Cyclic symmetry and Numerical invariant in Lagrangian Floer theory

Semi-Toric Case

1

Cyclic filter A_∞ alg. C $R = \mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$

$\Lambda_0 = \{ \sum a_i T^{\lambda_i} \mid a_i \in R, \lambda_i \geq 0 \quad \lambda_i \uparrow \infty \}$

$\Lambda = \Lambda_0[T^{-1}]$ field

$\Lambda_+ = \{ \sum a_i T^{\lambda_i} \mid \lambda_i > 0 \}$ maximal ideal of Λ_0

$\Lambda_0 / \Lambda_+ = R$

$\overline{C}$ R vect. space graded n dim fixed

$C = \overline{C} \hat{\otimes} \Lambda_0$ $(C[2])^k = C^{k+1}$

$\langle , \rangle : \overline{C}^k \otimes \overline{C}^{n-k} \longrightarrow R$

inner prod. perfect

$\dim_R \overline{C} < \infty$.

$\langle u, v \rangle = (-1)^{|u|+|v|} \langle v, u \rangle$

③ $\times$

$m_k : C(2)^{\otimes k} \longrightarrow C(2)$

①

$$\sum_{\substack{k_1+k_2=k+1 \\ l}} (-1)^{\deg x_1 + \cdots + \deg x_{l-1} + l} m_{k_1}(x_1 \cdots m_{k_2}(x_l \cdots) \cdots x_k) = 0$$

②

$$\langle m_k(x_1 \cdots x_k), x_0 \rangle$$

$$= (-1)^{(\deg x_0 + 1)(\deg x_1 + \cdots + \deg x_k + k)}$$

$$\langle m_k(x_0, x_1 \cdots x_{k-1}), x_k \rangle$$

③ $m_0(1) = 0$ mod Λ^+

③ X

$e \in C^0$ is unit

$\iff m_2(e\,x) = (-1)^{\deg x} m_2(x\,e) = x$

$b \in C^1$ MC $h \equiv 0$ mod Λ_+

$\iff \sum_{k=0}^{\infty} m_k(b \cdots b) = 0$ MC

$$m_k^b(x_1 \cdots x_k) = \sum_{l_0, \ldots, l_k} m_{k+\sum l_i}(b^{l_0} x_1 \cdots \cdots x_k b^{l_k})$$

$m_1^b m_1^b = 0$ $(\iff$ MC$)$

$$H(C, m_1^b) = \frac{\mathrm{Ker}\, m_1^b}{\mathrm{Im}\, m_1^b}$$

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Def

$\mathcal{H} = (H, v^0, e, \langle\,\rangle)$ unital Frob alg.

$Z(\mathcal{H}) \in \Lambda$

$e_0 = e,\ e_I\quad I = 1, \dots, N$ are

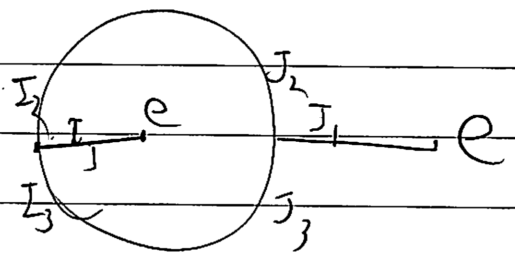
bases of H

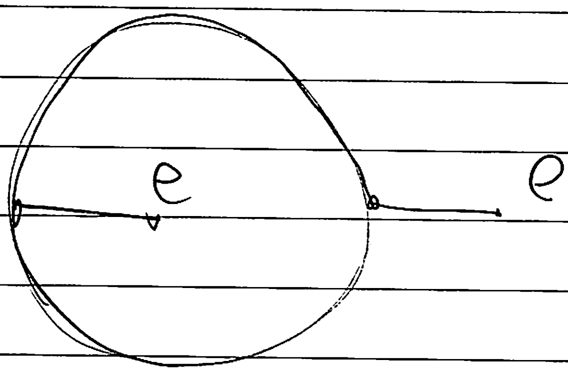
$g_{IJ} = \langle e_I, e_J \rangle \qquad (g^{IJ}) = (g_{IJ})^{-1}$

$m_{IJK} = \langle m_2(e_I, e_J), e_K \rangle$

6

$$Z_1 = \sum_{\substack{I_1 J_1 K_1 \\ I_2 J_2 K_2}} \pm g^{I_1 0} g^{J_1 0} g^{I_2 J_2} g^{I_3 J_3} m_{I_1 I_2 I_3} m_{J_1 J_2 J_3}$$





7

Lem

Z_1 is ind of lines ℓ_i

∵) easy super tensor calculus //

AMS TP 46

Thm (Hori-I)
arXive: 0907.4219

X sym $L \subset X$ Lag. rel. spin

$\exists$ ($H(L;\Lambda_0)$, $\langle\ \rangle$, m_k) cyclic A-alg

P.D.

