Chapter 9. Orientation.[1]

In this chapter, we give orientations to various moduli spaces in our construction of the filtered A_∞ algebra, the filtered A_∞ bimodules, the filtered A_∞ homomorphisms, etc. We use the notion of Kuranishi structure, hence we give orientations of the tangent bundles of the spaces with Kuranishi structures, see Definition A1.17. In our situation, the fiber products are taken with respect to weakly submersive strongly smooth maps in the sense of Kuranishi structures, see Definition A1.13. In this chapter, the symbol $\subset$ denotes an open inclusion, which respects orientations or covered by a canonical identification of orientation bundles.

§44. Orientation on the moduli space of unmarked pseudo-holomorphic discs.

44.1. The case of holomorphic discs.

Let (M,ω) be a symplectic manifold and L a Lagrangian submanifold. We pick an almost complex structure J compatible with ω. Then L is a totally real submanifold, namely $J(TL)\cap TL$ is the zero section of $TM|_L$. In this section, we consider the orientation problem on the moduli space of pseudo-holomorphic discs, $w : (D^2,\partial D^2) \to (M,L)$ with totally real boundary condition. First of all, we should note that the moduli space of pseudo-holomorphic discs is not always orientable. This is also observed by de Silva [Sil97] independently. In the case of the moduli space of pseudo-holomorphic curves without boundary, it is well known that it has a canonical orientation. In this chapter, we put an assumption on the second Stiefel-Whitney class $w_2(TL)$ in order to consider the orientation problem. Our main result in this section is the following:

Theorem 44.1. *The moduli space of pseudo-holomorphic discs is orientable, if $L \subset (M,\omega)$ is a relatively spin Lagrangian submanifold. Furthermore the choice of relative spin structure on L determines an orientation on $\mathcal{M}(L;\beta)$ canonically for all $\beta \in \pi_2(M,L)$.*

The definition of the notion of relative spin structure is in order. Let $L \subset M$ be a relatively spin Lagrangian submanifold and $st \in H^2(M;\mathbb{Z}_2)$ such that $st|_L = w_2(L)$. We first fix a triangulation of M such that L is a subcomplex. We choose an oriented real vector bundle V on the 3 skeleton $M_{[3]}$ of M such that $w_2(V) = st$. Then since

$$w_2(TL|_{L_{[2]}} \oplus V|_{L_{[2]}}) = 0,$$

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it follows that $TL|_{L_{[2]}} \oplus V|_{L_{[2]}}$ has a spin structure.

Definition 44.2. The choice of an orientation of L, a cohomology class $st \in H^2(M;\mathbb{Z}_2)$ and a spin structure σ on $TL \oplus V|_{L_{[2]}}$ is called a *relative spin structure* of the Lagrangian submanifold $L \subset M$.

A pair of Lagrangian submanifolds $(L^{(0)}, L^{(1)})$ is called *relatively spin*, if there exists a class st satisfying $st|_{L^{(i)}} = w_2(TL^{(i)})$ $(i = 0,1)$ simultaneously. A *relative spin structure* of the pair $(L^{(0)}, L^{(1)})$ is the choice of orientations of $L^{(i)}$, a cohomology class $st \in H^2(M;\mathbb{Z}_2)$ and spin structures on $(TL^{(i)} \oplus V)|_{L^{(i)}_{[2]}}$. Here $L^{(i)}_{[2]}$ is the two skeleton of $L^{(i)}$.

Remark 44.3. (1) In case L is spin, we can associate a relative spin structure to each of the spin structures of L as follows. We take $st = 0$, V is trivial bundle and take the spin structure of $TL \oplus V = TL$ induced by one on L.

(2) Definition 44.2 depends on the choice of V and the triangulation of M. Later in Proposition 44.6 we define stable conjugacy on the set of relative spin structures so that the set of stable conjugacy classes of relative spin structures is independent of such a choice.

Let $D : V \to W$ be a Fredholm operator. We say that the index of D is oriented, if the determinant line $\det D = \det(\text{Coker } D)^* \otimes \det \text{Ker } D$ is oriented. We use the following fact frequently.

Pick a finite dimensional subspace E of W such that $W = \text{Im } D + E$. Then $\det D$ is canonically isomorphic to $\det E^* \otimes \det D^{-1}(E)$. Thus, if E is oriented, e.g., as a complex vector space, then the orientation of $D^{-1}(E)$ determines the orientation of $\det D$. Note also that the orientation of E induces an orientation of E^*. Thus we sometimes consider an orientation of E instead of an orientation of E^*.

Firstly, we study the orientation in the linearized problem.

Proposition 44.4. *Let E be a complex vector bundle over the 2-disc D^2 and F a totally real subbundle of $E|_{\partial D^2}$ over ∂D^2. We denote by $\overline{\partial}_{(E,F)}$ the Dolbeault operator on D^2 with coefficient (E,F),*

$$\overline{\partial}_{(E,F)} : W^{1,p}(D^2, \partial D^2; E, F) \to L^p(D^2; E \otimes \Lambda^{0,1}(D^2)).$$

Then a trivialization of F over ∂D^2 determines an orientation of the index of $\overline{\partial}_{(E,F)}$, i.e., an orientation of $\det(\text{Coker } \overline{\partial}_{(E,F)})^* \otimes \det \text{Ker } \overline{\partial}_{(E,F)}$ *in a canonical way.*

Proof. By choosing a Hermitian connection appropriately, we may assume that the totally real subbundle F is parallel (especially, trivially flat) and the connection is "product" in a collar neighborhood of ∂D^2. Let C be a concentric circle in the collar neighborhood of ∂D^2. If we pinch C to a point, we have the union of a 2-disc D^2 and a 2-sphere $\mathbb{C}P^1$ with the center $O \in D^2$ and $S \in \mathbb{C}P^1$ identified. By the parallel translation along radials, the trivial vector bundle F extends up to C and its complexification gives a trivialization of $E|_C$. Thus the bundle E descends to $D^2 \cup \mathbb{C}P^1$. We denote this vector bundle by E' and the totally real subbundle of $E'|_{\partial D^2}$ by F'.

Figure 44.1.

We compare the indices of the following two operators:

$$\overline{\partial}_{(E,F)} : W^{1,p}(D^2, \partial D^2; E, F) \to L^p(D^2; E \otimes \Lambda^{0,1}(D^2))$$

and

$$\begin{aligned}\overline{\partial}_{D^2 \cup \mathbb{C}P^1} : \{(\xi_0, \xi_1) \in W^{1,p}(D^2, \partial D^2; E', F') \oplus W^{1,p}(\mathbb{C}P^1, E') \mid \xi_0(O) = \xi_1(S)\} \\ \to L^p(D^2, E' \otimes \Lambda^{0,1}(D^2)) \oplus L^p(\mathbb{C}P^1, E' \otimes \Lambda^{0,1}(\mathbb{C}P^1))\end{aligned}$$

using the sum formula for indices. (See for example Appendix A [McSa94].)

Before doing so, we take a finite dimensional subspace $\mathcal{E}$ of $L^p(\mathbb{C}P^1, E' \otimes \Lambda^{0,1}(\mathbb{C}P^1))$ as follows. Since the real vector bundle F is trivialized, we may identify

$$(D^2, \partial D^2; E', F') \cong (D^2, \partial D^2; \mathbb{C}^n, \mathbb{R}^n).$$

Hence the Dolbeault operator

$$\overline{\partial}_{(E',F')} : W^{1,p}(D^2, \partial D^2; E', F') \to L^p(D^2, E' \otimes \Lambda^{0,1}(D^2))$$

is surjective. Therefore, (using the unique continuation theorem), we can choose a finite dimensional *complex* linear subspace

$$\mathcal{E} \subset L^p(\mathbb{C}P^1, E' \otimes \Lambda^{0,1}(\mathbb{C}P^1))$$

consisting of smooth sections whose support do not contain S, such that

$$\mathrm{Im}(\overline{\partial}_{D^2\cup\mathbb{C}P^1}) + (0\oplus\mathcal{E}) = L^p(D^2, E'\otimes\Lambda^{0,1}(D^2)) \oplus L^p(\mathbb{C}P^1, E'\otimes\Lambda^{0,1}(\mathbb{C}P^1)).$$

(We note that $\mathcal{E}$ will turn out to be the obstruction bundle of the Kuranishi structure.)

Now let us choose small coordinate neighborhoods $D^2_\delta(O)$ and $D^2_\delta(S)$ of O and S. Here D^2_δ is the disc of radius δ. Fix a positive real number r. We glue D^2 and $\mathbb{C}P^1$ around O and S by identifying $z\in D^2_\delta(O)$ and $w\in D^2_\delta(S)$ whenever $zw = 1/r$. We denote the resulting bordered Riemann surface by X_r, which is biholomorphic to the unit disc. We also obtain the vector bundle on X_r from E' and denote it by E_r. The totally real subbundle F' over ∂D^2 induces a totally real subbundle F_r on ∂X_r.

We may identify $\mathcal{E} = 0\oplus\mathcal{E}$ and also denote by the same symbol $\mathcal{E}$ the subspace of $L^p(X_r; E_r\otimes\Lambda^{0,1}(X_r))$

We claim that

$$\mathrm{Im}(\overline{\partial}_{(E_r,F_r)}) + \mathcal{E} = L^p(X_r; E_r\otimes\Lambda^{0,1}(X_r)),$$

and that there is a canonical isomorphism

$$(\overline{\partial}_{D^2\cup\mathbb{C}P^1})^{-1}(\mathcal{E}) \cong (\overline{\partial}_{(E_r,F_r)})^{-1}(\mathcal{E})$$

for large r: In fact, let χ_r be a sequence of solutions $\overline{\partial}_{(E_r,F_r)}\chi_r \equiv 0 \mod \mathcal{E}$ on X_r. Then we may choose a subsequence of χ_r which converges to a section (ξ_0,ξ_1) on $D^2\cup\mathbb{C}P^1$ such that $\overline{\partial}_{D^2\cup\mathbb{C}P^1}(\xi_0,\xi_1)\equiv 0 \mod \mathcal{E}$. Note that if the L^p-norm of χ_r is 1 independent of r, then $(\xi_0,\xi_1)\neq(0,0)$. (Because the L^p-norm on the neck region $D^2_\delta(O)\cup D^2_\delta(S)\subset X_r$ is uniformly dominated by the L^p-norm on $D^2_\delta(O) - D^2_{\delta/2}(O)$ and $D^2_\delta(S) - D^2_{\delta/2}(S)$, it is impossible that the L^p-norm concentrates in the neck region around C.)

Conversely, we glue solutions (ξ_0,ξ_1) of $\overline{\partial}_{D^2\cup\mathbb{C}P^1}(\xi_0,\xi_1)\equiv 0 \mod \mathcal{E}$ to a solution χ_r of $\overline{\partial}_{(E_r,F_r)}\chi_r\equiv 0 \mod \mathcal{E}$ for a sufficiently large r.

Thus we have an isomorphism (canonical up to homotopy) of the indices. If necessary, we extend $\mathcal{E}$ to a larger *complex* subspace of $L^p(\mathbb{C}P^1, E'\otimes\Lambda^{0,1}(\mathbb{C}P^1))$ such that the evaluation map $ev_S : \overline{\partial}^{-1}_{\mathbb{C}P^1,E}(\mathcal{E})\to E_S$ at the south pole $S\in\mathbb{C}P^1$ is surjective.

Since the real vector bundle F' is trivialized, $(\overline{\partial}_{D^2\cup\mathbb{C}P^1})^{-1}(\mathcal{E})$ is the kernel of the surjective homomorphism:

$$Ev : (\xi_0,\xi_1)\in \mathrm{Hol}(D^2,\partial D^2;\mathbb{C}^n,\mathbb{R}^n)\times(\overline{\partial}_{\mathbb{C}P^1})^{-1}(\mathcal{E}) \mapsto \xi_0(O)-\xi_1(S)\in\mathbb{C}^n\simeq E_S.$$

Since $\mathcal{E}$ is a complex linear subspace and $\overline{\partial}_{\mathbb{C}P^1}$ is complex linear, it follows that $(\overline{\partial}_{\mathbb{C}P^1})^{-1}(\mathcal{E})$ is a complex linear subspace and hence is oriented. On the other hand, the linear space

$$\mathrm{Hol}(D^2,\partial D^2;\mathbb{C}^n,\mathbb{R}^n)\cong\mathbb{R}^n$$

is identified with the tangent space of L at one point and hence is oriented.

Therefore $\operatorname{Ker} Ev - \mathcal{E}$ has an orientation. See (4) in Convention 45.1 for our way to orient it. This proves Proposition 44.4. □

Proof of Theorem 44.1. We will apply Proposition 44.4 to the case that $E = w^*TM$ and $F = \ell^*TL$. Since L is oriented, ℓ^*TL is a trivial bundle. However, its trivialization is not unique and the choice of a relative spin structure provides a (stable) trivialization unique up to homotopy.

Before going into the proof, we give its outline. Using Proposition 44.4, we give an orientation of $\det D_w\overline{\partial}$ for each holomorphic disc w. Then we show that this orientation depends only on the relative spin structure. We proceed as follows. Firstly, we pick a holomorphic disc w_0 in each homotopy class of maps from $(D^2, \partial D^2) \to (M, L)$ and define an orientation of $\det D_{w_0}\overline{\partial}$. The orientation depends on the choice of the relative spin structure. We remark that, if we change the homotopy class of a stable trivialization of ℓ^*TL, then the orientation on the index bundles in the proof of Proposition 44.4 changes. If $\{w_t\}_{0\le t\le 1}$ is a homotopy from w_0 to $w = w_1$, the determinant line bundle $\det D_{w_t}\overline{\partial}$ is trivial, since the base space $[0,1]$ is contractible. Thus we obtain an orientation of $\det D_w\overline{\partial}$, which may depend on the homotopy $\{w_t\}_{0\le t\le 1}$. The remaining task is to prove that the induced orientation does not depend on the choice of a homotopy. Here we need the condition that L is relatively spin, i.e., there is a relative spin structure of $L \subset M$. (The second step, showing the orientability of a certain determinant line bundle over S^1, only requires existence of a relative spin structure but does not depend on a specific choice.)

Step 1. Assigning an orientation to $\det D_w\overline{\partial}$.

Denote by

$$D_w\overline{\partial} : W^{1,p}(D^2, \partial D^2; w^*TM, \ell^*TL) \to W^{0,p}(D^2; w^*TM \otimes \Lambda^{0,1}(D^2)) = L^p(D^2; w^*TM \otimes \Lambda^{0,1}(D^2))$$

the linearized operator of the pseudo-holomorphic curve equation, which is the first order elliptic differential operator with the same symbol as the Dolbeault operator. Here ℓ is the restriction of w to the boundary ∂D^2 and we take p as $p > 2$. To prove Theorem 44.1, it suffices to show that the index of the linearized operator $D_w\overline{\partial}$ is oriented.

Note that $D_w\overline{\partial} + A$ is always Fredholm for any zero-th order operator A and the space $\mathcal{F}_{D_w\overline{\partial}}$ of such operators is contractible. The index of the family $D_w\overline{\partial} + A \in \mathcal{F}_{D_w\overline{\partial}}$ is a virtual vector bundle over the contractible space. In particular, the determinant line bundle of the family is trivial. Since the zero-th order term does not affect the index problem, we may consider Dolbeault operator

$$\overline{\partial}_{(w^*TM, \ell^*TL)} : W^{1,p}(D^2, \partial D^2; w^*TM, \ell^*TL) \to L^p(D^2; w^*TM \otimes \Lambda^{0,1}(D^2)),$$

instead of $D_w\overline{\partial}$.

By the simplicial approximation theorem, any $w : (D^2, \partial D^2) \to (M, L)$ can be deformed to some $w_0 : (D^2, \partial D^2) \to (M_{[2]}, L_{[1]})$. (Pick such w_0 for each connected component of the space of maps $(D^2, \partial D^2) \to (M, L)$.) Then the relative spin structure determines the (stable) homotopy class of trivializations of $\ell_0^*(TL \oplus V)$. By Proposition 44.4, we can specify an orientation of

$$\det(\text{Coker } \overline{\partial}_{(w_0^*TM, \ell_0^*TL)})^* \otimes \det \text{Ker } \overline{\partial}_{(w_0^*TM, \ell_0^*TL)}.$$

For any given element w in the path component of the space of maps $(D^2, \partial D^2) \to (M, L)$ containing w_0, we consider a path of maps $w_t : (D^2, \partial D^2) \to (M, L)$ parameterized by $t \in [0, 1]$ starting from w_0 and ending at w. (Note that the linear operator $D_w\overline{\partial}$ does make sense for any smooth map $(D^2, \partial D^2) \to (M, L)$, once we pick a complex linear connection on TM.) By Proposition 44.4 and since [0,1] is contractible, we can provide a family of orientations of

$$\det(\text{Coker } \overline{\partial}_{(w_t^*TM, \ell_t^*TL)})^* \otimes \det \text{Ker } \overline{\partial}_{(w_t^*TM, \ell_t^*TL)},$$

which depends continuously on $t \in [0, 1]$, by taking a trivialization of ℓ_t^*TL over $S^1 \times [0, 1]$ where $\ell_t = w_t|_{\partial D^2}$. This will define an orientation for w.

Step 2. Independence of the choice of a path $\{w_t\}$.

We prove that this assignment of the orientation is independent of the choice of path $\{w_t\}_{t\in[0,1]}$. Here we need existence of the relative spin structure. Let st and V are as in Definition 44.2. Then the relative spin structure determines a spin structure σ of $TL \oplus V$ on $M_{[2]} \cap L$.

We show that any two paths $w_t^{(1)}, w_t^{(2)}$ from w_0 to w induce the same orientation of the index. The concatenation of the path $w_t^{(1)}$ and the reversed path of $w_t^{(2)}$ gives a family of maps $(D^2, \partial D^2) \to (M, L)$ parameterized by S^1. More generally, let $w_t : (D^2, \partial D^2) \to (M, L)$ be a $t \in S^1$ parameterized family of maps $(D^2, \partial D^2) \to (M, L)$. We define the map

$$\Phi : D^2 \times S^1 \to M; \quad \Phi(z, t) = w_t(z).$$

(In particular, we can apply the following argument to S^1 parameterized families of pseudo-holomorphic discs.)

By the simplicial approximation theorem, Φ can be homotoped to a map Φ' such that $\Phi'(D^2 \times S^1)$ is in the 3-skeleton $M_{[3]}$ of M and that $\Phi'(\partial D^2 \times S^1) \subset M_{[2]} \cap L = L_{[2]}$. Note that the family of Dolbeault type operators parameterized by S^1 corresponding to Φ' is homotoped to the one corresponding to Φ in the space of Fredholm operators. Hence it is enough to discuss the one for Φ' as far as the index, especially, the orientation issue is concerned. Hereafter we put $\Phi' = \Phi$.

We pull back V by Φ to a vector bundle over $D^2 \times S^1$, and denote it by Φ^*V. The spin structure chosen on $(TL \oplus V)|_{L_{[2]}}$ gives its stable trivialization on $L_{[2]}$

which induces a (stable) trivialization on $\Phi^*(TL \oplus V)|_{\partial D^2 \times S^1}$. On the other hand, $\Phi^* V$ is defined on $D^2 \times S^1 \simeq S^1$ and is also oriented. Since the base space is $D^2 \times S^1$, the orientation of $\Phi^* V$ guarantees the existence of a trivialization of $\Phi^* V$ on $D^2 \times S^1$, hence a trivialization of its restriction to $\partial D^2 \times S^1$. We like to mention that this trivialization restricted to $\partial D^2 \times \{t\}$ is independent of the choice of the spin structure on $(TL \oplus V)|_{M_{[3]} \cap L}$.

Combining these, we have a family of stable trivializations of $(\Phi|_{\partial D^2 \times \{t\}})^* TL$ which is continuous over $t \in S^1$. In particular, we find that the homotopy class of trivializations of $\ell^* TL \oplus V$ is independent of the choice of a path $w_t^{(i)}$, $i = 1, 2$. In sum, for any w, the spin structure on $(TL \oplus V)|_{L_{[2]}}$ induces a unique (homotopy class of) trivialization of $\ell^*(TL \oplus V)$ where $\ell = w|_{\partial D^2}$ and the existence of the oriented bundle V on $M_{[3]}$ induces a unique (homotopy class of) trivialization of $\ell^* V$. Namely, $\ell^* TL$ is stably trivialized for each $\ell = w|_{\partial D^2}$. Moreover the stable trivialization can be taken in a continuous way with respect to ℓ.

We put $V_{\mathbb{C}} = V \otimes \mathbb{C}$. Then we find that

$$\overline{\partial}_{(w^*(TM \oplus V_{\mathbb{C}}), \ell^*(TL \oplus V))} : W^{1,p}(w^*(TM \oplus V_{\mathbb{C}}), \ell^*(TL \oplus V)) \to L^p(w^*(TM \oplus V_{\mathbb{C}}))$$

is the direct sum of

$$\overline{\partial}_{(w^*TM, \ell^*TL)} : W^{1,p}(w^*TM, \ell^*TL) \to L^p(w^*TM)$$

and

$$\overline{\partial}_{(w^*V_{\mathbb{C}}, \ell^*V)} : W^{1,p}(w^*V_{\mathbb{C}}, \ell^*V) \to L^p(w^*(V_{\mathbb{C}})).$$

By Proposition 44.4, the orientation of $\ell^*(TL \oplus V)$ induces a canonical orientation of the index of $\overline{\partial}_{(w^*(TM \oplus V_{\mathbb{C}}), \ell^*(TL \oplus V))}$. The trivialization of $\ell^* V$, which extends to $w^* V$, induces a canonical orientation of $\overline{\partial}_{(w^*V_{\mathbb{C}}, \ell^*V)}$. Hence the index of $\overline{\partial}_{(w^*TM, \ell^*TL)}$ is also canonically oriented. This implies that $\mathcal{M}(L; \beta)$ is oriented.

From the above argument, the choice of orientation on L, $st \in H^2(M, \mathbb{Z}_2)$ and the vector bundle V on $M_{[3]}$ and the spin structure $(TL \oplus V)|_{M_{[2]} \cap L}$ canonically determine the orientations on $\mathcal{M}(L; \beta)$ for all β. This completes the proof of Theorem 44.1. □

In our later argument, it is important that the orientation provided in Step 1 of the proof of Theorem 44.1 is compatible with the gluing procedure of holomorphic discs.

Let us next clarify how the notion of relative spin structure depends on the choice of V and triangulation. First of all, we recall the following basic fact. Let E be an oriented vector bundle on X with $w_2(E) = 0$. A spin structure is equivalent to a fiberwise double covering space of the associated principal SO-bundle P_E, which are classified by homomorphisms $\pi_1(P_E) \to \mathbb{Z}_2$ such that its restriction to the fiber

$\pi_1(SO) \to \mathbb{Z}_2$ is non-trivial. Thus the set of spin structures on E on X is a principal homogeneous space of $H^1(X;\mathbb{Z}_2)$. A spin structure on E induces a spin structure on its stabilization $E \oplus \mathbb{R}^k$ by a trivialized vector bundle $\mathbb{R}^k$, namely, the one induced by the spin structure on E and the trivialization of $\mathbb{R}^k$. If we consider $\mathbb{R}^k$ as a trivial bundle without specific trivialization, then we get a coarse equivalence relation among (stabilized) spin structures, which are parameterized by the quotient of $H^1(X;\mathbb{Z}_2)$ by the gauge transformations of $\mathbb{R}^k$. (Cf. the description of spin structures above.) Now we introduce an analogous coarse equivalence relation on relative spin structure.

In the next definition we still fix a triangulation of M such that L is its subcomplex (a triangulation of M compatible with L).

Definition 44.5. Let (st_i, V_i, σ_i) $(i = 1, 2)$ be relative spin structures of L. We say that they are *stably conjugate* to each other if there exist integers k_i and an orientation preserving bundle isomorphism $\tau : V_1 \oplus \mathbb{R}^{k_1} \to V_2 \oplus \mathbb{R}^{k_2}$ such that by $1 \oplus \tau|_{L_{[2]}} : (TL \oplus V_1)_{L_{[2]}} \oplus \mathbb{R}^{k_1} \to (TL \oplus V_2)_{L_{[2]}} \oplus \mathbb{R}^{k_2}$, the spin structure $\sigma_1 \oplus 1$ induces the spin structure $\sigma_2 \oplus 1$.

We remark that if (st_1, V_1, σ_1) is stably conjugate to (st_2, V_2, σ_2), then they determine the same orientation on $\mathcal{M}(\beta)$. (This fact is obvious from the definition of the orientation given during the proof of Theorem 44.1.)

For a given triangulation $\mathfrak{T}$ of M (such that L is a subcomplex), we denote by $\mathrm{Spin}(M, L; \mathfrak{T})$ the set of all the stably conjugacy classes of relative spin structures on L with respect to the triangulation $\mathfrak{T}$.

Proposition 44.6. (1) *There exists a simply transitive action of $H^2(M, L; \mathbb{Z}_2)$ on* $\mathrm{Spin}(M, L; \mathfrak{T})$.

(2) *For two triangulation $\mathfrak{T}$ and $\mathfrak{T}'$ of M compatible with L, there exists a canonical isomorphism* $\mathrm{Spin}(M, L; \mathfrak{T}) \cong \mathrm{Spin}(M, L; \mathfrak{T}')$ *compatible with the above action.*

In particular if a spin structure of L is given there is a canonical isomorphism $\mathrm{Spin}(M, L; \mathfrak{T}) \cong H^2(M, L; \mathbb{Z}_2)$.

Proof. Let $[(st, V, \sigma)] \in \mathrm{Spin}(M, L; \mathfrak{T})$ and $\mathfrak{x} \in H^2(M, L; \mathbb{Z}_2)$. We regard the class $\mathfrak{x} \in H^2(M, L; \mathbb{Z}_2) \cong H^2(M_{[3]}, L_{[2]}; \mathbb{Z}_2)$. Since $M_{[3]}/L_{[2]}$ is a 3-dimensional cell complex, $(w_1, w_2) : \widetilde{KO}(M_{[3]}/L_{[2]}) \to H^1(M_{[3]}, L_{[2]}; \mathbb{Z}_2) \oplus H^2(M_{[3]}, L_{[2]}; \mathbb{Z}_2)$ is an isomorphism. Thus, $\mathfrak{x} \in H^2(M, L; \mathbb{Z}_2)$ determines a unique stable class of orientable vector bundle $E_{\mathfrak{x}}$ with $w_2(E_{\mathfrak{x}}) = \mathfrak{x}$. Pick and fix an orientation on $E_{\mathfrak{x}}$. We remark that the right hand side is independent of the orientation of $E_{\mathfrak{x}}$ we take, since there exists an orientation reversing involution on $E_{\mathfrak{x}} \oplus \mathbb{R}$, i.e., the multiplication by -1 on the second factor $\mathbb{R}$. We pull back $E_{\mathfrak{x}}$ by $M_{[3]} \to M_{[3]}/L_{[2]}$ to get a vector bundle $V_{\mathfrak{x}}$. Note that the restriction $V_{\mathfrak{x}}|_{L_{[2]}}$ to $L_{[2]}$ is endowed with the trivialization as the pull-back of the frame at the collapsed point, thus the spin structure $\sigma_{\mathfrak{x}}$ on $V_{\mathfrak{x}}|_{L_{[2]}}$. Now we put

$$\mathfrak{x} \cdot [(st, V, \sigma)] = [(st + \bar{\mathfrak{x}}, V \oplus V_{\mathfrak{x}}, \sigma \oplus \sigma_{\mathfrak{x}})],$$

where $\bar{\mathfrak{x}}$ is the image of $\mathfrak{x}$ by $H^2(M, L; \mathbb{Z}_2) \to H^2(M; \mathbb{Z}_2)$.

Conversely, for two relative spin structures $[(st, V, \sigma)]$ and $[(st', V', \sigma')]$, we define their *difference* as follows. Pick an oriented vector bundle E on $L_{[2]}$ such that $TL \oplus E$ is trivial and fix an isomorphism $I : TL \oplus E \cong L_{[2]} \times \mathbb{R}^N$. For V', we pick an oriented vector bundle W' on $M_{[3]}$ such that $V' \oplus W'$ is trivial and fix an isomorphism $I_{V',W'} : V' \oplus W' \cong M_{[3]} \times \mathbb{R}^{N'}$. Note that the spin structure σ' and the isomorphisms I, $I_{V',W'}$ induce a spin structure on $E \oplus W'$, which we denote by $\check{\sigma}'$. Then the vector bundle $V \oplus W'$ is a vector bundle equipped with a stable trivialization on $L_{[2]}$ as follows.

$$(V \oplus W')|_{L_{[2]}} \oplus \mathbb{R}^N \cong (V \oplus W')|_{L_{[2]}} \oplus (TL \oplus E) \cong (TL \oplus V|_{L_{[2]}}) \oplus (E \oplus W'|_{L_{[2]}}),$$

which is stably trivialized in a compatible way with the spin structure $\sigma \oplus \check{\sigma}'$. Denote this trivialization by $\mathfrak{I}_{\sigma,\sigma'}$. Then we use this trivialization to get an element $[V \oplus W', \mathfrak{I}_{\sigma,\sigma'}] \in \widetilde{KO}(M_{[3]}/L_{[2]})$.

Fix a relative spin structure (st, V, σ). Then it is straightforward to see that the above construction gives a one-to-one correspondence between stable conjugate classes of relative spin structures and $KO(M_{[3]}, L_{[2]})$. Hence, the action of $H^2(M, L; \mathbb{Z}_2)$ on $\mathrm{Spin}(M, L; \mathfrak{T})$ is simply transitive.

To prove (2) we first remark that, by taking common subdivision, it suffices to consider the case when $\mathfrak{T}'$ is a subdivision of $\mathfrak{T}$. In this case, the restriction defines a map $\mathrm{Spin}(M, L; \mathfrak{T}') \to \mathrm{Spin}(M, L; \mathfrak{T})$. By construction, this map is compatible with the action defined in (1). Hence it is an isomorphism. □

Proposition 44.6 states that the difference of relative spin structures is measured by an element $\mathfrak{x}$ in $H^2(M, L; \mathbb{Z}/2)$. When we change the relative spin structure by $\mathfrak{r}$, then the orientation of the index for $D_w\overline{\partial}$ changes by $(-1)^{\mathfrak{x}[w]}$. (For spin case, the example given in the next subsection illustrates this phenomenon.)

44.2. Examples of non-orientable family index.

In this subsection, we will show that the index bundle appearing in the proof of Proposition 44.4 can actually be unoriented, if we do not assume F to be trivial.

