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Floer homology and Gromov-Witten invariant over integer of general symplectic manifolds - summary -

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Abstract. In this article we give a summary of an improvement our earier result [FO2] on Arnold's conjecture about the number of periodic orbits of periodic Hamiltonian system. In [FO2], we gave an estimate in terms of Betti numbers. In this article, we include torsion coefficients. We also define an "integer part" of the Gromov-Witten invariant.

§1. Introduction.

Let (X^{2n},ω) be a compact symplectic manifold and $h : X \times S^1 \to \mathbb{R}$ be a smooth function. We put $h_t(x) = h(x,t)$. Let V_{h_t} be the Hamiltonian vector field generated by h_t. Let $\Phi_t : X \to X$ be the family of symplectic diffeomorphisms such that

$$\frac{d\Phi_t}{dt} = V_{h_t} \circ \Phi_t, \quad \Phi_0 = id.$$

We assume that the graph $\mathrm{Graph}(\Phi_1)$ of $\Phi_1 \subset X \times X$ is transversal to the diagonal Δ_X. The intersection $\Delta_X \cap \mathrm{Graph}(\Phi_1)$ can be identified with the fixed point set $\mathrm{Fix}(\Phi_1)$ of Φ_1. Our main result is an estimate of the order of $\mathrm{Fix}(\Phi_1)$ in terms of the Betti numbers and the torsion coefficients of X.

We define the universal Novikov ring Λ by

$$\Lambda = \left\{ \sum c_i T^{\lambda_i} \,\middle|\, c_i \in \mathbb{Z},\, \lambda_i \in \mathbb{R},\, \lim_{i\to\infty} \lambda_i = \infty \right\}.$$

Here T is a formal parameter. We remark that the modulo 2 Conley-Zehnder index μ of elements of $\mathrm{Fix}(\Phi_1)$ is well-defined (see [F].) We put (for $i \in \mathbb{Z}_2$),

$$CF_i(X,h) = \bigoplus_{p\in \mathrm{Fix}(\Phi_1),\, \mu(p)=i} \Lambda[p].$$

The main result explained in this article is the following theorem, which is a version of Arnold's conjecture [A1],[A2].

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THEOREM 1. *There exist homomorphisms $\partial_i : CF_i(X,h) \to CF_{i-1}(X,h)$ such that $\partial_i\partial_{i+1} = 0$ and*

$$\frac{\operatorname{Ker}\partial_i}{\operatorname{Im}\partial_{i+1}} \simeq \sum_{i\equiv k \mod 2} H_k(M;\mathbb{Z})\otimes_{\mathbb{Z}}\Lambda. \tag{1}$$

REMARK 1. If we replace $\mathbb{Z}$ by $\mathbb{Q}$, Theorem 1 was proved by Fukaya-Ono[FO1,2], Liu-Tian[LT], Ruan [R]. In case when X is semi-positive, Theorem 1 was proved by Hofer-Salamon[HS] and Ono[O]. (They are generalizations of celebrated results by Conley-Zehnder [CZ] and Floer [F].)

In this article, we show an outline of a proof of Theorem 1. The detail will appear elsewhere.

§2. A brief review of Floer homology and negative multiple cover problem.

It is known to experts that, if one can define the fundamental chain over $\mathbb{Z}$ of the moduli space of pseudoholomorphic curves with appropriate properties, then we can prove Theorem 1. We first explain it briefly. Let J_X be an almost complex structure on X compatible with ω. We put

$$\operatorname{Orb}(h) = \left\{ \ell : S^1 \to X \,\middle|\, \frac{d\ell}{dt} = V_{h_t}(\ell(t)) \right\}$$

We can identify $\operatorname{Orb}(h)$ with $\operatorname{Fix}(\Phi_1)$. For $\ell_1,\ell_2 \in \operatorname{Orb}(h)$, we put

$$\widetilde{\mathcal{M}}(\ell_1,\ell_2) = \left\{ \varphi : \mathbb{R}\times S^1 \to X \,\middle|\, \begin{array}{l} \dfrac{\partial\varphi}{\partial\tau} = J_X\left(\dfrac{\partial\varphi}{\partial t} - V_{h_t}\right), \\ \lim_{\tau\to-\infty}\varphi(\tau,t) = \ell_1(t), \\ \lim_{\tau\to+\infty}\varphi(\tau,t) = \ell_2(t). \end{array} \right\}$$

$\mathbb{R}$ acts on $\widetilde{\mathcal{M}}(\ell_1,\ell_2)$ by the translation along $\mathbb{R}$ factor. Let $\mathcal{M}(\ell_1,\ell_2)$ be the quotient space. For $\varphi \in \widetilde{\mathcal{M}}(\ell_1,\ell_2)$, we define its energy by

$$E_h(\varphi) = \frac{1}{2}\int \left(\left\| \frac{\partial\varphi}{\partial\tau} \right\|^2 + \left\| \frac{\partial\varphi}{\partial t} - V_{h_t} \right\|^2 \right) dt d\tau.$$

We put

$$\mathcal{M}(\ell_1,\ell_2;E) = \{\varphi \in \mathcal{M}(\ell_1,\ell_2) | E_h(\varphi) = E\}.$$

Gromov's compactness theorem [G] implies that $\mathcal{M}(\ell_1,\ell_2;E)$ is nonempty only for $E = E_1, E_2, \cdots$ such that $0 = E_1 < E_2 < E_3 < \cdots$, $\lim E_i \to \infty$.