Unit

⑨

$$\tilde{\mathcal{M}}(L) = \{ b \in H^{odd}(L;\Lambda_0) \mid \sum \mathfrak{m}_k(b,\dots,b) = 0 \}$$

$HF((L,b);\Lambda)$, $\cup$, $\langle\ \rangle$, e

unital Frob alg

<u>Cor</u>

$Z_1(HF(L,b);\Lambda)$ is an inv

of (L,b).

Rmk

$h \vdash H'$

If $c^1 M = 0$ Massey $L = 0$

h

$M_2 : H_*(L; \mathbb{Z}) \to \mathbb{Z}$

$\Rightarrow Z_1(\quad) = 0$

$\because M_2 = 0$

$\Rightarrow m_r^b : C(\mathbb{Z})^{\otimes n} \to C(\mathbb{Z})$

alg. deg 1

$m_2^b : H\Omega^k \otimes H\Omega^l \to H\Omega^{k+l}$

(10)

$$g^{I_1 0}\, g^{J_1 0}\, g^{I_2 J_2}\, g^{I_3 J_3} \quad m_{I_1 I_2 I_3}\, m_{J_1 J_2 J_3} \quad \neq 0$$

unless

$|I_1| + 0 = m$

$|J_1| + 0 = m$

$|I_2| + |J_2| = m$

$|I_3| + |J_3| = m$

$|I_1| + |I_2| + |I_3| = m$

$|J_1| + |J_2| + |J_3| = m$

$(x \vee y, \circ\, ?)$

$(|I_1| + \cdots + |J_3|)$

$= 4m$

$2n$

$Z_1(\quad)$ is $\cdots \neq 0$ in Fano

or turn case

Ex.

$(H, \langle\ \rangle, \nu^0)$ cyclic clifford

$e_1, \dots, e_n \subset H$ $\quad\quad$ $\mathrm{rk}\, H = 2^n$

$$e_i \nu^0 e_j + e_j \nu^0 e_i = \begin{cases} 0 & i \neq j \\ 2 d_i & i = j \end{cases}$$

$d_i \in \Lambda$

$I = \{i_1, \dots, i_p\} \subset \{1, \dots, n\}$

$e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$

$e_I \quad I \in 2^n$ is a basis

of H

Clifford

$e_\emptyset = 1 \quad I = \phi$ unit

$$\langle e_I , e_J \rangle = \begin{cases} 0 & I \cap J = \phi \\ & I \cup J = \{1 \cdots n\} \\ \pm 1 & I \cap J = \phi \\ & I \cup J = \{1 \cdots n\} \\ \quad \operatorname{sign}\begin{pmatrix} 1 & \cdots & n \\ I & & J \end{pmatrix} & \end{cases}$$

13

Lem $(H, \subset, \mathfrak{z}, v^\theta)$ cyclic Clifford alg

$\Rightarrow Z_1(\quad) = d_1 \cdots d_n$

- $L = T^n \subset M$ Lag torus

☆

$\beta \in \pi_2(M, L) \quad \beta \neq 0$

$\mathcal{M}(\beta) \neq \emptyset$

$\shortparallel$

(group) $\{u : (D^2, \partial D^2) \to (M, L) \mid$ hol, $u(D^2) \in \beta \} / M$

14

Morlov.