We can find such a pair (E, F) as follows. Take the trivial bundle $\tilde{E} = (D^2 \times [0,1]) \times \mathbb{C}^n$ and $\tilde{F} = \mathbb{R}^n \times (\partial D^2 \times [0,1]) \times \mathbb{C}^n$. Identifying $D^2 \times \{0\}$ and $D^2 \times \{1\}$, we get $D^2 \times S^1$. We lift this identification to the vector bundle $\tilde{E}$ such that its restriction to $\partial D^2 \times \{i\}$ $(i = 0, 1)$ preserves the real part $\tilde{F}$ and identify them by homotopically non-trivial loop $\gamma : \partial D^2 \to SO(n)$. Such identification exists, because the image of the loop by $SO(n) \to U(n)$ is null-homotopic and extends to a continuous map $D^2 \to U(n)$. Let (E, F) be the pair obtained by this identification.

Proposition 44.7 below implies that the index bundle is not orientable for such (E,F). This means that the change of a stable trivialization by γ reverses the orientation on the moduli space of holomorphic discs.

Let us consider the case $n=2$. Another description of the resulting vector bundle pair (E,F) over $D^2\times S^1$ is the following. Take a trivial vector bundle $(D^2\times[0,1])\times\mathbb{H}$, where $\mathbb{H}$ is the quaternions considered as a left $\mathbb{C}$-module. Identify the fibers over $D^2\times\{0\}$ and $D^2\times\{1\}$ by multiplication of $\mathbf{i}\in\mathbb{H}^*$, that is, $(z,0,\zeta\mathbf{i})\sim(z,1,\zeta)$ for $z\in D^2$ and $\zeta\in\mathbb{H}$ to get E. For $(z,t)\in\partial D^2\times[0,1]$, the subspace

$$F_{z,t}=(\mathbb{R}\mathbf{1}\oplus\mathbb{R}\mathbf{j})\left\{t\left(\frac{z+\overline{z}}{2}+\frac{z-\overline{z}}{2i}\mathbf{j}\right)+(1-t)\mathbf{k}\right\}$$

gives a totally real subspace in $\mathbb{H}$, hence we have a totally real subbundle of $(D^2\times[0,1])\times\mathbb{H}$ over $\partial D^2\times[0,1]$ which descends to a totally real subbundle F of E over $\partial D^2\times S^1$.

Proposition 44.7. *The family index of Dolbeault operators twisted by (E,F) is a non-orientable virtual vector bundle over S^1.*

Proof. By using the Fourier expansion, we calculate the kernels and cokernels of Dolbeault operators explicitly. We take a basis $\langle 1+\mathbf{k},1-\mathbf{k}\rangle$ of the left $\mathbb{C}$-module $\mathbb{H}$. By regarding $\mathbb{H}$ as $\mathbb{C}\oplus\mathbb{C}\mathbf{j}$, we write the basis as column vectors $\left\langle\begin{pmatrix}1\\ i\end{pmatrix},\begin{pmatrix}1\\ -i\end{pmatrix}\right\rangle$. Let f be a holomorphic map from D^2 to $\mathbb{C}^2=\mathbb{H}$. Then we have the Fourier expansion of f such that

$$f(z)=\sum_{n=0}^{\infty}z^n\left\{\alpha_n\begin{pmatrix}1\\ i\end{pmatrix}+\beta_n\begin{pmatrix}1\\ -i\end{pmatrix}\right\}$$

for $z\in D^2$ and $\alpha_n,\beta_n\in\mathbb{C}$. Now suppose that the image of $f|_{\partial D^2}$ is in the totally real subspace

$$(\mathbb{R}\mathbf{1}\oplus\mathbb{R}\mathbf{j})\left\{t\left(\frac{z+\overline{z}}{2}+\frac{z-\overline{z}}{2i}\mathbf{j}\right)+(1-t)\mathbf{k}\right\}$$

of $\mathbb{H}$. This condition is equivalent to

$$f(z)\left\{t\left(\frac{z+\overline{z}}{2}-\frac{z-\overline{z}}{2i}\mathbf{j}\right)-(1-t)\mathbf{k}\right\}\in\mathbb{R}\mathbf{1}\oplus\mathbb{R}\mathbf{j}\qquad\text{for }|z|=1.\tag{44.8}$$

When we describe the multiplication from the right by $t(\frac{z+\overline{z}}{2}-\frac{z-\overline{z}}{2i}\mathbf{j})-(1-t)\mathbf{k}$ as a 2×2 complex matrix acting on column vectors of $\mathbb{H}=\mathbb{C}\oplus\mathbb{C}\mathbf{j}$ from the left, we find that the multiplication can be described by

$$\begin{aligned}&t\frac{z+\overline{z}}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-t\frac{z-\overline{z}}{2i}\begin{pmatrix}0&-1\\1&0\end{pmatrix}-(1-t)\begin{pmatrix}0&i\\i&0\end{pmatrix}\\=&z\frac{t}{2}\begin{pmatrix}1&-i\\i&1\end{pmatrix}+\overline{z}\frac{t}{2}\begin{pmatrix}1&i\\-i&1\end{pmatrix}-(1-t)\begin{pmatrix}0&i\\i&0\end{pmatrix}.\end{aligned}$$

It follows that we have

$$f(z)\left\{t\left(\frac{z+\overline{z}}{2}-\frac{z-\overline{z}}{2i}\mathbf{j}\right)-(1-t)\mathbf{k}\right\}$$

$$=\sum_{n=0}^{\infty}z^n\left\{z\frac{t}{2}\alpha_n\begin{pmatrix}1&-i\\i&1\end{pmatrix}\begin{pmatrix}1\\i\end{pmatrix}+\overline{z}\frac{t}{2}\alpha_n\begin{pmatrix}1&i\\-i&1\end{pmatrix}\begin{pmatrix}1\\i\end{pmatrix}-(1-t)\alpha_n\begin{pmatrix}0&i\\i&0\end{pmatrix}\begin{pmatrix}1\\i\end{pmatrix}\right.$$

$$\left.+\,z\frac{t}{2}\beta_n\begin{pmatrix}1&-i\\i&1\end{pmatrix}\begin{pmatrix}1\\-i\end{pmatrix}+\overline{z}\frac{t}{2}\beta_n\begin{pmatrix}1&i\\-i&1\end{pmatrix}\begin{pmatrix}1\\-i\end{pmatrix}-(1-t)\beta_n\begin{pmatrix}0&i\\i&0\end{pmatrix}\begin{pmatrix}1\\-i\end{pmatrix}\right\}.$$

Note that $\overline{z}=z^{-1}$ for $|z|=1$. So we can put it as $\displaystyle\sum_{n=-1}^{\infty}z^n\vec{a}_n$ with $\vec{a}_n\in\mathbb{C}\oplus\mathbb{C}\mathbf{j}$. In this expression, we have

$$\vec{a}_{-1}=t\beta_0\begin{pmatrix}1\\-i\end{pmatrix},$$

$$\vec{a}_0=-(1-t)\beta_0\begin{pmatrix}1\\i\end{pmatrix}+\{(1-t)\alpha_0+t\beta_1\}\begin{pmatrix}1\\-i\end{pmatrix},$$

$$\vec{a}_n=\{t\alpha_{n-1}-(1-t)\beta_n\}\begin{pmatrix}1\\i\end{pmatrix}+\{(1-t)\alpha_n+t\beta_{n+1}\}\begin{pmatrix}1\\-i\end{pmatrix}\quad\text{for } n\geq 1.$$

Since $\mathbb{R}\mathbf{1}\oplus\mathbb{R}\mathbf{j}$ is a totally real subspace in $\mathbb{H}$, the condition (44.8) implies that $\overline{\vec{a}_n}=\vec{a}_{-n}$ for any n. These conditions yield that

(44.9.1) $\overline{\vec{a}_{-1}}=\vec{a}_1$
(44.9.2) $\vec{a}_0\in\mathbb{R}^2$
(44.9.3) $\vec{a}_n=0$ for $n\geq 2$.

Now let us consider the case $t\neq 0,1$. From (44.9.3) we have $\alpha_n=0$ $(n\geq 1)$ and $\beta_n=0$ $(n\geq 2)$. The condition (44.9.2) is equivalent to the equality $-(1-t)\overline{\beta_0}=(1-t)\alpha_0+t\beta_1$. Combining it with (44.9.1), we easily find that α_0 and β_1 are determined by β_0. Explicitly we have

$$\alpha_0=\frac{2t-1}{2t^2-2t+1}\overline{\beta_0},\qquad \beta_1=-\frac{1-t}{t}(\overline{\beta_0}+\alpha_0).$$

Hence our f is determined by only $\beta_0\in\mathbb{C}$. Therefore we find that the real dimension of the kernel of the Dolbeault operator for $t\neq 0,1$ is equal to 2. On the other hand, we can see that the Fredholm index of the operator is 2. So we can find that the operator is surjective.

Next, we describe the kernels for the cases $t=0,1$. In both cases, we can see that $\alpha_n=0$ $(n\geq 1)$ and $\beta_n=0$ $(n\geq 2)$ by using (44.9.3). Moreover, by using (44.9.2), we find that $\beta_1=0$ in both cases and

$$\alpha_0=\begin{cases}-\overline{\beta_0} & \text{for } t=0,\\ \overline{\beta_0} & \text{for } t=1.\end{cases}$$

Thus β_0 determines the kernels of the operators in both cases.

With these explicit descriptions of the kernels understood, we can show the non-orientability of our index bundle. For $t \in [0,1]$, the kernel is isomorphic to $\{\beta_0 \in \mathbb{C}\}$. So we have the (complex) orientation. What we want to show is that the identification map between the kernels on $t = 0$ and $t = 1$ is an orientation reversing map. When $t = 0$, the kernel V_0 is given by

$$V_0 = \left\{ -\overline{\beta_0} \begin{pmatrix} 1 \\ i \end{pmatrix} + \beta_0 \begin{pmatrix} 1 \\ -i \end{pmatrix} \middle| \ \beta_0 \in \mathbb{C} \right\}$$

When $t = 1$, the kernel V_1 is given by

$$V_1 = \left\{ \overline{\beta_0} \begin{pmatrix} 1 \\ i \end{pmatrix} + \beta_0 \begin{pmatrix} 1 \\ -i \end{pmatrix} \middle| \ \beta_0 \in \mathbb{C} \right\}$$

Recall that we identify V_0 with V_1 by the right multiplication of $\mathbf{i}$. As before, this multiplication can be written by a 2×2 complex matrix acting on $\mathbb{C} \oplus \mathbb{C}\mathbf{j}$ from the left as

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Thus V_0 is identified with

$$\begin{aligned} V_1 \cdot \mathbf{i} &= \left\{ \overline{\beta_0} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} + \beta_0 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\} \\ &= \left\{ \overline{\beta_0} \begin{pmatrix} i \\ 1 \end{pmatrix} + \beta_0 \begin{pmatrix} i \\ -1 \end{pmatrix} \middle| \ \beta_0 \in \mathbb{C} \right\}. \end{aligned}$$

We find that this identification map is an orientation reversing map. Because when we put $\beta_0 = 1$ and i, we have bases

$$\left\langle \begin{pmatrix} 0 \\ -2i \end{pmatrix}, \begin{pmatrix} 2i \\ 0 \end{pmatrix} \right\rangle, \qquad \left\langle \begin{pmatrix} 2i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2i \end{pmatrix} \right\rangle$$

of the real vector spaces V_0 and $V_1 \cdot \mathbf{i}$ respectively. Clearly, the orientations on these spaces are opposite. Therefore we have finished the proof of Proposition 44.7. □

Note that the totally real subbundle F above is orientable, but *not spin*.

44.3. The case of connecting orbits in the Floer theory.

Let $(L^{(0)}, L^{(1)})$ be a relative spin pair of Lagrangian submanifolds, (see Definition 44.2), intersecting transversely. A connecting orbit in the Floer complex for $L^{(0)}$,

$L^{(1)}$ is a pseudo-holomorphic map $u : \mathbb{R} \times [0,1] \to M$, which maps $\mathbb{R} \times \{i\}$ to $L^{(i)}$, $u(-\infty, *) = p^-$ and $u(+\infty, *) = p^+$.

In this chapter, we adopt a different notation from §4 in Chapter 2. Namely, we denote by $\mathcal{M}^{\rm reg}(p^+, p^-)$ instead of $\mathcal{M}^{\rm reg}(p^-, p^+)$. In other words, $u(-\infty, *) = p^-$, $u(+\infty, *) = p^+$, for an element $u \in \mathcal{M}^{\rm reg}(p^+, p^-)$. This is because the former is more suitable in the orientation business under our convention and makes it consistent with the orientation of the moduli spaces for defining the filtered A_∞ algebra in §47.

Consider the pull back of the tangent bundle TM by the projection $M \times [0,1] \to M$. For $p \in L^{(0)} \cap L^{(1)}$, we take a path of oriented Lagrangian subspaces $\lambda_p : [0,1] \to \Lambda^{ori}(T_pM)$ so that

$$\lambda_p(0) = T_pL^{(0)}, \quad \lambda_p(1) = T_pL^{(1)}$$

and a trivialization (relative to the fixed trivialization at end points) of the bundle

$$\widetilde{\lambda_p} = \bigcup_{t \in [0,1]} \{t\} \times \lambda_p(t) \to [0,1].$$

(See §3.2.) Then we have a Lagrangian subbundle

$$\mathcal{L} \to K(L^{(0)}, L^{(1)}) = (L^{(0)} \times \{0\}) \cup ((L^{(0)} \cap L^{(1)}) \times [0,1]) \cup (L^{(1)} \times \{1\})$$

of the pull back of TM.

Let $st \in H^2(M; \mathbb{Z}_2)$ such that $w_2(TL^{(i)}) = st|_{L^{(i)}}$, for $i = 0, 1$. As in Definition 44.2 we choose an oriented vector bundle V on the 3-skeleton of M with $w_2(V) = st$. Denote by $P_{spin}(L^{(i)}, V)$ the principal spin bundle of $(TL^{(i)} \oplus V)|_{L^{(i)}_{[2]}}$, which is a fiberwise double cover of its oriented frame bundle $P_{SO}(L^{(i)}, V)$. We may assume that $L^{(0)} \cap L^{(1)}$ is contained in the 3-skeleton of M. In order to specify the spin structure on $(\mathcal{L} \oplus pr^*V)\,|_{K(L^{(0)}, L^{(1)})_{[2]}}$, where $K(L^{(0)}, L^{(1)})_{[2]}$ is the 2-skeleton of $K(L^{(0)}, L^{(1)})$ and pr is the projection to $M \times [0,1] \to M$, we need additional information. Let $\sigma : [0,1] \times \mathbb{R}^n \cong \widetilde{\lambda}_p$ be a trivialization. Since $\lambda_p(i) = T_pL^{(i)}$, $\sigma|_{t=i}$ induces a framing of $T_pL^{(i)}$, hence, we obtain embeddings $\sigma_{t=i*} : SO(n) \times SO(V_p) \to P_{SO}(L^{(i)}, V)|_p$ for $i = 0, 1$. Let $\iota_i : (Spin(n) \times Spin(V_p))\,/\{\pm 1\} \to P_{spin}(L^{(i)}, V)|_p$, which covers $\sigma_{t=i*}$. (Here ± 1 acts on $Spin(n) \times Spin(V_p)$ diagonally.) Now we consider the triple $(\sigma, \iota_0, \iota_1)$. For the trivialization σ, we pick a bundle isomorphism $\widetilde{\sigma} : [0,1] \times Spin(n+k) \cong P_{spin}(\mathcal{L} \oplus V)|_{\{p\} \times [0,1]}$, which covers σ_*. (There are two choices of $\widetilde{\sigma}$.) Gluing $P_{spin}(\mathcal{L} \oplus V)|_{\{p\} \times [0,1]}$ with $P_{spin}(L^{(i)}, V)$, $i = 0, 1$, by $\iota_i \circ (\widetilde{\sigma})^{-1}$, we obtain a spin structure of $\mathcal{L} \oplus V$ on $K(L^{(0)}, L^{(1)})_{[2]}$. It is easy to see that the resulting spin structure does not depend on the choice of a lift $\widetilde{\sigma}$ of σ_*.

Let $\mathbf{I}_{\lambda_p}$ be the space of trivializations σ of $\widetilde{\lambda}_p$, which respect the orientation, and $\widetilde{\mathbf{I}}_{\lambda_p}$ the space of triples $(\sigma, \iota_0, \iota_1)$ as above. It is clear that $\mathbf{I}_{\lambda_p}$ is homotopy equivalent to $SO(n)$ and $\widetilde{\mathbf{I}}_{\lambda_p}$ is homotopy equivalent to

$$\{(g_0, g_1) \in Spin(n) \times Spin(n) \mid \pi(g_0) = \pi(g_1)\}/\{\pm 1\},$$

where $\pi : Spin(n) \to SO(n)$ is the canonical projection and ± 1 acts on $Spin(n) \times Spin(n)$ diagonally. Then we find that $\widetilde{\mathbf{I}}_{\lambda_p} \to \mathbf{I}_{\lambda_p}$ is a trivial double covering space, i.e., $\widetilde{\mathbf{I}}_{\lambda_p}$ is the union of two copies of $\mathbf{I}_{\lambda_p}$.

Now we consider the space $\mathcal{P}_p(TL^{(0)}, TL^{(1)})$ of all paths in $\Lambda^{ori}(T_pM)$ from $T_pL^{(0)}$ to $T_pL^{(1)}$. Define

$$\mathcal{I}(p) = \bigcup_{\lambda_p \in \mathcal{P}_p(TL^{(0)}, TL^{(1)})} \mathbf{I}_{\lambda_p}$$

and

$$\widetilde{\mathcal{I}}(p) = \bigcup_{\lambda_p \in \mathcal{P}_p(TL^{(0)}, TL^{(1)})} \widetilde{\mathbf{I}}_{\lambda_p}.$$

As we saw in §44.2, there is an oriented *non-spin* Lagrangian subbundle in $(S^1 \times S^1) \times \mathbb{C}^n$. Hence $\widetilde{\mathcal{I}}(p) \to \mathcal{I}(p)$ is a non-trivial double covering space.

For each p, we pick $\lambda_p \in \mathcal{P}_p(TL^{(0)}, TL^{(1)})$ and consider the elliptic operators

$$\overline{\partial}_{\lambda_p, Z_\pm} : W^{1,p}_{\lambda_p}(Z_\pm; T_pM) \to L^p(Z_\pm; T_pM \otimes \Lambda^{0,1}(Z_\pm)).$$

on the capped half infinite cylinders

$$\begin{aligned} Z_- &= \{z \in \mathbb{C} \mid |z| \le 1\} \cup \{z \in \mathbb{C} \mid \mathrm{Re} z \ge 0, |\mathrm{Im} z| \le 1\} \\ Z_+ &= \{z \in \mathbb{C} \mid \mathrm{Re} z \le 0, |\mathrm{Im} z| \le 1\} \cup \{z \in \mathbb{C} \mid |z| \le 1\}. \end{aligned}$$

as in §12.5. (See right before Definition 12.62.)

We define the Maslov-Morse index $\mu(p, \lambda_p)$ of the intersection point $p \in L^{(0)} \cap L^{(1)}$ with a path λ_p of Lagrangian subspaces in T_pM by

$$\mu(p, \lambda_p) = \mathrm{Index}(\overline{\partial}_{\lambda_p, Z_-}). \tag{44.11}$$

For $R > 0$, we define the space

$$Z_- \#_R (\mathbb{R} \times [0,1]) \#_R Z_+$$

as follows. We consider

$$\begin{aligned} Z_{-,R} &= \{z \in Z_- \mid \mathrm{Re}\, z \le R\} \\ Z_{+,R} &= \{z \in Z_- \mid \mathrm{Re}\, z \ge -R\}. \end{aligned}$$

We glue 3 spaces $Z_{-,R}$, $[-R,R]\times[0,1]$, $Z_{+,R}$, by identifying $(R,t)\in Z_{-,R}$ with $(-R,t)\in[-R,R]\times[0,1]$ and $(R,t)\in[-R,R]\times[0,1]$ with $(-R,t)\in Z_{+,R}$, respectively.

Figure 44.2

We consider operators $\overline{\partial}_{\lambda_p,Z_\pm}$ on $Z_\pm$ and $D_u\overline{\partial}$ on $\mathbb{R}\times[0,1]$. (Here $D_u\overline{\partial}$ is the linearization of the Cauchy-Riemann equation at u.) We can use appropriate partition of unity to glue these 3 operators and obtain an operator $\overline{\partial}_{u;\lambda_{p^+},\lambda_{p^-}}$ on $Z_-\#_R(\mathbb{R}\times[0,1])\#_RZ_+$. Now index sum formula implies

$$\mathrm{Index}(\overline{\partial}_{\lambda_p,Z_-})+\mathrm{Index}(D_u\overline{\partial})+\mathrm{Index}(\overline{\partial}_{\lambda_p,Z_+})=\mathrm{Index}(\overline{\partial}_{u;\lambda_{p^+},\lambda_{p^-}}). \tag{44.12}$$

We define $\hat{u}:Z_-\#_R(\mathbb{R}\times[0,1])\#_RZ_+\to M$ by gluing u with constant maps to $p^\pm$, by appropriate partition of unity. We trivialize the bundle

$$\hat{u}^*TM\cong(Z_-\#_R(\mathbb{R}\times[0,1])\#_RZ_+)\times\mathbb{C}^n.$$

We then obtain a path

$$\lambda:\partial(Z_-\#_R(\mathbb{R}\times[0,1])\#_RZ_+)\to\Lambda(\mathbb{C}^n)$$

in Lagrangian Grassmannian by

$$\begin{aligned}&\lambda(e^{\pi\sqrt{-1}(1/2\pm t)})=\lambda_{p^\pm}(t),\quad\text{where } e^{\pi\sqrt{-1}(1/2\pm t)}\in\partial Z_\pm,\\&\lambda(\tau,i)=T_{u(\tau,i)}L^{(i)}\quad i=0,1.\end{aligned}$$

It is easy to see

$$\mathrm{Index}(\overline{\partial}_{u;\lambda_{p^+},\lambda_{p^-}})=\mathrm{Index}(\overline{\partial}_{\mathbb{C}^n,\lambda}) \tag{44.13}$$

where the right hand side is as in Proposition 44.4. Pick $(\sigma, \iota_0, \iota_1) \in \widetilde{\mathbf{I}}_{\lambda_p}$, then the spin structure of $\mathcal{L} \oplus V$ induces an orientation of the virtual vector space $\operatorname{Index}(\overline{\partial}_{\mathbb{C}^n,\lambda})$ in the same way as the proof of Theorem 44.1.

Therefore, *once we choose an orientation on* $\operatorname{Index}\overline{\partial}_{\lambda_{p^+},Z_\pm}$ and $\operatorname{Index}\overline{\partial}_{\lambda_{p^-},Z_\pm}$ as well as $(\sigma^\pm, \iota_0^\pm, \iota_1^\pm) \in \widetilde{\mathbf{I}}_{\lambda_{p^\pm}}$, an orientation of Index $D_u\overline{\partial}$ is induced by (44.12), (44.13). Thus (using Remark 44.15 (1) also) we have proved the following:

Theorem 44.14. *Suppose that a pair of Lagrangian submanifolds $(L^{(0)}, L^{(1)})$ is relatively spin. Then for any $p^\pm \in L^{(0)} \cap L^{(1)}$, the moduli space $\mathcal{M}(p^+, p^-)$ of connecting orbits in Lagrangian intersection Floer cohomology is orientable. Furthermore, orientations on* $\operatorname{Index}\overline{\partial}_{\lambda_{p^\pm},Z_-}$, $(\sigma, \iota_0, \iota_1) \in \widetilde{\mathbf{I}}_{\lambda_p}$ *and relative spin structures for the pair $(L^{(0)}, L^{(1)})$ canonically determine the orientation on $\mathcal{M}(p^+, p^-)$.*

Remark 44.15. (1) We consider u_p (where $u_p(\tau, t) \equiv p$) and glue $\overline{\partial}_{\lambda_p,Z_-}$, $D_{u_p}\overline{\partial}$ and $\overline{\partial}_{\lambda_p,Z_+}$ as above to obtain an operator $\overline{\partial}_{u_p;\lambda_p,\lambda_p}$. The index of $\overline{\partial}_{u_p;\lambda_p,\lambda_p}$ is canonically isomorphic to $T_pL^{(0)}$ and hence is oriented. The index of $D_{u_p}\overline{\partial}$ is trivial and is also oriented. Therefore by (44.12), an orientation of $\operatorname{Index}(\overline{\partial}_{\lambda_p,Z_-})$, determines an orientation of $\operatorname{Index}\overline{\partial}_{\lambda_p,Z_+}$. Hence in Theorem 44.14, it is enough to choose an orientation on Index $\overline{\partial}_{(\overline{\partial}_{\lambda_p,Z_-})}$ for each $p = p^\pm$.

(2) The orientation determined by the relative spin structure (Theorems 44.1 and 44.14) automatically satisfies the coherence condition [FlHo93], i.e., the compatibility under gluing process. See Proposition 46.3 and Lemma 46.5 below for compatibility of orientation on moduli spaces of holomorphic discs. The compatibility for the moduli space of connecting orbits is also proven based on Lemma 46.5, see the proof of Proposition 50.3. For the procedure of giving orientation in Theorem 44.14, we follow the argument from §21 [FuOn99II]. (See also [FlHo93] for argument on the coherent orientation problem.)

We remark that coherent system of orientations in the sense of [FlHo93] is *not* unique. In fact, there are coherent orientation systems, which derive non-isomorphic cohomologies. In finite dimensional case, if the manifold is not simply connected, one can twist the Morse complex by local systems to obtain the cohomology with coefficients in local systems. We then obtain a twisted cohomology which are not isomorphic to the untwisted cohomology, in general. The holonomy of a local system contributes to signs in "counting connecting orbits". In Floer theory for Lagrangian submanifolds, Cho found spin structures, which derives non-isomorphic Floer cohomologies [Cho04].

In other words, to define Floer cohomology, it is *not* enough to prove existence of orientation satisfying the coherence condition, but we need to specify the way to orient the moduli spaces satisfying the coherence condition.

(3) The determinant line bundle of $\operatorname{Index}\overline{\partial}_{\cdot,Z_-}$ over each connected component of $\mathcal{P}_p(TL^{(0)}, TL^{(1)})$ is non-orientable , but its pull-back to $\widetilde{\mathcal{I}}(p)$ is orientable. To see this, we use the argument in (1). Pick and fix $(\lambda_p, \sigma, \iota_0, \iota_1) \in \widetilde{\mathcal{I}}(p)$. Glue the family

$\overline{\partial}_{\cdot,Z_-}$ with $\overline{\partial}_{\lambda_p,Z_+}$ to obtain a family of Dolbeault type operators with totally real boundary condition. Then the example in §44.2 implies that the determinant line bundle of the family index is not orientable. Since $\overline{\partial}_{\lambda_p,Z_+}$ does not depend on the elements in $\mathcal{P}_p(TL^{(0)},TL^{(1)})$, the index bundle of $\overline{\partial}_{\cdot,Z_-}$ is not orientable. It is easy to see that the pull-back by $\mathcal{I}(p) \to \mathcal{P}_p(TL^{(0)},TL^{(1)})$ is also non-orientable on each connected component, since the totally real boundary condition does not depend on the elements in $\mathbf{I}_{\lambda_p}$. Note that $\widetilde{\mathcal{I}}(p) \to \mathcal{I}(p)$ is the unique double covering space, which is non-trivial over each connected component of $\mathcal{I}(p)$. Therefore the pull-back of the index bundle should be orientable, hence the claim. (We can also directly find the second claim, since $\widetilde{\mathcal{I}}(p)$ provides a consistent family of spin structures on the totally real subbundles.)

Hence, we can rephrase "a choice of an orientation on $\operatorname{Index}\overline{\partial}_{\lambda_{p^{\pm}},Z_-}$ as well as $(\sigma^{\pm},\iota_0^{\pm},\iota_1^{\pm})$" by the orientation of the determinant line bundle of the family $\operatorname{Index}\overline{\partial}_{\cdot,Z_-}$ on $\widetilde{\mathcal{I}}(p)$. This interpretation is more canonical and can be adopted to other cases, e.g., Bott-Morse case, see Proposition 51.1.

(4) In [Oh97II], the second named author previously studied the orientation problem for the Floer homology of Lagrangian intersection for the case where $L^{(0)} = \phi(o_N)$ and $L^{(1)} = o_N$ on $M = T^*N$ (or more generally $L^{(0)} = \nu^*S$ the conormal bundle of $S \subset N$). He proved that in this case there exists a coherent orientation on the Floer complex for any compact manifold N *irrespective of orientability of* N. In this case, the condition on w_2 is automatically satisfied and there are no holomorphic discs with boundary on $L^{(0)}$ nor $L^{(1)}$.

§45. Conventions and Preliminaries.

In this section, we will fix some basic conventions and prepare some formulae concerning orientation. At the beginning, we assume that spaces are smooth (so manifolds), and mappings are submersions. Later we incorporate Kuranishi structure and deal with the space with Kuranishi structure and the fiber product of weakly submersive strongly smooth maps in the sense of Kuranishi structure. In this section, we denote dimensions of spaces X or Y by corresponding small letters x or y respectively. We will denote by $*$ a generic point without specifying it from now on.

Convention 45.1.

(1) Let X be an oriented smooth manifold with boundary ∂X. Then we define

an orientation on ∂X so that

$$T_*X \simeq \mathbb{R}_{\mathrm{out}} \times T_*(\partial X) \tag{45.1.1}$$

is an isomorphism of oriented vector spaces. Here $\mathbb{R}_{\mathrm{out}}$ is $\mathbb{R}$ oriented by an outer normal vector.

In order to discuss the case of spaces with Kuranishi structure, we prepare some notations. A Kuranishi neighborhood around $p \in X$ is a quintuple $(V_p, E_p, \Gamma_p, \psi_p, s_p)$ as in Definition A1.1. We denote the quotient V_p/Γ_p by U_p. By definition of the orientation of the Kuranishi structure (see Definition A1.17 and [FuOn99II] Definition 5.8), we have an orientation on $\det E_p \otimes \det T_*V_p$ and the finite group Γ_p acts on E_p and V_p preserving orientation of $\det E_p^* \otimes \det TV_p$. Then we find that the group action does not effect on orientation problem. So we may assume here the action is trivial in the argument below. Namely, we consider the case that Γ_p is trivial and identify V_p with U_p. In this chapter we denote by an (oriented) Kuranishi neighborhood (local chart of a space with oriented Kuranishi structure) by $(s; E \to U)$, where $E \to U$ be a vector bundle and s is a section of it. Here s is a section of a vector bundle E over U. When s fails to be transversal to the zero section, we use multi-valued perturbations to get the virtual fundamental chain. Our task here is to assign a canonical orientation to each zero locus. So we assume, after perturbations, that s is transversal to the zero section. Then we have the exact sequence

$$0 \longrightarrow T_*s^{-1}(0) \longrightarrow T_*U \overset{D_*s}{\longrightarrow} E_* \longrightarrow 0 \tag{45.2}$$

where $D_*s : T_*U \to E_*$ is the covariant derivative of $s : U \to E$ at $*$ with respect to a connection. Note that this derivative does not depend on the choice of connection as long as $* \in s^{-1}(0)$. We arrange the basis of T_*U so that *the basis of E_* comes first and then that of $T_*s^{-1}(0)$ next.* We then define the orientation on $s^{-1}(0)$ by

$$E_* \times T_*s^{-1}(0) = T_*U.$$

(Note that since an orientation on $\det E_* \otimes \det T_*U$ in given, the above equality determines the orientation on $T_*s^{-1}(0)$.)