The virtual dimension of $\mathcal{M}(\ell_1,\ell_2;E)$ depends on the component. Let $\mathcal{M}(\ell_1,\ell_2;E;k)$ be the union of the components of virtual dimension k.

Suppose we have a "perturbation" of $\mathcal{M}(\ell_1,\ell_2;E;k)$ for $k = 0,1$ with the following properties.

(2.1) $\mathcal{M}(\ell_1,\ell_2;E;0)$ consists of finitely many points. Each point φ of $\mathcal{M}(\ell_1,\ell_2;E;0)$ is given an orientation $\epsilon_\varphi = \pm 1$.

(2.2) $\mathcal{M}(\ell_1,\ell_2;E;1)$ can be compactified to an oriented one dimensional manifold whose boundary is

$$\bigcup_{E'+E''=E}\bigcup_{\ell_3}\mathcal{M}(\ell_1,\ell_3;E';0)\times\mathcal{M}(\ell_3,\ell_2;E'';0).$$

We then put

$$\partial[\ell_1]=\sum_i\sum_{\ell_2}\sum_{[\varphi]\in\mathcal{M}(\ell_1,\ell_2;0;E_i)}\epsilon_\varphi T^{E_i}[\ell_2].$$

(2.1) implies that the coefficient of the right hand side belongs to Λ. Then (2.2) implies $\partial\partial=0$. We need some more properties to show the isomorphism (1). We omit the discussion about it in this article.

There is a trouble to find a perturbed moduli space satisfying (2.1) and (2.2). The main problem is the equivariant transversality at infinity, which we recall very briefly here. (A bit more detailed summary is in the introduction of [FO2].)

Let us consider a divergent sequence $\varphi_i\in\mathcal{M}(\ell_1,\ell_2;E;1)$. One possibility of its "limit" is an element of $\mathcal{M}(\ell_1,\ell_3;E';0)\times\mathcal{M}(\ell_3,\ell_2;E'';0)$. This is the component of the boundary of a compactification of $\mathcal{M}(\ell_1,\ell_2;E;1)$ described in (2.2). However there is another possibility. Namely φ_i may "converege" to a map $\varphi\sharp(\psi\circ\pi)$. Here $\varphi\in\mathcal{M}(\ell_1,\ell_2;E';*)$, $\psi:S^2\to M$ is a pseudoholomorphic map, and $\pi:S^2\to S^2$ is a degree k holomorphic map. ($E=E'+k([S^2]\cap\varphi^*\omega)$.) We assume also that $\psi(S^2)$ intersects with the image of φ. $\sharp$ denotes the connected sum. The trouble is especially serious in the case when $\varphi\sharp(\psi\circ\pi)$ has a nontrivial symmetry. If moreover $c^1(M)\cap\psi(S^2)$ is negative, we find that there is no perturbation, in the usual sense, to make $\varphi\sharp(\psi\circ\pi)$ transversal. This trouble is called the negative multiple cover problem. We studied it in [FO2], where we used a multivalued perturbation and hence we worked over rational coefficient. The purpose of this article is to explain an outline of a way to overcome this trouble without using rational coefficient.

§3. Period-doubling bifurcation and Stiefel-Whitney class.

Let us describe a toy model which shows how the rational coefficient occurs in a natural way. In this toy model, we consider a moduli space of maps $S^1\to Y$ in place of $\Sigma^2\to X$. Let Y be the Möbius band $\mathbb{R}\times[0,1]/\sim$ where $(x,1)=(f_\epsilon(x),0)$ and $f_\epsilon(x)$ is a diffeomorphism of $\mathbb{R}$ such that $f_\epsilon(x)=-(1+\epsilon)x+x^3$ in a neighborhood of 0. We consider the vector field $V_\epsilon=\partial/\partial y$. (Here y is the coordinate of the second factor.) Let $\mathcal{M}_\epsilon(2)$ be the moduli space of the solutions of

$$\frac{d\ell}{dt}=V_\epsilon$$

whose homology class is 2 times the generator of $H_1(Y;\mathbb{Z})\simeq\mathbb{Z}$. $\mathcal{M}_\epsilon(2)$ can be identified with the fixed point set of $f_\epsilon\circ f_\epsilon$ divided by the $\mathbb{Z}_2$ action induced by f_ϵ on it. Since

$$f_\epsilon\circ f_\epsilon(x)=(1+\epsilon)^2x-(4+\epsilon)x^3+\cdots,$$

in a neighborhood of 0, it follows that the fixed point set of $f_\epsilon\circ f_\epsilon$ consists of one point for $\epsilon<0$ and of 3 points for $\epsilon>0$. Taking into acount $\mathbb{Z}_2$ action, we find that $\mathcal{M}_\epsilon(2)$ consists of one point with multiplicity $-1/2$ for $\epsilon<0$, and of two points with multiplicity -1,$+1/2$, respectively, for $\epsilon>0$. Hence the total multiplicity is preserved. (Namely $-1/2 = -1 + 1/2$.) At first sight, it seems impossible to keep this independence of total multiplicity without introducing rational coefficient.