$\exists \ \mu_L(\beta) \geq 2$

Thus

1) $\dim_{\mathbb{C}} M = 2$

or

2) M ref toric L divisor

or

(3) M toric L divisor after T^n inv perturb)

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15

Thm $L \cong T^n \subset M$ ☆

$\Rightarrow \quad \forall b \in H^1(L)$

$\sum m_k(b \cdots b) = \underset{W(b)}{c}\,[L]^{e}$

(enough to consider Frob alg)

If $m_1^b : H^*(L;\Lambda)\circlearrowleft$ is zero

$\Rightarrow HF(L) = (H(L), m_2^b, \langle\ \rangle)$

is a cyclic Clifford alg

(4)

$W : H^1(L) \longrightarrow \Lambda$

$$W(x) = \sum_{k=0}^{\infty} \int_L m_k(x, \cdots, x) \in \Lambda$$

$x \in H^1(L)$

① $m_1^b = 0 \iff \frac{\partial W}{\partial x}(b) = 0$

③ $d_1, \cdots, d_n$ is eigen value of $\left(\frac{\partial^2 W}{\partial x_i \partial x_j}\right)$

$x = \sum x_i e_i$ $\quad e_1, \cdots, e_n$ render basis of $H^1(L) \simeq H^1(T^n)$

17

$$m_2 = \sum m_{2,\beta} T^{E(\beta)}$$

$$\deg m_{2,\beta}(x, y) = \deg x + \deg y - \mu_L(\beta)$$

$\exists$ (...) $m_{2,\beta} \neq 0$

$\Rightarrow \mu(\beta) \geq 0$

$\cup$

$\mu(\beta) = 0$ iff if $\beta = 0$

$\Rightarrow m_{2,\beta}(x, y) = \pm x \cup y$ $\beta = 0$

small deg $\beta \neq 0$

(18)

$x v^\ell y = x v y$ + elem of smaller degree

(Prop)

$e_1, \dots, e_n$ standard basis of $H(T^n)$

$$m_2^b(e_i, e_j) + m_2^b(e_j, e_i) = \frac{\partial^2 W}{\partial x_i \partial x_j}$$

Change basis of $H(T^n; \Lambda)$

19

My choice

$$e_i v^\theta e_j + e_j v^\theta e_i = \begin{cases} 0 & i \neq j \\ d_i & i = j \end{cases}$$

e_i gen clifford alg

$\Rightarrow$ e_i gen $HF = H(T')$ is a alg

(basically due to Chs)

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∂y γ

$\langle e_I, e_J \rangle = ?$

not $\cup^Q$

$\langle e_I, e_J \rangle = \int_L e_I \cup e_J$

$\|$

$\langle e_I, e_J \cup^Q 1 \rangle$

$\|$ cyclic sym.

$\langle e_I \cup^Q e_J, 1 \rangle = \int_L e_I \cup^Q e_J$

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例 7

$$e_I \cup e_J = \pm c\, e_K \qquad K = I \cup J \setminus (I \cap J)$$

$$c = d^{i_1 \cdots i_\ell} \qquad i_1 \cdots i_\ell = I \cap J$$

$$\int_L e_K = 0 \quad \text{unless} \quad K = \{1, \dots, n\}$$

$$K = k_1 \cdots k_m$$

$$e_K = \underline{e_{k_1} \cup \cdots \cup e_{k_m}} + \text{lower order}$$

$$\langle e_I, e_J \rangle = \begin{cases} 0 & \text{unless } I \cap J = \emptyset \quad I \cup J = \{1 \cdots n\} \\ \pm 1 & I \cap J = \emptyset \quad I \cup J = \{1 \cdots n\} \end{cases}$$

$B^{\gamma^c} H \otimes Y$

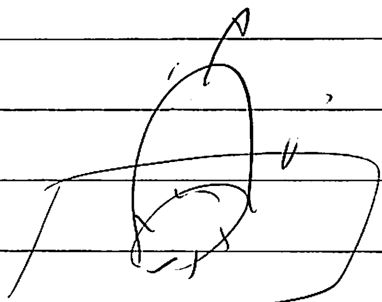
$\shortparallel$

$p_* : \otimes H(L)^{\otimes n} \longrightarrow H(X)$

$\wr$

$|$

cycl.



