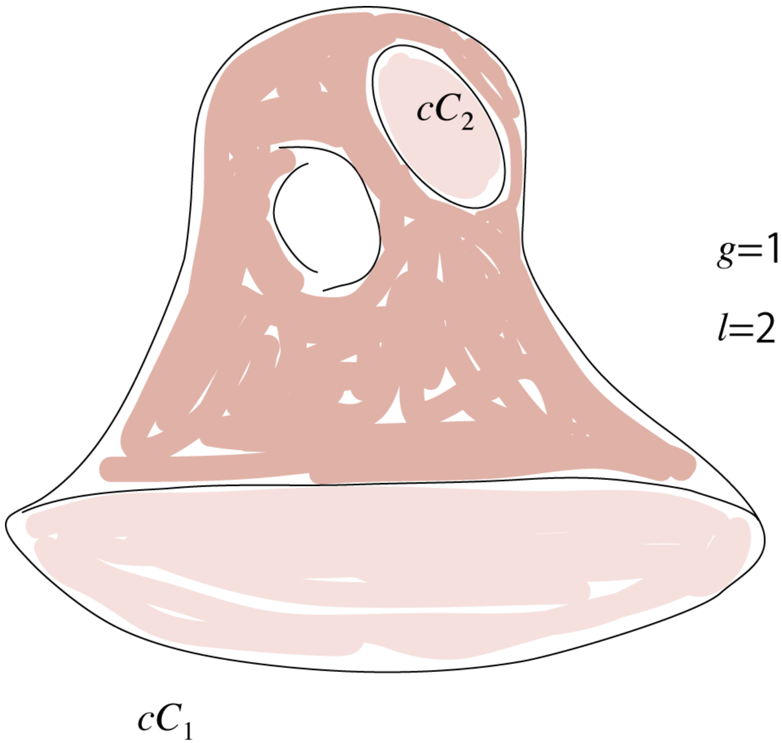
etc.

Moduli of genus g Riemann surface with l marked points

This isomorphism was used in Kontsevich's proof of Witten conjecture

$\mathscr{M}_{l,g} \times \mathbb{R}_+^l$ is identified with moduli space of bordered Riemann surface.

Fix $C_1, \cdots, C_\ell$ and consider the family $\vec{R}(c) = (cC_1, \cdots, cC_\ell)$



g=1

l=2

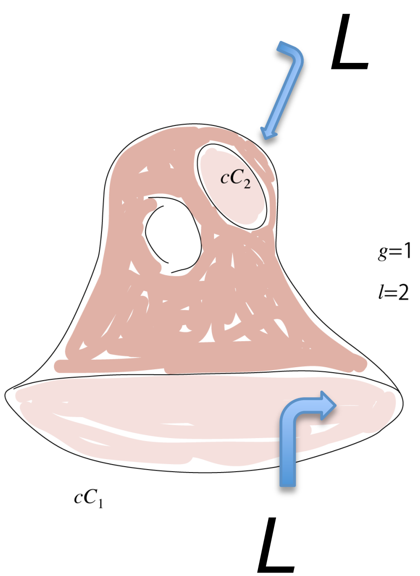
$\Sigma(c)$ parametrised by $c \in \mathbb{R}_{>0}$

$\Sigma \in \mathscr{M}_{l,g}$

Consider one parameter family of moduli spaces

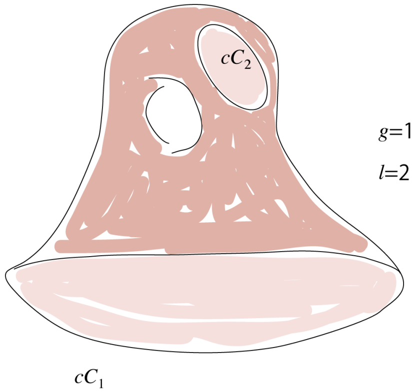
$$\mathcal{M}_{g,l}(c;\beta) = \{(u,\Sigma) \mid \; u:(\Sigma(c),\partial\Sigma(c)) \to (X,L)$$