Figure 1

This phenomenon is called the period-doubling bifurcation and is famous in the study of dynamical system. (Taubes [T] also discussed it in the context of pseudohomolomorhpic tori in 4 manifolds.) Moreover period-doubling bifurcation can occur repeatedly and multiplicity will become 2^{-m}.

There is also a similar bifurcation related to cyclic groups of order ≥ 3. We will discuss it later in §5.

Let us now go back to our problem. First we compactify $\mathcal{M}(\ell_1, \ell_2; E; k)$ by adding isomorphism classes of maps from singular Rieman surfaces. (See [FO2] §19, where it is called stable connecting orbits.) We denote by $\mathcal{CM}(\ell_1, \ell_2; E; k)$ the compactification. Now the main technical result established in [FO2] is :

Theorem 2. ([FO2] Theorem 19.14.) *$\mathcal{CM}(\ell_1, \ell_2; E; k)$ has Kuranishi structure with corners.*

The precise definition of Kuranishi structure is in [FO2] §5. We birefly recall it here for reader's convenience. $\mathcal{CM}(\ell_1, \ell_2; E; k)$ is said to have a Kuranishi structure if, for each $x \in \mathcal{CM}(\ell_1, \ell_2; E; k)$, there exists an open subset $U_x \in \mathbb{R}^{m_x}$, a finite group Γ_x (the group of automorphisms of x) such that Γ_x acts on U_x and the action is linear. We also assume that there exist a Γ_x module $\mathcal{E}_x$ and a Γ_x equivariant map $s_x : U_x \to \mathcal{E}_x$, such that

$$s_x^{-1}(0)/\Gamma_x \simeq \text{a neighborhood of } x \text{ in } \mathcal{CM}(\ell_1, \ell_2; E; k).$$

We need to assume various compatibility conditions for these deta, which are omitted here. We call U_x the Kuranishi neighborhood, $\mathcal{E}_x$ the obstruction bundle and s_x the Kuranishi map.

The idea in [FO2] to find a $\mathbb{Q}$ chain is to perturb s_x by using multivalued perturbation. This method does not work for the purpose of this article. So we first try to go as much as single valued perturbation goes. We then obtain the following Proposition 1. To state it we need some notations. Let s'_x be a (single valued) perturbation of s_x satisfying appropriate compatibility conditions. (See [FO] §6.) We put

$$\mathcal{CM}'(\ell_1, \ell_2; E; k) = \bigcup s_x'^{-1}(0)/\Gamma_x.$$

We write it $\mathcal{CM}'$ in case no confusion can occur. Let G be a finite group. We put

$$\mathcal{CM}'(G) = \{x \in \mathcal{CM}' | \Gamma_x \simeq G\},$$
$$\mathcal{G}(G) = \bigcup_{x \in \mathcal{CM}'(G)} \Gamma_x.$$

$\mathcal{G}(G)$ is a local system on $\mathcal{CM}'(G)$.

Proposition 1. *The following holds for generic s'_x.*

(3.1) *$\mathcal{CM}'(G)$ is a smooth manifold with corners.*

(3.2) *There exists two vector bundles $\mathcal{E}_1(G)$, $\mathcal{E}_2(G)$ on $\mathcal{CM}'(G)$. $\mathcal{G}(G)$ acts on them. There exists also a $\mathcal{G}(G)$ equivariant bundle map $s_G : \mathcal{E}_1(G) \to \mathcal{E}_2(G)$ between them. (s_G may not be linear in general.)*

(3.3) *Let $x \in \mathcal{CM}'(G)$ and $\mathcal{E}_{1,x}(G)$, $\mathcal{E}_{2,x}(G)$ be fibers. We regard them as G vector spaces. Then they do not contain trivial component.* (*Note that this condition implies that s_G sends zero section to zero section.*)
(3.4) *The intersection of $s_G^{-1}(0)/\mathcal{G}(G)$ and a neighborhood of zero section in $\mathcal{E}_1(G)$ is identified to a neighborhood of $\mathcal{CM}'(G)$ in $\mathcal{CM}'$.*
(3.5) *Moreover, for each $x \in \mathcal{CM}'(G)$, its Kuranishi neighborhood U_x is identified to a neighborhood of x in $\mathcal{E}_1(G)$. The obstruction bundle is isomorphic to $\mathcal{E}_2(G)$ and the Kuranishi map is identified to the restriction of s_G to U_x.*

The proof will be given in [FO3]. Hereafter we write $\mathcal{CM}(G)$ etc. in place of $\mathcal{CM}'(G)$ etc.

Remark 2. We remark that, to show Proposition 1, we need to use abstract perturbation. In fact, the conclusion of Proposition 1 is not satisfied by any perturbation of the almost complex structure of M. The reason is that, if we perturb only almost complex structure, then multiple covered spheres may not be made transversal even in the case when its automorphism group is trivial.

Note the condition that the peudoholomorphic sphere is somewhere injective in the sense of McDuff [M] is related to but is different from the condition that pseudoholomorphic sphere does not have nontrivial symmetry.

We are going to show how we use Proposition 1 to avoid period-doubling bifurcations.