$x_1 \otimes \cdots \otimes x_n \sim \pm\, x_n \otimes x_1 \otimes \cdots \otimes x_{n-1}$

Thm $e \in H(L)$ with $e \in B_1^{\gamma^c} H$

$\langle p_*(ee^b)\ p_*(ee^b) \rangle_{PD_M}$ $\sum_i e \tilde{b}^i \cdot b$

$= e \cdot e^b_i$

$=$

$\Sigma(H(L))$

23

(written in the toric case

hold in general (not yet written)

$\Sigma_1 \longleftrightarrow$ Poincaré duality

M toric

$H(M)$ is gen by $P_*(e\,e^b)$

over all

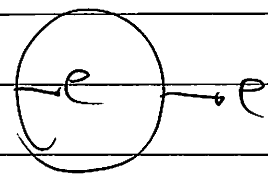
(L, b) $HF(L,b)$

$\cong H(T^n)$

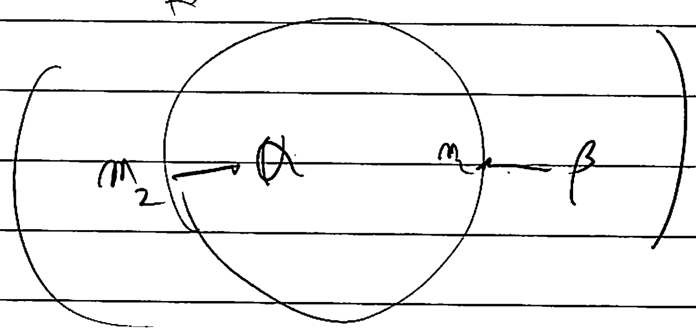
Mn (PMW) $\frac{1}{2}$

$\sum_{(L,b)} P_*(e\,e^b) = 1$ wt in $H(X)$

Z_1



$\langle\, n_*(\alpha\, e^h),\ n_*(\beta\, e^h) \,\rangle$



~ 1

$$\sum \frac{p_+(e\, e')}{2(L,p)} = 1 \quad \text{wt is } H(X)$$

Los

$$\sum \frac{1}{2(L,p)} = 0$$

Residue formula

$$\sum \left\langle \frac{p_+(e,e')}{2}, \frac{p_-(e,e')}{2} \right\rangle$$

$$= \langle 1, 1 \rangle = 0$$

Ex

2 pt bl-up of $\mathbb{C}P^2$

X — 2 Monot

$T^2 \subset X$ moment fiber

$$W = T^{1/3}\left(y_1 + y_2 + (y_1 y_2)^{-1} + y_1^{-1}\right)$$

$y_i = e^{x_i}$

x_i : cor of $H^1(T^2)$

$Z = \det \mathrm{Hess}\, W$

(27)

$$\frac{\partial W}{\partial y} = 0 \iff \begin{cases} y_1 = \dfrac{1}{y_2^2} \\ y_2^4 + y_2^3 - 1 = 0 \end{cases}$$

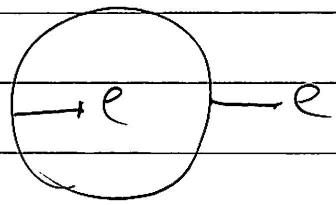
$$Z = T^{2/3} \frac{4 - y_2^3}{y_2}$$

$$\ast \iff \sum_{y^4+y^3-1=0} \frac{4 - y^3}{y} = 0$$

$$y^4 + y^3 - 1 = 0$$

(20)

Z_1



higher Z_n ?

HC

$m_1 = 0$

$(\mathfrak{g}, m_2) \to A_\infty$ alg

$B^{cyc}_n C = \overbrace{\underbrace{cA - ac}_{\sim}}^{n}$

$x_1 a - a x_b$

$\sim x_n a x_1 \cdots x_{n-1}$

$\exists\, B^{cyc}_n C = B^{cyc} C \circlearrowright \delta_c$

$\delta_c [x_1 \cdots x_k]$

$= \sum [\cdots m_\ell(x_i \cdots x_{i+\ell-1}) \cdots x_k]$

$+ \sum [m_\ell(x_j \cdots x_k x_1 \cdots x_\cdot) \cdots]$

$\delta_c \delta_c = 0$

$$HC(H, m) = \frac{\operatorname{Ker} \delta_c}{\operatorname{Im} \delta_c}$$

Def the $F^i_\lambda HC(H, m)$

Im of $\bigoplus_{\ell \leq h} B^{cyc}_\ell H$

$m_1 \geq 0$

$(R_f), m_2, (?)$

Thm (Kassel + ε)

C is $C\{f\}/f^m$ $d_1 \geq 0$

$$\Rightarrow \frac{I^k HC(C[H, m])}{I_{k-1} HC(\ )} \cong \begin{cases} A & k \text{ even} \\ 0 & \text{odd} \end{cases}$$

