Cyclic Symmetry and Numerical Invariant in Lagrangian Floer theory

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0908.0148

Cyclic filtered A_∞ alg.

C or $\Omega(L)$ ~ de Rham cpx

finite dimensional

L cpt mfd

oriented

$\langle\ \rangle : C^k \otimes C^{n-k} \to \Lambda$

$$\Lambda = \{\sum a_i T^{\lambda_i} \mid a_i \in K,\ \lambda_i \uparrow \infty\}$$

$\lambda_i \in \mathbb{R}$

$$\cup\!\!\mid \qquad \Lambda_0 = \{ \qquad \mid \qquad \lambda_i \geq 0\}$$

$$\cup\!\!\mid \qquad \Lambda_+ = \qquad \lambda_i > 0\}$$

$$\langle u, v\rangle = (-1)^{\deg' u \deg' v + 1} \langle v, u\rangle$$

$$\deg' u = \deg u + 1$$

product or $(-1)^{\deg u(\deg v + 1)} \int_L u \wedge v$

Cyclic A_∞ alg

$$\mathfrak{m}_{k,\beta} : \quad C[2]^{\otimes k} \to C[2]$$

$$\deg \quad 1 - \mu(\beta)$$

$$\mathfrak{m}_k = \sum_{\beta \in G} T^{E(\beta)} \mathfrak{m}_{k,\beta} \qquad \mathfrak{m}_{0,\beta_0} = 0$$

$$E : G \to \mathbb{R}_{\geq 0} \quad \text{proper}$$

$$E(\beta_0) = 0 \qquad E(\beta) > 0 \quad \beta \neq \beta_0$$

① A_∞ rel $\quad \sum \pm \mathfrak{m}_{k_1}(\cdots \mathfrak{m}_{k_2}(\cdots) \cdots) = 0$

② $\langle \mathfrak{m}_k(x_1 \cdots x_k), x_0 \rangle$

$$= (-1)^{*} \langle \mathfrak{m}_k(x_0 \cdots x_{k-1}), x_k \rangle$$

$* = \deg' x_0 (\deg' x_1 + \cdots + \deg' x_k)$

Thm (Fooo+α)

$L \subset M$ compact, Lag. subman.

$\subset$ compact symplectic mfd

rel. spin

$\Rightarrow$ $\Omega(L) \otimes_{\mathbb{R}} \Lambda_0$ has cyclic

filtered A_∞ structure

well define up to pseudo iso.

④

$(C, \langle\,\rangle, m)$, $(C', \langle\,\rangle, m')$ cyclic A_∞ alg

$f = (f_k)$ hom. equivalence

$f_k : C[1]^{\otimes k} \longrightarrow C'[1]$ deg 0

st

① $\sum m'_\ell (f_{k_1}(x_1 \cdots) f_{k_2}(\cdots) \cdots f_{k_\ell}(\cdots))$

$= \sum \pm f_{k_1}(\cdots m_{k_2}(\cdots) \cdots)$

② $\sum_{k_1+k_2=k} \langle f_{k_1}(x_1 \cdots x_{k_1}), f_{k_2}(x_{k_1+1} \cdots x_k) \rangle$

$= \begin{cases} 0 & \text{if } k \neq 2 \\ \langle x_1, x_2 \rangle & k = 2 \end{cases}$

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pseudo-isotopy

$(C, \{m^0_*\}, \langle\ \rangle) \sim (C, \{m^1_*\}, \langle\ \rangle)$ pseudo isotopic