Let $(s; E \to U)$ be a local chart of a space with oriented Kuranishi structure with boundary. Recall that the orientation of $(s; E \to U)$ is the orientation of $\det E \otimes \det TU$. If necessary, we choose U smaller so that we may assume that U is an oriented manifold with boundary and E is an oriented vector bundle on it. We define the orientation of the boundary of $(s; E \to U)$ by

$$\partial(s; E \to U) = (-1)^{\mathrm{rank} E}(s|_{\partial U}; E|_{\partial U} \to \partial U).$$

Here ∂U is oriented as the boundary of U.

(2) Let G be a Lie group given an orientation. When G acts from the right on an oriented manifold X smoothly and freely, then we define an orientation of the quotient space X/G so that

$$T_*X \simeq T_*(X/G) \times \text{Lie } G \tag{45.1.2}$$

is an isomorphism of oriented vector spaces. Here Lie G is the Lie algebra of G.

If G acts on the space X with oriented Kuranishi structure such that the isotropy subgroup at any point is finite, the quotient space inherits a structure of oriented Kuranishi structure. Namely, for a G-equivariant Kuranishi neighborhood $(s; E \to U)$, the quotient space U/G is oriented as above. The bundle E and its section s descend to an (orbi)bundle $E/G \to U/G$ and its section $\underline{s}$. Then $(\underline{s}; E/G \to U/G)$ is oriented, since

$$\begin{aligned} &\det E_x \otimes \det T_xU \\ \cong &\det E_{[x]} \otimes \det\Big(T_{[x]}(U/G) \oplus \text{Lie } G\Big) \\ \cong &\det E_{[x]} \otimes \det T_{[x]}(U/G) \otimes \det \text{Lie } G, \end{aligned}$$

where $x \in U$ and $[x]$ is the equivalence class in U/G.

(3) (Fiber product orientation). Let X_i $(i = 1, 2)$ and Y be oriented smooth manifolds and $f_i : X_i \to Y$ two submersions. We denote the fiber product by $X_1 {}_{f_1}\times_{f_2} X_2$ or $X_1 \times_Y X_2$. Take a point $q \in f_1(X_1) \cap f_2(X_2)$. Let $\langle u_1, \cdots, u_y\rangle$ be an oriented basis of T_qY. By transversality, we have $\langle v_{X_i,1}, \cdots, v_{X_i,y}\rangle \in T_{p_i}X_i$ for $p_i \in f_i^{-1}(q)$ such that $(df_i)_{p_i}(v_{X_i,k}) = u_k$ for $k = 1, \cdots, y$. We can choose a basis $\langle \eta_{X_i,1}, \cdots, \eta_{X_i,x_i-y}\rangle$ in $\text{Ker}(df_i)_{p_i}$ so that

$$\langle \eta_{X_1,1}, \cdots, \eta_{X_1,x_1-y}, v_{X_1,1}, \cdots, v_{X_1,y}\rangle \in T_{p_1}X_1 \simeq \text{Ker}(df_1)_{p_1} \times \text{Im}(df_1)_{p_1}$$

is a basis compatible with the given orientation of $T_{p_1}X_1$ and

$$\langle v_{X_2,1}, \cdots, v_{X_2,y}, \eta_{X_2,1}, \cdots, \eta_{X_2,x_2-y}\rangle \in T_{p_2}X_2 \simeq \text{Im}(df_2)_{p_2} \times \text{Ker}(df_2)_{p_2}$$

is a basis compatible with the given orientation of $T_{p_2}X_2$ respectively. Then we define an orientation on the fiber product $X_1 \times_Y X_2$ so that

$$\langle \eta_{X_1,1}, \cdots, \eta_{X_1,x_1-y}, u_1, \cdots, u_y, \eta_{X_2,1}, \cdots, \eta_{X_2,x_2-y}\rangle$$

is an oriented basis at $T_{[p_1,p_2]}(X_1 \times_Y X_2)$. Here we identify T_qY with $T_{[p_1,p_2]}((s_1 \times s_2)Y)$, where s_i is a local section of $f_i : X_i \to Y$.

In other words, we identify $T_{[p_1,p_2]}(X_1 \times_Y X_2)$ and $\text{Ker}(df_1)_{p_1} \times T_qY \times \text{Ker}(df_2)_{p_2}$, as vector spaces and define an orientation on $T_{[p_1,p_2]}(X_1 \times_Y X_2)$ so that

$$T_{[p_1,p_2]}(X_1 \times_Y X_2) \simeq \text{Ker}(df_1)_{p_1} \times T_qY \times \text{Ker}(df_2)_{p_2}$$

is an isomorphism of oriented vector spaces. Here we define the orientation on $\mathrm{Ker}(df_i)_{p_i}$ such that

$$T_{p_1}X_1 \simeq \mathrm{Ker}(df_1)_{p_1} \times \mathrm{Im}(df_1)_{p_1} \tag{45.1.3}$$

$$T_{p_2}X_2 \simeq \mathrm{Im}(df_2)_{p_2} \times \mathrm{Ker}(df_2)_{p_2} \tag{45.1.4}$$

are isomorphisms as oriented vector spaces respectively. Here $\mathrm{Im}(df_i)_{p_i}$ is isomorphic to T_qY as oriented vector spaces. (When $(df_i)_{p_i}$ $(i = 1, 2)$ is bijective, we define the orientation on the one-point set $\mathrm{Ker}(df_i)_{p_i}$ by ϵ_i, where $\epsilon_i = +1$ if $(df_i)_{p_i}$ is an orientation preserving isomorphism, and $\epsilon_i = -1$ if $(df_i)_{p_i}$ is an orientation reserving isomorphism.) In this sense, when we consider orientations of fiber products, we will hereafter write oriented isomorphisms (45.1.3)-(45.1.4) such as

$$X_1 = X_1^\circ \times Y \tag{45.1.5}$$

$$X_2 = Y \times {}^\circ X_2. \tag{45.1.6}$$

Then we will write the fiber product $X_1 \times_Y X_2$ as $X_1^\circ \times Y \times {}^\circ X_2$. (Of course, these equalities make sense as equalities at tangent space level.)

(4) (Fiber product of Kuranishi structure.) Let $(s_1; E_1 \to U_1)$ and $(s_2; E_2 \to U_2)$ be Kuranishi neighborhoods of X_1 and X_2. Then we can take the product $(s_1 \oplus s_2; E_1 \oplus E_2 \to U_1 \times U_2)$, which gives a Kuranishi structure on the product $X_1 \times X_2$. Note that the orientation on $(s_1 \oplus s_2)^{-1}(0)$ defined by $(s_1 \oplus s_2; E_1 \oplus E_2 \to U_1 \times U_2)$ is different from the product orientation on $X_1 \times X_2 = s_1^{-1}(0) \times s_2^{-1}(0)$. The difference is given by

$$\begin{aligned} X_1 \times X_2 &= (-1)^{\mathrm{rank} E_2 \dim X_1} (s_1 \oplus s_2)^{-1}(0) \\ &= (-1)^{\mathrm{rank} E_2 (\dim U_1 - \mathrm{rank} E_1)} (s_1 \oplus s_2)^{-1}(0). \end{aligned}$$

As for the general case of fiber products, we define the orientation on the fiber product with Kuranishi structures as follows. Here we assume that the maps to define the fiber product are weakly submersive (see §A1). Let $f_1 : U_1 \to B$ and $f_2 : U_2 \to B$ be submersions. Take the fiber product of them, i.e., $U_1 \times_B U_2 = U_1 {}_{f_1}\times_{f_2} U_2$. We have the orientation on $U_1 \times_B U_2$ defined by Convention 45.1 (3). We restrict the bundle $E_1 \oplus E_2$ on $U_1 \times U_2$ to $U_1 \times_B U_2$. The orientation on $E_1 \oplus E_2$ is also induced. The Kuranishi neighborhood $(s_1 \oplus s_2; E_1 \oplus E_2 \to U_1 \times_B U_2)$ defines an orientation on $(s_1 \oplus s_2)^{-1}(0)$ by

$$(E_1 \oplus E_2)_* \times T_*(s_1 \oplus s_2)^{-1}(0) = T_*(U_1 \times_B U_2).$$

Then we define the fiber product $X_1 \times_B X_2$ by

$$\begin{aligned} X_1 \times_B X_2 &= (-1)^{\mathrm{rank} E_2 (\dim X_1 - \dim B)} (s_1 \oplus s_2)^{-1}(0) \\ &= (-1)^{\mathrm{rank} E_2 (\dim U_1 - \mathrm{rank} E_1 - \dim B)} (s_1 \oplus s_2)^{-1}(0). \end{aligned}$$

Note that this convention is independent of the choice of Kuranishi neighborhoods $(s_i, E_i \to U_i)$.

As consequences of these conventions, we have the following fundamental formulae which will be used later. Hereafter (-1)-*oriented isomorphism* (resp. $(+1)$-*oriented isomorphism*) stands for orientation reversing (resp. preserving) isomorphism.

Lemma 45.3. (1) *Assume X_1 and X_2 have boundaries and the boundary of Y is empty. For $X_1 \to Y$ and $X_2 \to Y$, we have*

$$\partial(X_1 \times_Y X_2) = \partial X_1 \times_Y X_2 \bigsqcup (-1)^{x_1+y}(X_1 \times_Y \partial X_2).$$

(2) (*Associativity.*) *For $X_1 \to Y_1, X_2 \to Y_1 \times Y_2$ and $X_3 \to Y_2$, we have*

$$(X_1 \times_{Y_1} X_2) \times_{Y_2} X_3 = X_1 \times_{Y_1} (X_2 \times_{Y_2} X_3).$$

(3) (*Iteration formula.*) *For $X_1 \to Y_1 \times Y_2, X_2 \to Y_1$ and $X_3 \to Y_2$, we have*

$$X_1 \times_{Y_1 \times Y_2} (X_2 \times X_3) = (-1)^{y_2(y_1+x_2)}(X_1 \times_{Y_1} X_2) \times_{Y_2} X_3.$$

(4) *Let $f_i : X_i \to X'_i$ $(i = 1, 2)$ be an $\epsilon(f_i)$-oriented isomorphism and $g : Y \to Y'$ an $\epsilon(g)$-oriented isomorphism. Then for $X_i \to Y$ and $X'_i \to Y'$, the induced map*

$$f_1 \times_g f_2 \ : \ X_1 \times_Y X_2 \longrightarrow X'_1 \times_{Y'} X'_2$$

is an $\epsilon(f_1)\epsilon(f_2)\epsilon(g)$-oriented isomorphism.

Proof: All claims are trivial except signs. So we only check signs. Firstly we consider the case that all maps are submersions between manifolds.

(1) We put $X_1 = X_1^\circ \times Y$ and $X_2 = Y \times {}^\circ X_2$ as in (45.1.5)-(45.1.6). Moreover we put $\partial X_1 = (\partial X_1)^\circ \times Y$ and $\partial X_2 = Y \times {}^\circ(\partial X_2)$. From the convention of the boundary orientation (45.1.1), we have

$$\mathbb{R}_{X_1} \times (\partial X_1)^\circ \times Y = X_1^\circ \times Y$$
$$\mathbb{R}_{X_2} \times (Y \times {}^\circ(\partial X_2)) = Y \times {}^\circ X_2,$$

where $\mathbb{R}_{X_i}$ is oriented by the outward normal vector. Thus we have

$$\mathbb{R}_{X_1} \times (\partial X_1)^\circ = X_1^\circ$$
$$(-1)^y(\mathbb{R}_{X_2} \times {}^\circ(\partial X_2)) = {}^\circ X_2.$$

Hence we have

$$\begin{aligned}\mathbb{R}_{X_1\times_Y X_2}\times\partial(X_1\times_Y X_2) &= X_1\times_Y X_2 = X_1^\circ\times Y\times{}^\circ X_2\\ &= \mathbb{R}_{X_1}\times(\partial X_1)^\circ\times Y\times{}^\circ X_2\\ &\quad\bigsqcup X_1^\circ\times Y\times(-1)^y(\mathbb{R}_{X_2}\times{}^\circ(\partial X_2))\\ &= \mathbb{R}_{X_1}\times(\partial X_1\times_Y X_2)\bigsqcup(-1)^{x_1+y}\mathbb{R}_{X_2}\times(X_1\times_Y\partial X_2)),\end{aligned}$$

which proves (1).

(2) Put

$$\begin{aligned}&X_1 = X_1^\circ\times Y_1\\ &X_2 = Y_1\times{}^\circ X_2 = X_2^\circ\times Y_2 = Y_1\times{}^\circ X_2^\circ\times Y_2\\ &X_3 = Y_2\times{}^\circ X_3.\end{aligned}$$

Then we have ${}^\circ X_2 = {}^\circ X_2^\circ\times Y_2$ and $X_2^\circ = Y_1\times{}^\circ X_2^\circ$. Since

$$X_1\times_{Y_1}X_2 = X_1^\circ\times Y_1\times{}^\circ X_2 = X_1^\circ\times Y_1\times{}^\circ X_2^\circ\times Y_2,$$

we have

$$(X_1\times_{Y_1}X_2)\times_{Y_2}X_3 = X_1^\circ\times Y_1\times{}^\circ X_2^\circ\times Y_2\times{}^\circ X_3.$$

On the other hand, since

$$X_2\times_{Y_2}X_3 = X_2^\circ\times Y_2\times{}^\circ X_3 = Y_1\times{}^\circ X_2^\circ\times Y_2\times{}^\circ X_3,$$

the right hand side in (2) can be written as

$$X_1\times_{Y_1}(X_2\times_{Y_2}X_3) = X_1^\circ\times Y_1\times{}^\circ X_2^\circ\times Y_2\times{}^\circ X_3,$$

which implies the associativity.

(3) We put

$$X_1 = X_1^\circ\times Y_1\times Y_2,\qquad X_2 = Y_1\times{}^\circ X_2,\qquad X_3 = Y_2\times{}^\circ X_3.$$

Then since we have $X_2\times X_3 = (-1)^{y_2(x_2-y_1)}Y_1\times Y_2\times{}^\circ X_2\times{}^\circ X_3$, the left hand side is written as

$$X_1\times_{Y_1\times Y_2}(X_2\times X_3) = (-1)^{y_2(x_2-y_1)}X_1^\circ\times Y_1\times Y_2\times{}^\circ X_2\times{}^\circ X_3. \tag{45.4}$$

On the other hand, since we have

$$\begin{aligned}&X_1\times_{Y_1}X_2 = (-1)^{y_1y_2}X_1^\circ\times Y_2\times Y_1\times{}^\circ X_2\\ =&(-1)^{y_1y_2}(-1)^{y_2x_2}X_1^\circ\times Y_1\times{}^\circ X_2\times Y_2,\end{aligned}$$

the right hand side is written as

$$\begin{aligned}
&(X_1 \times_{Y_1} X_2) \times_{Y_2} X_3 \\
=&(-1)^{y_2(y_1+x_2)} X_1^\circ \times Y_1 \times {}^\circ X_2 \times Y_2 \times {}^\circ X_3 \\
=&(-1)^{y_2(y_1+x_2)}(-1)^{y_2(x_2-y_1)} X_1^\circ \times Y_1 \times Y_2 \times {}^\circ X_2 \times {}^\circ X_3. \qquad (45.5)
\end{aligned}$$

Comparing (45.4) and (45.5), we get the formula.

(4) We put $X_1 = X_1^\circ \times Y$ and $X_2 = Y \times {}^\circ X_2$. Then we can write as

$$X_1 \times_Y X_2 = X_1^\circ \times Y \times {}^\circ X_2.$$

Similarly we have $X_1' \times_{Y'} X_2' = X_1'^\circ \times Y' \times {}^\circ X_2'$. Since $Y' = \epsilon(g)Y$ and $X_i' = \epsilon(f_i)X_i$, we have $X_1'^\circ = \epsilon(f_1)\epsilon(g)X_1^\circ$ and ${}^\circ X_2' = \epsilon(f_2)\epsilon(g){}^\circ X_2$. Thus we get (4).

The same conclusions hold for spaces with Kuranishi structures. In fact, Convention 45.1 (1) and (4) are designed so that the above argument works with the following modification. If $\dim X < \dim Y$, we regard X as the space with Kuranishi structure $(s; E \to U)$. When X is a submanifold of Y, X° plays the role of the obstruction bundle E on $U = Y$.

In general, the statements (1), (2), (3) and (4) in the case of spaces of Kuranishi structure are proved as follows.

(1) Let $X_i = (s_i; E_i \to U_i)$. By Convention 45.1 (1) and (4), we have

$$\partial(X_1 \times_Y X_2) = (-1)^{\delta_1}\Big((s_1 \oplus s_2)|_{\partial(U_1 \times_Y U_2)}; (E_1 \oplus E_2)|_{\partial(U_1 \times_Y U_2)} \to \partial(U_1 \times_Y U_2)\Big),$$

where $\delta_1 = \mathrm{rank} E_1 + \mathrm{rank} E_2 + \mathrm{rank} E_2(\dim X_1 - \dim Y)$.

Similarly, we have

$$\partial X_1 \times_Y X_2 = (-1)^{\delta_2}\Big((s_1 \oplus s_2)|_{\partial U_1 \times_Y U_2}; (E_1 \oplus E_2)|_{\partial U_1 \times_Y U_2} \to \partial U_1 \times_Y U_2\Big),$$

where $\delta_2 = \mathrm{rank} E_2(\dim X_1 - 1 - \dim Y) + \mathrm{rank} E_1$, and

$$X_1 \times_Y \partial X_2 = (-1)^{\delta_3}\Big((s_1 \oplus s_2)|_{U_1 \times_Y \partial U_2}; (E_1 \oplus E_2)|_{U_1 \times_Y \partial U_2} \to U_1 \times_Y \partial U_2\Big),$$

where $\delta_3 = \mathrm{rank} E_2 + \mathrm{rank} E_2(\dim X_1 - \dim Y)$.

Note that, for oriented manifolds U_1, U_2, we have

$$\partial(U_1 \times_Y U_2) = \partial U_1 \times_Y U_2 \bigsqcup (-1)^{\delta_4} U_1 \times_Y \partial U_2,$$

where $\delta_4 = \dim U_1 + \dim Y$.

Then we find $\delta_1 \equiv \delta_2$ and $\delta_1 + \delta_3 + \delta_4 \equiv \dim X_1 + \dim Y$ modulo 2. Here we used the equality $\dim U_1 - \mathrm{rank} E_1 = \dim X_1$, which is the virtual dimension of

$X_1 = (s_1; E_1 \to U_1)$ as the space with Kuranishi structure. (See Definition A1.5.) Hence we obtain

$$\partial(X_1 \times_Y X_2) = \partial X_1 \times_Y X_2 \bigsqcup (-1)^{\dim X_1 + \dim Y} X_1 \times_Y \partial X_2.$$

(2) By Convention 45.1 (4), we have

$$\begin{aligned}(X_1 \times_{Y_1} X_2) \times_{Y_2} X_3 &= (-1)^{\delta_5} (s_1 \oplus s_2; E_1 \oplus E_2 \to U_1 \times_{Y_1} U_2) \times_{Y_2} X_3 \\ &= (-1)^{\delta_5 + \delta_6} \Big(s_1 \oplus s_2 \oplus s_3; E_1 \oplus E_2 \oplus E_3 \to (U_1 \times_{Y_1} U_2) \times_{Y_2} U_3 \Big),\end{aligned}$$

where

$$\delta_5 = \text{rank } E_2 (\dim X_1 - \dim Y_1)$$

and

$$\delta_6 = \text{rank } E_3 (\dim X_1 + \dim X_2 - \dim Y_1 - \dim Y_2).$$

Similarly, we have

$$\begin{aligned}X_1 \times_{Y_1} (X_2 \times_{Y_2} X_3) &= (-1)^{\delta_7} X_1 \times_{Y_1} (s_2 \oplus s_3; E_2 \oplus E_3 \to U_2 \times_{Y_2} U_3) \\ &= (-1)^{\delta_7 + \delta_8} \Big(s_1 \oplus s_2 \oplus s_3; E_1 \oplus E_2 \oplus E_3 \to U_1 \times_{Y_1} (U_2 \times_{Y_2} U_3) \Big),\end{aligned}$$

where

$$\delta_7 = \text{rank } E_3 (\dim X_2 - \dim Y_2)$$

and

$$\delta_8 = (\text{rank } E_2 + \text{rank } E_3)(\dim X_1 - \dim Y_1).$$

For oriented manifolds U_1, U_2, U_3, we showed that $(U_1 \times_{Y_1} U_2) \times_{Y_2} U_3 = U_1 \times_{Y_1} (U_2 \times_{Y_2} U_3)$. Since $\delta_5 + \delta_6 = \delta_7 + \delta_8$, we obtain

$$(X_1 \times_{Y_1} X_2) = X_1 \times_{Y_1} (X_2 \times_{Y_2} X_3).$$

(3) By Convention 45.1 (4), we have

$$\begin{aligned}X_1 \times_{Y_1 \times Y_2} (X_2 \times X_3) &= (-1)^{\delta_9} X_1 \times_{Y_1 \times Y_2} (s_2 \oplus s_3; E_2 \oplus E_3 \to U_2 \times U_3) \\ &= (-1)^{\delta_9 + \delta_{10}} \Big(s_1 \oplus s_2 \oplus s_3; E_1 \oplus E_2 \oplus E_3 \to U_1 \times_{Y_1 \times Y_2} (U_2 \times U_3) \Big),\end{aligned}$$

where

$$\delta_9 = \text{rank } E_3 \cdot \dim X_2$$

and

$$\delta_{10} = (\text{rank } E_2 + \text{rank } E_3)(\dim X_1 - \dim Y_1 - \dim Y_2).$$

On the other hand, we have

$$\begin{aligned}(X_1 \times_{Y_1} X_2) \times_{Y_2} X_3 &= (-1)^{\delta_{11}}(s_1 \oplus s_2; E_1 \oplus E_2 \to U_1 \times_{Y_1} U_2) \times_{Y_2} X_3 \\ &= (-1)^{\delta_{11}+\delta_{12}}\Big(s_1 \oplus s_2 \oplus s_3; E_1 \oplus E_2 \oplus E_3 \to (U_1 \times_{Y_1} U_2) \times_{Y_2} U_3\Big),\end{aligned}$$

where

$$\delta_{11} = \text{rank } E_2(\dim X_1 - \dim Y_1)$$

and

$$\delta_{12} = \text{rank } E_3(\dim X_1 + \dim X_2 - \dim Y_1 - \dim Y_2).$$

Recall that, for oriented manifolds U_1, U_2, U_3, we showed

$$U_1 \times_{Y_1 \times Y_2} (U_2 \times U_3) = (-1)^{\delta_{13}}(U_1 \times_{Y_1} U_2) \times_{Y_2} U_3,$$

where

$$\delta_{13} = \dim Y_2(\dim Y_1 + \dim U_2).$$

Since $\delta_9 + \delta_{10} + \delta_{11} + \delta_{12} + \delta_{13} \equiv \dim Y_2(\dim Y_1 + \dim X_2)$ modulo 2, we obtain

$$X_1 \times_{Y_1 \times Y_2} (X_2 \times X_3) = (-1)^{\dim Y_2(\dim Y_1 + \dim X_2)}(X_1 \times_{Y_1} X_2) \times_{Y_2} X_3.$$

(4) By Convention 45.1 (4), we defined the orientation of the fiber product $X_1 \times_Y X_2$ of spaces with Kuranishi structure depends only on the orientations of X_1, X_2 in the sense of Kuranishi structure and the orientation of Y. (In other words, it does not depend on the choice of local charts with orientation.) Thus (4) clearly holds. □

Remark 45.6. By using our expression of a fiber product, we can write the fiber product in a simple manner as follows. Let $f_k : X \to Y_k$ and $g_k : X_k \to Y_k$ $(k = 1, \dots, \ell)$. For $k = 1, \dots, \ell$, we put

$$X =: X^\circ \times \prod_{k=1}^{\ell} Y_k, \qquad X_k =: Y_k \times {}^\circ X_k.$$

Then the orientation on the fiber product is given by

$$X_{(f_1,\dots,f_\ell)} \times_{g_1 \times \cdots \times g_\ell} \left(\prod_{k=1}^{\ell} X_k\right) = X^\circ \times \prod_{k=1}^{\ell}(Y_k \times {}^\circ X_k). \tag{45.7}$$

To prove (45.7), we put

$$\prod_{k=1}^{\ell} X_k =: \left(\prod_{k=1}^{\ell} Y_k\right) \times {}^\circ \left(\prod_{k=1}^{\ell} X_k\right).$$

Then by definition of the orientation on the fiber product, we have

$$X_{(f_1,\dots,f_\ell)} \times_{g_1\times\cdots\times g_\ell} (\prod_{k=1}^{\ell} X_k) = X^\circ \times \left(\prod_{k=1}^{\ell} Y_k\right) \times {}^\circ \left(\prod_{k=1}^{\ell} X_k\right).$$

On the other hand, it is easy to see that

$${}^\circ\left(\prod_{k=1}^{\ell} X_k\right) = (-1)^{\delta_1} \prod_{k=1}^{\ell} {}^\circ X_k, \quad \text{where } \delta_1 = \sum_{k=2}^{\ell} y_k \sum_{j=1}^{k-1} (x_j - y_j).$$

Thus we have

$$\begin{aligned} X_{(f_1,\dots,f_\ell)} \times_{g_1\times\cdots\times g_\ell} \left(\prod_{k=1}^{\ell} X_k\right) &= (-1)^{\delta_1} X^\circ \times \left(\prod_{k=1}^{\ell} Y_k\right) \times \left(\prod_{k=1}^{\ell} {}^\circ X_k\right) \\ &= (-1)^{\delta_1+\delta_2} X^\circ \times \prod_{k=1}^{\ell} (Y_k \times {}^\circ X_k). \end{aligned}$$

But it is clear that $\delta_1 = \delta_2$, so we find the formula (45.7).

§46. Orientation on the moduli space of marked pseudo-holomorphic discs and on the singular strata of the moduli space.

In this section, we firstly give an orientation on the moduli space of marked pseudo-holomorphic discs and, then, we describe the orientation on the moduli space of pseudo-holomorphic maps from the union of 2 discs glued at one point, by regarding it as the boundary of the moduli space of (irreducible) pseudo-holomorphic discs.

First of all, we fix the orientation on $PSL(2;\mathbb{R})$ as follows.

Convention 46.1. Recall that we adopt our orientation convention for ∂D^2 by the counter-clock-wise orientation. We pick three distinct marked points z_0, z_1 and z_2 on ∂D^2, whose order respects the counter-clock-wise orientation of the boundary. We embed $PSL(2;\mathbb{R})$ in $\partial D^2 \times \partial D^2 \times \partial D^2$ by $g \mapsto (g\cdot z_0, g\cdot z_1 . g\cdot z_2)$ and orient $PSL(2;\mathbb{R})$ as an open subset of $\partial D^2 \times \partial D^2 \times \partial D^2$. If we fix the first two marked points z_0 and z_1, the common stabilizer of z_0 and z_1 is $\mathbb{R}$, whose orientation is given as follows. The orbit $t \mapsto t\cdot p$, where $p \neq z_0, z_1$, converges to z_0 as t tends to $+\infty$ and converges to z_1 as t tends to $-\infty$. To make the $PSL(2;\mathbb{R})$-action on ∂D^2 into the right action, we set $z \cdot g := g^{-1}z$. *Thus we adopt the opposite orientation on* $PSL(2;\mathbb{R})$ *and* $\mathbb{R}$.

Let $\widetilde{\mathcal{M}}^{\rm reg}(\beta) = \widetilde{\mathcal{M}}^{\rm reg}(L;\beta)$ be the space of pseudo-holomorphic maps from the unit disc representing the homotopy class $\beta \in \pi_2(M,L)$. (For simplicity of notations, we will often omit L.) In §46 and §47, we deal with all the components of $\widetilde{\mathcal{M}}^{\rm reg}(\beta)$. In §48, we restrict ourselves to the main component and give signs in the filtered A_∞-operations. Recall that relative spin structure determine an orientation on $\widetilde{\mathcal{M}}^{\rm reg}(\beta)$ by Theorem 44.1. We write

$$\mathcal{M}^{\rm reg}_{m+1}(\beta) = \Big(\widetilde{\mathcal{M}}^{\rm reg}(\beta) \times (\partial D^2)^{m+1}\Big)/PSL(2;\mathbb{R}).$$

We assign the point $+1 \in \partial D^2$ to the 0-th marked point z_0 and -1 to the first one z_1 and set

$$\widehat{\mathcal{M}}^{\rm reg}_{m+1}(\beta) = \widetilde{\mathcal{M}}(\beta) \times (\partial D^2)^{m-1},$$

which can be regarded as the space of pseudo-holomorphic maps from the unit disc representing the homotopy class β with $m+1$ marked points $(z_0,\cdots,z_m)$ on the boundary of the disc, such that $z_0 = 1$, $z_1 = -1$ are fixed.

Strictly speaking, we require all the marked points are distinct. So $\widehat{\mathcal{M}}^{\rm reg}_{m+1}(\beta)$ is not $\widetilde{\mathcal{M}}^{\rm reg}(\beta) \times (\partial D^2)^{m-1}$ itself but is its open subset. However when we consider the orientation problem, we often write it as $\widehat{\mathcal{M}}^{\rm reg}_{m+1}(\beta) = \widetilde{\mathcal{M}}^{\rm reg}(\beta) \times (\partial D^2)^{m-1}$ for simplicity.

Then $\mathcal{M}^{\rm reg}_{m+1}(\beta)$ is identified with $\widehat{\mathcal{M}}^{\rm reg}_{m+1}(\beta)/\operatorname{Aut}(D^2;z_0,z_1)$. Here $\operatorname{Aut}(D^2;z_0,z_1)$ is the biholomorphic automorphisms group of D^2 fixing z_0 and z_1.

Since $PSL(2;\mathbb{R})/\operatorname{Aut}(D^2;z_0,z_1)$ is even dimensional, the orientation of $\mathcal{M}^{\rm reg}_{m+1}(\beta)$ as the quotient by $PSL(2;\mathbb{R})$ coincides with one as the quotient by $\operatorname{Aut}(D^2;z_0,z_1)$, cf. (45.1.2). We denote by $\mathbb{R}_\beta$ the automorphism group $\operatorname{Aut}(D^2;z_0=+1,z_1=-1)$ acting on $\widehat{\mathcal{M}}^{\rm reg}_{m+1}(\beta)$. (Under our definition of the orientation on $PSL(2;\mathbb{R})$, which acts on D^2 from the right, orbits of this $\mathbb{R}$-action converge to -1 (resp. $+1$), as $t \in \mathbb{R}$ tends to $+\infty$ (resp. $-\infty$).) Note that $\dim \widehat{\mathcal{M}}^{\rm reg}_{m+1}(\beta) = n + \mu(\beta) + m - 1$.