To clarify the idea, we first consider the simplest case. Namely we assume that $\mathcal{CM}(G)$ is nonempty only for $G = 1$ or $G = \mathbb{Z}_2$. We put $\mathcal{CM}(1) = N, \mathcal{CM}(\mathbb{Z}_2) = M$.

We first remark that (3.3) of Proposition 1 implies that $\mathcal{E}_1(1)$, $\mathcal{E}_2(1)$ are trivial. Namely N is transversal. In other words, the actual dimension of N is equal to its virtual dimension. On the other hand, the dimension of M can be higher than that.

We have $\mathbb{Z}_2$ vector bundles $\mathcal{E}_1(\mathbb{Z}_2)$, $\mathcal{E}_2(\mathbb{Z}_2)$ over M and $s_{\mathbb{Z}_2} : \mathcal{E}_1(\mathbb{Z}_2) \to \mathcal{E}_2(\mathbb{Z}_2)$. (The local system is trivial in this case.) We write $\mathcal{E}_1$, $\mathcal{E}_2$, s in place of $\mathcal{E}_1(\mathbb{Z}_2)$, $\mathcal{E}_2(\mathbb{Z}_2)$, $s_{\mathbb{Z}_2}$ for simplicity. Note that the action of $\mathbb{Z}_2$ on the fibers of $\mathcal{E}_1$, $\mathcal{E}_2$ is $\times -1$. ((3.3) of Proposition 1.) Hence the leading term of $\mathbb{Z}_2$ equivariant map $s : \mathcal{E}_1 \to \mathcal{E}_2$ is linear. So, by replaicing s, we may assume that s is linear in a neighborhood of 0 section. (This is not the case when the group G is more complicated.) We put

$$\Xi = \{x \in M | s_x : \mathcal{E}_{1x} \to \mathcal{E}_{2x} \text{ is not injective}\}. \tag{4}$$

By definition, it is easy to see that $M \cap \overline{N} = \Xi$. Namely Ξ is the set of points where period-doubling bifurcation occurs.

We can prove the following lemma by an easy dimension counting.

Lemma 1.
$$\operatorname{codim} \Xi = \operatorname{rank} \mathcal{E}_2 - \operatorname{rank} \mathcal{E}_1 + 1.$$

Note the virtual dimension of our moduli space is $\dim M + \operatorname{rank}\mathcal{E}_1 - \operatorname{rank}\mathcal{E}_2$. Therefore

$$\dim N = \dim M + \operatorname{rank}\mathcal{E}_1 - \operatorname{rank}\mathcal{E}_2 = \dim \Xi + 1.$$

It follows that $\dim \partial N = \dim \Xi$. In other words, N contains other boundary components than those stated in (2.2).

To clarify the topological backgroud, we prove the following :

Proposition 2. *Let M be an oriented closed manifold, $\mathcal{E}_1$, $\mathcal{E}_2$ be oriented vector bundles on it, and $s : \mathcal{E}_1 \to \mathcal{E}_2$ be a generic bundle homomorphism. (s is linear.) We assume that* $\mathrm{rank}\mathcal{E}_2 - \mathrm{rank}\mathcal{E}_1$ *is even. Define Ξ by* (4)*. Then we have the following :*

(5.1) *Ξ has an orientation and determines a cycle over $\mathbb{Z}$.*
(5.2) *The Poincaré dual to $[\Xi]$ is δy. Here*

$$\delta : H^k(M;\mathbb{Z}_2) \to H^{k+1}(M;\mathbb{Z})$$

is the Bockstein operator associated to the exact sequence $0 \to \mathbb{Z} \overset{\times 2}{\to} \mathbb{Z} \to \mathbb{Z}_2 \to 0$, and y is a polynomial of the Stiefel-Whitney classes of $\mathcal{E}_1$, $\mathcal{E}_2$.

Proof. First we define an orientation of Ξ. We put

$$\Xi_2 = \{x \in M \mid \dim \mathrm{Ker}\, s_x \geq 2\}.$$

It is easy to see that $\dim \Xi - \dim \Xi_2 \geq 2$. So it suffices to define an orientation only on $\Xi - \Xi_2$. (It is also easy to see that $\Xi - \Xi_2$ is a smooth manifold for generic s.) Let $x \in \Xi - \Xi_2$. *Choose* an orientation of $\mathrm{Im}\, s_x \subset \mathcal{E}_{2,x}$. Take $V_x \subset \mathcal{E}_{1,x}$ such that $s_x : V_x \to \mathrm{Im}\, s_x$ is an isomorphism. ($\mathrm{rank} V_x = \mathrm{rank}\mathcal{E}_{1,x} - 1$.) The orientation of $\mathrm{Im}\, s_x$ induces one on V_x. This orientation together with the orientation on $\mathcal{E}_{1,x}$ determine an orientation of one dimensional vector space $\mathcal{E}_{1,x}/V_x$. Let e_x be the oriented basis of the complement of $\mathcal{E}_{1,x}$ in V_x. We extend V_x and e_x to a neighborhood of x and denote it by V and e. Then $s(e)$ determines a section $\overline{e}$ of the bundle $\mathcal{E}_2/s(V)$. It is easy to see that the intersection of Ξ and a neighborhood of x is $\overline{e}^{-1}(0)$. Since the orientations of V and $\mathcal{E}_2$ determine the orientation of $\mathcal{E}_2/s(V)$, we obtain an orientation of $\overline{e}^{-1}(0)$ and of Ξ in a neighborhood of x.