$\Leftrightarrow$ $\exists\ m^t_*,\ c^t_*$ $\quad m^0 = m^0 \quad m^1 = m^1$

depending smoothly on t

m^t_*, c^t_* define cyclic A_∞ alg

on $C \otimes_{\mathbb{C}} C^\infty([0,1])[1, dt]$

$\alpha_i = x_i(t) + y_i(t)\,dt$

$m_k(\alpha_1 \cdots \alpha_k) = x(t) + y(t)$

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$$x(t) = m_k^t(x_1(t), \ldots, x_k(t))$$

$$y(t) = \sum_i t\, m_k^t(x_1(t) \quad y_i(t) \quad \cdots x_k(t))$$

$$+ \sum t\, c_k^t(x_1(t) \cdots x_k(t))$$

$$k \neq 1$$

$$m_1(x_1(t) + y_1(t)dA)$$

$$= m_1^t(x_1(t)) + m_1^t(y_1(t))\, dA$$

$$+ c_1^t(x_1(t))\, dA + \frac{\partial x_1}{\partial t}(t)\, dA$$

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<u>Lemma</u>

$(C, \{m^0\}, \langle\,\rangle)$: pseudo isotopic

$(C, \{m^1\}, \langle\,\rangle)$

$\Rightarrow$ homotopy equiv. as cyclic
A_∞ alg

$\exists\ f_t : (C, \{m^0\}, \langle\,\rangle)$

$\to (C, \{m^t\}, \langle\,\rangle)$

homotopy equiv.
depending smoothly on t

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$(C, \{m_k\})$ filtered A_∞ algebra

$$\widetilde{\mathcal{M}}(C, \{m_k\})$$

$$= \{ b \in C^{odd} \mid \sum_{k=0}^{\infty} m_k(b \cdots b) = 0 \}$$

gauge equivalence

$$\widetilde{\mathcal{M}}(C, \{m_k\}) / \sim \; = \mathcal{M}(C, \{m_k\})$$

If $m_1 \; G \to 0$

$$\Rightarrow \widetilde{\mathcal{M}}(C, \{m_k\})$$

$$= \{ b \in C^1 \mid \sum m_k(b \cdots b) = 0 \}$$

Gau…

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$n=3, \quad \mu=0$

$\Psi' : L^{\prime} \longrightarrow \Lambda_0$

$\Psi'(b)$

$$= \sum_{k=0}^{\infty} \frac{1}{k+1} \langle m_k(b, \ldots, b), b \rangle,$$

<u>Lemma</u>

$\sum m_k(b, \ldots, b) = 0$

$\Leftrightarrow (\nabla' \Psi)(b) = 0$

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$f : (C, \{m\}) \longrightarrow (C', \{m'\})$

homotopy equivi

$f_* : M(C, \{m\}) \longrightarrow M(C', \{m'\})$

$\psi \qquad\qquad\qquad \psi$

$b \longmapsto \sum_{k=0}^{\infty} f_k(b \cdots b)$

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Let $(C, \{m^0\}, \langle\,\rangle) \sim (C, \{m^1\}, \langle\,\rangle)$
pseud iso.

$m \geq 3, \; n \geq 0$

then

$$
\begin{tikzcd}
\mathcal{M}(C, \{m^0\}) \arrow[rr, "\simeq"] \arrow[dr, "\psi'"'] & & \mathcal{M}(C, \{m^1\}) \arrow[dl, "\psi"] \\
& \Lambda_0 &
\end{tikzcd}
$$

X ms

(13)

$f_t : (C, \{m^0\}) \to (C, \{m^t\})$

homotopy equiv

$b_t = f_{t*}(b)$

$$\frac{d}{dt} \Psi'_t(b_t)$$

$$= \sum_k \frac{1}{k+1} \langle m^t(b_t \cdots b_t), b_t \rangle$$

$\vdots$

$$= \sum_{k_1+k_2 \geq 1} \langle c^t_{k_1}(b_t \cdots b_t), m^t_{k_2}(b_t - b_t) \rangle$$

$= \langle c_0^t(1), m_0^t(1) \rangle$ (MC of b_t)

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<u>Defn</u> Inhomogeneous cyc. fil A_∞ alg

$(C, \{m_k\}_{k=-1}^{\infty}, \langle\ \rangle)$

st $\circ$ $(C, \{m_k\}_{k\geq 0}^{\infty}, \langle\ \rangle)$ is a cyclic filtered A_∞ alg