Orientation on these spaces are defined as the product of oriented spaces and the quotient by the automorphism group (see Convention (45.1.2) in §45). So when we consider orientations, we shall simply write as

$$\widehat{\mathcal{M}}^{\rm reg}_{m+1}(\beta) = \mathcal{M}^{\rm reg}_{m+1}(\beta) \times \mathbb{R}_\beta, \tag{46.2}$$

and $\dim \mathcal{M}^{\rm reg}_{m+1}(\beta) = n + \mu(\beta) + m - 2 \equiv n + m \pmod 2$. Recall the Maslov index is even since we assume that Lagrangian submanifolds are oriented.

We denote by ev^β_j the evaluation map : $\mathcal{M}^{\rm reg}_{m+1}(\beta) \to L$ at the marked point z_j $(j = 0,\ldots,m)$.

Proposition 46.3. *We have an isomorphism*

$$\begin{aligned}&\partial \mathcal{M}^{\text{reg}}_{m+1}(\beta'+\beta'')\\ &= \bigcup (-1)^{(m_1-1)(m_2-1)+(n+m_1)} \mathcal{M}^{\text{reg}}_{m_1+1}(\beta')\,{}_{ev_1^{\beta'}}\times_{ev_0^{\beta''}} \mathcal{M}^{\text{reg}}_{m_2+1}(\beta''),\end{aligned}$$

where $n = \dim L$, as oriented spaces with Kuranishi structures. Here the union is taken over $\beta = \beta'+\beta''$ and $m = m_1+m_2-1$.

Remark 46.4. Here the order of marked points on the boundary of the unit disc is specified as follows. After gluing holomorphic discs, the first marked point $z_{1,\beta'}$ of the β'-disc is glued with the 0-th marked point $z_{0,\beta''}$ of the β''-disc and the marked points disappear after smoothing. After gluing, the 0-th marked point $z_{0,\beta'}$ of the β'-disc becomes the 0-th marked point $z_{0,\beta'+\beta''}$ of the $(\beta'+\beta'')$-disc. The rest of the marked points of β''-disc are numbered from 1 to m_2 and then the rest of the marked points of β'-disc are numbered from m_2+1 to $m = m_1+m_2-1$. Namely, $z_{j,\beta'+\beta''} = z_{j,\beta''}$ $(j = 1,\dots,m_2)$ and $z_{m_2+j,\beta'+\beta''} = z_{j+1,\beta'}$ $(j = 1,\dots,m_1-1)$. We write this convention as

$$\partial D^2_{2,\beta'+\beta''} \times \cdots \times \partial D^2_{m,\beta'+\beta''} = \partial D^2_{2,\beta''} \times \cdots \times \partial D^2_{m_2,\beta''} \times \partial D^2_{2,\beta'} \times \cdots \times \partial D^2_{m_1,\beta'},$$

or simply

$$(\partial D^2)^{m-1} = (\partial D^2)^{m_2-1} \times (\partial D^2)^{m_1-1}.$$

Figure 46.1

Proof of Proposition 46.3: Firstly we state the following.

Lemma 46.5. *The gluing map*

$$\widetilde{\mathcal{M}}^{\rm reg}(\beta')_{ev_1^{\beta'}} \times_{ev_0^{\beta''}} \widetilde{\mathcal{M}}^{\rm reg}(\beta'') \to \widetilde{\mathcal{M}}^{\rm reg}(\beta'+\beta'')$$

is orientation preserving in the sense of Kuranishi structure.

We will simply write

$$\widetilde{\mathcal{M}}^{\rm reg}(\beta'+\beta'') = \widetilde{\mathcal{M}}^{\rm reg}(\beta')_{ev_1^{\beta'}} \times_{ev_0^{\beta''}} \widetilde{\mathcal{M}}^{\rm reg}(\beta''),$$

when we consider the orientation problem.

The proof of Lemma 46.5 is given at the end of this section. Using this lemma, the proof of Proposition 46.3 goes as follows.

$$\begin{aligned}
&\widehat{\mathcal{M}}^{\rm reg}_{m+1}(\beta'+\beta'') = \widetilde{\mathcal{M}}^{\rm reg}(\beta'+\beta'') \times (\partial D^2)^{m-1}\\
=&(\widetilde{\mathcal{M}}^{\rm reg}(\beta')_{ev_1^{\beta'}} \times_{ev_0^{\beta''}} \widetilde{\mathcal{M}}^{\rm reg}(\beta'')) \times (\partial D^2)^{m-1}\\
=&(-1)^{(m_1-1)(m_2-1)}(\widetilde{\mathcal{M}}^{\rm reg}(\beta') \times (\partial D^2)^{m_1-1})_{ev_1^{\beta'}} \times_{ev_0^{\beta''}} (\widetilde{\mathcal{M}}^{\rm reg}(\beta'') \times (\partial D^2)^{m_2-1})\\
=&(-1)^{(m_1-1)(m_2-1)}(\mathcal{M}^{\rm reg}_{m_1+1}(\beta') \times \mathbb{R}_{\beta'})_{ev_1^{\beta'}} \times_{ev_0^{\beta''}} (\mathcal{M}^{\rm reg}_{m_2+1}(\beta'') \times \mathbb{R}_{\beta''}) \qquad \text{by (46.2)}\\
=&(-1)^{(m_1-1)(m_2-1)+(n+m_1)}(\mathbb{R}_{\beta'} \times \mathcal{M}^{\rm reg}{}_{m_1+1}(\beta'))_{ev_1^{\beta'}} \times_{ev_0^{\beta''}} (\mathcal{M}^{\rm reg}_{m_2+1}(\beta'') \times \mathbb{R}_{\beta''})\\
=&(-1)^{(m_1-1)(m_2-1)+(n+m_1)}\mathbb{R}_{\beta'} \times (\mathcal{M}^{\rm reg}_{m_1+1}(\beta')_{ev_1^{\beta'}} \times_{ev_0^{\beta''}} \mathcal{M}^{\rm reg}_{m_2+1}(\beta'')) \times \mathbb{R}_{\beta''}\\
=&(-1)^{(m_1-1)(m_2-1)+(n+m_1)}\mathbb{R}_{out} \times (\mathcal{M}^{\rm reg}_{m_1+1}(\beta')_{ev_1^{\beta'}} \times_{ev_0^{\beta''}} \mathcal{M}^{\rm reg}_{m_2+1}(\beta'')) \times \mathbb{R}_{\beta'+\beta''}.
\end{aligned}$$

Here we explain the proof of some of the equalities above. To prove the equality in the third line, we put

$$\widetilde{\mathcal{M}}^{\rm reg}(\beta') = \widetilde{\mathcal{M}}^{\rm reg}(\beta')^\circ \times L, \qquad \widetilde{\mathcal{M}}^{\rm reg}(\beta'') = L \times {}^\circ\widetilde{\mathcal{M}}^{\rm reg}(\beta'').$$

Then since

$$\widetilde{\mathcal{M}}^{\rm reg}(\beta') \times (\partial D^2)^{m_1-1} = (-1)^{(m_1-1)n}\widetilde{\mathcal{M}}^{\rm reg}(\beta')^\circ \times (\partial D^2)^{m_1-1} \times L,$$

we have

$$\begin{aligned}
&(\widetilde{\mathcal{M}}^{\rm reg}(\beta') \times (\partial D^2)^{m_1-1})_{ev_1^{\beta'}} \times_{ev_0^{\beta''}} (\widetilde{\mathcal{M}}^{\rm reg}(\beta'') \times (\partial D^2)^{m_2-1})\\
=&(-1)^{(m_1-1)n}\widetilde{\mathcal{M}}^{\rm reg}(\beta')^\circ \times (\partial D^2)^{m_1-1} \times L \times {}^\circ\widetilde{\mathcal{M}}^{\rm reg}(\beta'') \times (\partial D^2)^{m_2-1}\\
=&(-1)^{(m_1-1)n+(m_1-1)(n+m_2-1)}\widetilde{\mathcal{M}}^{\rm reg}(\beta')^\circ \times L \times {}^\circ\widetilde{\mathcal{M}}^{\rm reg}(\beta'') \times (\partial D^2)^{m_2-1} \times (\partial D^2)^{m_1-1}\\
=&(-1)^{(m_1-1)(m_2-1)}(\widetilde{\mathcal{M}}^{\rm reg}(\beta')_{ev_1^{\beta'}} \times_{ev_0^{\beta''}} \widetilde{\mathcal{M}}^{\rm reg}(\beta'')) \times (\partial D^2)^{m-1}.
\end{aligned}$$

In the last equality, $\mathbb{R}_{out}$ and $\mathbb{R}_{\beta'+\beta''}$ are anti-diagonal and diagonal oriented submanifolds in $\mathbb{R}_{\beta'} \times \mathbb{R}_{\beta''}$, i.e., $\mathbb{R}_{out}$ is generated by $(1,-1) \in \mathbb{R}_{\beta'} \times \mathbb{R}_{\beta''}$ and $\mathbb{R}_{\beta'+\beta''}$ is generated by $(1,1)$. So the orientation on $\mathbb{R}_{\beta'} \times \mathbb{R}_{\beta''}$ coincides with that on $\mathbb{R}_{out} \times \mathbb{R}_{\beta'+\beta''}$. The factor $\mathbb{R}_{out}$ can be regarded as the space of the gluing parameter of holomorphic discs and the $\mathbb{R}_{\beta'+\beta''}$ can be regarded as an $\mathbb{R}$-action on $\widetilde{\mathcal{M}}^{\rm reg}_{m+1}(\beta'+\beta'')$. (See Figure 46.2.) From our definition of orientations on the boundary and the quotient, we get Proposition 46.3. □

$(1,-1)$ $\mathbb{R}_{out}$, gluing

$(1,1)$ $\mathbb{R}_{\beta'+\beta''}$

Figure 46.2

Proof of Lemma 46.5: Let $u \in \widetilde{\mathcal{M}}^{\rm reg}(\beta')$ and $v \in \widetilde{\mathcal{M}}^{\rm reg}(\beta'')$ be holomorphic discs with $u(-1) = v(1)$. For a fixed sufficiently large $R > 0$, we glue u and v in $1/R$-neighborhoods of -1 and 1, respectively, using a partition of unity and implicit function theorem. (See §29.3.) Denote by $u\#_R v \in \widetilde{\mathcal{M}}^{\rm reg}(\beta'+\beta'')$ the glued holomorphic disc. A relative spin structure on L gives a stable trivialization of $(u|_{\partial D^2})^*TL$, $(u|_{\partial D^2})^*TL$ and $((u\#_R v|_{\partial D^2})^*TL$, which are compatible under the gluing. Hence Lemma 46.5 reduces to the linearized problem. Namely Lemma 46.5 follows from Lemma 46.10 below. □

To state Lemma 46.10 we need some notations. Let (E_1, F_1) and (E_2, F_2) be complex vector bundles over the unit disc with totally real subbundles on the boundary. (Namely it is a complex bundle pair in the sense of Definition 2.9.) We have Dolbeault operators :

$$\overline{\partial}_{E_i,F_i} : W^{1,p}(D^2, \partial D^2; E_i, F_i) \to L^p(D^2; E_i)$$

$(i = 1,2)$ which are Fredholm maps. For $p > 2$ and $z \in \partial D^2$ we define evaluation maps

$$ev_z : W^{1,p}(D^2, \partial D^2; E_i, F_i) \to F_i|_z$$

by

$$ev_z(\zeta) = \zeta(z).$$

Suppose we have an identification $F_1|_{-1} \cong F_2|_1$. We can use it to glue bundle pairs (E_1, F_1), (E_2, F_2) to obtain a complex bundle pair (E, F) on $(D^2, \partial D^2)$. (Here D^2 is obtained by gluing two copies D_1^2, D_2^2 of D^2 at $-1 \in D_1^2$ and $+1 \in D_2^2$.

We define the fiber product

$$\mathrm{Index}(\overline{\partial}_{E_1,F_1})_{ev_{-1}} \times_{ev_1} \mathrm{Index}(\overline{\partial}_{E_2,F_2})$$

of indices as follows. We take finite dimensional complex linear subspaces $\mathcal{E}_i \subset L^p(D^2; E_i)$ with the following properties.

(46.6) $\quad \mathrm{Im}(\overline{\partial}_{E_i,F_i}) + \mathcal{E}_i = L^p(D^2; E_i) \quad (i = 1, 2).$

(46.7.1) $\quad ev_1 : \overline{\partial}^{-1}_{E_1,F_1}(\mathcal{E}_1) \to F_1|_1$ is surjective.

(46.7.2) $\quad ev_{-1} : \overline{\partial}^{-1}_{E_2,F_2}(\mathcal{E}_2) \to F_2|_{-1}$ is surjective.

Definition 46.8. We put

$$\begin{aligned}
&\overline{\partial}^{-1}_{E_1,F_1}(\mathcal{E}_1)\,{}_{ev_1}\times_{ev_{-1}} \overline{\partial}^{-1}_{E_2,F_2}(\mathcal{E}_2)\\
&= \{(v_1, v_2) \in W^{1,p}(D^2, \partial D^2; E_1, F_1) \oplus W^{1,p}(D^2, \partial D^2; E_2, F_2)\\
&\qquad\qquad \mid \overline{\partial}_{E_i,F_i}(v_i) \in \mathcal{E}_i, (i = 1, 2), \quad ev_1(v_1) = ev_{-1}(v_2)\},
\end{aligned}$$

and define

$$\begin{aligned}
&\mathrm{Index}(\overline{\partial}_{E_1,F_1})_{ev_{-1}} \times_{ev_1} \mathrm{Index}(\overline{\partial}_{E_2,F_2})\\
&= \overline{\partial}^{-1}_{E_1,F_1}(\mathcal{E}_1)\,{}_{ev_1}\times_{ev_{-1}} (\overline{\partial}^{-1}_{E_2,F_2})(\mathcal{E}_2) - \mathcal{E}_1 - \mathcal{E}_2.
\end{aligned} \tag{46.9}$$

Here (46.9) is an equality of oriented virtual vector spaces. It is easy to see that the right hand side of (46.9) is independent of the choice $\mathcal{E}_i$.

Lemma 46.10. *There exists an isomorphism of oriented virtual vector spaces*

$$\mathrm{Index}(\overline{\partial}_{E,F}) = \mathrm{Index}(\overline{\partial}_{E_1,F_1})_{ev_{-1}} \times_{ev_1} \mathrm{Index}(\overline{\partial}_{E_2,F_2}).$$

Proof. First of all, we consider the case when (E_1, F_1) and (E_2, F_2) are both trivial, that is the case when F_1 and F_2 extend to totally real subbundles of E_1 and E_2 over the disc, respectively. In this case, $\mathrm{Index}(\overline{\partial}_{E,F})$ can be identified with the fiber $F|_*$ of F at $* \in \partial D^2$. Moreover, we may take $\mathcal{E}_i = 0$ and then (46.7.1) and (46.7.2) are orientation preserving isomorphisms. Thus Lemma 46.10 holds in this case.

In the general case , we use the (stable) trivialization of the totally real subbundle over the boundary to push down the bundle to the one-point union of the disc and the Riemann sphere in the same way as the proof of Proposition 44.4. Note that the points which are identified are interior points of the disc and the Riemann sphere. Therefore the fiber product is taken over a fiber of a complex vector bundle, which has complex orientation.(See Figure 46.3.)

Figure 46.3

In order to check that orientations on $\overline{\partial}^{-1}_{(E,F)}(\mathcal{E}_1 \oplus \mathcal{E}_2)$ and $\overline{\partial}^{-1}_{E_1,F_1}(\mathcal{E}_1)\,{}_{ev_1}\times_{ev_{-1}} \overline{\partial}^{-1}_{E_2,F_2}(\mathcal{E}_2)$ are compatible, it is enough to show that the following linear gluing problem : The real index of the Dolbeault operator on the genus 0 bordered Riemann surface with two interior double points (i.e., two copies of $\mathbb{C}P^1$ are attached at two interior points on the disc), is isomorphic to the real index of the Dolbeault operator on the glued Riemann sphere with the glued vector bundle.

Since they carry natural orientations coming from the complex structure, it is obvious that they are compatible. □

§47. The orientation on $\mathcal{M}_{\ell+1}(\beta; P_1, \dots, P_\ell)$.

§47.1. Definition of the orientation on $\mathcal{M}_{\ell+1}(\beta; P_1, \dots, P_\ell)$.

Let L be a relatively spin Lagrangian submanifold of (M, ω). Theorem 44.1 shows that the moduli space $\mathcal{M}^{\rm reg}_{\ell+1}(\beta)$ of pseudo-holomorphic maps from the unit disc representing the homotopy class $\beta \in \pi_2(M, L)$ with $(\ell+1)$-marked points on the boundary is oriented in a canonical way. When we consider the orientation problem, it is convenient to fix two marked points. Because the group of biholomorphic automorphisms of the disc fixing two points is $\mathbb{R}$, it can be interpreted as the gluing parameter of two holomorphic discs. (See §46.) In general, we can choose the fixed marked points arbitrarily. For example, if we choose any z_i and z_j, as those marked points we fix, then we can define an orientation on $\mathcal{M}^{\rm reg}_{\ell+1}(\beta)$ by the following equalities (*). In order to define A_∞ operations, it is enough to give a canonical orientation on the top dimensional stratum $\mathcal{M}^{\rm reg}_{\ell+1}(\beta; P_1, \ldots, P_\ell)$ of $\mathcal{M}_{\ell+1}(\beta; P_1, \ldots, P_\ell)$. Thus, from now on, we simply denote by $\mathcal{M}_{\ell+1}(\beta)$ and $\mathcal{M}_{\ell+1}(\beta; P_1, \ldots, P_\ell)$ their top dimensional strata.

$$(*) \qquad \begin{cases} \widehat{\mathcal{M}}_{\ell+1}(\beta) := \widetilde{\mathcal{M}}(\beta) \times \partial D^2_0 \times \cdots \times \partial \check{D}^2_i \times \cdots \times \partial \check{D}^2_j \times \cdots \times \partial D^2_\ell, \\ \mathcal{M}_{\ell+1}(\beta) := \widehat{\mathcal{M}}_{\ell+1}(\beta)/\mathbb{R}. \end{cases}$$

If we change the order of marked points by an element σ of $(\ell+1)$-th symmetric group $\mathfrak{S}_{\ell+1}$, then it exchanges connected components of the moduli space. This diffeomorphism is orientation preserving if and only if σ is even permutation. (See Proposition 2.22.) *Hereafter we chose the* 0*-th marked point* z_0 *and the first marked point* z_1*, as the marked points we fix.* This convention is the same as in §46. Note in the case we choose any other z_i, z_j, the orientation we obtain on $\mathcal{M}_{\ell+1}(\beta)$ by this choice can be compared to the case we choose z_0, z_1, by using the action of $\mathfrak{S}_{\ell+1}$ mentioned above.

We note that, when we consider the moduli space $\mathcal{M}_1(\beta)$ with only one marked point, we add another *second* marked point arbitrarily and we follow the convention explained above to define an orientation on $\mathcal{M}_2(\beta)$. After that, we consider the map $\mathcal{M}_2(\beta) \to \mathcal{M}_1(\beta)$ of forgetting the second marked point. Since the fiber of this map is homeomorphic to an interval which has a canonical orientation determined by ∂D^2, we have an orientation on $\mathcal{M}_1(\beta)$.

Now let $[P_k, f_k]$ be a smooth singular simplex in L with $\dim P_k = p_k$. We put $\deg P_k = n - p_k$, which is the degree as a *cochain*. By Proposition 29.1, $ev : \mathcal{M}_{\ell+1}(\beta) \to L^{\ell+1}$ is weakly submersive. Therefore the fiber product in Definition 47.1 below is "transversal" in the sense of Convention 45.1 (4). We define a space $\mathcal{M}_{\ell+1}(\beta; P_1, \ldots, P_\ell)$ with Kuranishi structure as the following;

Definition 47.1.

$$\mathcal{M}_{\ell+1}(\beta; P_1, \ldots, P_\ell) := (-1)^{\epsilon_1} \mathcal{M}_{\ell+1}(\beta)_{(ev_1, \ldots, ev_\ell)} \times_{f_1 \times \cdots \times f_\ell} \left(\prod_{k=1}^{\ell} P_k \right),$$

where

$$\epsilon_1 = (n+1) \sum_{k=1}^{\ell-1} \sum_{j=1}^{k} \deg P_j.$$

Remark 47.2. (1) Using the iteration formula in Lemma 45.3 (3) or the formula (45.7) in Remark 45.6, we can rewrite the right hand side as

$$\begin{aligned}&\mathcal{M}_{\ell+1}(\beta; P_1, \dots, P_\ell)\\ =&(-1)^{\sum_{k=1}^{\ell-1}\sum_{j=1}^{k}\deg P_j}\left(\cdots((\mathcal{M}_{\ell+1}(\beta)_{ev_1}\times_{f_1} P_1)_{ev_2}\times_{f_2} P_2)\times\cdots\right)_{ev_\ell}\times_{f_\ell} P_\ell.\end{aligned}$$

(2). If all P_k are bounding chains $\mathcal{B}(L;\beta_k)$ with $\beta_k \in \pi_2(M,L)$ (see §5.6), then we have $\deg \mathcal{B}(L;\beta_k) \equiv 1 \pmod 2$. In this case, ϵ_1 is given by

$$\frac{(n+1)\ell(\ell-1)}{2}.$$

When we change the ordering of marked points, we can find the difference of orientations by the following key lemma, which plays a fundamental role in our argument later.

Lemma 47.3. *Let σ be the transposition element $(i, i+1)$ in the ℓ-th symmetric group $\mathfrak{S}_\ell$. ($i = 1, \dots, \ell-1$). Then the action of σ on $\mathcal{M}_{\ell+1}(\beta; P_1, \dots, P_i, P_{i+1}, \dots, P_\ell)$ by changing the order of marked points is described by the following.*

$$\begin{aligned}&\sigma(\mathcal{M}_{\ell+1}(\beta; P_1, \dots, P_i, P_{i+1}, \dots, P_\ell))\\ =&(-1)^{(\deg P_i+1)(\deg P_{i+1}+1)}\mathcal{M}_{\ell+1}(\beta; P_1, \dots, P_{i+1}, P_i, \dots, P_\ell).\end{aligned}$$

Proof: By definition we have

$$\begin{aligned}&\mathcal{M}_{\ell+1}(\beta; P_1, \dots, P_i, P_{i+1}, \dots, P_\ell)\\ &= (-1)^{\epsilon_1}\mathcal{M}_{\ell+1}(\beta)_{(ev_1,\dots,ev_i,ev_{i+1},\dots,ev_\ell)}\times_{f_1\times\cdots f_i\times f_{i+1}\cdots\times f_\ell}\left(\prod_{k=1}^{\ell} P_k\right),\end{aligned} \tag{47.4.1}$$

and

$$\begin{aligned}&\mathcal{M}_{\ell+1}(\beta; P_1, \dots, P_{i+1}, P_i, \dots, P_\ell)\\ &= (-1)^{\epsilon_2}\mathcal{M}_{\ell+1}(\beta)_{(ev_1,\dots,ev_i,ev_{i+1},\dots,ev_\ell)}\times_{f_1\times\cdots f_{i+1}\times f_i\cdots\times f_\ell}\\ &\qquad\left(\prod_{k=1}^{i-1} P_k \times P_{i+1}\times P_i \times \prod_{k=i+2}^{\ell} P_k\right),\end{aligned} \tag{47.4.2}$$

where ϵ_1 is the same as in Definition 47.1 and ϵ_2 is given by

$$\epsilon_2 = \epsilon_1 - (n+1)(\deg P_i - \deg P_{i+1}).$$

To compute the difference between the fiber product orientations on the right hand sides above, we recall Lemma 45.3 (4). The element σ acts on $\mathcal{M}_{\ell+1}(\beta)$ by changing the order of marked points. Thus σ induces a (-1)-oriented isomorphism on $\mathcal{M}_{\ell+1}(\beta)$. Clearly we have

$$\prod_{k=1}^{\ell} P_k = (-1)^{p_i p_{i+1}} \prod_{k=1}^{i-1} P_k \times P_{i+1} \times P_i \times \prod_{k=i+2}^{\ell} P_k.$$

Here $p_i = \dim P_i$ and $p_{i+1} = \dim P_{i+1}$. Moreover, as for the orientation of base spaces of the fiber products, we have

$$\prod_{k=1}^{\ell} L_k = (-1)^{n} \prod_{k=1}^{i-1} L_k \times L_{i+1} \times L_i \times \prod_{k=i+2}^{\ell} L_k.$$

Here L_k is a copy of L. Therefore by Lemma 45.3 (4), we can find that

(47.5)

$$\begin{aligned}
&\mathcal{M}_{\ell+1}(\beta)_{(ev_1,\dots,ev_i,ev_{i+1},\dots,ev_\ell)} \times_{f_1\times\cdots f_i\times f_{i+1}\cdots\times f_\ell} \left(\prod_{k=1}^{\ell} P_k\right)\\
&= (-1)^{1+p_ip_{i+1}+n}\mathcal{M}_{\ell+1}(\beta)\\
&\quad {}_{(ev_1,\dots,ev_i,ev_{i+1},\dots,ev_\ell)} \times_{f_1\times\cdots f_{i+1}\times f_i\cdots\times f_\ell} \left(\prod_{k=1}^{i-1} P_k \times P_{i+1} \times P_i \times \prod_{k=i+2}^{\ell} P_k\right)
\end{aligned}$$

By combining (47.4) and (47.5), we can see that

$$\epsilon_1 + \epsilon_2 + 1 + p_ip_{i+1} + n \equiv (\deg P_i + 1)(\deg P_{i+1} + 1) \pmod 2,$$

which proves Lemma 47.3. □

§47.2. Anti-symplectic involution and orientation.

In this subsection we will prove Proposition 38.7 and Lemma 38.17 in Chapter 8. We briefly recall the situation in §38. Let $\tau : (M,\omega) \to (M,\omega)$ be an anti-symplectic involution on a compact symplectic manifold (M,ω). Assume that $L = \mathrm{Fix}\,\tau$ is nonempty. We denote by $\mathcal{J}_\omega^\tau$ the set of all τ-anti-invariant compatible almost complex structures. Pick $J \in \mathcal{J}_\omega^\tau$. For a J holomorphic curve $w : (D^2, \partial D^2) \to (M, L)$, we define $\widetilde{w}$ by

$$\widetilde{w}(z) = (\tau \circ w)(\overline{z}).$$

Moreover for $[(D^2,w)] \in \mathcal{M}^{\rm reg}(J;\beta)$, $[((D^2,\vec z,\vec z^+),w)] \in \mathcal{M}^{\rm reg}_{k+1,m}(J;\beta)$ we define

$$\tau_*([(D^2,w)]) = [(D^2,\widetilde w)], \qquad \tau_*([((D^2,\vec z,\vec z^+),w)]) = [((D^2,\vec{\overline z},\vec{\overline z}^+),\widetilde w)],$$

where

$$\vec{\overline z} = (\overline z_0,\cdots,\overline z_k), \qquad \vec{\overline z}^+ = (\overline z_0^+,\cdots,\overline z_m^+).$$

Then by Lemma 38.6 τ induces the maps

$$\tau_* : \mathcal{M}^{\rm reg}(J;\beta) \to \mathcal{M}^{\rm reg}(J;\beta), \quad \tau_* : \mathcal{M}^{\rm reg}_{k+1,m}(J;\beta) \to \mathcal{M}^{\rm reg}_{k+1,m}(J;\beta)$$

for $\beta \in \Pi(L) = \pi_2(M,L)/\sim$. Then Proposition 38.7 is stated as follows.

Proposition 38.7. *Let $J \in \mathcal{J}_\omega^\tau$. The map $\tau_* : \mathcal{M}^{\rm reg}(J;\beta) \to \mathcal{M}^{\rm reg}(J;\beta)$ induces an involution of the space with Kuranishi structure. It is orientation preserving if $\mu_L(\beta) \equiv 0 \mod 4$ and is orientation reversing if $\mu_L(\beta) \equiv 2 \mod 4$.*

Proof. Let $[D^2,w] \in \mathcal{M}(J;\beta)$. We consider the deformation complex

$$D_w\overline\partial : \Gamma(D^2,\partial D^2 : w^*TM, w|_{\partial D^2}^*TL) \to \Gamma(D^2;\Lambda^{0,1}\otimes w^*TM), \tag{47.6.1}$$

and

$$D_{\widetilde w}\overline\partial : \Gamma(D^2,\partial D^2 : \widetilde w^*TM, \widetilde w|_{\partial D^2}^*TL) \to \Gamma(D^2;\Lambda^{0,1}\otimes \widetilde w^*TM). \tag{47.6.2}$$

(Here and hereafter, $\Lambda^1 = \Lambda^{1,0}\oplus\Lambda^{0,1}$ is the decomposition of the complexified cotangent bundle of the *domain* (that is D^2 or S^2).)

We have the commutative diagram of bundle pairs

$$\begin{array}{ccc} (w^*TM, w|_{\partial D^2}^*TL) & \xrightarrow{T\tau} & (\widetilde w^*TM, \widetilde w|_{\partial D^2}^*TL) \\ \downarrow & & \downarrow \\ (D^2,\partial D^2) & \xrightarrow{c} & (D^2,\partial D^2) \end{array}$$

Diagram 47.1.

where $c(z) = \overline z$ and we denote by $T\tau$ the differential of τ. It induces a bundle map

$$\mathrm{Hom}_{\mathbb R}(TD^2, w^*TM) \to \mathrm{Hom}_{\mathbb R}(TD^2, \widetilde w^*TM), \tag{47.7}$$

which covers $z \mapsto \overline z$. The map (47.7) is anti-complex linear. Therefore it preserves the decomposition

$$\mathrm{Hom}_{\mathbb R}(TD^2, w^*TM)\otimes\mathbb C = (\Lambda^{1,0}\otimes w^*TM)\oplus(\Lambda^{0,1}\otimes w^*TM), \tag{47.8}$$

since (47.8) is the decomposition to the complex and anti-complex linear parts. Hence we obtain a map

$$T_{w,1}\tau_* : \Gamma(D^2; \Lambda^{0,1} \otimes w^*TM) \to \Gamma(D^2; \Lambda^{0,1} \otimes \widetilde{w}^*TM) \tag{47.9}$$

which is anti-complex linear. In the similar way, we obtain an anti-complex linear map :

$$T_{w,0}\tau_* : \Gamma(D^2, \partial D^2 : w^*TM, w|^*_{\partial D^2}TL) \to \Gamma(D^2, \partial D^2 : \widetilde{w}^*TM, \widetilde{w}|^*_{\partial D^2}TL). \tag{47.10}$$

Since τ is an isometry, it commutes with the covariant derivative. This gives rise to the following commutative diagram.

$$\begin{array}{ccc}
\Gamma(D^2, \partial D^2 : w^*TM, w|^*_{\partial D^2}TL) & \xrightarrow{D_w\overline{\partial}} & \Gamma(D^2; \Lambda^{0,1} \otimes w^*TM) \\
{\scriptstyle T_{w,0}\tau_*}\downarrow & & {\scriptstyle T_{w,1}\tau_*}\downarrow \\
\Gamma(D^2, \partial D^2 : \widetilde{w}^*TM, \widetilde{w}|^*_{\partial D^2}TL) & \xrightarrow{D_{\widetilde{w}}\overline{\partial}} & \Gamma(D^2; \Lambda^{0,1} \otimes \widetilde{w}^*TM)
\end{array}$$

Diagram 47.2.