We remark that this orientation of Ξ is independent of the orientation of $\mathrm{Im}\, s_x$ we have chosen. In fact, if we change the orientation of $\mathrm{Im}\, s_x$, then the orientation of V will be reversed. Hence we need to replace e by $-e$. On the other hand, the orientation of $\mathcal{E}_2/s(V)$ also will be reversed. Therefore, the orientation on $(-\overline{e})^{-1}(0) = \Xi$ does not change.

It follows that we obtain a global orientation of $\Xi - \Xi_2$.

Next we show the property (5.2). We choose a generic section t of $\mathcal{E}_2$. It induces a section $\overline{t}$ of $\mathcal{E}_2/s(\mathcal{E}_1)$. We remark that $\mathcal{E}_2/s(\mathcal{E}_1)$ is a vector bundle on $M - \Xi$. We put

$$Y = \overline{t}^{-1}(0) \cap (M - \Xi).$$

Let $\overline{Y}$ be its closure in M. Since $\mathcal{E}_2/s(\mathcal{E}_1)$ is oriented, it follows that Y is oriented.

Lemma 2. *$\overline{Y}$ is a $\mathbb{Z}$ chain and satisfies $\partial\overline{Y} = 2\Xi$.*

Proof. Let $x \in \Xi - \Xi_2$. Let U be an neighborhood of x. We choose V_x, V, e_x and e as before. We then obtain an isomorphism $\mathcal{E}_2/s(V)|_{\Xi\cap U} \simeq N_\Xi M$. (Here N denotes the normal bundle.) Hence the restriction of $\mathcal{E}_2/s(\mathcal{E}_1) \simeq (\mathcal{E}_2/s(V))/\overline{e}$ to $\partial N_\Xi M$ is isomorphic to the fiberwise tangent bundle of $\partial N_\Xi M \to \Xi$. The fiber is $S^{\mathrm{rank}\mathcal{E}_2 - \mathrm{rank}\mathcal{E}_1}$. Hence the Euler number of the fiber is 2. (Here we use the assumption that $\mathrm{rank}\mathcal{E}_2 - \mathrm{rank}\mathcal{E}_1$ is even.) t induces a section of $(\mathcal{E}_2/s(V))/\overline{e}$. The induced section is close to constant on U. The lemma follows.

Lemma 2 implies that $[\overline{Y}]$ is a $\mathbb{Z}_2$ cycle and that $[\Xi]$ is a Bockstein image of $[\overline{Y}]$. The proof of Proposition 2 is complete.

The following figure illustrates the relation of Lemma 2 to the period-doubling bifurcation. We remark that t is *not* $\mathbb{Z}_2$ equivariant. Hence it is a multisection in the sense of [FO2]. Therefore $\overline{t}^{-1}(0)$ has the multiplicity $1/2$. $\overline{t}^{-1}(0)/2 + N$ is the $\mathbb{Q}$ cycle constructed in [FO2]. The orientation of $\overline{t}^{-1}(0)$ changes at the point where $\overline{N}$ intersect with it. Hence $\overline{t}^{-1}(0)/2 + N$ becomes a $\mathbb{Q}$ cycle in a similar way as the toy model we discussed before.

Figure 2

§4. Normally complex linear perturbation.

Proposition 2 suggests, to avoid period-doubling bifurcation, we need to lift $\mathbb{Z}_2$ characteristic classes to a class defined over $\mathbb{Z}$. This is impossible for general oriented vector bundle. However, for complex vector bundle, any $\mathbb{Z}_2$ characteristic class can be lifted to a class defined over $\mathbb{Z}$ in a canonical way, since the cohomology group of complex Grassmannian is torsion free. In fact, we need to perform the construction in the chain level in order to define Floer homology. (Compare [FO2] §20.) For this purpose, we proceed as follows.

Let $\mathcal{E}_1, \mathcal{E}_2 \to M$ be complex vector bundles on an oriented manifold M. (We do not need to assume that M has a complex structure.) Let $s : \mathcal{E}_1 \to \mathcal{E}_2$ be a generic *complex linear* bundle homomorphism. We put

$$\Xi = \{x \in M | s_x : \mathcal{E}_{1,x} \to \mathcal{E}_{2,x} \text{ is not injective.}\}.$$

Lemma 3.

$$\operatorname{codim}_{\mathbb{R}} \Xi = \operatorname{rank}_{\mathbb{R}}\mathcal{E}_2 - \operatorname{rank}_{\mathbb{R}}\mathcal{E}_1 + 2.$$

The proof is a simple dimension counting. We remark that the right hand side of Lemma 3 is the right hand side of Lemma 1 plus 1. This is a good news.

Now we go back to the Kuranishi structure of Theorerm 2.

Proposition 3. $[\mathcal{E}_1(G)]-[\mathcal{E}_2(G)] \in KO(\mathcal{CM}(G))$ *is in the image of* $K(\mathcal{CM}(G))$.