$I_k' \rightsquigarrow I_{k-2}'$

Connes exact seq

$$HC(H, m) \xrightarrow{S} HC(H, m) \xrightarrow{B} HH(H, m)$$

31

$F_0 HC(H, m) \cong HH(H, m) = K$

$\frac{F_3}{F_0} = S\, HH(H, m)$

m

e

$(H, m_\lambda^t, \langle\,\rangle, e)$ def of cycl

Ar alg

(H, m_2^0) diff

$Z(\quad) = Z_e$

$$\left(\frac{d m_h^j}{ds}\Big|_{t=0}\right) \in HC(H, m^0)$$

α

(def of cyc. tr. alg

$\Leftrightarrow$ HC)

$$\alpha \longmapsto \frac{dZ_j}{ds}\Big|_{t=0} \in \Lambda$$

$$HC(\ ) \xrightarrow{\nabla Z} \Lambda$$

Prop

D2 give an iso

$$\frac{F_2}{F_0} \cong \Lambda$$

$\|$

Λ

is (C, m_2^u) to clsfnd

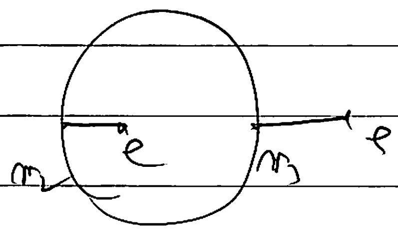
$$m_0 \ldots = W \qquad \frac{\partial W}{\partial \tau}$$

$$F_0 \longrightarrow \Lambda$$

$$\frac{F_{2n}}{F_{2n-1}} \simeq \Lambda$$

$p=0$ $\quad e$ —— m_0 $\quad W$

$p=1$



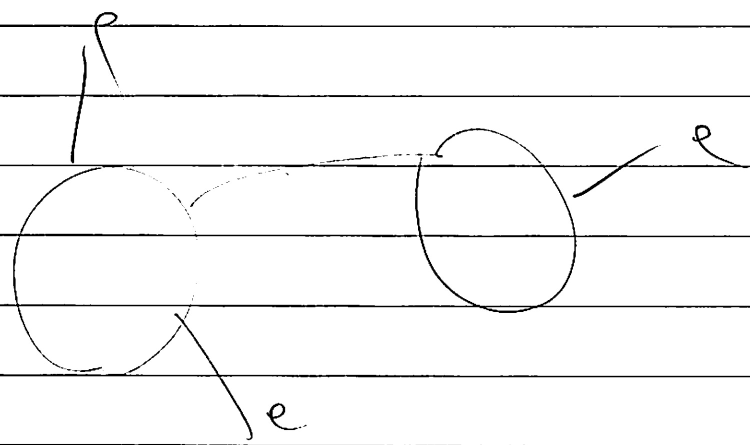
Cyc $\quad Z_k$: hom. ass. of Cyc. Ass. alg. $\wedge$ wk

(C, m_2)

$\frac{dZ_k}{dA}$ gives an iso $\frac{F_{2n}}{F_{2m}} \to \Lambda$

BJ

all (s m?) ✓ cbdmd



?

X toric

W

Gvl

mirror sym

GW

X

$$\int e^{\sqrt{-1}\,W(y_1,\dots,y_n)/\hbar}\ \frac{dy_1}{y_1}\cdots\frac{dy_n}{y_n}$$

$$\sim \sum e^{-W(b)/\hbar}\ \frac{1}{\sqrt{\det \mathrm{Hess}\, W}}\left(1+\sum c_k \hbar^k\right)$$

critical pt $(b_1 b_2)$

$\shortparallel$
$2,$

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Calabi-Yau $\quad L \subset M \quad c_1(X) = 0$

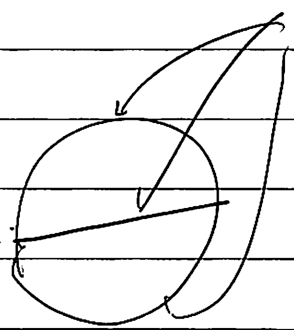
$\eta_L = 0$

~~Solution~~ ...

$Z_L = 0$

Wh

projection



no e is used!