To define a Kuranishi chart in a neighborhood of $[D^2, w]$ we need to take a finite dimensional subspace $E_{[D^2,w]}$ of $\Gamma(D^2; \Lambda^{0,1} \otimes w^*TM)$ such that

$$\operatorname{Im} D_w\overline{\partial} + E_{[D^2,w]} = \Gamma(D^2; \Lambda^{0,1} \otimes w^*TM).$$

We choose $E_{[D^2,w]}$ so that it is invariant under $T_{w,1}\tau_*$, i.e.,

$$E_{[D^2,\widetilde{w}]} = T_{w,1}\tau_*(E_{[D^2,w]}). \tag{47.11}$$

Let $w' : (D^2, \partial D^2) \to (M, L)$ be a map C^0 close to w. By definition it is easy to see that

$$\overline{\partial}\widetilde{w}' = (T_{w',0}\tau_*)(\overline{\partial}w'). \tag{47.12}$$

We may take an isomorphism

$$I_{w,w'} : \Gamma(D^2, \partial D^2 : w^*TM, w|^*_{\partial D^2}TL) \cong \Gamma(D^2, \partial D^2 : (w')^*TM, w'|^*_{\partial D^2}TL)$$

so that it is complex linear and satisfies

$$T_{w',0}\tau_* \circ I_{w,w'} = I_{\widetilde{w},\widetilde{w}'} \circ T_{w,0}\tau_*. \tag{47.13}$$

Now a Kuranishi neighborhood $V_{[D^2,w]}$ was defined in §29 so that it is the set of solutions of the equation

$$\overline{\partial}\widetilde{w}' \equiv 0 \mod I_{w,w'}(E_{[D^2,w]}). \tag{47.14}$$

Hence by (47.13) $w' \mapsto \widetilde{w}'$ defines a diffeomorphism

$$\tau_* : V_{[D^2,w]} \cong V_{[D^2,\widetilde{w}]}.$$

Moreover the Kuranishi map $w' \mapsto s(w') = \overline{\partial}w'$ commutes with τ_*. Hence τ_* induces an involution of the Kuranishi structure.

We next study the orientation. Let $w \in \widetilde{\mathcal{M}}(J;\beta)$ and $\widetilde{w} \in \widetilde{\mathcal{M}}(J;\beta)$ be the corresponding element. We consider commutative Diagram 47.1. A trivialization

$$\Phi : (w^*TM, w|_{\partial D^2}^*TL) \to (D^2, \partial D^2; \mathbb{C}^n, \Lambda)$$

naturally induces a trivialization

$$\widetilde{\Phi} : (\widetilde{w}^*TM, \widetilde{w}|_{\partial D^2}^*TL) \to (D^2, \partial D^2; \mathbb{C}^n, \widetilde{\Lambda}),$$

where $\Lambda : S^1 \simeq \partial D^2 \to \Lambda(\mathbb{C}^n)$ is a loop of Lagrangian subspaces given by $\Lambda(z) := T_{w(z)}L$ in the trivialization and $\widetilde{\Lambda}$ is defined by

$$\widetilde{\Lambda}(z) = \Lambda(\overline{z}).$$

With respect to these trivializations, we have the commutative diagram

$$\begin{array}{ccc} (D^2, \partial D^2; \mathbb{C}^n, \Lambda) & \overset{\widetilde{\Phi}\circ T\tau\circ\Phi^{-1}}{\longrightarrow} & (D^2, \partial D^2; \mathbb{C}^n, \widetilde{\Lambda}) \\ \downarrow & & \downarrow \\ (D^2, \partial D^2) & \overset{c}{\longrightarrow} & (D^2, \partial D^2) \end{array}$$

Diagram 47.3.

and the elliptic complex (47.6). Note that the map

$$\tau_z := \widetilde{\Phi} \circ T\tau \circ \Phi^{-1}(z, \cdot) : (\mathbb{C}^n, \Lambda(z)) \to (\mathbb{C}^n, \Lambda(z))$$

defines an involution with the Lagrangian subspace $\Lambda(z)$ fixed. Now we deform the metric on D^2 and the trivialization Φ so that $\Lambda(z) \equiv \mathbb{R}^n$ as in the proof of Proposition 44.4 in this chapter. Recall that we assume L is orientable and so the bundle $w|_{\partial D^2}^*TL \to S^1$ is trivial. After deforming further the Cauchy-Riemann operator on $(D^2, \partial D^2; \mathbb{C}^n, \Lambda)$, we are reduced to considering the case

$$\begin{aligned} C : \text{Hol}\ (D^2, \partial D^2 : \mathbb{C}^n, \mathbb{R}^n) \times \text{Hol}\ (\mathbb{C}P^1 : E) \\ \to \text{Hol}\ (D^2, \partial D^2 : \mathbb{C}^n, \mathbb{R}^n) \times \text{Hol}\ (\mathbb{C}P^1 : E) \end{aligned}$$

where E is a holomorphic vector bundle whose topology is determined by Λ (see the proof of Proposition 44.4) and C is the natural map induced from the conjugation on $\mathbb{C}^n$ and E. In particular, we have $\dim_{\mathbb{C}} \mathrm{Hol}\ (\mathbb{C}P^1 : E) = \frac{1}{2}\mu(\Lambda)$. The first factor is invariant under the conjugation because $\mathrm{Hol}\ (D^2, \partial D^2 : \mathbb{C}^n, \mathbb{R}^n) \simeq \mathbb{R}^n$. For the second factor, we have $\mathrm{Hol}\ (\mathbb{C}P^1 : E) \simeq \mathbb{C}^{\frac{1}{2}\mu(\Lambda)}$ with C is reduced to the standard conjugation on $\mathbb{C}^{\frac{1}{2}\mu(\Lambda)}$. Therefore it boils down to considering the conjugation

$$C : \mathbb{C}^{\frac{1}{2}\mu(\Lambda)} \to \mathbb{C}^{\frac{1}{2}\mu(\Lambda)}.$$

It is easy to see that this map is orientation preserving if and only if $\frac{1}{2}\mu(\Lambda) \equiv 0 \mod 2$, i.e., $\mu(\Lambda) \equiv 0 \mod 4$. This finishes the proof. □

We next prove Lemma 38.17. By considering the assignment given in (38.13)

$$(w, (z_1, z_2, \cdots, z_{k-1}, z_k, z_0)) \mapsto (\widetilde{w}, (\overline{z}_k, \overline{z}_{k-1}, \cdots, \overline{z}_2, \overline{z}_1, \overline{z}_0)),$$

an anti-symplectic involution τ induces

$$\tau_*^{\mathrm{main}} : \mathcal{M}_{k+1}^{\mathrm{main}}(J;\beta) \to \mathcal{M}_{k+1}^{\mathrm{main}}(J;\beta)$$

which is an involution of the space with Kuranishi structure. See Lemma 38.14. Let $P_1, \cdots, P_k$ be smooth singular simplexes on L. Then τ_*^{main} induces the involution

$$\tau_*^{\mathrm{main}} : \mathcal{M}_{k+1}^{\mathrm{main}}(J;\beta;P_1,\cdots,P_k) \to \mathcal{M}_{k+1}^{\mathrm{main}}(J;\beta;P_k,\cdots,P_1). \tag{47.15}$$

See (38.16). We put

$$\epsilon = \mu_L(\beta)/2 + k + 1 + \sum_{1\le i<j\le k} \deg' P_i \deg' P_j.$$

Here $\deg'$ is the shifted degree. Then by using Lemma 47.3 in the previous subsection, we can show the following.

Lemma 38.17. *The involution* (47.15) *preserves orientation if ϵ is even, and reverses orientation if ϵ is odd.*

Proof. By Proposition 38.7, $\tau_* : \mathcal{M}^{\mathrm{reg}}(J;\beta) \to \mathcal{M}^{\mathrm{reg}}(J;\beta)$ is orientation preserving if and only if $\mu_L(\beta)/2$ is even. By the involution τ_*, each boundary marked point z_i is mapped to $\overline{z}_i$. Denote by $\mathcal{M}'^{\mathrm{reg}}_{k+1}(J;\beta)$ the moduli space with the boundary marked points $(z_0, z_1, \ldots, z_k)$ respect the clock-wise orientation. Since $z \mapsto \overline{z}$ reverses the orientation on the boundary, $\tau_* : \mathcal{M}^{\mathrm{reg}}_{k+1}(J;\beta) \to \mathcal{M}'^{\mathrm{reg}}_{k+1}(J;\beta)$ respects the orientation if and only if $\mu_L(\beta)/2 + k + 1$ is even. Thus we have

$$\mathcal{M}^{\mathrm{reg}}_{k+1}(\beta;P_1,\ldots,P_k) = (-1)^{\mu_L(\beta)/2+k+1}\mathcal{M}'^{\mathrm{reg}}_{k+1}(\beta;P_1,\ldots,P_k).$$

Combining Lemma 47.3, we obtain Lemma 38.17. □

§47.3. Cyclic symmetry and orientation.

In this subsection, we describe the behavior of the orientation under the cyclic symmetry and complete the proof of Proposition 37.27. First of all, we introduce a version of the intersection pairing $\langle , \rangle$ used in (37.26).

Definition 47.16. *For a pair P_1, P_2 of smooth singular simplexes, which are transversal and of complementary dimension, we define*

$$\langle P_1, P_2 \rangle := \#(P_1 \times_L P_2).$$

Remark 47.17. (1) Let N_i be the normal bundle of P_i in L. Extend N_i to a tubular neighborhood of P_i, $i = 1, 2$. Then P_i is written as the zero locus of the tautological section s_i of E_i on the tubular neighborhood. Then by Convention 45.1 (4), we find that

$$\begin{aligned} P_1 \times_L P_2 &= (s_1; N_1 \to L) \times_L (s_2; N_2 \to L) \\ &= (-1)^{\deg P_2 \cdot \deg P_1} (s_1 \oplus s_2)^{-1}(0). \end{aligned}$$

Taking Convention 49.1 into account, we obtain

$$\langle P_1, P_2 \rangle = (-1)^{\deg P_1 \cdot \deg P_2} \#(P_1 \cap P_2).$$

Regard P_1, P_2 as currents $T(P_1), T(P_2)$, respectively. Since we assumed that P_1 and P_2 intersects transversally, the product $T(P_1) \wedge T(P_2)$ is defined. Then the above observation is rephrased as

$$\langle P_1, P_2 \rangle = (-1)^{\deg P_1 \cdot \deg P_2} \int_L T(P_1) \wedge T(P_2) = \int_L T(P_2) \wedge T(P_1).$$

(2) In order to define the pairing on $C^\bullet(L)$, we take the intersection number after perturbation in a similar way to the A_∞-structure.

Proposition 37.27 is a direct consequence of the following

Proposition 37.27'.

$$\langle P_0, \mathfrak{m}_{k,\beta}(P_1, \ldots, P_k) \rangle = (-1)^{\deg' P_k (\deg' P_0 + \cdots + \deg' P_{k-1})} \langle P_k, \mathfrak{m}_{k,\beta}(P_0, \ldots, P_{k-1}) \rangle.$$

As in Lemma 47.3, we use the moduli space of bordered stable maps, which is not the main component, i.e., the marked points do not respect the canonical cyclic ordering on the boundary. Here we apply the permutation of the zero-th and first marked points. Let $\mathcal{M}'_{k+1}(\beta)$ be the moduli space with $(z_1, z_0, z_2, \ldots, z_k)$ respects the counter clockwise orientation and write

$$\mathcal{M}'_{k+1}(\beta; P_0, P_2, \ldots, P_k) = (-1)^{\epsilon'} \mathcal{M}'_{k+1}(\beta) \times_{L\times\cdots\times L} (P_0 \times P_2 \times \ldots P_k),$$

where

$$\epsilon' = (n+1)\Big((k-1)\deg P_0 + \sum_{j=2}^{k-1}\sum_{i=2}^{j} \deg P_j\Big),$$

see Definition 47.1.

Lemma 47.18.

$$P_0 \times_L \mathcal{M}_{k+1}(\beta; P_1, P_2, \ldots, P_k) = (-1)^{(\deg P_0+1)(\deg P_1+1)} P_1 \times_L \mathcal{M}'_{k+1}(\beta; P_0, P_2, \ldots, P_k).$$

Proof. Denote by L_i the target of the evaluation map corresponding to P_i and write

$$\mathcal{M}_{k+1}(\beta) = L_0 \times {}^\circ\mathcal{M}_{k+1}(\beta)^\circ \times L_1 \times L_2 \cdots \times L_k,$$

$$\mathcal{M}'_{k+1}(\beta) = L_1 \times {}^\circ\mathcal{M}'_{k+1}(\beta)^\circ \times L_0 \times L_2 \times \ldots L_k.$$

Note that $\mathcal{M}_{k+1}(\beta) = -\mathcal{M}'_{k+1}(\beta)$, since the first two marked points are exchanged and

$${}^\circ\mathcal{M}_{k+1}(\beta)'^\circ = (-1)^{\gamma}\,{}^\circ\mathcal{M}_{k+1}(\beta)^\circ,$$

where $\gamma = (\dim L)^2 + 1 \equiv n+1 \mod 2$. Taking (45.7) into account, by Definition 47.1, we have

$$\mathcal{M}_{k+1}(\beta; P_1, \ldots P_k) = (-1)^{\epsilon} L_0 \times {}^\circ\mathcal{M}_{k+1}(\beta)^\circ \times L_1 \times {}^\circ P_1 \times \cdots \times L_k \times {}^\circ P_k,$$

where

$$\epsilon = (n+1)\sum_{j=1}^{k-1}\sum_{i=1}^{j} \deg P_i$$

and

$$\mathcal{M}'_{k+1}(\beta; P_0, P_2, \ldots P_k) = (-1)^{\epsilon'} L_1 \times {}^\circ\mathcal{M}'_{k+1}(\beta)^\circ \times L_0 \times {}^\circ P_0 \times P_2 \times {}^\circ P_2 \times \cdots \times L_k \times {}^\circ P_k.$$

Then we have

$$\begin{aligned}
&P_0 \times_L \mathcal{M}_{k+1}(\beta; P_1, P_2, \ldots, P_k)\\
=&(-1)^{\epsilon}(P_0^\circ \times L_0) \times {}^\circ\mathcal{M}_{k+1}(\beta)^\circ \times (L_1 \times {}^\circ P_1) \times (L_2 \times {}^\circ P_2) \times \cdots \times (L_k \times {}^\circ P_k)\\
=&(-1)^{\epsilon+\delta}(L_1 \times {}^\circ P_1) \times {}^\circ\mathcal{M}_{k+1}(\beta)^\circ \times (P_0^\circ \times L_0) \times (L_2 \times {}^\circ P_2) \times \cdots \times (L_k \times {}^\circ P_k)\\
=&(-1)^{\epsilon+\delta+\gamma}(P_1^\circ \times L_1) \times {}^\circ\mathcal{M}'_{k+1}(\beta)^\circ \times (L_0 \times {}^\circ P_0) \times (L_2 \times {}^\circ P_2) \times \cdots \times (L_k \times {}^\circ P_k)\\
=&(-1)^{\epsilon+\delta+\gamma+\epsilon'} P_1 \times_L \mathcal{M}'_{k+1}(\beta; P_0, P_2, \ldots, P_k),
\end{aligned}$$

where

$$\delta = p_0p_1 + (n+1)(p_0+p_1)k, \quad (p_i = \dim P_i).$$

In the second equality, we note that

$$\dim {}^\circ\mathcal{M}_{k+1}(\beta)^\circ = n + \mu(\beta) + k + 1 - 3 - k(n+1) \equiv (n+1)k \mod 2.$$

We used

$$L \times {}^\circ P = P = L \times P^\circ$$

in the third equality. Noting that

$$\epsilon + \epsilon' \equiv (n+1)(k+1)(p_0+p_1) \mod 2,$$

we have

$$\epsilon + \delta + \gamma + \epsilon' \equiv (n+1+p_0)(n+1+p_1) \equiv (\deg P_0 + 1)(\deg P_1 + 1) \mod 2.$$

Hence the proof of Lemma 47.18 is complete. □

Proof of Proposition 37.27'. Proposition 37.27' follows from Lemma 47.3 and Lemma 47.18. □

§48. The filtered A_∞ algebra case.

We recall that, when we study the structure of the filtered A_∞ algebra associated to a Lagrangian submanifold L, we use the main component of the moduli space. (See §2.2 for the definition of the main component.) So we choose the order of the marked points on the boundary so that it is consistent with the counter clockwise orientation of the boundary. Let $[P_i, f_i] \in C^{g_i}(L, \Lambda_{0,nov})$.

By Definition 47.1 we have :

$$\mathcal{M}^{\text{main}}_{k+1}(\beta; P_1, \dots, P_k) := (-1)^{\epsilon_1}\mathcal{M}^{\text{main}}_{k+1}(L;\beta)_{(ev_1,\dots,ev_k)} \times_{f_1\times\dots\times f_k} \left(\prod_{i=1}^k P_i\right),$$

where

$$\epsilon_1 = (n+1)\sum_{j=1}^{k-1}\sum_{i=1}^{j} \deg P_i.$$

We will check signs and complete the proof of Theorem 10.11, which states that $(C(L;\Lambda_{0,nov}),\mathfrak{m})$ is a filtered A_∞ algebra. There are two points where we need to check signs. The first one is in the proof of $\widehat{d}\circ\widehat{d}=0$, and the second one is in the comparison of our orientation on $(\mathcal{M}_3^{\text{main}}(L,\beta_0;P_1,P_2),ev_0)$ with the geometric orientation on $P_1\cap P_2$. (Note $\beta_0=0$.) We will check the first part in this subsection and the second one in the next subsection. To prove (10.17.1) and (10.17.2), we need the following Proposition 48.1 (2) and (1), respectively.

Proposition 48.1. (1) *For $\beta=\beta_1+\beta_2$, we have*

$$\begin{aligned}&\mathcal{M}^{\text{main}}_{k-k_2+2}(\beta_1;P_1,\dots,P_{i-1},\mathcal{M}^{\text{main}}_{k_2+1}(\beta_2;P_i,\dots,P_{i+k_2-1}),P_{i+k_2},\dots,P_k)\\&\subset(-1)^{\epsilon_2}\partial\mathcal{M}^{\text{main}}_{k+1}(\beta;P_1,\dots,P_{i-1},P_i,\dots,P_{i+k_2-1},P_{i+k_2},\dots,P_k),\end{aligned}$$

where

$$\epsilon_2=n+1+\sum_{j=1}^{i-1}(\deg P_j+1).$$

(2) *We have*

$$\mathcal{M}^{\text{main}}_{k+1}(\beta;P_1,\dots,\partial P_i,\dots,P_k)\subset(-1)^{\epsilon_3}\partial\mathcal{M}^{\text{main}}_{k+1}(\beta;P_1,\dots,P_i,\dots,P_k),$$

where

$$\epsilon_3=1+\sum_{j=1}^{i-1}(\deg P_j+1).$$

Proof. The proof of (1) is divided into several steps.
Step 1. We note that the main component is not preserved by the change the order of marked points except by cyclic permutations. However we will use the components other than main components at the intermediate stage of the calculation. From Proposition 10.2, we note that

$$\deg\mathcal{M}^{\text{main}}_{k_2+1}(\beta_2;P_i,\dots,P_{i+k_2-1})\equiv\sum_{j=i}^{i+k_2-1}(\deg P_j+1)\quad(\bmod 2).$$

Then by using Lemma 47.3 repeatedly, we find

$$\begin{aligned}&\mathcal{M}^{\text{main}}_{k-k_2+2}(\beta_1;P_1,\dots,\mathcal{M}^{\text{main}}_{k_2+1}(\beta_2;P_i,\dots,P_{i+k_2-1}),P_{i+k_2},\dots,P_k)\\&\subseteq(-1)^{\delta_1}\mathcal{M}_{k-k_2+2}(\beta_1;\mathcal{M}^{\text{main}}_{k_2+1}(\beta_2;P_i,\dots,P_{i+k_2-1}),P_1,\dots,P_{i-1},P_{i+k_2},\dots,P_k),\end{aligned}$$

where

$$\delta_1=\Big(1+\sum_{j=i}^{i+k_2-1}(\deg P_j+1)\Big)\Big(\sum_{j=1}^{i-1}(\deg P_j+1)\Big).$$

Step 2. Next, we compare the orientation on

$$\mathcal{M}_{k-k_2+2}(\beta_1;\mathcal{M}^{\text{main}}_{k_2+1}(\beta_2;P_i,\dots,P_{i+k_2-1}),P_1,\dots,P_{i-1},P_{i+k_2},\dots,P_k)$$

with that on

$$\partial\mathcal{M}_{k+1}(\beta;P_i,\dots,P_{i+k_2-1},P_1,\dots,P_{i-1},P_{i+k_2},\dots,P_k).$$

By Definition 47.1, we have

$$\begin{aligned}&\mathcal{M}_{k+1}(\beta;P_i,\dots,P_{i+k_2-1},P_1,\dots,P_{i-1},P_{i+k_2},\dots,P_k)\\ =&(-1)^{\gamma_1}\mathcal{M}_{k+1}(\beta)_{(ev_1,\dots,ev_k)}\times\left(\prod_{j=i}^{i+k_2-1}P_j\times\prod_{j=1}^{i-1}P_j\times\prod_{j=i+k_2}^{k}P_j\right),\end{aligned}$$

where

$$\begin{aligned}\gamma_1=(n+1)\Big(&\sum_{j=i}^{i+k_2-1}\sum_{\ell=i}^{j}\deg P_\ell+\sum_{j=1}^{i-1}\Big(\sum_{m=i}^{i+k_2-1}\deg P_m+\sum_{\ell=1}^{j}\deg P_\ell\Big)\\ &+\sum_{j=i+k_2}^{k-1}\Big(\sum_{m=i}^{i+k_2-1}\deg P_m+\sum_{m=1}^{i-1}\deg P_m+\sum_{\ell=i+k_2}^{j}\deg P_\ell\Big)\Big).\end{aligned}$$

Using the iteration formula Lemma 45.3 (3), we have

$$\begin{aligned}&\mathcal{M}_{k+1}(\beta)_{(ev_1,\dots,ev_k)}\times\left(\prod_{j=i}^{i+k_2-1}P_j\times\prod_{j=1}^{i-1}P_j\times\prod_{j=i+k_2}^{k}P_j\right)\\ =&(-1)^{\gamma_2}\left(\mathcal{M}_{k+1}(\beta)_{(ev_1,\dots,ev_{k_2})}\times\left(\prod_{j=i}^{i+k_2-1}P_j\right)\right)\\ &\qquad{}_{(ev_{k_2+1},\dots,ev_k)}\times\left(\prod_{j=1}^{i-1}P_j\times\prod_{j=i+k_2}^{k}P_j\right),\end{aligned}$$

where

$$\gamma_2=(k-k_2)n\Big(k_2n+\sum_{j=i}^{i+k_2-1}(n-\deg P_j)\Big)\equiv(k-k_2)n\sum_{j=i}^{i+k_2-1}\deg P_j.$$

When we glue two holomorphic discs together with marked points, our convention (see Remark 46.4) of the ordering of marked points and of the boundary orientation (and Proposition 46.3) show that

$$\partial\left(\mathcal{M}_{k+1}(\beta)_{(ev_1,\dots,ev_{k_2})}\times\left(\prod_{j=i}^{i+k_2-1}P_j\right)\right)_{(ev_{k_2+1},\dots,ev_k)}\times(\prod_{j=1}^{i-1}P_j\times\prod_{j=i+k_2}^{k}P_j)$$
$$\supset(-1)^{\gamma_3}\left(\left(\mathcal{M}_{k-k_2+2}(\beta_1)_{ev_1^{\beta_1}}\times_{ev_0^{\beta_2}}\mathcal{M}_{k_2+1}(\beta_2)\right)_{(ev_1^{\beta},\dots,ev_{k_2}^{\beta})}\times\left(\prod_{j=i}^{i+k_2-1}P_j\right)\right)$$
$$_{(ev_{k_2+1}^{\beta},\dots,ev_k^{\beta})}\times\left(\prod_{j=1}^{i-1}P_j\times\prod_{j=i+k_2}^{k}P_j\right),$$

where

$$\gamma_3=(k-k_2)(k_2-1)+n+k-k_2+1\equiv n+1+kk_2+k_2\pmod 2.$$

By the associativity property Lemma 45.3 (2) and Definition 53.1, we have

$$\left(\left(\mathcal{M}_{k-k_2+2}(\beta_1)_{ev_1^{\beta_1}}\times_{ev_0^{\beta_2}}\mathcal{M}_{k_2+1}(\beta_2)\right)_{(ev_1^{\beta},\dots,ev_{k_2}^{\beta})}\times\left(\prod_{j=i}^{i+k_2-1}P_j\right)\right)$$
$$_{(ev_{k_2+1}^{\beta},\dots,ev_k^{\beta})}\times\left(\prod_{j=1}^{i-1}P_j\times\prod_{j=i+k_2}^{k}P_j\right)$$
$$=\left(\mathcal{M}_{k-k_2+2}(\beta_1)_{ev_1^{\beta_1}}\times_{ev_0^{\beta_2}}\left(\mathcal{M}_{k_2+1}(\beta_2)_{(ev_1^{\beta_2},\dots,ev_{k_2}^{\beta_2})}\times\left(\prod_{j=i}^{i+k_2-1}P_j\right)\right)\right)$$
$$_{(ev_{k_2+1}^{\beta},\dots,ev_k^{\beta})}\times\left(\prod_{j=1}^{i-1}P_j\times\prod_{j=i+k_2}^{k}P_j\right)$$
$$=(-1)^{\gamma_4}\left(\mathcal{M}_{k-k_2+2}(\beta_1)_{ev_1^{\beta_1}}\times_{ev_0^{\beta_2}}\mathcal{M}_{k_2+1}(\beta_2;P_i,\dots,P_{i+k_2-1})\right)$$
$$_{(ev_{k_2+1}^{\beta},\dots,ev_k^{\beta})}\times\left(\prod_{j=1}^{i-1}P_j\times\prod_{j=i+k_2}^{k}P_j\right),$$

where

$$\gamma_4=(n+1)\sum_{j=i}^{i+k_2-2}\sum_{\ell=i}^{j}\deg P_\ell.$$

Again by using the iteration formula, we find that

$$\begin{aligned}&\left(\mathcal{M}_{k-k_2+2}(\beta_1)_{ev_1^{\beta_1}} \times_{ev_0^{\beta_2}} \mathcal{M}_{k_2+1}(\beta_2; P_i, \dots, P_{i+k_2-1})\right) \\ &\qquad\qquad\qquad {}_{(ev^\beta_{k_2+1},\dots,ev^\beta_k)}\times \left(\prod_{j=1}^{i-1} P_j \times \prod_{j=i+k_2}^{k} P_j\right) \\ =&(-1)^{\gamma_5}\mathcal{M}_{k-k_2+2}(\beta_1)_{(ev_1^{\beta_1},\dots,ev^{\beta_1}_{k-k_2+1})}\times \\ &\quad\left(\mathcal{M}_{k_2+1}(\beta_2; P_i, \dots, P_{i+k_2-1}) \times \left(\prod_{j=1}^{i-1} P_j \times \prod_{j=i+k_2}^{k} P_j\right)\right) \\ =&(-1)^{\gamma_5+\gamma_6}\mathcal{M}_{k-k_2+2}(\beta;\mathcal{M}_{k_2+1}(\beta_2;P_i,\dots,P_{i+k_2-1}),P_1,\dots,P_{i-1},P_{i+k_2},\dots,P_k),\end{aligned}$$

where

$$\begin{aligned}\gamma_5 &= n(k-k_2)(2n - \sum_{j=i}^{i+k_2-1}(\deg P_j+1)) \equiv n(k-k_2)\left(k_2 + \sum_{j=i}^{i+k_2-1}\deg P_j\right), \\ \gamma_6 &= (n+1)\Bigg(\deg\mathcal{M}_{k_2+1}(\beta_2;P_i,\dots,P_{i+k_2-1}) \\ &\quad + \sum_{j=1}^{i-1}\Big(\deg\mathcal{M}_{k_2+1}(\beta_2;P_i,\dots,P_{i+k_2-1}) + \sum_{\ell=1}^{j}\deg P_\ell\Big) \\ &\quad + \sum_{j=i+k_2}^{k-1}\Big(\deg\mathcal{M}_{k_2+1}(\beta_2;P_i,\dots,P_{i+k_2-1}) + \sum_{m=1}^{i-1}\deg P_m + \sum_{\ell=i+k_2}^{j}\deg P_\ell\Big)\Bigg) \\ &= (n+1)\left((k-k_2)\left(k_2 + \sum_{m=i}^{i+k_2-1}\deg P_m\right) + (k-i-k_2)\left(\sum_{m=1}^{i-1}\deg P_m\right)\right. \\ &\quad \left.+\sum_{j=1}^{i-1}\sum_{\ell=1}^{j}\deg P_\ell + \sum_{j=i+k_2}^{k-1}\sum_{\ell=i+k_2}^{j}\deg P_\ell\right).\end{aligned}$$

Then an elementary calculation shows that

$$\gamma_1+\gamma_2+\gamma_3+\gamma_4+\gamma_5+\gamma_6 \equiv n+1 \pmod 2 .$$

Hence we have found that

$$\begin{aligned}&\mathcal{M}_{k-k_2+1}(\beta_1;\mathcal{M}^{\text{main}}_{k_2+1}(\beta_2;P_i,\dots,P_{i+k_2-1}),P_1,\dots,P_{i-1},P_{i+k_2},\dots,P_k) \\ &\subset (-1)^{n+1}\partial\mathcal{M}_{k+1}(\beta;P_i,\dots,P_{i+k_2-1},P_1,\dots,P_{i-1},P_{i+k_2},\dots,P_k).\end{aligned}$$

Step 3. On the other hand, by using Lemma 47.3 again, we can see that

$$\begin{aligned}&\mathcal{M}_{k+1}(\beta; P_i, \dots, P_{i+k_2-1}, P_1, \dots, P_{i-1}, P_{i+k_2}, \dots, P_k)\\ &= (-1)^{\delta_2}\mathcal{M}^{\rm main}_{k+1}(\beta; P_1, \dots, P_{i-1}, P_i, \dots, P_{i+k_2-1}, P_{i+k_2}, \dots, P_k),\end{aligned}$$

where

$$\delta_2 = \Big(\sum_{j=i}^{i+k_2-1}(\deg P_j + 1)\Big)\Big(\sum_{j=1}^{i-1}(\deg P_j + 1)\Big).$$

Therefore we have

$$\delta_1 + n + 1 + \delta_2 \equiv n + 1 + \sum_{j=1}^{i-1}(\deg P_j + 1) \quad (\bmod 2),$$

which proves Proposition 48.1 (1).