We proved in [FO2] §16 that Kuranishi structure on the moduli space of stable pseudoholomorphic maps is stably almost complex. (See [FO2] §5 for the definition of stably almost complexity.) In case of the moduli space of stable connecting orbits, the same is true. (We can reduce its proof to the case of closed Rieman surface. We will discuss it in [FO3].) Proposition 3 is a consequence of this fact.

Proposition 3 implies that there exists a vector bundle $\mathcal{F}$ over $\mathcal{CM}(G)$ such that $\mathcal{E}_1(G) \oplus \mathcal{F}$ and $\mathcal{E}_2(G) \oplus \mathcal{F}$ are complex vector bundles. In fact, we can choose $\mathcal{F}$ so that if $x = [\Sigma, \varphi] \in \mathcal{CM}(G)$, then the fiber $\mathcal{F}_x$ is a subspace of $\Gamma(\Sigma, \varphi^*TX \otimes \Lambda^{0,1}(\Sigma))$. So the construction of Kuranishi structure in [FO2] implies that we may change it such that $\mathcal{E}_1(G)$, $\mathcal{E}_2(G)$ will become complex vector bundles for the new Kuranishi structure.

Now we modify s in a neighborhood of 0 section so that it is complex linear there. (We can not change s outside a neighborhood of 0 section, because we need to modify s so that its zero point sets can be patched with N.)

We remark that the modified s is also $\mathbb{Z}_2$ equivariant. The following lemma then is an immediate consequence of Lemma 3.

Lemma 4. *We assume that the virtual dimension of* $\mathcal{CM}$ *is* 0 *or* 1. *We modify* s *so that it is complex linear in a neighborhood of* 0 *section. Then* Ξ *is empty.*

It follows from Lemma 4 that

$$\overline{N} \cap M = \emptyset.$$

We will write $N(\ell_1, \ell_2; k; E)$, $M(\ell_1, \ell_2; k; E)$ in place of N, M, in case they are components of $\mathcal{CM}(\ell_1, \ell_2; k; E)$. We then have

$$\sharp\mathcal{CM}(\ell_1, \ell_2; 0; E) = \sharp N(\ell_1, \ell_2; 0; E) + \frac{[M(\ell_1, \ell_2; 0; E)] \cap e(\mathcal{E}_2(\mathbb{Z}_2)/s(\mathcal{E}_1(\mathbb{Z}_2)))}{2}.$$

Here the left hand side is the fundamental chain of $\mathcal{CM}(\ell_1, \ell_2; 0; E)$ (which is a rational number) in the sense of Kuranishi structure. $\sharp$ in the right hand side is the order counted with sign. $e(\mathcal{E}_2(\mathbb{Z}_2)/s(\mathcal{E}_1(\mathbb{Z}_2))$ is the Euler class of the bundle. In fact, it is not precise to use this notation, since $M(\ell_1, \ell_2; 0; E)$ may have a boundary. So, to be precise, by using generic sectin $\overline{t}$ of $\mathcal{E}_2(\mathbb{Z}_2)/s(\mathcal{E}_1(\mathbb{Z}_2))$, we obtain

$$\sharp\mathcal{CM}(\ell_1, \ell_2; 0; E) = \sharp N(\ell_1, \ell_2; 0; E) + \frac{\sharp(\overline{t}^{-1}(0))}{2}.$$

We recall that the boundary operator we defined in [FO2] §20 is

$$\partial^{old}[\ell_1] = \sum_{\ell_2, E} \sharp\mathcal{CM}(\ell_1, \ell_2; 0; E) T^E [\ell_2].$$

The coefficient in the right hand side is in $\Lambda \otimes \mathbb{Q}$. We define our new boundary operator by

$$\partial^{new}[\ell_1] = \sum_{\ell_2, E} \sharp N(\ell_1, \ell_2; 0; E) T^E [\ell_2].$$

By applying Lemma 4 to $N(\ell_1, \ell_2; 1; E)$, we can prove $\partial^{new}\partial^{new} = 0$.

We thus explained the definition of the boundary operator in the case when $\mathcal{CM}(G)$ is nonempty only for $G = 1, \mathbb{Z}_2$.

The following figure shows how the moduli space in Figure 1 will be modified.

Figure 3.

§5. Another example of bifurcation in the case of cyclic group.

Before discussing the case when the group G is general, we mention another example of bifurcation. We consider the case when $\mathcal{CM}(G)$ is empty unless $G = 1, \mathbb{Z}_3$. We put $N = \mathcal{CM}(1)$, $M = \mathcal{CM}(\mathbb{Z}_3)$. Let $\dim N = \operatorname{virdim} \mathcal{CM} = 1$, and $M = [0, 1]$. Let us assume that $\mathcal{E}_1(\mathbb{Z}_3) = \mathcal{E}_2(\mathbb{Z}_3) = M \times \mathbb{C}$. We suppose also that the generator of $\mathbb{Z}_3$ acts by $\times \exp(4\pi\sqrt{-1}/3)$ on $\mathcal{E}_1(\mathbb{Z}_3)$, and by $\times \exp(2\pi\sqrt{-1}/3)$ on $\mathcal{E}_2(\mathbb{Z}_3)$. Let τ be the coordinate of M. We consider $s_\tau : \mathbb{C} \to \mathbb{C}$ such that

$$s_\tau(z) = \begin{cases} \overline{z} & \tau = 0, \\ z^2 & \tau = 1. \end{cases}$$

if z is in a neighborhood of 0 and $s_\tau(z) = z^2$ if $|z| > 1$. s_τ determines a $\mathbb{Z}_3$-equivariant map $s : \mathcal{E}_1(\mathbb{Z}_3) \to \mathcal{E}_2(\mathbb{Z}_3)$. A neighborhood of M in $\mathcal{CM}$ is identified with

$$\{(z, \tau) | s_\tau(z) = 0\}/\mathbb{Z}_3.$$

It is easy to see that this moduli space is described as in Figure 4 below.