$\circ$ $m_{-1} \in \Lambda_0$

<u>Defn</u> $n=3$, $\mu=0$, pseudo isotopy of inhomogeneous cyc. fil A_∞ alg

$(C, \{m_k^{t}\}_{k\geq -1}^{\infty}, \{c_k^{t}\}_{k\geq 0}^{\infty}, \langle\ \rangle)$

st

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$$\left(C, \{m_k^\tau\}_{k\geq 0}^{b}, \{c_k^\tau\}_{k\geq 0}^{b}, \langle\ ,\ \rangle\right)$$

is a pseudo isotopy of cyclic filtered A_∞ alg

$$\frac{dm_k^\tau}{d\tau} + \langle c_0^\tau(1), m_0^\tau(1)\rangle = 0$$

$$=$$

$$\Psi(b) = m_{-1} + \sum_{k\geq 0}^{\infty} \frac{\langle m_k(b \cdots b), b\rangle}{k+1}$$

Lemma $(C, \{m_k^\tau\}, \{\varphi^\tau\}, \langle\ \rangle)$

pseudo iso. of inhomogeneous cyclic

filtered A_∞ alg

$\Downarrow$

$$
\begin{tikzcd}
\mathcal{M}(C, \{m_k^0\}, \langle\ \rangle) \arrow[rr, "\simeq"] \arrow[dr, "\Psi"'] & \arrow[d, phantom, "\circlearrowright"] & \mathcal{M}(C, \{m_k^1\}, \langle\ \rangle) \arrow[dl, "\Psi"] \\
& \Lambda_0 &
\end{tikzcd}
$$

super potential

Thm $\dim M = 3$ $L \subset M$

Maslov : $\pi_2(M, L) \to \mathbb{Z}$

$\parallel$

0 (e.g. special)

J ☆

$\Rightarrow$ $\Omega(M)$ has a str of

inhomogeneous cyclic filtered

A_∞ algebra

depends on $M, L, \underline{\underline{J}}$

$\Psi_J : \mathcal{M}(L) \longrightarrow A_\infty$

Note $\mathcal{M}(L) \subseteq H^1(L;\Lambda_0)$

$H_1(L;\mathbb{Q}) = 0$

$\Rightarrow \mathcal{M}(L) = \text{one point}$

$\Psi_J \in \Lambda_0$

☆ $\alpha \in \pi_2(M)$

$$\mathcal{M}_1(\alpha;J) = \frac{\left\{ u : S^2 \to M \;\middle|\; \begin{array}{l} J\text{ holomorphic} \\ [u] = \alpha \end{array} \right\}}{\mathrm{Aut}(S^2, z_0)}$$

$$\begin{array}{ccc} u & & \mathcal{M}_1(\alpha;J) \\ \downarrow & & \downarrow{\scriptstyle ev} \\ u(z_0) & & M \end{array}$$

☆ $ev(\mathcal{M}_1(\alpha)) \cap L = \phi$ for $\alpha \neq 0$.

J dependence $\leftarrow$ wall crossing

J_0, J_1 ☆

$\mathcal{J} = \{ J_t \mid t \in [0,1] \}$

$$\mathcal{M}_1(\alpha, \mathcal{J}) = \bigsqcup_t \mathcal{M}_1(\alpha, J_t) \xrightarrow{ev_1} M$$