(2) We prove Proposition 48.1 (2). We recall that

$$\mathcal{M}^{\rm main}_{k+1}(\beta; P_1, \dots, P_i, \dots, P_k) = (-1)^{\epsilon_1}\mathcal{M}^{\rm main}_{k+1}(\beta)_{(ev_1,\dots,ev_k)} \times_{f_1\times\cdots\times f_k} \left(\prod_{j=1}^k P_j\right)$$

with

$$\epsilon_1 = (n+1)\sum_{j=1}^{k-1}\sum_{\ell=1}^{j}\deg P_\ell.$$

By Lemma 45.3 (1), we find that

$$\begin{aligned}&\partial\Big(\mathcal{M}^{\rm main}_{k+1}(\beta)_{(ev_1,\dots,ev_k)} \times_{f_1\times\cdots\times f_k} \left(\prod_{j=1}^k P_j\right)\Big)\\ =&\partial\mathcal{M}^{\rm main}_{k+1}(\beta)_{(ev_1,\dots,ev_k)} \times_{f_1\times\cdots\times f_k} \left(\prod_{j=1}^k P_j\right)\\ &\qquad \bigsqcup(-1)^{n+k+nk}\mathcal{M}^{\rm main}_{k+1}(\beta)_{(ev_1,\dots,ev_k)} \times_{f_1\times\cdots\times f_k} \partial\left(\prod_{j=1}^k P_j\right),\end{aligned}$$

because $\dim\mathcal{M}^{\rm main}_{k+1}(\beta) = n + \mu(\beta) - 3 + k + 1 \equiv n + k$. Moreover by using Lemma 45.3 again, it is easy to see that

$$\partial\left(\prod_{j=1}^k P_j\right) = \bigsqcup_{i=1}^k(-1)^{\sum_{j=1}^{i-1}\dim P_j}P_1\times\cdots\times\partial P_i\times\cdots\times P_k.$$

On the other hand, Definition 47.1 yields that

$$\begin{aligned}&\mathcal{M}^{\text{main}}_{k+1}(\beta; P_1, \dots, \partial P_i, \dots, P_k)\\ =&(-1)^{\epsilon_2}\mathcal{M}^{\text{main}}_{k+1}(\beta)_{(ev_1,\dots,ev_k)} \times_{f_1\times\cdots\times f_k} (\prod_{j=1}^{i-1} P_j \times \partial P_i \times \prod_{j=i+1}^{k} P_j)\end{aligned}$$

with

$$\epsilon_2 = (n+1)\sum_{j=1}^{k-1}\sum_{\ell=1}^{j}(\deg P_\ell)',$$

where

$$(\deg P_\ell)' = \begin{cases} \deg P_\ell & \text{for } \ell \neq i \\ \deg P_\ell + 1 & \text{for } \ell = i.\end{cases}$$

Therefore we can see that

$$\epsilon_1 + n + k + nk + \sum_{j=1}^{i-1}(n - \deg P_j) + \epsilon_2 \equiv \sum_{j=1}^{i-1}(\deg P_j + 1) + 1 \pmod 2,$$

which proves Proposition 48.1 (2). □

Then Proposition 48.1 (1) and (2) imply (10.17.2) and (10.17.1) respectively. Therefore we complete proving that

$$\widehat{d} \circ \widehat{d} = 0$$

in $(C(L; \Lambda_{0,nov}), \mathfrak{m})$.

§49. Orientation on the moduli space of constant maps.

Next, let us assume that P_1 and P_2 are oriented submanifolds. Considering the moduli space of constant maps $\mathcal{M}^{\text{main}}_3(L, \beta_0; P_1, P_2)$, we have $\mathcal{M}^{\text{main}}_3(L, \beta_0; P_1, P_2) = P_1 \cap P_2$ as sets. We have to study the difference of orientations between them. To do this, we firstly confirm the convention on orientation on $P_1 \cap P_2$.

Convention 49.1. We assume P_1 and P_2 are submanifolds in L and intersect transversally. We denote the dimensions of L, P_1 and P_2 by n, p_1 and p_2. Suppose that L, P_1 and P_2 are oriented. Take $x \in P_1 \cap P_2$. The orientation on the normal bundle $N_{P_i}L$ of P_i $(i = 1, 2)$ in L is defined by

$$T_x L = N_{x,P_i} L \times T_x P_i.$$

Since P_1 and P_2 intersect transversally, we may assume that $N_{x,P_1}L \subset T_x P_2$ and $N_{x,P_2} \subset T_x P_1$. Then we define the orientation on $P_1 \cap P_2$ by

$$T_x L = N_{x,P_1} L \times N_{x,P_2} L \times T_x (P_1 \cap P_2).$$

Remark 49.2. The convention defines "cohomology orientation" on $P_1 \cap P_2$. Namely we have $P_1 \cap P_2 = (-1)^{(n-p_1)(n-p_2)} P_2 \cap P_1$.

Then we can show the following;

Proposition 49.3. *We can give an oriented Kuranishi structure on* $\mathcal{M}_3^{\text{main}}(L; \beta_0)$ *so that we have the following oriented isomorphism*

$$ev_0 : \mathcal{M}_3^{\text{main}}(L; \beta_0)_{(ev_1, ev_2)} \times_{f_1 \times f_2} (P_1 \times P_2) \to (-1)^{n(p_1+p_2)+p_1(n-p_2)} P_1 \cap P_2.$$

Note that since we have from Definition 47.1

$$\mathcal{M}_3^{\text{main}}(L; \beta_0; P_1, P_2) = (-1)^{(n+1)\deg P_1} \mathcal{M}_3^{\text{main}}(L; \beta_0)_{(ev_1, ev_2)} \times_{f_1 \times f_2} (P_1 \times P_2),$$

we can immediately obtain the following consequence.

Corollary 49.4. *Using the oriented Kuranishi structure on* $\mathcal{M}_3^{\text{main}}(L; \beta_0)$ *in Proposition* 49.3, *we have*

$$(\mathcal{M}_3^{\text{main}}(L; \beta_0; P_1, P_2), ev_0) = (-1)^{\deg P_1 (\deg P_2 + 1)} P_1 \cap P_2.$$

Proof of Proposition 49.3. In order to show Proposition 49.3, we are going to explicitly describe (weakly submersive) Kuranishi structures on $\mathcal{M}_3^{\text{main}}(L; \beta_0)$, $P_1 \times P_2$ and their fiber product. Firstly, we describe the Kuranishi structure on $\mathcal{M}_3^{\text{main}}(L; \beta_0)$. We like to study orientation on the Kuranishi structure on $\mathcal{M}_3^{\text{main}}(L; \beta_0)$. Since $\mathcal{M}_3^{\text{main}}(L; \beta_0)$ consists of constant maps, the argument is a local problem. Here we shall consider the case $L = \mathbb{R}^n$ and $M = \mathbb{C}^n$ with the standard orientations. We put three marked points $z_0 = 1, z_1 = \sqrt{-1}, z_2 = -1$ on the boundary of the unit disc D^2. The maps ev_1 and ev_2 on $\mathcal{M}_3^{\text{main}}(L; \beta_0)$ are the evaluation maps at z_1 and z_2, respectively.

Now we put

$$U := \left\{ u_{a,b} : D^2 \to \mathbb{C}^n \;\middle|\; u_{a,b}(z) = \frac{1}{2} a \sqrt{-1} \overline{z} + b - \frac{1}{2} a \sqrt{-1} z, a, b \in \mathbb{R}^n \right\}$$
$$\mathcal{E} := \{ (u_{a,b}, c) \mid z \in D^2, a, b, c \in \mathbb{R}^n \}$$

and define a section $s : U \to \mathcal{E}$ by $s(u_{a,b}) = (u_{a,b}, a)$. Then the moduli space $\mathcal{M}_3^{\text{main}}(L;\beta_0)$ of constant maps to L is nothing but $s^{-1}(0) \simeq \mathbb{R}^n_b$.

Of course, U can be identified with $\mathbb{R}^n_a \times \mathbb{R}^n_b$ which can be regarded as $T_*L \times T_*L$, and $\mathcal{E}$ is a trivial vector bundle over U with fiber $\mathbb{R}^n \cong T_*L$. Applying the argument in §44.1 to $\mathcal{M}_3^{\text{main}}(L;\beta_0)$, the orientation in this case is given by the orientation of L after identifying it with the target L via ev_0. We must provide the orientation on U and the fiberwise orientation on $\mathcal{E}$ respectively so that the orientation of $\mathcal{M}_3^{\text{main}}(L;\beta_0)$ is consistent with the orientation of $(s;\mathcal{E} \to U)$ in the sense of Kuranishi structure. From now on, we give U an orientation through identification with $T_*L \times T_*L$ and the fiber orientation of $\mathcal{E}$ through identification with $\mathbb{R}^n \cong T_*L$ and $(s;\mathcal{E} \to U)$ gives a Kuranishi structure on $\mathcal{M}_3^{\text{main}}(L;\beta_0)$.

Note that the evaluation map $(ev_1, ev_2) : U \to L \times L$, which is given by

$$(ev_1, ev_2)(u_{a,b}) = (a + b, b),$$

is an orientation preserving map. We have an orientation preserving identification

$$\mathcal{E}_* \times T_*\mathcal{M}_3^{\text{main}}(L;\beta_0) = T_*U = T_*L \times T_*L.$$

Secondly, we describe the Kuranishi structure on $P_1 \times P_2$. Let $U_i \subset L$ be a tubular neighborhood of P_i in L $(i = 1, 2)$. Let N_{P_i} be the normal bundle of P_i in L and $s_i : U_i \to N_{P_i}$ the canonical section. Then $(s_i; N_{P_i} \to U_i)$ gives a Kuranishi structure on P_i. Namely $P_i = s_i^{-1}(0)$ and the orientation is given by

$$N_{P_i}|_{x_i} \times T_{x_i} s_i^{-1}(0) = T_{x_i} U_i \cong T_{x_i} L$$

for $x_i \in P_i$. Thus $P_i = s_i^{-1}(0)$ as an oriented space. The product of the Kuranishi structures $(s_1 \oplus s_2; N_{P_1} \oplus N_{P_2} \to U_1 \times U_2)$ gives a Kuranishi structure on $P_1 \times P_2$. Then by Convention 45.1.(4), the relation between the orientation defined by the Kuranishi structure and the product orientation on $P_1 \times P_2$ is as following :

$$P_1 \times P_2 = (-1)^{p_1(n-p_2)} (s_1 \oplus s_2)^{-1}(0),$$

because we have

$$\begin{aligned}
&N_{P_1 x_1} \times N_{P_2 x_2} \times T(s_1 \oplus s_2)^{-1}(0)_{(x_1,x_2)} = T_{x_1}U_1 \times T_{x_2}U_2 \\
&= (N_{P_1 x_1} \times T_{x_1}P_1) \times (N_{P_2 x_2} \times T_{x_2}P_2) \\
&= (-1)^{p_1(n-p_2)} N_{P_1 x_1} \times N_{P_2 x_2} \times T_{x_1}P_1 \times T_{x_2}P_2.
\end{aligned}$$

(See Convention 45.1 (4).)

Now, for two submersions;

$$(ev_1, ev_2) : U \to U_1 \times U_2, \quad id = \text{identity} : U_1 \times U_2 \to U_1 \times U_2,$$

we consider the fiber product of Kuranishi structures

$$(s \oplus s_1 \oplus s_2; \mathcal{E} \oplus N_{P_1} \oplus N_{P_2} \to U_{(ev_1, ev_2)} \times_{id} (U_1 \times U_2)),$$

which gives a Kuranishi structure on $\mathcal{M}_3^{\rm main}(L;\beta_0)_{(ev_1,ev_2)} \times_{f_1 \times f_2} (P_1 \times P_2)$. As for the orientation, we have from Convention 45.1 (4),

$$\begin{aligned}
&\mathcal{M}_3^{\rm main}(L;0)_{(ev_1,ev_2)} \times_{f_1 \times f_2} (P_1 \times P_2) \\
&= (-1)^{\mathrm{rank}(N_{P_1} \oplus N_{P_2})(\dim \mathcal{M}_3^{\rm main}(L;0) - \dim U_1 \times U_2)} \\
&\qquad \times (-1)^{p_1(n-p_2)} (s \oplus s_1 \oplus s_2)^{-1}(0) \\
&= (-1)^{n(p_1+p_2)+p_1(n-p_2)} (s \oplus s_1 \oplus s_2)^{-1}(0).
\end{aligned} \tag{49.5}$$

Now we are going to compare the orientation on $(s \oplus s_1 \oplus s_2)^{-1}(0)$ with that on $P_1 \cap P_2$. The orientation on $(s \oplus s_1 \oplus s_2)^{-1}(0)$ is defined by

$$\mathcal{E}_* \times N_{P_1 *} \times N_{P_2 *} \times T_*(s \oplus s_1 \oplus s_2)^{-1}(0) = T_*(U_{(ev_1,ev_2)} \times_{\rm identity} (U_1 \times U_2)).$$

Then we will show

Lemma 49.6. *The isomorphism*

$$ev_0 : (s \oplus s_1 \oplus s_2)^{-1}(0) \to P_1 \cap P_2$$

is orientation preserving.

Proof of Lemma 49.6: The orientations of $P_1 \cap P_2$ and $(s \oplus s_1 \oplus s_2)^{-1}(0)$ are given by the following exact sequences respectively:

$$0 \to T_*(P_1 \cap P_2) \to T_*L \to N_{P_1} \oplus N_{P_2} \to 0$$

and

$$0 \to T_*(s \oplus s_1 \oplus s_2)^{-1}(0) \to T_*\Big(U_{ev_1,ev_2} \times_{\rm id} (U_1 \times U_2)\Big) \to \mathcal{E} \oplus \pi_1^* N_{P_1} \oplus \pi_2^* N_{P_2} \to 0.$$

We consider the following commutative diagram, which relates two exact sequences above.

$$\begin{array}{ccccccc}
 & & & & \mathcal{E}_* & & \xrightarrow{\rm id} \\
 & & & & \oplus & & \\
0 & \longrightarrow & T_*(P_1 \cap P_2) & \longrightarrow & T_*L & & \longrightarrow \\
 & & I \uparrow & & \Phi \uparrow & & \\
0 & \longrightarrow & T_*(s \oplus s_1 \oplus s_2)^{-1}(0) & \longrightarrow & T_*\Big(U_{ev_1,ev_2} \times_{\rm id} (U_1 \times U_2)\Big) & & \longrightarrow
\end{array}$$

$$\begin{array}{ccccc} \xrightarrow{\mathrm{id}} & \mathcal{E}_* & & \\ & \oplus & & \\ \longrightarrow & N_{P_1} \oplus N_{P_2} & \longrightarrow & 0 \\ & \Psi \uparrow & & \\ \longrightarrow & \mathcal{E} \oplus \pi_1^* N_{P_1} \oplus \pi_2^* N_{P_2} & \longrightarrow & 0 \end{array}$$

where vertical isomorphisms are given as follows: I is the restriction to $T_*(s \oplus s_1 \oplus s_2)^{-1}(0)$ of the differential of the evaluation map from the Kuranishi neighborhood, i.e., $ev_0 : \Big(U_{ev_1,ev_2} \times_{\mathrm{id}} (U_1 \times U_2)\Big) \to L$. In other words,

$$I = T_* ev_0 : T_*(s \oplus s_1 \oplus s_2)^{-1}(0) \to T_*(P_1 \cap P_2)$$

and can be explicitly written as

$$I((0,v),v,v) = v.$$

The maps Φ and Ψ are defined as

$$\Phi((a,b),v_1,v_2) = (a, \mathrm{pr}_1(v_1) + \mathrm{pr}_2(v_2) + \mathrm{pr}_3(b)) \quad \text{with} \quad v_1 = a + b, v_2 = b,$$

and

$$\Psi(u,v_1,v_2) = (u,v_1,v_2).$$

Here pr_i is the i-th factor projection of $T_*L = N_{P_1} \oplus N_{P_2} \oplus (P_1 \cap P_2)$. Because the isomorphisms Φ and Ψ preserve their orientations, it follows from the diagram chasing argument that the isomorphism I also preserves orientations. This finishes the proof. □

Therefore, combining (49.5) and Lemma 49.6, we complete the proof of Proposition 49.3.

§50. Orientation of the moduli space of connecting orbits.

In §44.3, we described the orientation on the moduli space of connecting orbits. In this section, we will give orientations on the moduli spaces of marked pseudo-holomorphic strips and their fiber products with smooth singular simplexes on $L^{(i)}$. They are used to define the filtered A_∞ bimodule structure.

Suppose that a pair $(L^{(0)}, L^{(1)})$ of Lagrangian submanifolds of (M,ω) is relatively spin and intersect transversally. (See Definition 44.2 for the relative spin structure for the pair $(L^{(0)}, L^{(1)})$.) Choose and fix the path λ_p of oriented Lagrangian linear subspaces and a trivialization of the bundle $\widetilde{\lambda}_p$ for $p \in L^{(0)} \cap L^{(1)}$. (See §44.3.) From now on, we abbreviate these data in the notation of the moduli spaces.

Now let $[\ell_p, w], [\ell_q, w'] \in Cr(L^{(1)}, L^{(0)})$. (See §3.2 for the definition of $Cr(L^{(1)}, L^{(0)})$.) We denote by $\widetilde{\mathcal{M}}^{\rm reg}([\ell_q, w'], [\ell_p, w]) = \widetilde{\mathcal{M}}^{\rm reg}(L^{(1)}, L^{(0)}; [\ell_q, w'], [\ell_p, w])$ the space of pseudo-holomorphic maps u from the infinite cylinder $\mathbb{R} \times [0,1] \subset \mathbb{C}$ to M such that

$$u(-\infty, t) = p, u(+\infty, t) = q, \; u(\mathbb{R} \times \{i\}) \subset L^{(i)} \; (i = 0, 1) \text{ and } w\#u \sim w.'$$

Note that this notation is different from one in §12 in Chapter 3, where the above moduli space is denoted by $\widetilde{\mathcal{M}}^{\rm reg}([\ell_p, w], [\ell_q, w'])$. Recall that we adopt different notation only for the orientation business. See the top of §44.3. (For simplicity of notations, we often omit $L^{(0)}$ and $L^{(1)}$ in $\widetilde{\mathcal{M}}(L^{(1)}, L^{(0)}; [\ell_q, w'], [\ell_p, w])$ etc.) From now on, we identify the infinite cylinder $\mathbb{R} \times [0,1]$ with $D^2 \setminus \{\pm 1\} \subset \mathbb{C}$ so that the ends $\{+\infty\} \times [0,1]$ and $\{-\infty\} \times [0,1]$ correspond to $z_0 = 1$ and $z_1 = -1$ respectively. Then the relative spin structure of $(L^{(0)}, L^{(1)})$ determines the orientation on $\widetilde{\mathcal{M}}^{\rm reg}([\ell_q, w'], [\ell_p, w])$ by Theorem 44.14 in §44.

We put

$$\partial_1 D^2 = \{z \in \partial D^2 \mid \operatorname{Im} z > 0\}, \quad \partial_0 D^2 = \{z \in \partial D^2 \mid \operatorname{Im} z < 0\}.$$

We denote by $\widetilde{\mathcal{M}}^{\rm reg}_{\ell,m}([\ell_q, w'], [\ell_p, w])$ the space of connecting orbits

$$u : (D^2 \setminus \{\pm 1\}; \partial_1 D^2, \partial_0 D^2) \to (M; L^{(1)}, L^{(0)})$$

with ℓ marked points on $\partial_1 D^2$ and m marked points are on $\partial_0 D^2$ such that the following holds. $u(-1) = p$ and $u(1) = q$. The marked points are distinct. 2nd, 3rd, ... ,$(\ell+1)$-th marked points $z_2, \cdots, z_{\ell+1}$ are on $\partial_1 D^2$ and $\ell+2$-th, ... ,$\ell+m+1$-th marked points $z_{\ell+2}, \ldots, z_{\ell+m+1}$ are on $L^{(0)}$. If we put $z_0 = 1$, $z_1 = -1$, then the order of the marked points $z_0, z_2, \cdots, z_{\ell+1}, z_1, z_{\ell+2}, \ldots, z_{\ell+m+1}$ respects the counter clockwise orientation of ∂D^2. (See Remark 22.23 (2) and Figure 12.2.)

The space $\widetilde{\mathcal{M}}^{\rm reg}_{\ell,m}([\ell_q, w'], [\ell_p, w])$ is an open subset of $\widetilde{\mathcal{M}}^{\rm reg}([\ell_q, w'], [\ell_p, w]) \times (\partial D^2)^{\ell+m}$. So, under this convention, the orientation on $\widetilde{\mathcal{M}}^{\rm reg}_{\ell,m}([\ell_q, w'], [\ell_p, w])$ is given by

$$\widetilde{\mathcal{M}}^{\rm reg}_{\ell,m}([\ell_q, w'], [\ell_p, w]) \subset \widetilde{\mathcal{M}}^{\rm reg}([\ell_q, w'], [\ell_p, w]) \times (\partial D^2_2 \times \cdots \times \partial D^2_{\ell+m+1}),$$

where ∂D^2_i is a parameter space of the marked points z_i $(i = 2, \ldots, \ell+m+1)$.

We denote by $\mathcal{M}_{\ell,m}([\ell_q, w'], [\ell_p, w])$ the quotient space of $\widetilde{\mathcal{M}}_{\ell,m}([\ell_q, w'], [\ell_p, w])$ by the biholomorphic automorphism group $\mathbb{R}$ of D^2 fixing the two marked points $z_0 = +1$ and $z_1 = -1$. (See Definition 5.1.)

The orientation on $\mathcal{M}_{\ell,m}([\ell_q, w'], [\ell_p, w])$ is defined as the quotient orientation defined in §45. As in §46, we simply write as

$$\widetilde{\mathcal{M}}^{\rm reg}_{\ell,m}([\ell_q, w'], [\ell_p, w]) = \mathcal{M}^{\rm reg}_{\ell,m}([\ell_q, w'], [\ell_p, w]) \times \mathbb{R}.$$

Let $P^{(1)}_1, \dots, P^{(1)}_\ell \in C(L^{(1)}; \mathbb{Q})$, $P^{(0)}_1, \dots, P^{(0)}_m \in C(L^{(0)}; \mathbb{Q})$ be smooth singular simplexes on $L^{(i)}$, $i = 0, 1$.

We define

Definition 50.1.

$$\begin{aligned} &\mathcal{M}^{\rm reg}_{\ell,m}(L^{(1)}, L^{(0)}; [\ell_q, w'], [\ell_p, w] : P^{(1)}_1, \dots, P^{(1)}_\ell; P^{(0)}_1, \dots, P^{(0)}_m) \\ &\quad = \mathcal{M}^{\rm reg}_{\ell,m}([\ell_q, w'], [\ell_p, w] : P^{(1)}_1, \dots, P^{(1)}_\ell; P^{(0)}_1, \dots, P^{(0)}_m) \\ &\quad := (-1)^\epsilon \mathcal{M}^{\rm reg}_{\ell,m}([\ell_q, w'], [\ell_p, w])_{(ev_2, \dots, ev_{\ell+1}, ev_{\ell+2}, \dots, ev_{\ell+m+1})} \times \\ &\qquad\qquad \left(\prod_{k=1}^{\ell} P^{(1)}_k \times \prod_{k=1}^{m} P^{(0)}_k \right), \end{aligned}$$

where we define ϵ as follows. Let $\mu([\ell_p, w])$ be the Maslov-Morse index in Definition 3.12. (We omit λ^0 from the notation $\mu([\ell_p, w]; \lambda^0)$, since the parity of $\mu([\ell_p, w]; \lambda^0)$ is independent of λ^0. It is independent of w also.) We put :
if $m = 0$,

$$\begin{aligned} \epsilon =&(n+1) \sum_{k=1}^{\ell-1} \sum_{j=1}^{k} \deg P^{(1)}_j + \ell(n+1)\mu([\ell_p, w]) \\ &+ \mu([\ell_q, w']) + (\mu([\ell_p, w]) + 1) \sum_{k=1}^{\ell} \left(\deg P^{(1)}_k + 1 \right), \end{aligned}$$

and if $m > 1$,

$$\begin{aligned} \epsilon =&(n+1) \left(\sum_{k=1}^{\ell} \sum_{j=1}^{k} \deg P^{(1)}_j + \sum_{k=1}^{m-1} \left(\left(\sum_{h=1}^{\ell} \deg P^{(1)}_h \right) + \sum_{j'=1}^{k} \deg P^{(0)}_{j'} \right) \right) \\ &+ \mu([\ell_q, w']) + (\mu([\ell_p, w]) + 1) \sum_{k=1}^{\ell} \left(\deg P^{(1)}_k + 1 \right) + (\ell + m)(n+1)\mu([\ell_p, w]). \end{aligned}$$

Remark 50.2. (1) When we regard the space of connecting orbits as a space of pseudo-holomorphic maps from the unit disc, this sign is nothing but one determined by the rule in Definition 47.1. Note that the way we treat the first two marked points z_0, z_1 is different from the way we treat the other marked points. This is because we need to do so when we define the orientation on the space $\widetilde{\mathcal{M}}([\ell_q, w'], [\ell_p, w])$. Namely, we glue half discs at these marked points and after gluing these marked points disappear.

(2) We discuss the case when $L^{(0)}$ and $L^{(1)}$ intersect cleanly, in §51.

The space $\mathcal{M}_{\ell,m}([\ell_q, w'], [\ell_p, w] : P_1^{(1)}, \dots, P_\ell^{(1)}; P_1^{(0)}, \dots, P_m^{(0)})$ is the stable map compactification of $\mathcal{M}^{\rm reg}_{\ell,m}([\ell_q, w'], [\ell_p, w]) : P_1^{(1)}, \dots, P_\ell^{(1)}; P_1^{(0)}, \dots, P_m^{(0)})$ and is a space with Kuranishi structure with corners. Note that the orientation bundle of the Kuranishi structure naturally extends to the stable map compactification. Hence $\mathcal{M}_{\ell,m}([\ell_q, w'], [\ell_p, w]) : P_1^{(1)}, \dots, P_\ell^{(1)}; P_1^{(0)}, \dots, P_m^{(0)})$ is also canonically oriented. Using these oriented moduli spaces of dimension 0, we define $\mathfrak{n}_{\ell,m}$ by

$$\begin{aligned}&\langle \mathfrak{n}_{\ell,m}(P_1^{(1)} \otimes \cdots \otimes P_\ell^{(1)} \otimes [\ell_p, w] \otimes P_1^{(0)} \otimes \cdots \otimes P_m^{(0)}), [\ell_q, w'] \rangle \\ :=&\# \mathcal{M}_{\ell,m}([\ell_q, w'], [\ell_p, w]) : P_1^{(1)}, \dots, P_\ell^{(1)}; P_1^{(0)}, \dots, P_m^{(0)})\end{aligned}$$

as in Definition 12.41. Then we can show

Proposition 50.3. *The operators $\mathfrak{n}_{\ell,m}$ satisfy the A_∞ bimodule formulae.*

Proof: We have three kinds of ends of the moduli space

$$\mathcal{M}_{\ell,m}([\ell_q, w'], [\ell_p, w] : P_1^{(1)}, \dots, P_\ell^{(1)}; P_1^{(0)}, \dots, P_m^{(0)}). \tag{50.4}$$

The ends corresponding to bubbling off of holomorphic discs are treated in a similar way to the proof of Proposition 48.1 (1). (These contributions cancel with the A_∞ algebra operations on $L^{(i)}$, $i = 0, 1$.) We see that other boundary contributions cancel one another.

We find that

$$\begin{aligned}&(-1)^{\epsilon_1} \mathcal{M}_{\ell,m}([\ell_q, w'], [\ell_p, w] : P_1^{(1)}, \dots, \partial P_j^{(1)}, \dots, P_\ell^{(1)}; P_1^{(0)}, \dots, P_m^{(0)}) \\ \subset &\partial \mathcal{M}_{\ell,m}([\ell_q, w'], [\ell_p, w] : P_1^{(1)}, \dots, P_\ell^{(1)}; P_1^{(0)}, \dots, P_m^{(0)}),\end{aligned} \tag{50.5}$$

where $\epsilon_1 = \mu([\ell_q, w']) + n + 1 + \sum_{i=1}^{j-1} (\deg P_i^{(1)} + 1)$.

$$\begin{aligned}&(-1)^{\epsilon_2} \mathcal{M}_{\ell,m}([\ell_q, w'], [\ell_p, w] : P_1^{(1)}, \dots, P_\ell^{(1)}; P_1^{(0)}, \dots, \partial P_j^{(0)}, \dots, P_m^{(0)}) \\ \subset &\partial \mathcal{M}_{\ell,m}([\ell_q, w'], [\ell_p, w] : P_1^{(1)}, \dots, P_\ell^{(1)}; P_1^{(0)}, \dots, P_m^{(0)}),\end{aligned} \tag{50.6}$$

where

$$\epsilon_2 = \mu([\ell_q, w']) + n + 1 + \sum_{i=1}^{\ell} (\deg P_i^{(1)} + 1) + (\mu([\ell_p, w]) + 1) + \sum_{i=1}^{j-1} (\deg P_i^{(0)} + 1).$$

For the proof of (50.5) and (50.6), see (51.12.2) and (51.12.3) and the proof of Theorem 51.10 (2) in the next section, where we will prove more general statement, combined with Definitions 51.8 and 51.11.

Other ends of the moduli space (50.4) correspond to the splitting of a connecting orbit into the sum of two connecting orbits. As for the orientation given in §44.3 and Convention 49.1, we can show that the gluing map

$$\widetilde{\mathcal{M}}([\ell_q,w'],[\ell_r,w'']) \times \widetilde{\mathcal{M}}([\ell_r,w''],[\ell_p,w]) \to \widetilde{\mathcal{M}}([\ell_q,w'],[\ell_p,w''])$$

respects their orientations. Note that the orientations we use are ones in §44.3. These orientations are modified in Definition 50.1.