Figure 4.

From this example, it is easy to see that, in the case when the group G is not $\mathbb{Z}_2$, we may not be able to take s so that it is complex linear in a neighborhood of zero section.

We also remark that, if we take s to be generic, then $z \mapsto c\overline{z}$ is the leading term. However we insist s to be holomorphic (or complex polynomial) at each fiber. For example, in this particular case, we take $z \mapsto cz^2$.

§6. The general case.

We now go back to the study of $\mathcal{CM}$. The proof of the general case is based on the following Proposition 4. We need some notations. Let M be a manifold and $\mathcal{G}$ be a local system of finite group. Let $\mathcal{E}_1$, $\mathcal{E}_2$ be complex vector bundles on which $\mathcal{G}$ acts. We assume (3.3). We assume moreover that the action of $\mathcal{G}$ on $\mathcal{E}_1$ is effective. Let D be a sufficiently large integer.

Proposition 4. *Let $s : \mathcal{E}_1 \to \mathcal{E}_2$ be a smooth bundle map such that $s_x : \mathcal{E}_{1x} \to \mathcal{E}_{2x}$ is a (complex)polynomial map of degree $\le D$ for each $x \in M$ at a neighborhood is* 0 *section. We assume that s is generic among such maps. We put*

$$N = \{v \in \mathcal{E}_1 | s(v) = 0,\, I_v = \{1\}\}.$$

(Here $I_v = \{g \in \mathcal{G}_x | gv = v.\}$, $v \in \mathcal{G}_x$.) Then we have

$$\dim(\overline{N} - N) \le \dim M + \mathrm{rank}\mathcal{E}_1 - \mathrm{rank}\mathcal{E}_2 - 2.$$

Sketch of the proof. Let $V_1 = \mathcal{E}_{1x}$, $V_2 = \mathcal{E}_{2x}$ be fibers. Let $\mathrm{Poly}^D_G(V_1, V_2)$ be the set of all G-equivariant polynomial maps $P : V_1 \to V_2$ of degree $\le D$. There is an evaluation map $ev : \mathrm{Poly}^D_G(V_1, V_2) \times V_1 \to V_2$. We put

$$V_{1free} = \{v \in V_1 | I_v = \{1\}.\}$$
$$Y = ev^{-1}(0) \cap (\mathrm{Poly}^D_G(V_1, V_2) \times V_{1free}).$$

Lemma 5. *If the action of G on V_1 is effective, then, for sufficiently large D, the space Y is a smooth manifold of dimension*

$$\dim Y = \dim V_1 + \dim \mathrm{Poly}^D_G(V_1, V_2) - \dim V_2.$$

In other words, ev is a submersion on $\mathrm{Poly}^D_G(V_1, V_2) \times V_{1free}$.

Lemma 5 follows easily from the following sublemma :

Sublemma. *Let $p \in V_1$ and $w \in V_2$. We assume $I_p = \{1\}$. Then there exists a G equivariant polynomial map $P : V_1 \to V_2$ such that $P(p) = w$.*

Proof. We may assume that V_2 is an irreducible G module. We put

$$W = \bigoplus_{\gamma \in G} \mathbb{C}[\gamma].$$

and define a G action on it by

$$g(\sum c_\gamma[\gamma]) = \sum c_\gamma[\gamma g^{-1}].$$

Since W is a regular representation of G, there exists a surjective G linear map $\Psi : W \to V_2$. We choose $w_\gamma \in \mathbb{C}$ such that :

$$\Psi\left(\sum w_\gamma[\gamma]\right) = w.$$

Since $I_p = \{1\}$, there exists a ($\mathbb{C}$ valued) polynomial f on V_1 such

$$f(\gamma p) = w_\gamma$$

for each $\gamma \in G$. We put

$$P(x) = \Psi\left(\sum_\gamma f(\gamma x)[\gamma]\right).$$

It is straightforward to see that P has the required property[1].