$$\text{holomorphic}, [u] = \beta\}$$



Study the limit when $c \to 0$ $c \to \infty$

$\Sigma(c)$



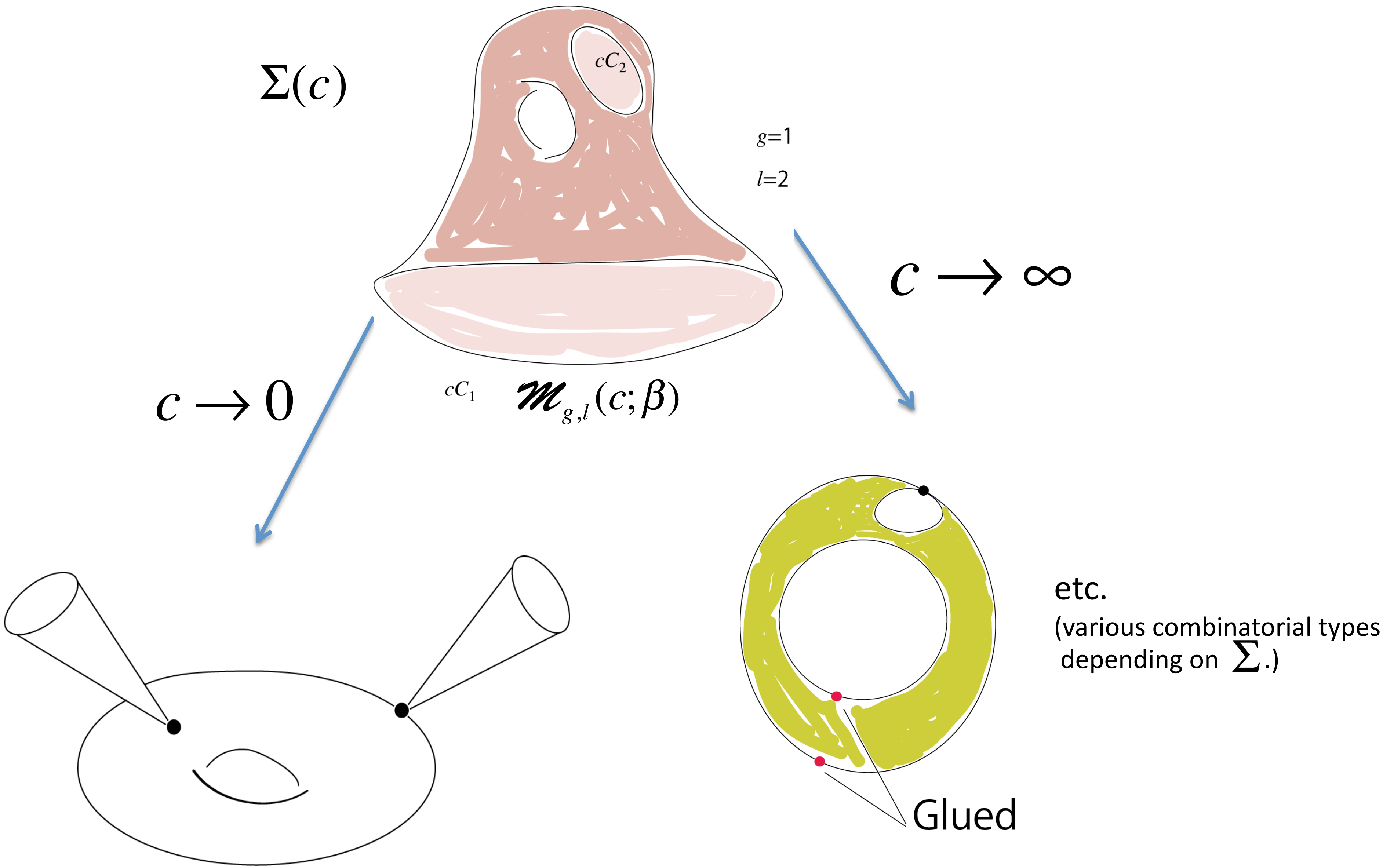
$c \to 0$

$\mathcal{M}_{g,l}(c;\beta)$

$c \to \infty$

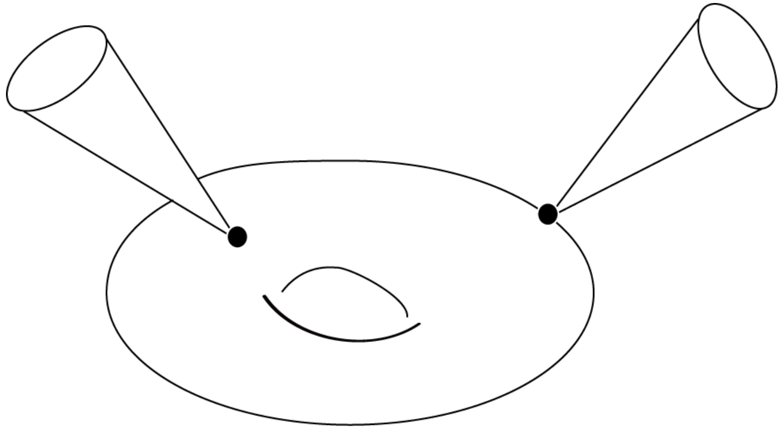
?

?



$\Sigma(c)$
cC_2
$g=1$
$l=2$
$c \to \infty$
$c \to 0$
cC_1
$\mathcal{M}_{g,l}(c;\beta)$
etc.
(various combinatorial types
depending on Σ.)
Glued

Counting



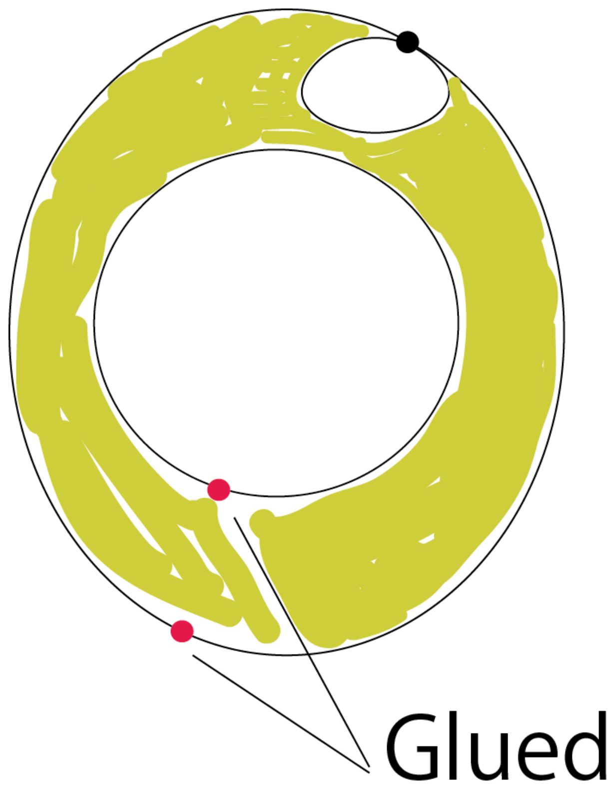
gives $$GW_{1,2}\left(p\left(\mathbf{x}_1\right),p\left(\mathbf{x}_2\right)\right)$$

More precisely integrating the forms $\mathbf{x}_1,\mathbf{x}_2,\ldots$

on the moduli space $\lim_{c\to 0}\mathcal{M}(c)$

by the evaluation map using the boundary marked points

Counting



We obtain numbers that can be calculated from $\{\mathscr{M}_{\ell,g}\}$

Actually we need to work it out more carefully.

Let $\mathbf{x} \in HH_*(\mathscr{L}, \mathscr{L})$

and try to compute $GW_{g,1}(p(\mathbf{x}))$

Need to use actually

$$\mathcal{M}^*_{g,1}(c,\beta) = \{(u,\Sigma,z) \mid u : (\Sigma(c), \partial\Sigma(c)) \to (X,L) : \text{holomorphic}$$
$$[u] = \beta, \text{ gunus of } \Sigma = 1, \quad \partial\Sigma(c) = S^1$$
$$z \in \partial\Sigma(c) : \text{ boundary marked point.}\}$$

$$\mathcal{M}^*_{g,1}(\beta) = \bigcup_{c \geq 0} \mathcal{M}^*_{g,1}(c,\beta)$$

Forgetting u defines $\pi : \mathcal{M}^*_{g,1}(\beta) \to \mathcal{M}^*_{g,1}$

$\mathcal{M}^*_{g,1}$ is identified with the total space of complex line bundle over