Now we consider the $\mathbb{R}$-action defined as the translation in τ-variable. By our convention in §45, we have the following analog of Lemma 46.5.

$$(-1)^{\epsilon_3}\mathcal{M}([\ell_q,w'],[\ell_r,w'']) \times \mathcal{M}([\ell_r,w''],[\ell_p,w]) \subset \partial\mathcal{M}([\ell_q,w'],[\ell_p,w]),$$

where $\epsilon_3 = \mu([\ell_q,w']) - \mu([\ell_r,w'']) - 1$. We will find a more general claim in the proof of Theorem 51.10 (3). Using this fact, we find that

$$\begin{aligned}
&(-1)^{\epsilon_4}\mathcal{M}_{\ell_1,m_1}([\ell_q,w'],[\ell_r,w''] : P^{(1)}_1,\dots,P^{(1)}_{\ell_1};P^{(0)}_1,\dots,P^{(0)}_{m_1})\\
&\times \mathcal{M}_{\ell_2,m_2}([\ell_r,w''].[\ell_p,w] : P^{(1)\prime}_1,\dots,P^{(1)\prime}_{\ell_2};P^{(0)\prime}_1,\dots,P^{(0)\prime}_{m_2})\\
&\subset\partial\mathcal{M}_{\ell.m}([\ell_q,w'],[\ell_p,w] : P^{(1)}_1,\dots,P^{(1)}_{\ell_1},P^{(1)\prime}_1,\dots,P^{(1)\prime}_{\ell_2};P^{(0)\prime}_1,\dots,P^{(0)\prime}_{m_2},P^{(0)}_1,\dots,P^{(0)}_{m_1}),
\end{aligned}$$

where $\epsilon_4 = \mu([\ell_q,w']) + 1 + \sum_{i=1}^{\ell_1}(\deg P^{(1)}_i + 1)$, cf. (51.12.4).

The rest of the argument is now standard. □

§51. The Bott-Morse case.

We discuss the orientation problem for the space of connecting orbits in general Bott-Morse setting. First of all, we review some notations in §12. Let $L^{(0)}$ and $L^{(1)}$ be two Lagrangian submanifolds which intersect cleanly. We denote by R_h a connected component of $L^{(0)} \cap L^{(1)}$. We define

$$V_h = \left.\frac{(TL^{(0)} + TL^{(1)})}{(TL^{(0)} + TL^{(1)})^{\perp_\omega}}\right|_{R_h}.$$

V_h is a vector bundle on R_h. We regard V_h as a subbundle of $TM|_{R_h}$.

For each point p in R_h, we denote by $\mathcal{P}_{R_h}(T_pL^{(0)}, T_pL^{(1)})$ the space of all paths in the oriented Lagrangian Grassmannian of T_pM such that it is of the form $t \mapsto \lambda(t) \oplus T_pR_h$ and satisfies $\lambda(0) \oplus T_pR_h = T_pL^{(0)}$, $\lambda(1) \oplus T_pR_h = T_pL^{(1)}$. Here $\lambda(t)$ is a path of Lagrangian subspaces in $V_h|_p$. In Chapter 3, we consider the case that

$$\lambda(t) \oplus T_pR_h = \lambda_{w,\lambda^0}.$$

We like to remark that R_h itself may not be orientable. We write

$$\mathcal{P}_{R_h}(TL^{(0)}, TL^{(1)}) = \bigcup_{p \in R_h} \mathcal{P}_{R_h}(T_pL^{(0)}, T_pL^{(1)}).$$

Denote by $\widetilde{\lambda}_p$ the Lagrangian subbundle in $[0,1] \times T_pM$ corresponding to the path $\lambda \oplus T_pR_h$.

In this section we consider a connecting orbit as a holomorphic map $u : D^2 \to M$ such that $u(\partial_0 D^2) \subset L^{(0)}$, $u(\partial_1 D^2) \subset L^{(1)}$, $u(1) = q \in R_{h'}$ and $u(-1) = p \in R_h$, where $\partial_0 D^2$ is the arc with negative imaginary part and $\partial_1 D^2$ is the arc with positive imaginary part. We denote by

$$\mathcal{M}(R_{h'}, R_h)$$

the moduli space of such maps u. We remark that the order of $R_{h'}$, R_h in the notation above is opposite to the convention in §12. (See the top of §44.3.) Namely

$$\mathcal{M}(R_{h'}, R_h) = \bigcup_{w'} \mathcal{M}_{0,0}([h,w],[h',w'])$$

where the right hand side is as in Proposition 12.55. (Note we fix w and take a sum over w' in the right hand side.)

We recall

$$\begin{aligned} Z_- &= (D^2 \cap \{\mathrm{Re} z \le 0\}) \cup ([0,\infty) \times [0,1]) \\ Z_+ &= ((-\infty,0] \times [0,1]) \cup (D^2 \cap \{\mathrm{Re} z \ge 0\}). \end{aligned}$$

The Dolbeault operators

$$\overline{\partial}_{\lambda \oplus TR_h, Z_\pm} : W^{1,p}_{\lambda \oplus TR_h}(Z_\pm; T_0M) \to L^p(Z_\pm; T_pM \oplus \Lambda^{0,1}(Z_\pm))$$

are defined in §12.5. (See right before Definition 12.62.)

$\{\overline{\partial}_{\lambda \oplus TR_h, Z_\pm}\}_{\lambda \oplus TR_h \in \mathcal{P}_{R_h}(TL^{(0)}, TL^{(1)})}$ is a family of elliptic operators parameterized by $\mathcal{P}_{R_h}(TL^{(0)}, TL^{(1)})$. We will show that there exists a fiber bundle $\widetilde{\mathcal{I}}(R_h) \to \mathcal{P}_{R_h}(TL^{(0)}, TL^{(1)})$ such that the pull-back of the determinant line bundle descends to R_h.

We note that Remark 44.15 (1) is generalized to our situation in the following way. Using the notations there, and putting $\lambda_p(t) = \lambda(t) \oplus TR_h$, we have isomorphisms

$$\mathrm{Index}(\overline{\partial}_{u_p;\lambda_p,\lambda_p}) \cong \mathrm{Index}(\overline{\partial}_{\lambda_p,Z_+}) \oplus T_pR_h \oplus \mathrm{Index}(\overline{\partial}_{\lambda_p,Z_-})$$

and

$$\mathrm{Index}(\overline{\partial}_{u_p;\lambda_p,\lambda_p}) \cong T_pL^{(0)}.$$

Therefore the orientation of $T_pR_h \oplus \mathrm{Index}(\overline{\partial}_{\lambda\oplus TR_h,Z_-})$ determines the orientation on $\mathrm{Index}(\overline{\partial}_{\lambda\oplus TR_h,Z_+})$.

Pick and fix $\lambda_p \in \mathcal{P}_{R_h}(T_pL^{(0)}, T_p^{(1)})$. Gluing the operator $\overline{\partial}_{\lambda_p,Z_+}$ and a family of operators $\overline{\partial}_{\lambda'_p,Z_-}$ parameterized by $\lambda'_p \in \mathcal{P}_{R_h}(T_pL^{(0)}, T_p^{(1)})$, we obtain a family of Dolbeault operators on D^2 with totally real boundary condition parameterized by $\mathcal{P}_{R_h}(T_pL^{(0)}, T_p^{(1)})$. In a similar way as in Remark 44.15 (3), we find that the determinant line bundle of this family is non-trivial on each connected component of $\mathcal{P}_{R_h}(T_pL^{(0)}, T_p^{(1)})$. Since $T_pL^{(0)}$ is oriented and the operator $\overline{\partial}_{\lambda_p,Z_+}$ does not depend on λ'_p, the determinant line bundle of the family $\overline{\partial}_{\lambda'_p,Z_-}$ is non-trivial.

Now we introduce a family version of $\mathcal{I}(p)$ and $\widetilde{\mathcal{I}}(p)$ in §44.3. Denote by $\mathcal{I}(R_h)$ the space of paths λ_p and trivializations $\sigma : [0,1] \times \mathbb{R}^n \to \widetilde{\lambda}_p$. For σ, we consider $\iota_i : (Spin(n) \times Spin(V_p))/\{\pm\} \to P_{spin}(L^{(i)}, V)|_p$, which is a lift of $\sigma_{t=i*}$, $i = 0, 1$. We define $\widetilde{\mathcal{I}}(R_h)$ as the space of quadruples $(\lambda_p, \sigma, \iota_0, \iota_1)$. Note that there is a sequence of natural projections

$$\Pi : \widetilde{\mathcal{I}}(R_h) \to \mathcal{I}(R_h) \to \mathcal{P}_{R_h}(TL^{(0)}, TL^{(1)}) \to R_h.$$

Denote by $\mathcal{D}(R_h)^-$ the pull-back of the family $\{\overline{\partial}_{\lambda'_p,Z_-}\}$ of operators to $\widetilde{\mathcal{I}}(R_h)$. Then we can show the following:

Proposition 51.1. *The determinant line bundle of the index bundle of the family $\mathcal{D}(R_h)^-$ descends to a real line bundle on R_h.*

Proof. Note that $\Pi^{-1}(p) \subset \widetilde{\mathcal{I}}(R_h)$ is not connected. (This is because

$$\mathcal{P}_{R_h}(T_pL^{(0)}, T_pL^{(1)}) \subset \mathcal{P}_{R_h}(TL^{(0)}, TL^{(1)}), \quad p \in R_h$$

is not connected.) Thus we firstly present the way to compare the determinant lines of the indices for λ and λ' in different connected components.

Let $\lambda \oplus T_pR_h \in \mathcal{P}_{R_h}(T_pL^{(0)}, T_pL^{(1)})$. We can twist the trivial pair $Z_- \times T_pM$ by gluing a complex vector bundle E of rank n on $\mathbb{C}P^1$ as follows. Pick an isomorphism from T_pM to the fiber E_S at the south pole $S \in \mathbb{C}P^1$ and identify the fiber of $Z_- \times T_pM$ at $O \in Z_-$ and E_S to obtain a vector bundle $(Z_- \times T_pM) \vee E$ over the one-point union $Z_- \vee \mathbb{C}P^1$. We denote by

$$cont : Z_- \to Z_- \vee \mathbb{C}P^1$$

the mapping obtained by collapsing a small circle around $O \in Z_-$ and write

$$(Z_- \times T_pM, \lambda \oplus T_pR_h)\#(\mathbb{C}P^1, E) = cont^*((Z_- \times T_pM, \lambda \oplus T_pR_h) \vee E).$$

For each $\lambda \oplus T_pR_h, \lambda' \oplus T_pR_h \in \mathcal{P}_{R_h}(T_pL^{(0)}, T_pL^{(1)})$, there exists a complex vector bundle E on $\mathbb{C}P^1$ such that

$$(Z_- \times T_pM, \lambda' \oplus T_pR_h) \cong (Z_- \times T_pM, \lambda \oplus T_pR_h)\#(\mathbb{C}P^1, E).$$

Therefore, by the index sum formula, we have

$$\mathrm{Index}(\overline{\partial}_{\lambda' \oplus TR_h, Z_-}) \cong \mathrm{Index}(\overline{\partial}_E) \oplus \mathrm{Index}(\overline{\partial}_{\lambda \oplus TR_h, Z_-}).$$

Note that the index of the Dolbeault operator $\overline{\partial}_E$ has a canonical orientation as a complex virtual vector bundle. Hence the orientation of $\mathrm{Index}(\overline{\partial}_{\lambda' \oplus TR_h, Z_-})$ determines the orientation of $\mathrm{Index}(\overline{\partial}_{\lambda \oplus TR_h, Z_-})$ in the way independent of the choice of E.

Next we prove that the determinant line bundle of $\mathcal{D}(R_h)$ restricted to each connected component of $\Pi^{-1}(p)$ is trivial. Combining the above argument, the determinant line bundle descends to R_h. (Since the determinant line bundle is a real line bundle, its structure group can be reduced to $\{\pm 1\}$.) Pick and fix $(\lambda_p, \sigma, \iota_0, \iota_1) \in \Pi^{-1}(p)$. Consider the family $\mathcal{D}_p = \{\overline{\partial}_{\lambda'_p, Z_-}\}$ parameterized by $(\lambda'_p, \sigma', \iota'_0, \iota'_1) \in \Pi^{-1}(p)$ through $\mathcal{P}_{R_h}(T_pL^{(0)}, T_pL^{(1)})$. Glue $\mathcal{D}_p$ with $\overline{\partial}_{\lambda_p, Z_+}$ to obtain a family of Dolbeault operators on D^2. Since $(\lambda_p, \sigma, \iota_0, \iota_1)$ and $(\lambda'_p, \sigma', \iota'_0, \iota'_1)$ determine the spin structure on the family of totally real subbundles in a consistent way, the determinant line bundle of the family of Dolbeault operators is trivial on $\Pi^{-1}(p)$. Hence also the determinant line bundle of $\mathcal{D}_p$. Proposition 51.1 follows. □

Definition 51.2. We denote by $\Theta^{\pm}_{R_h}$ the local system on R_h, which is obtained by Proposition 51.1 from the determinant of the index bundle of $\overline{\partial}_{\lambda \oplus TR_h, Z_{\pm}}$.

In the situation of finite dimensional Bott-Morse theory, the local systems $\Theta^{+}_{R_h}$ and $\Theta^{-}_{R_h}$ correspond to the orientation bundles of the positive definite part (stable direction) and the negative definite part (unstable direction) of the restriction of the Hessian to the normal bundle, respectively.

Note that R_h is not necessarily orientable and the space of connecting orbits may not be orientable. Hence, we must deal with fiber product over a non-orientable space. In order to treat them, we introduce the notion of the orientation bundle of a space with Kuranishi structure and present its fundamental properties.

Definition 51.3. For a local chart $(s; E \to U)$ of a Kuranishi structure with tangent bundle, we call $\det E \otimes \det TU$ the *orientation bundle*.

The orientation bundles of local charts (Kuranishi neighborhoods) are naturally glued to a line bundle over the space with Kuranishi structure, if the Kuranishi structure has a tangent bundle. (Definition A1.14.) We call it the *orientation bundle of the space with Kuranishi structure.*

Next we give our convention on the identification of orientation bundles of fiber products.

Convention 51.4. Let O_X, O_Y denote the orientation bundles of the spaces X, Y with Kuranishi structure with tangent bundle. Let $f : X \to B$ and $g : Y \to B$ be weakly submersive strongly continuous maps to a manifold B. Let O_B be the orientation bundle of B.

We identify the orientation bundle $O_{X_f \times_g Y}$ of the fiber product $X_f \times_g Y$ with

$$pr_X^*(O_X \otimes f^*O_B) \otimes \pi^*O_B \otimes pr_Y^*(g^*O_B \otimes O_Y).$$

The above real line bundle is naturally identified also with

$$pr_X^*O_X \otimes \pi^*O_B \otimes pr_Y^*O_Y$$

where pr_X and pr_Y are projections to X and Y respectively and π is the projection to B.

We emphasize that the order of factors in the above tensor products is essential.

For instance, in the case when X and Y are manifolds, the identification above corresponds to the isomorphism : $O_X \otimes O_Y \to O_{X\times Y}$ such that $s \otimes t \mapsto s \wedge t$.

We can prove the following properties in the same way as Lemma 45.3.

Lemma 51.5. *We have the following identifications.*

(1) $O_{\partial(X\times_B Y)}|_{\partial X\times_B Y} \cong O_{\partial X\times_B Y}, \quad O_{\partial(X\times_B Y)}|_{X\times_B \partial Y} \cong (-1)^{x+y} O_{X\times_B \partial Y}.$

(2) $O_{(X_1\times_{B_1} X_2)\times_{B_2} X_2} \cong O_{X_1\times_{B_1}(X_2\times_{B_2} X_3)}.$

(3) $O_{X_1\times_{B_1\times B_2}(X_2\times X_3)} \cong (-1)^{b_2(b_1+x_2)} O_{(X_1\times_{B_1} X_2)\times_{B_2} X_3}.$ *More generally, we have*

$$O_{X_1\times_{B_1\times\cdots\times B_l}(X_2\times\cdots X_{l+1})} \cong (-1)^{\sum_{k=2}^{l} b_k \sum_{j=2}^{k}(b_{j-1}+x_j)} O_{(\cdots(X_1\times_{B_1} X_2)\cdots)\times_{B_l} X_{l+1}}.$$

We use Lemma 51.5 to study orientation bundle of the moduli space $\mathcal{M}(R_{h'}, R_h)$ introduced at the beginning of this section.

Proposition 51.6. *A relative spin structure for the pair* $(L^{(0)}, L^{(1)})$ *induces an isomorphism*

$$O_{\widetilde{\mathcal{M}}(R_{h'},R_h)} \cong ev_0^*\Theta^+_{R_{h'}} \otimes ev_1^*\Theta^-_{R_h}$$

in a canonical way. We also have a canonical isomorphism

$$O_{\mathcal{M}(R_{h'},R_h)} \cong ev_0^*\Theta^+_{R_{h'}} \otimes ev_1^*\Theta^-_{R_h} \otimes O_{\mathbb{R}}.$$

Proof. We may assume that the evaluation maps $ev_0, ev_1 : \mathcal{M}(R_{h'}, R_h) \to L$ are weakly submersive. (This follows from a Bott-Morse analog of Proposition 29.1).

Let $u \in \mathcal{M}(R_{h'}, R_h)$. The linearization $D_u\overline{\partial}$ of the Cauchy-Riemann operator is a Fredholm operator. (See §29.2 Lemma 29.10 where $D_u\overline{\partial}$ is defined in the case $R_h = R_{h'} = L$. It is straightforward to generalize it to the general Bott-Morse situation.) We have a homomorphism from the domain of $D_u\overline{\partial}$ to $T_pR_h \oplus T_qR_{h'}$, by identifying T_pR_h, (resp. $T_qR_{h'}$) with the zero eigenspace of $J\frac{d}{dt}$ at z_1, (resp. z_0).

Let $p' = u(+\infty, t) \in R_{h'}$, $p = u(-\infty, t) \in R_h$ and let $\lambda_p \oplus R_h \in \mathcal{P}_{R_h}(T_pL^{(0)}, T_pL^{(1)})$, $\lambda_{p'} \oplus R_{h'} \in \mathcal{P}_{R_{h'}}(T_{p'}L^{(0)}, T_{p'}L^{(1)})$. We trivialize u^*TM and define a path $\lambda(\lambda_{p'}, u, \lambda_p)$ of Lagrangian linear subspaces of the fiber of this trivial bundle by gluing $\tau \mapsto T_{u(\tau,i)}L^{(i)}$, $\lambda_p \oplus R_h$, $\lambda_{p'} \oplus R_{h'}$ in an obvious way.

Then, by a family version of the index sum formula, we have the following isomorphism of the fiber product of the indices. (See Convention 45.1 (4) for the convention of the orientation of the fiber product and (46.9) for the definition of fiber product of index bundles.)

$$\begin{aligned}\text{Index}(\overline{\partial}_{\lambda(\lambda_{p'},u,\lambda_p)}) \cong (&\text{Index } \overline{\partial}_{\lambda_{p'} \oplus TR_{h'}, Z_+} \oplus T_qR_{h'}) \\ &\times_{T_qR_{h'}} \text{Index } D_u\overline{\partial} \times_{T_pR_h} (T_pR_h \oplus \text{Index } \overline{\partial}_{\lambda_p \oplus TR_h, Z_-}).\end{aligned}$$

In the same way as the proof of Theorem 44.1 (and of Theorem 44.14) we can prove that $\text{Index}(\overline{\partial}_{\lambda(\lambda_{p'},u,\lambda_p)})$ has a canonical orientation.

Therefore the orientation of Index $D_u\overline{\partial}$ is determined by the orientations of Index $\overline{\partial}_{\lambda_{p'} \oplus T_qR_{h'}, Z_+} \oplus T_qR_{h'}$, $T_qR_{h'}$, T_pR_h and $T_pR_h \oplus$ Index $\overline{\partial}_{\lambda_p \oplus TR_h, Z_-}$. Since each of T_pR_h and $T_qR_{h'}$ appears twice, the orientations of Index $D_u\overline{\partial}$ is independent of the orientation of $T_qR_{h'}$, T_pR_h. Hence we obtain the first half of Proposition 51.6. The second half follows immediately from the first half and Convention 45.1 (2). □

Recall that we are working with cohomological convention. We adopt the following convention. The orientation on the standard simplex is the usual one. Let (Δ, σ) be a singular simplex in a manifold X, which is not necessarily orientable, with a coefficient in the orientation bundle $O_X = \det TX$. For example, an embedded submanifold with an oriented normal bundle represents an ordinary cochain.

Now, we explain how to work with chains with local coefficients in our construction. Let S be a singular simplex in R_h with coefficient in $O_{R_h} \otimes \Theta^-_{R_h} = \det TR_h \otimes \Theta^-_{R_h}$, which is canonically isomorphic to $\Theta^+_{R_h}$. Recall that a connecting orbit (that is an element of $\mathcal{M}(R_{h'}, R_h)$) is a pseudo-holomorphic map $u : D^2 \to M$ such that $u(\partial_0 D^2) \subset L^{(0)}$, $u(\partial_1 D^2) \subset L^{(1)}$, $u(1) \in R_{h'}$ and $u(-1) \in R_h$, where $\partial_0 D^2$ is the arc with negative imaginary part and $\partial_1 D^2$ is the arc with positive imaginary part.

Proposition 51.7. *Let S be a singular simplex with coefficients in $O_{R_h} \otimes \Theta^-_{R_h}$. Then*

$$(\mathcal{M}(R_{h'}, R_h) \times_{R_h} S, ev_0)$$

is a chain with coefficients in $O_{R_{h'}} \otimes \Theta^-_{R_{h'}}$.

Proof. It is enough to consider the case that S is a singular simplex with coefficient in $O_{R_h} \otimes \Theta^-_{R_h}$. Write $S = (\phi, s)$, where $\phi : \Delta^k \to R_h$ and s is a non-zero flat section of $\phi^*(O_{R_h} \otimes \Theta^-_{R_h})$. Note that s gives an orientation of $T_pR_h \oplus \operatorname{Index} \overline{\partial}_{\lambda_p \oplus TR_h, Z_-}$. Now we take the fiber product of $\mathcal{M}(R_{h'}, R_h)$ and S over R_h. Note that S is equipped with a non-zero flat section of the local system $O_{R_h} \otimes \Theta^-_{R_h}$. Based on Convention 51.4 and Proposition 51.6, we find

$$O_{\widetilde{\mathcal{M}}(R_{h'},R_h)\times_{R_h} S} \cong (\Theta^+_{R_{h'}} \otimes \Theta^-_{R_h}) \otimes O_{R_h} \otimes (O_{R_h} \otimes \Theta^-_{R_h}) \otimes O_{\Delta^k} \cong \Theta^+_{R_{h'}} \otimes O_{\Delta^k}.$$

Note that O_{Δ^k} is trivialized by the standard orientation. Since $O_{R_h} \otimes \Theta^-_{R_h}$ is trivialized by s, we have

$$O_{\widetilde{\mathcal{M}}(R_{h'},R_h)\times_{R_h} S} \cong \Theta^+_{R_{h'}} \cong O_{R_{h'}} \otimes \Theta^-_{R_{h'}}.$$

Finally, we find that

$$O_{\mathcal{M}(R_{h'},R_h)\times_{R_h} S} \cong \Theta^+_{R_{h'}} \otimes O_{\mathbb{R}} \cong O_{R_{h'}} \otimes \Theta^-_{R_{h'}} \otimes O_{\mathbb{R}}.$$

A choice of orientation on $\mathcal{M}(R_{h'}, R_h) \times_{R_h} S$ at a point z (in the sense of Kuranishi structure) determines a non-zero element of the fiber of $ev_0^*\Theta^+_{R_{h'}} \cong ev_0^*(O_{R_{h'}} \otimes \Theta^-_{R_{h'}})$ at z. If we reverse the choice of the orientation, the element of $ev_0^*(O_{R_{h'}} \otimes \Theta^-_{R_{h'}})|_z$ is multiplied by -1. Note also that we have a canonical orientation on $O_{\mathbb{R}}$. Hence we obtain, from the space $\mathcal{M}(R_{h'}, R_h) \times_{R_h} S$ with Kuranishi structure, a chain with coefficients in $ev_0^*(O_{R_{h'}} \otimes \Theta^-_{R_{h'}})$. □

Taking Proposition 51.7 into account, we can forget the effects of local coefficients and pretend as if we can work with ordinary oriented chains from now on. We will further modify the orientation to define the filtered A_∞ bimodule operation (see Definition 51.11).

Hereafter $P_1, \dots, P_k$ stand for smooth singular simplexes in either $L^{(0)}$ or $L^{(1)}$. Here the order of the marked points on the boundary of the infinite cylinder $\mathbb{R} \times [0, 1]$ is arbitrary and may not respect the standard ordering. Namely we require no particular rule on the order of the marked points on $\mathbb{R} \times \{0, 1\}$. Moreover whether $P_i \subset L^{(0)}$ or $P_i \subset L^{(1)}$ has no relation to the order i. We denote this moduli space by

$$(*) \qquad \mathcal{M}_{k+2}((R_{h'}, w'), (R_h, w)) \times_{R_h \times L \times \cdots \times L} (S \times P_1 \times \cdots \times P_k).$$

See Figure 51.1. We remark that in case

$$(P_1, \dots, P_k) = (P^{(1)}_1, \dots, P^{(1)}_{k_1}, P^{(0)}_1, \dots, P^{(0)}_{k_0})$$

with $P_j^{(i)}$ being a smooth singular simplex of $L^{(i)}$, the moduli space $(*)$ contains (as an open susbset) the moduli space

$$\mathcal{M}_{k_1,k_0,(+\infty)}(L^{(1)}, L^{(0)}; [h', w'], [h, w]; \vec{P}^{(1)}, S, \vec{P}^{(0)}),$$

which is defined in §12.5 right after Remark 12.58 and is used to define filtered A_∞ bimodule structure in Definition 12.71. (We like to remark again the the order of $[h, w]$, $[h'w']$ is reversed.) We are going to define an orientation on the moduli space (*).

Figure 51.1

Let S be a singular simplex in R_h. We define its degree by

$$\deg S = \mu([h, w]) + \dim R_h - \dim S. \tag{51.8}$$

(See just before Definition 12.52.) We remark that

$$\mu([h, w]) \equiv \operatorname{Index}(\overline{\partial}_{\lambda \oplus TR_h, Z_-}) \mod 2$$

for any $\lambda \oplus TR_h \in \mathcal{P}_{R_h}(T_pL^{(0)}, T_pL^{(1)})$. (This is the consequence of the independence (modulo 2) of the right hand side and the definition of $\mu([h, w])$, Definition 12.62.)

For computation, we introduce a temporary convention as follows. The final orientation convention will be given in Definition 51.11. We put

$$\begin{aligned}
&\mathcal{M}_1((R_{h'}, w'), (R_h, w); S, P_1, \dots, P_k) \\
&= (-1)^\epsilon \mathcal{M}_{k+2}((R_{h'}, w'), (R_h, w)) \times_{R_h \times L \times \dots \times L} (S \times P_1 \times \dots \times P_k),
\end{aligned}$$

where

$$\epsilon = k(n+1)\mu([h, w]) + \mu([h', w']) + k(n+1)\deg S + (n+1)\sum_{j=1}^{k-1}\sum_{i=1}^{j} \deg P_i.$$

Remark 51.9. When we exchange the marked points z_i, z_{i+1} with $i > 1$ (note that $z_1 = -\infty$) and P_{i-1}, P_i at the same time, we have an isomorphism

$$\begin{aligned}&\mathcal{M}_1((R_{h'}, w'), (R_h, w); S, P_1, \dots, P_{i-1}, P_i, \dots, P_k)\\ &\to \mathcal{M}_1((R_{h'}, w'), (R_h, w); S, P_1, \dots, P_i, P_{i-1}, \dots, P_k),\end{aligned}$$

which is a $(-1)^{(\deg P_{i-1}+1)(\deg P_i+1)}$-oriented isomorphism. The proof of this fact is similar to the proof of Lemma 47.3.

From now on, we abbreviate the data w, w', etc., and simply write

$$\mathcal{M}_1(R_{h'}, R_h; S, P_1, \dots, P_k) = \mathcal{M}_1((R_{h'}, w'), (R_h, w); S, P_1, \dots, P_k).$$

We also write $\mu(R_h)$ in place of $\mu([h, w])$ mod 2.

There are four types of boundaries of the moduli space $\mathcal{M}_1(R_{h'}, R_h; S, P_1, \dots, P_k)$. The first type appears when S is replaced by ∂S. The second type appears when P_i is replaced by ∂P_i. The third type appears when connecting orbits split and become broken connecting orbits. The third type is described by

$$\mathcal{M}_1(R_{h'}, R_{h''}; \mathcal{M}_1(R_{h''}, R_h; S, P_1, \dots, P_{k_1}), P_{k_1+1}, \dots, P_{k_1+k_2}),$$

after interchanging the marked points and P_i's. In (3) of the next theorem we put $k = k_1$, $\ell = k_2$, $Q_i = P_{k_1+i}$, and write the above moduli space

$$\mathcal{M}_1(R_{h'}, R_{h''}; \mathcal{M}_1(R_{h''}, R_h; S, P_1, \dots, P_k), Q_1, \dots, Q_\ell).$$

The fourth type appears in case of the bubbling-off of holomorphic discs.

Theorem 51.10. *Let $R_{h_1}, R_{h_2}, R_{h_3}$ be connecting components of $L^{(0)} \cap L^{(1)}$. Write $r_i = \dim R_{h_i}$ and $\mu_i \equiv \mu(R_{h_i}) \mod 2$. Then we have the following :*

(1)

$$(-1)^{\epsilon_1}\mathcal{M}_1(R_{h_2}, R_{h_1}; \partial S, P_1, \dots, P_k) \subset \partial\mathcal{M}_1(R_{h_2}, R_{h_1}; S, P_1, \dots, P_k),$$

where $\epsilon_1 = (\mu_2 + r_2) + (\mu_1 + r_1) + 1$.

(2)

$$(-1)^{\epsilon_2}\mathcal{M}_1(R_{h_2}, R_{h_1}; S, P_1, \dots, \partial P_i, \dots, P_k) \subset \partial\mathcal{M}_1(R_{h_2}, R_{h_1}; S, P_1, \dots, P_i, \dots, P_k),$$

where $\epsilon_2 = (\mu_2 + r_2) + n + (\deg S + 1) + \sum_{j=1}^{i-1}(\deg P_j + 1) + 1$.