We put $X = \overline{Y} - Y$. The space X is an algebraic variety. We have :

$$\dim_{\mathbb{C}} X \le \dim_{\mathbb{C}} V_1 + \dim_{\mathbb{C}} \mathrm{Poly}_G^D(V_1, V_2) - \dim_{\mathbb{C}} V_2 - 1.$$

Two bundles $\mathcal{E}_1, \mathcal{E}_2 \to M$ induce a bundle $\mathrm{Poly}_G^D(\mathcal{E}_1, \mathcal{E}_2) \to M$ whose fiber is $\mathrm{Poly}_G^D(V_1, V_2)$. We also have a bundle $\mathcal{X} \to M$ whose fiber is X. The projection $X \subset \mathrm{Poly}_G^D(V_1, V_2) \times V_{1free} \to \mathrm{Poly}_G^D(V_1, V_2)$ induces a bundle map

$$\pi : \mathcal{X} \to \mathrm{Poly}_G^D(\mathcal{E}_1, \mathcal{E}_2).$$

Since X is an algebraic variety, it has simplicial decomposition. Using it we can find a section $\mathfrak{s} : M \to \mathrm{Poly}_G^D(\mathcal{E}_1, \mathcal{E}_2)$ which is of general position to $\pi(\mathcal{X})$. It follows that

$$\dim_{\mathbb{R}}\{x \in M | \mathfrak{s}(x) \in \pi(\mathcal{X})\} \le \mathrm{rank}_{\mathbb{R}}\mathcal{E}_1 + \dim_{\mathbb{R}} M - \mathrm{rank}_{\mathbb{R}}\mathcal{E}_2 - 2.$$

(Note that dimension and rank here are real dimension and real rank.) $\mathfrak{s}$ induces a bundle map $s : \mathcal{E}_1 \to \mathcal{E}_2$ which is a polynomial map on each fibers. It is easy to see that

$$\{x \in M | \mathfrak{s}(x) \in \pi(\mathcal{X})\} \simeq \overline{N} - N.$$

Proposition 4 follows.

We apply Proposition 4 to $\mathcal{E}_1(G)$, $\mathcal{E}_2(G)$, $\mathcal{CM}(G)$. We remark that $\mathrm{rank}\mathcal{E}_1 + \dim M - \mathrm{rank}\mathcal{E}_2$ is the virtual dimension of $\mathcal{CM}$. We modify $s_G : \mathcal{E}_1(G) \to \mathcal{E}_2(G)$, so that it will be the bundle map constructed by Proposition 4 in a neighborhood of 0 section. Then it is easy to see that $N/\mathcal{G}(G)$ is identified with the intersection of $\mathcal{CM}(1)$ and a neighborhood of $\mathcal{CM}(G)$. Therefore, Proposition 4 implies

$$\dim_{\mathbb{R}} \overline{\mathcal{CM}(1)} \cap \mathcal{CM}(G) \le \dim_{\mathbb{R}} \mathcal{CM}(1) - 2, \tag{6}$$

for $G \neq 1$. (Note $\dim_{\mathbb{R}} \mathcal{CM}(1)$ is equal to the virtual dimension of $\mathcal{CM}$.)

We modify s_G by an induction of the stratum so that (6) is satisfied.

[1] Our first idea of the proof of Theorem 1 was to show Lemma 5 under additional assumption that G is abelian, and then use resolution of singularity to reduce the general case to this case. After Theorem 1 had been anounced by the first named author in several conferences, we realized that there is a simpler argument (which we gave above) without using resolution of singularity. We thank Prof. Hambleton who suggested that Proposition 4 may hold without assuming G to be abelian.

Now let us consider the case when the virtual dimension of $\mathcal{CM}$ is 0 or 1. Then (6) means that $\mathcal{CM}(1)$ is compact. Hence using it in place of $\mathcal{CM}$, we obtain ∂ such that $\partial^2 = 0$. This is an outline of the proof of Theorem 1.

§7. Gromov-Witten invariant.

Our construction in this article can be applied to the moduli space of marked stable maps also. Then we obtain a homology class defined over integer. The result can be summarized as in Theorem 3 below. Let X be an $2n$-dimensional compact symplectic manifold and $\beta \in H_2(X;\mathbb{Z})$. Let

$$GW_{g,m}(X;\beta) \in H_{2m+2\beta c^1+2(3-n)(g-1)}(\mathcal{CM}_{g,m} \times X^m;\mathbb{Q})$$

be the Gromov-Witten invariant. (Here g is the genus m is the number of marked point. $\mathcal{CM}_{g,m}$ is the Deligne-Mumford compactification of the moduli space of stable curves.) (See [FO2] §17 for a definition of Gromov-Witten invariant.)

Theorem 3. *There exists a decomposition*

$$GW_{g,m}(X;\beta) = GW_{g,m}(X;\beta)_{simple} + GW_{g,m}(X;\beta)_{multiple}$$

with the following properties.

(1) $GW_{g,m}(X;\beta)_{simple}$ *is a homology class defined over integer.*
(2) $GW_{g,m}(X;\beta)_{simple}$ *is invariant of the deformation of* X (*as far as it is smooth*).
(3) $GW_{0,3}(X;\beta)_{simple}$ *defines an associative product on* $H^*(X;\Lambda)$.

Some of the other axioms by Kontsevich-Manin [KM] (see also [FO2] §23) may hold for $GW_{g,m}(X;\beta)_{simple}$. The authors did not check yet which holds and which does not hold.

Problem. Are there any universal formula to calculate $GW_{g,m}(X;\beta)_{multiple}$ in terms of $GW_{g',m'}(X;\beta')_{simple}$ with $g' \le g$, $m' \le m$?

We remark that such a formula is known in the case when X is a Calabi-Yau 3 fold and $g = 0$. (See [Ma].)

Our method of this article can be used also in the case of moduli space of pseudoholomorphic disks. Combined with [FKO_3], it gives applications to the problem of Lagrangian intersection. We will discuss it later in [FO3].

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