<u>Thm</u> $f_\# : \mathcal{M}(L, J_0) \xrightarrow{\sim} \mathcal{M}(L, J_1)$

$$\Psi_{J_1}(f_*(h)) - \Psi_{J_0}(h)$$

$$= \sum_\alpha \#(\mathcal{M}_1(\alpha \mathcal{J}) \cap L) \, T^{E(\alpha)}$$

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Relation other works

(1) Eyjolfsson (Welschinger) 0606429

M^3 $\tau : M \to M$ anti-holo

$L = \{x \in M \mid \tau(x) = x\}$ $\tau^* J = -J$

Solution inv. (M, L) GW

# of disks

Iwao 0912.2646

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Thm

$M,\ \tau : M \to M$ anti. holo

$L = \int x \tau$ $\quad x_i \in H(L)$

$m_{k,\beta}(x_1 \cdots x_k)$

$= (-1)^{*} m_{k,\beta}(x_k \cdots x_1)$

$* = \frac{\mu(\beta)}{2} + k + 1 + \sum_{i<j} \deg' x_i \deg' x_j$

$k = 0,\ \mu = 0$

$m_0(1) = -m_0(1)$

$\Rightarrow m_0(1) = 0$

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$0 \in M(L)$ MC of

$\Psi_J(0) \in \Lambda_0$

Prop J_0, J_1 $\tau^* J_0 = -J_0$
$\tau^* J_1 = -J_1$

$\Rightarrow$ $\exists J = (J_t)$ $\tau^* J_t = -J_t$

$\Rightarrow$ $GW(\alpha, J) = 0$

Conj $\Psi_J(0) =$ Solomon's inv

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3 M, Liu

02/02/57

$M \supset L$

S^1 act on M S^1 prev. L

free on L

$\beta \in \pi_2(M, L)$

$\mathcal{M}(\beta) = \{ u : (D^2, \partial) \to (M, L)$

$\mid$ hol $[u] = \beta \} / \mathrm{Aut}(D^2)$

Case $\dim \mathcal{M}(\beta) = 0$

$\#\mathcal{M}(\beta)$ is well defined.

(taking S^2 equiv. perturbation)

$\dim_{\mathbb{C}} M = 3$, $\mathcal{M} = 0$ $\dim \mathcal{M}(\beta) = 0$

$\#\mathcal{M}(\beta) =$ by fixed pt loc

$S^1 \curvearrowright \mathcal{M}(\beta)$ has a fixed pt

$\Rightarrow$ $\partial\beta \sim k S^1$ orbit

$$\gamma = \sum w_i c_i \in H_1(L)$$

$$\gamma^+ = \lambda : H^1(L) \longrightarrow \mathbb{R}$$

$$y : H^1(L;\Lambda) \longrightarrow \Lambda_0$$

$$y(u) = e^{\gamma(u)}$$

$$\Psi : H^1(L;\Lambda_0) \longrightarrow \Lambda_0$$

$$\Psi(b) = \sum_\beta \# M(\beta)\, T^{E(\beta)}\, y(b)^k$$

$$\partial\beta = k\gamma$$

Conj

$M(L)$ = critical pts of Ψ

Ψ coincide to our inv on $M(L)$
($\exists$ as S^1 inv.)

Lem Mll crossing does not occur as for as we take S^2 equiv. J

Conifold transition

$M \supset S^2$ lag

$\downarrow$

$\bar{M} \ni p$ sing S^3 to a pt

M' small resolu

$M' \to \bar{M}$ p iso outside p

$\pi^{-1}(p) \cong \mathbb{P}^1$

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Property Smith Thom Yan JDG 2002

0209319

$M \supset L_1, \ldots, L_n$

$\exists \lambda_i \in \mathbb{Q}$ $\sum \lambda_i L_i = 0$ in $H_3(M)$

$\Rightarrow M'$ conif trans. along L_i

$\supset P'_1 \ldots P'_n$

Chose C

$\partial C = \sum \lambda_i L_i$

$C' \subset M'$ cycle

$GW(C')_M$

$= \sum T^{E(\alpha)} \#(M_1(\alpha) \cap C')$

$M_1(\alpha) = \{(S^2, v_1) \xrightarrow{u} M \mid \begin{matrix} \text{hol} \\ [u] = \alpha \end{matrix}\} / \mathrm{Aut}(S^2, v_1)$