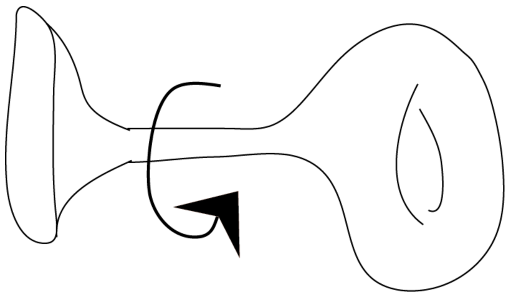
$\mathcal{M}_{g,1}$ (the fiber of $\pi : \mathcal{M}^*_{g,1} \to \mathcal{M}_{g,1}$

is identified to the tangent space of the unique interior marked point.)

$$\mathcal{M}^*_{g,1}(\beta) = \bigcup_{c \geq 0} \mathcal{M}^*_{g,1}(c,\beta)$$

(The absolute value corresponds to c

the phase S^1 corresponds to the extra freedom to glue.)



$$\mathcal{M}^*_{g,1} \supset \mathcal{M}_{g,1} = \text{Zero Section}$$

$$\uparrow \pi \qquad\qquad \uparrow \pi$$

$$\mathcal{M}^*_{g,1}(\beta) \supset \text{The space in tegration on which calculate}$$

$$\text{Gromov-Witten invariant}$$

$P : \mathcal{M}^*_{g,1} \to \mathcal{M}_{g,1}$ is not a trivial bundle. (Its chern class is Mumford-Morita class.)

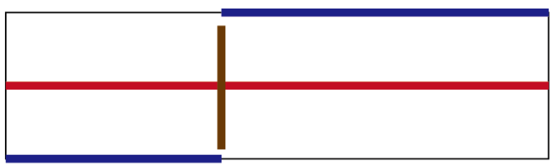
So $\mathcal{M}_{g,1} = \text{Zero Section}$ is not homlogous to a class in the boundary.

$\mathcal{M}_{g,1}$ = Zero Section

is homologous to a class on the boundary $\partial\mathcal{M}^*_{g,1}(\beta)$
and $\pi^{-1}(D)$

$$\begin{array}{ccc} \pi^{-1}(D) & \to & D \\ \uparrow & & \uparrow \\ \mathcal{M}^*_{g,1} & \to & \mathcal{M}_{g,1} \end{array}$$

here D is Poincare dual to the c^1



$\mathcal{M}_{g,1}$ = Zero Section

a class on the boundary

$\pi^{-1}(D)$

$$\int_{\pi^{-1}P^{-1}(D)} \mathrm{ev}^*(\mathbf{x}) =$$ something coming from boundary.

Because $\pi^{-1}P^{-1}(D)$ is a union of S^1 orbits, $\pi^{-1}P^{-1}(D) = S^1 W$

then

$$\int_{S^1 W} \mathrm{ev}^*(\mathbf{x}) = \int_W \mathrm{ev}^*(B\mathbf{x}) = \int_W \mathrm{ev}^*(\delta K\mathbf{x}) = \int_{\partial W} \mathrm{ev}^*(K\mathbf{x})$$

$$\because \quad \delta K + K\delta = B$$

Hodge to de Rham degeneration

$$\partial W = \partial_1 W + S^1 W'$$

$\partial_1 W \subset \partial \mathcal{M}^*_{g,1}(\beta)$ and $\int_{S^1 W'} \mathrm{ev}^*(K\mathbf{x}) = \int_{W'} \mathrm{ev}^*(BK\mathbf{x}) = 0$

$$\because \quad KB = 0$$

$$\int_{\pi^{-1}(\mathcal{M}_{g,1})} \mathrm{ev}^*(\mathbf{x}) = \int_A \mathrm{ev}^*(\mathbf{x}) + \int_{\partial W_1} \mathrm{ev}^*(K\mathbf{x})$$

A a class on the boundary

$\int_A \mathrm{ev}^*(\mathbf{x})$ and $\int_{\partial W_1} \mathrm{ev}^*(K\mathbf{x})$

are determined by $\{\mathscr{M}_{\ell,g}\}$ and $\mathbf{x} \in HH_*(\mathscr{L},\mathscr{L})$

QED