$\downarrow ev$

M

$M_1(\alpha) {}_{ev}\times C'$

$L_1 = S^3 \qquad \Psi_J(U; L_1) \quad G\Lambda_0$

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∘ tr knot mdl curves

(Joyce, Lecorvino) → 0907.5225

$$GW(C; M, J)$$

$$= \sum T^{I_\omega(\alpha)} \#(M_1(\alpha) \cap C)$$

dep on J for $\partial C \neq \emptyset$

$$\sum \lambda_i \Psi_J(v, L_i) + GW(C; M, J)$$

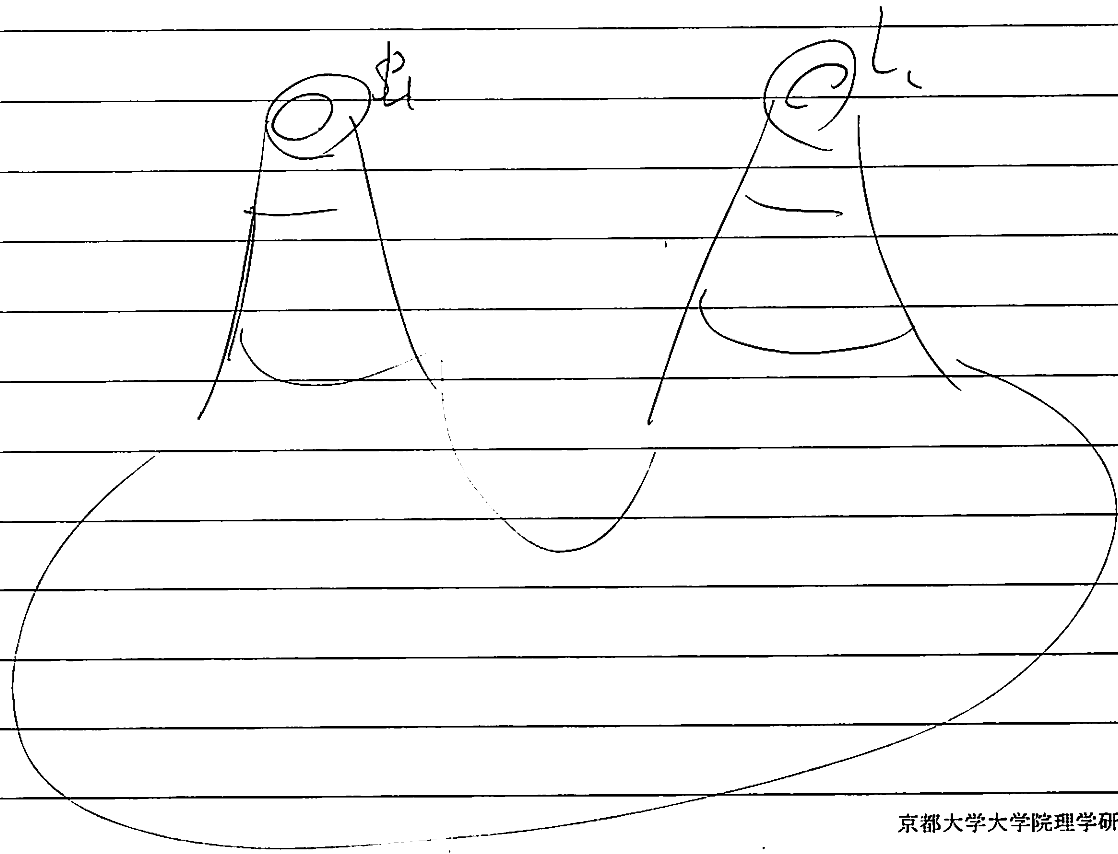
is ind of J

Cor (Joyce)

$$GW(C', M)$$

$$= \sum \lambda_i \Psi_J(C, L_i) + GW(C, M, J)$$

??



Take family T_2 of almost

cpx str

nbd of $L_0 = S^3$

on

$S^3 \times [0, T_2]$

$T_2 \to \infty$

claim $\lim_{\alpha \to 0} \Psi(L_1, J_2) = 0$

//

The

$$\sum \Lambda_j \Psi_j (u, L_j) + GW(C, M, J)$$

$$= \lim_{\alpha \to \infty} GW(C, M, J_\alpha)$$

$$= GW(C', M')$$

//