(3)

$$\begin{aligned}&(-1)^{\epsilon_3}\mathcal{M}_1(R_{h_3}, R_{h_2}; \mathcal{M}_1(R_{h_2}, R_{h_1}; S, P_1, \dots, P_k), Q_1, \dots, Q_\ell)\\ \subset&\partial\mathcal{M}_1(R_{h_3}, R_{h_1}; S, P_1, \dots, P_k, Q_1, \dots, Q_\ell),\end{aligned}$$

where $\epsilon_3 = (\mu_3 + r_3) + 1.$
(4)

$$
\begin{aligned}
&(-1)^{\epsilon_4}\mathcal{M}_1(R_{h_2}, R_{h_1}; S, \mathcal{M}_{k+1}(\beta; P_1, \dots, P_k), Q_1, \dots, Q_\ell) \\
\subset &\partial\mathcal{M}_1(R_{h_2}, R_{h_1}; S, P_1, \dots, P_k, Q_1, \dots, Q_\ell),
\end{aligned}
$$

where $\epsilon_4 = (\mu_2 + r_2) + \deg S.$

Proof. First of all, we note that $\dim \mathcal{M}(R_2, R_1) = \mu_2 - \mu_1 + r_2 - 1$.

(1) By (51.8), we have

$$
\mathcal{M}_1(R_{h_2}, R_{h_1}; \partial S, P_1, \dots, P_k) = (-1)^{a_1}\mathcal{M}_{k+2}(R_{h_2}, R_{h_1}) \times_{R_{h_1}\times L\times\cdots\times L} \left(\partial S \times \prod_{i=1}^{k} P_i\right),
$$

where

$$
a_1 = k\mu_1(n+1) + \mu_2 + k(n+1)\deg \partial S + (n+1)\sum_{j=1}^{k-1}\sum_{i=1}^{j} \deg P_i.
$$

We also have

$$
\mathcal{M}_1(R_{h_2}, R_{h_1}; S, P_1, \dots, P_k) = (-1)^{a_2}\mathcal{M}_{k+2}(R_{h_2}, R_{h_1}) \times_{R_{h_1}\times L\times\cdots\times L} \left(S \times \prod_{i=1}^{k} P_i\right),
$$

where

$$
a_2 = k\mu_1(n+1) + \mu_2 + k(n+1)\deg S + (n+1)\sum_{j=1}^{k-1}\sum_{i=1}^{j} \deg P_i.
$$

By Lemma 45.3 (1), we have

$$
\begin{aligned}
&(-1)^{a_3}\mathcal{M}_{k+2}(R_{h_2}, R_{h_1}) \times_{R_{h_1}\times L\times\cdots\times L} \left(\partial S \times \prod_{i=1}^{k} P_i\right) \\
\subset &\partial\mathcal{M}_{k+2}(R_{h_2}, R_{h_1}) \times_{R_{h_1}\times L\times\cdots\times L} \left(S \times \prod_{i=1}^{k} P_i\right),
\end{aligned}
$$

where

$$
a_3 = (r_2 + \mu_2) - \mu_1 + k - 1 + r_1 + kn.
$$

Hence we have $\epsilon_1 \equiv a_1 + a_2 + a_3 \equiv (r_2 + \mu_2) + (r_1 + \mu_1) + 1$.

(2) Similarly, we have

$$\begin{aligned}&\mathcal{M}_1(R_{h_2}, R_{h_1}; S, P_1, \dots \partial P_i, \dots, P_k)\\&= (-1)^{b_1}\mathcal{M}_{k+2}(R_{h_2}, R_{h_1}) \times_{R_{h_1}\times L\times\cdots\times L} \left(S \times \prod_{j=1}^k P'_j\right),\end{aligned}$$

where $P'_j = P_j, j \neq i$, $P'_i = \partial P_i$,

$$b_1 = k\mu_1(n+1) + \mu_2 + k(n+1)\deg S + (n+1)\sum_{j=1}^{k-1}\sum_{h=1}^{j}\deg P'_h.$$

Note that $(-1)^c S \times \prod_{j=1}^k P'_j \subset \partial(S \times \prod_{j=1}^k P_j)$, with $c = \dim S + \sum_{j=1}^{i-1}\dim P_j$. By Lemma 45.3 (1), we have

$$\begin{aligned}&(-1)^{b_3}\mathcal{M}_{k+2}(R_{h_2}, R_{h_1}) \times_{R_{h_1}\times L\times\cdots\times L} \left(S \times \prod_{j=1}^k P'_j\right)\\&\subset\partial\mathcal{M}_{k+2}(R_{h_2}, R_{h_1}) \times_{R_{h_1}\times L\times\cdots\times L} \left(S \times \prod_{j=1}^k P_j\right),\end{aligned}$$

where

$$b_3 = a_3 + c = (r_2 + \mu_2) - \mu_1 + k - 1 + r_1 + kn + \dim S + \sum_{j=1}^{i-1}\dim P_j.$$

Hence we have $\epsilon_2 \equiv b_1 + a_2 + b_3 \equiv (r_2 + \mu_2) + n + (\deg S + 1) + \sum_{j=1}^{i-1}(\deg P_j + 1) + 1$.

(3) Based on our orientation convention for the index of the linearized operator of connecting orbits and Convention 45.1 (4), we can see that

$$\widetilde{\mathcal{M}}(R_{h_3}, R_{h_1}) = \widetilde{\mathcal{M}}(R_{h_3}, R_{h_2}) \times_{R_{h_2}} \widetilde{\mathcal{M}}(R_{h_2}, R_{h_1}).$$

Then, an analogue of Proposition 46.3 holds, i.e., if we put the ℓ marked points before the k marked points as in §46, we have

$$(-1)^{d_0}\mathcal{M}_{k+2}(R_{h_3}, R_{h_2}) \times_{R_{h_2}} \mathcal{M}_{\ell+2}(R_{h_2}, R_{h_1}) \subset \partial\mathcal{M}_{k+\ell+2}(R_{h_3}, R_{h_1}),$$

where $d_0 = k\ell + k(\mu_2 - \mu_1) + (\mu_3 - \mu_2) + k - 1 + r_3$.

By the definition, we have

$$\begin{aligned}&\mathcal{M}_1(R_{h_3}, R_{h_2}; \mathcal{M}_1(R_{h_2}, R_{h_1}; S, P_1, \ldots, P_\ell), Q_1, \ldots, Q_k)\\ =&(-1)^{d_1}\mathcal{M}_{k+2}(R_{h_3}, R_{h_2}) \times_{R_{h_2}\times L\times\cdots\times L} \left(\mathcal{M}_1(R_{h_2}, R_{h_1}; S, P_1, \ldots, P_\ell) \times \prod_{i=1}^{k} Q_i\right),\end{aligned}$$

where

$$\begin{aligned}d_1 =&k(n+1)\mu_2 + \mu_3 + k(n+1)\{\deg \mathcal{M}_1(R_{h_2}, R_{h_1}; S, P_1, \ldots, P_\ell)\}\\ &+ (n+1)\sum_{i=1}^{k-1}\sum_{j=1}^{i} \deg Q_j.\end{aligned}$$

Note that

$$\begin{aligned}&\deg \mathcal{M}_1(R_{h_2}, R_{h_1}; S, P_1, \ldots, P_\ell)\\ &= r_2 + \mu_2 - \Big(r_2 + \mu_2 - \mu_1 + \ell - 1 - (\deg S - \mu_1) - \sum \deg P_i\Big)\\ &= -\ell + 1 + \deg S + \sum \deg P_i.\end{aligned}$$

We also have, by the definition,

$$\mathcal{M}_1(R_{h_2}, R_{h_1}; S, P_1, \ldots, P_\ell) = (-1)^{d_2}\mathcal{M}_{\ell+2}(R_{h_2}, R_{h_1}) \times_{R_{h_1}\times L\times\cdots\times L} (S \times \prod_{i=1}^{\ell} P_i),$$

where $d_2 = \ell(n+1)\mu_1 + \mu_2 + \ell(n+1)\deg S + (n+1)\sum_{j=1}^{\ell-1}\sum_{i=1}^{j} \deg P_i$.

Using Lemma 45.3 (3) several times, we find that

$$\begin{aligned}&\mathcal{M}_{k+2}(R_{h_3}, R_{h_2})\times_{R_{h_2}\times L\times\cdots\times L}\\ &\qquad\left(\mathcal{M}_{\ell+2}(R_{h_2}, R_{h_1}) \times_{R_{h_1}\times L\times\cdots\times L} \left(S \times \prod_{i=1}^{\ell} P_i\right)\right) \times \prod_{j=1}^{k} Q_j\\ =&(-1)^{d_3}\mathcal{M}_{k+2}(R_{h_3}, R_{h_2}) \times_{R_{h_2}} \mathcal{M}_{\ell+2}(R_{h_2}, R_{h_1}) \times_{R_{h_1}\times L\times\cdots\times L} \left(S \times \prod_{i=1}^{\ell} P_i \times \prod_{j=1}^{k} Q_j\right),\end{aligned}$$

where $d_3 = kn(\mu_2 - \mu_1 + \ell - 1)$.

$$\begin{aligned}&\mathcal{M}_1(R_{h_3}, R_{h_1}; S, P_1, \ldots, P_\ell, Q_1, \ldots, Q_k)\\ =&(-1)^{d_4}\mathcal{M}_{k+\ell+2}(R_{h_3}, R_{h_1}) \times_{R_{h_1}\times L\times\cdots\times L} \left(S \times \prod_{i=1}^{\ell} P_i \times \prod_{j=1}^{k} Q_j\right),\end{aligned}$$

where

$$d_4 =(k+\ell)(n+1)\mu_1 + \mu_3 + (k+\ell)(n+1)\deg S + (n+1)\sum_{j=1}^{\ell-1}\sum_{i=1}^{j}\deg P_i + k(n+1)\sum_{i=1}^{\ell}\deg P_i + (n+1)\sum_{j=1}^{k-1}\sum_{i=1}^{j}\deg Q_i.$$

Combining these contributions, we find that $\epsilon_3 = d_0+d_1+d_2+d_3+d_4 = (\mu_3+r_3)+1$.

The last statement (4) can be proved in a similar way to Proposition 48.1 (1). □

In order to define the filtered A_∞ bimodule structures, we consider the case where the marked points respect the standard ordering, i.e., $z_0 = +\infty$, $z_1, \dots, z_{k_1}$ are on $\mathbb{R} \times \{1\}$ such that $\mathrm{Re} z_1 > \cdots > \mathrm{Re} z_{k_1}$, $z_{k_1+1} = -\infty$, and $z_{k_1+2}, \dots, z_{k_1+k_2+1}$ are on $\mathbb{R} \times \{0\}$ such that $\mathrm{Re} z_{k_1+2} < \cdots < \mathrm{Re} z_{k_1+k_2+1}$. See Remark 12.23 (2). Taking Remark 51.9 and Theorem 51.10 into account, we adopt the following:

Definition 51.11. (1) Let $P_1^{(1)}, \dots, P_{k_1}^{(1)}$ be smooth singular simplexes in $L^{(1)}$ and $P_1^{(0)}, \dots, P_{k_0}^{(0)}$ smooth singular simplexes in $L^{(0)}$. We define the orientation of the moduli space $\mathcal{M}_{k_1,k_0,(+\infty)}(L^{(1)}, L^{(0)}; [h,w],[h',w']; \vec{P}^{(1)}, S, \vec{P}^{(0)})$ which is used in §12.5 by the following formula.

$$\begin{aligned}&\mathcal{M}_{k_1,k_0,(+\infty)}(L^{(1)}, L^{(0)}; [h,w],[h',w']; \vec{P}^{(1)}, S, \vec{P}^{(0)})\\ &= (-1)^{\delta}\mathcal{M}_1(R_{h'}, R_h; S, P_1, \dots, P_k),\end{aligned}$$

where

$$\delta = (\deg S + 1)\sum_{i=1}^{k_1}(\deg P_i^{(1)} + 1)$$

and

$$(P_1, \dots, P_k) = (P_1^{(1)}, \dots, P_{k_1}^{(1)}, P_1^{(0)}, \dots, P_{k_0}^{(0)}).$$

The moduli space in the right hand side is used to define $\mathfrak{n}_{k_1,k_0}$ in Definition 12.71.
(2) For the classical contribution $\overline{\mathfrak{n}}_{0,0}$ to $\mathfrak{n}_{0,0}$, we define

$$\overline{\mathfrak{n}}_{0,0} = (-1)^{\dim R_h + \mu(R_h)}\partial$$

for chains in R_h, where ∂ is the usual boundary operator for chains in R_h.

In this new convention (Definition 51.11), Theorem 51.10 then implies the following.

(51.12.1)

$$\begin{aligned}&(-1)^{\delta_1}\mathcal{M}_{k_1,k_0,(+\infty)}(R_{h_2}, R_{h_1}; P_1^{(1)}, \dots, P_{k_1}^{(1)}, \partial S, P_1^{(0)}, \dots, P_{k_0}^{(0)})\\ \subset&\partial\mathcal{M}_{k_1,k_0,(+\infty)}(R_{h_2}, R_{h_1}; P_1^{(1)}, \dots, P_{k_1}^{(1)}, S, P_1^{(0)}, \dots, P_{k_0}^{(0)}),\end{aligned}$$

where

$$\delta_1 = (\mu_2 + r_2) + (\mu_1 + r_1) + \sum_{j=1}^{k_1} (\deg P_j^{(1)} + 1) + 1.$$

(51.12.2)

$$\begin{aligned}&(-1)^{\delta_2}\mathcal{M}_{k_1,k_0,(+\infty)}(R_{h_2}, R_{h_1}; P_1^{(1)}, \dots, \partial P_i^{(1)}, \dots, P_{k_1}^{(1)}, S, P_1^{(0)}, \dots, P_{k_0}^{(0)}) \\ \subset &\partial\mathcal{M}_{k_1,k_0,(+\infty)}(R_{h_2}, R_{h_1}; P_1^{(1)}, \dots, P_{k_1}^{(1)}, S, P_1^{(0)}, \dots, P_{k_0}^{(0)}),\end{aligned}$$

where

$$\delta_2 = (\mu_2 + r_2) + n + \sum_{j=1}^{i-1} (\deg P_j^{(1)} + 1) + 1.$$

(51.12.3)

$$\begin{aligned}&(-1)^{\delta_3}\mathcal{M}_{k_1,k_0,(+\infty)}(R_{h_2}, R_{h_1}; P_1^{(1)}, \dots, P_{k_1}^{(1)}, S, P_1^{(0)}, \dots, \partial P_j^{(0)}, \dots, P_{k_0}^{(0)}) \\ \subset &\partial\mathcal{M}_{k_1,k_0,(+\infty)}(R_{h_2}, R_{h_1}; P_1^{(1)}, \dots, P_{k_1}^{(1)}, S, P_1^{(0)}, \dots, P_{k_0}^{(0)}),\end{aligned}$$

where

$$\delta_3 = (\mu_2 + r_2) + n + \sum_{j=1}^{k_1} (\deg P_j^{(1)}) + (\deg S + 1) + \sum_{j=1}^{i-1} (\deg P_j^{(0)} + 1) + 1.$$

$$\begin{aligned}&(-1)^{\delta_4}\mathcal{M}_{i-1,k_0-\ell,(+\infty)}(R_{h_3}, R_{h_2}; P_1^{(1)}, \dots, P_{i-1}^{(1)}, \\ &\mathcal{M}_{k_1-i+1,\ell,(+\infty)}(R_{h_2}, R_{h_1}; P_i^{(1)}, \dots, P_{k_1}^{(1)}, S, P_1^{(0)}, \dots, P_\ell^{(0)}), P_{\ell+1}^{(0)}, \dots, P_{k_0}^{(0)}) \\ &\subset \partial\mathcal{M}_{k_1,k_0,(+\infty)}(R_{h_3}, R_{h_1}; P_1^{(1)}, \dots, P_{k_1}^{(1)}, S, P_1^{(0)}, \dots, P_{k_0}^{(0)}),\end{aligned} \tag{51.12.4}$$

where

$$\delta_4 = (\mu_3 + r_3) + \sum_{j=1}^{i-1} (\deg P_j^{(1)} + 1) + 1.$$

$$\begin{aligned}&(-1)^{\delta_5}\mathcal{M}_{k_1-\ell+i,k_0,(+\infty)}(R_{h_2}, R_{h_1}; P_1^{(1)}, \dots, P_{i-1}^{(1)}, \mathcal{M}_{\ell-i+2}(\beta; P_i^{(1)}, \dots, P_\ell^{(1)}), P_{\ell+1}^{(1)}, \dots, \\ &P_{k_1}^{(1)}, S, P_1^{(0)}, \dots, P_{k_0}^{(0)}) \\ &\subset \partial\mathcal{M}_{k_1,k_0,(+\infty)}(R_{h_2}, R_{h_1}; P_1^{(1)}, \dots, P_{k_1}^{(1)}, S, P_1^{(0)}, \dots, P_{k_0}^{(0)}),\end{aligned} \tag{51.12.5}$$

where

$$\delta_5 = (\mu_2 + r_2) + \sum_{j=1}^{i-1} (\deg P_j^{(1)} + 1) + 1.$$

and

$$\begin{aligned}(-1)^{\delta_6} \mathcal{M}_{k_1, k_0 - \ell + i, (+\infty)}(R_{h_2}, R_{h_1}; &P_1^{(1)}, \dots, P_{k_1}^{(1)}, S, P_1^{(0)}, \dots, P_{i-1}^{(0)}, \\ &\mathcal{M}_{\ell - i + 2}(\beta; P_i^{(0)}, \dots, P_\ell^{(0)}), P_{\ell+1}^{(0)}, \dots, P_{k_0}^{(0)}) \\ \subset \partial \mathcal{M}_{k_1, k_0, (+\infty)}(R_{h_2}, R_{h_1}; &P_1^{(1)}, \dots, P_{k_1}^{(1)}, S, P_1^{(0)}, \dots, P_{k_0}^{(0)}),\end{aligned} \tag{51.12.6}$$

where

$$\delta_6 = (\mu_2 + r_2) + \sum_{j=1}^{k_1} (\deg P_j^{(1)} + 1) + (\deg S + 1) + \sum_{j=1}^{i-1} (\deg P_j^{(0)} + 1) + 1.$$

Hence $\mathfrak{n}_{k,\ell}$ satisfies (12.2) including *sign.* □

§52. Orientation of $\mathcal{M}_{k+1}^{\text{main}}(M', L', \{J_\rho\}_\rho : \beta; \text{top}(\rho))$.

In §19 Proposition 19.1, we used the moduli space $\mathcal{M}_{k+1}^{\text{main}}(M', L', \{J_\rho\}_\rho : \beta; \text{top}(\rho))$ to the define the filtered A_∞ homomorphism. We can reduce the general case to the case that the symplectic diffeomorphism φ is the identity and $\{J_\rho\}_\rho$ is an arbitrary family of almost complex structures compatible with ω. (However, we used (M, L) and (M', L') in order to clarify the domain and the target.) The goal of this section is to give a canonical orientation of

$$\mathcal{M}_{k+1}^{\text{main}}(\{J_\rho\}_\rho : \beta; \text{top}(\rho); P_1, \dots, P_k) := \mathcal{M}_{k+1}^{\text{main}}(M', L', \{J_\rho\}_\rho : \beta; \text{top}(\rho); P_1, \dots, P_k)$$

so that the resulting $\mathfrak{f}$ is a filtered A_∞ homomorphism. We first recall an element of $\mathcal{M}_{k+1}^{\text{main}}(M', L', \{J_\rho\}_\rho : \beta; \text{top}(\rho))$. Let $((\Sigma, \vec{z}), (u_\alpha), (\rho_\alpha))$ be a system satisfying the following properties

(19.7.1) $u_\alpha : (\Sigma_\alpha, \partial\Sigma_\alpha) \to (M, L)$ is a J_{ρ_α} holomorphic map.
(19.7.2) $\rho_\alpha \in [0, 1]$. If $\alpha_1 \le \alpha_2$, then $\rho_{\alpha_1} \le \rho_{\alpha_2}$
(19.7.3) $((\Sigma, \vec{z}), (u_\alpha))$ is stable in the sense of Definition 2.24.
(19.7.4) The homology class of u_α is $\beta(\alpha)$ and $\sum_\alpha \beta(\alpha) = \beta$.
(19.7.5) If $z \in \Sigma_\alpha \cap \Sigma_{\alpha'}$, then $u_\alpha(z) = u_{\alpha'}(z)$.

We denote by $\mathcal{M}_{k+1}^{\rm main}(\beta; P_1, \dots, P_k; J_\rho)$ the moduli space of objects in (10.1) with respect to the almost complex structure J_ρ. One of top dimensional strata of $\mathcal{M}_{k+1}^{\rm main}(M', L', \{J_\rho\}_\rho : \beta; {\rm top}(\rho))$ is the stratum consisting of objects in (19.7) with the domain being irreducible. When the domain is irreducible, we define the orientation by

$$(-1)^{n+1} \bigcup_{0\le\rho\le 1} \{\rho\} \times \mathcal{M}_{k+1}^{\rm main,reg}(\beta; P_1, \dots, P_k; J_\rho), \tag{52.1}$$

which we denote by

$$\mathcal{M}_{k+1}^{\rm main,reg}(\{J_\rho\}_\rho; \beta; {\rm top}(\rho); P_1, \dots, P_k) \subset \mathcal{M}_{k+1}^{\rm main}(\{J_\rho\}_\rho : \beta; {\rm top}(\rho); P_1, \dots, P_k).$$

Recall that, for each fixed J_ρ,

$$\begin{aligned}&\mathcal{M}_{k+1}^{\rm main,reg}(\beta; P_1, \dots, P_k; J_\rho)\\ &= (-1)^{\epsilon_1} \mathcal{M}_{k+1}^{\rm main,reg}(L; \beta; J_\rho)_{(ev_1,\dots,ev_k)} \times_{f_1\times\dots\times f_k} \left(\prod_{i=1}^k P_i\right)\end{aligned}$$

where

$$\epsilon_1 = (n+1)\sum_{j=1}^{k-1}\sum_{i=1}^{j} \deg P_i.$$

See Definition 47.1. Note that P_i appearing in $\mathcal{M}_{k+1}^{\rm main,reg}(\beta; P_1, \dots, P_k; J_\rho)$ are regarded as singular simplexes in L by $P_i \to L$. Then Proposition 48.1 (1) implies that

$$\begin{aligned}&(-1)^{\epsilon_2'+1} \bigcup_{0\le\rho\le 1} \{\rho\}\times\mathcal{M}_{k-\ell+2}^{\rm main,reg}(\beta'; P_1, \dots, P_{i-1}, \mathcal{M}_{\ell+1}^{\rm main,reg}(\beta''; P_i, \dots, P_{i+\ell-1}; J_\rho),\\ &\qquad\qquad\qquad\qquad\qquad\qquad P_{i+\ell}, \dots, P_k; J_\rho)\\ &\subset \partial\mathcal{M}_{k+1}^{\rm main,reg}(\{J_\rho\}_\rho : \beta; {\rm top}(\rho); P_1, \dots, P_k)\end{aligned}$$

where

$$\epsilon_2' = \epsilon_2 - (n+1) = \sum_{j=1}^{i-1}(\deg P_j + 1), \quad \text{and} \quad \beta = \beta' + \beta''.$$

On the other hand, Proposition 48.1 (2) implies that

$$\begin{aligned}&\bigcup_{0\le\rho\le 1} \{\rho\} \times \mathcal{M}_{k-\ell+2}^{\rm main,\ reg}(\beta'; P_1, \dots, P_{i-1},\\ &\qquad\qquad \mathcal{M}_{\ell+1}^{\rm main,reg}(\beta''; P_i, \dots, P_{i+\ell-1}; J_\rho), P_{i+\ell}, \dots, P_k; J_\rho)\\ &\subset \mathcal{M}_{k-\ell+2}^{\rm main,reg}(\{J_\rho\}_\rho : \beta; {\rm top}(\rho); P_1, \dots, P_{i-1}, \partial Q_{i,\ell}^{\rho}(\beta''), P_{i+\ell}, \dots, P_k)\\ &\subset (-1)^{\epsilon_3+1}\partial\mathcal{M}_{k-\ell+2}^{\rm main}(\{J_\rho\}_\rho : \beta; {\rm top}(\rho); P_1, \dots, P_{i-1}, Q_{i,\ell}^{\rho}(\beta''), P_{i+\ell}, \dots, P_k),\end{aligned}$$

where

$$\epsilon_3 = 1 + \sum_{j=1}^{i-1} (\deg P_j + 1)$$

and

$$Q^{\rho}_{i,\ell}(\beta'') := (-1)^{n+1} \bigcup_{0 \le \sigma \le \rho} \{\sigma\} \times \mathcal{M}^{\text{main,reg}}_{\ell+1}(\beta''; P_i, \dots, P_{i+\ell-1}; J_\sigma). \tag{52.2}$$

Hence the orientation of

$$\begin{aligned} &\bigcup_{0 \le \rho \le 1} \{\rho\} \times \mathcal{M}^{\text{main,reg}}_{k-\ell+2}(\beta'; P_1, \dots, P_{i-1}, \\ &\qquad \mathcal{M}^{\text{main,reg}}_{\ell+1}(\beta''; P_i, \dots, P_{i+\ell-1}; J_\rho), P_{i+\ell}, \dots, P_k; J_\rho) \end{aligned} \tag{52.3}$$

as the boundary of

$$\mathcal{M}^{\text{main,reg}}_{k+1}(\{J_\rho\}_\rho : \beta; \text{top}(\rho); P_1, \dots, P_k)$$

and its orientation as the boundary of

$$\mathcal{M}^{\text{main}}_{k-\ell+2}(\{J_\rho\}_\rho : \beta; \text{top}(\rho); P_1, \dots, P_{i-1}, Q^{\rho}_{i,\ell}(\beta''), P_{i+\ell}, \dots, P_k)$$

are opposite. Thus the orientation on

$$\mathcal{M}^{\text{main,reg}}_{k+1}(\{J_\rho\}_\rho : \beta; \text{top}(\rho); P_1, \dots, P_k)$$

and the one on

$$\mathcal{M}^{\text{main}}_{k-\ell+2}(\{J_\rho\}_\rho : \beta; \text{top}(\rho); P_1, \dots, P_{i-1}, Q^{\rho}_{i,\ell}(\beta''), P_{i+\ell}, \dots, P_k)$$

match on the codimension 1 stratum (52.3).

More generally, for the moduli space of objects in Definition 19.8, we define the orientation on $\mathcal{M}^{\text{main}}_{k+1}(\{J_\rho\}_\rho : \beta; \text{top}(\rho); P_1, \dots, P_k)$ by the induction on the number of singular points, in other words, the number of irreducible components. For $c \in [0,1]$ and $(\beta, k) \neq (\beta_0, 1)$, we denote by

$$\mathcal{M}^{\text{main},\le c}_{k+1}(\{J_\rho\}_\rho : \beta; \text{top}(\rho); P_1, \dots, P_k)$$

the moduli space of objects as in (19.7) such that all $\rho_\alpha \in [0, c]$.

Suppose that the orientation on $\mathcal{M}^{\text{main},\le c}_{k+1}(\{J_\rho\}_\rho : \beta; \text{top}(\rho); P_1, \dots, P_k)$ is determined for all elements with the domain consisting of at most d irreducible components and all $c \in [0,1]$. Let u be as in (19.7) with $d+1$ irreducible components. Then u belongs to

$$\{\sigma\} \times \mathcal{M}^{\text{main, reg}}_{\ell+1}(\beta'; Q_1, \dots, Q_\ell; J_\sigma),$$

where each Q_j is either (a) one of P_i's or (b) an element in

$$\mathcal{M}^{\text{main},\le\sigma}_{k_{i_j}+1}(\{J_\rho\}_\rho : \beta_i; \text{top}(\rho); P_{i_{j-1}+1}, \dots, P_{i_j}).$$

By the assumption, there must be at least one Q_j of type (b). Note also that the number of irreducible components appearing in the element of Q_j of type (b) is bounded by d. Thus all Q_j are oriented by the hypothesis of the induction. Then we adopt (52.1) for the orientation on

$$\bigcup_{0\le\rho\le c} \{\rho\} \times \mathcal{M}^{\text{main, reg}}_{\ell+1}(\beta'; Q_1, \dots, Q_\ell; J_\rho),$$

which determines the orientation on $\mathcal{M}^{\text{main},\le c}_{k+1}(\{J_\rho\}_\rho : \beta; \text{top}(\rho); P_1, \dots, P_k)$ at u.

In a similar way to the above discussion (the case that $d = 2$), we find that the orientations above match on codimension 1 strata and give a compatible orientation on the moduli space $\mathcal{M}^{\text{main}}_{k+1}(M', L', \{J_\rho\}_\rho : \beta; \text{top}(\rho); P_1, \dots, P_k)$.

Next we discuss the sign problem for the map $\hat{\mathfrak{f}}$ in Theorem 19.1 and Proposition 19.14. We state the following

Proposition 52.4. *Suppose that* $(\beta, k) \neq (\beta_0, 1)$. *The boundary of the moduli space* $\mathcal{M}^{\text{main}}_{k+1}(\{J_\rho\}_\rho : \beta; \text{top}(\rho); P_1, \dots, P_k)$ *is the closure of the strata as follows. Let* $k_\alpha + 1$ *be the number of special points* (*i.e., marked points or singular points*) *on* Σ_α.
Case (1) $\rho_{\alpha_0} = 1$ *and* $(\beta', \ell) \neq (\beta_0, 1)$.

$$(-1)^{n+1}\mathcal{M}^{\text{main}}_{\ell+1}(\beta'; Q_1, \dots, Q_\ell; J') \subset \partial\mathcal{M}^{\text{main}}_{k+1}(\{J_\rho\}_\rho : \beta; \text{top}(\rho); P_1, \dots, P_k),$$

where Q_j *is either one of* P_i*'s* (*then we write* $\beta^{(j)} = \beta_0$) *or*

$$Q_j = \mathcal{M}^{\text{main}}_{k_j+1}(\{J_\rho\}_\rho : \beta^{(j)}; \text{top}(\rho); P_{i_{j-1}+1}, \dots, P_j)$$

with $\beta^{(j)} \neq \beta_0$ *and* $\beta = \beta' + \sum_j \beta^{(j)}$.
Case (2). *Some of* $\rho_\alpha = 0$.

$$\begin{aligned}(-1)^{n+1+\epsilon_3}\mathcal{M}^{\text{main}}_{k-\ell+2}(\{J_\rho\}_\rho : \beta'; \text{top}(\rho); P_1, \dots, P_{i-1}, \\ \mathcal{M}^{\text{main}}_{\ell+1}(\beta''; P_i, \dots, P_{i+\ell-1}), P_{i+\ell}, \dots, P_k) \\ \subset \partial\mathcal{M}^{\text{main}}_{k+1}(\{J_\rho\}_\rho : \beta; \text{top}(\rho); P_1, \dots, P_k),\end{aligned}$$

where

$$\beta'' = \sum_{\alpha'\le\alpha} \beta(\alpha')$$

and

$$\epsilon_3 = 1 + \sum_{j=1}^{i-1} (\deg P_j + 1).$$

Case (3). *P_i is replaced by its boundary ∂P_i.*

$$\begin{aligned}&(-1)^{1+\epsilon_3}\mathcal{M}^{\text{main}}_{k+1}(\{J_\rho\}_\rho : \beta; \text{top}(\rho); P_1, \dots, P_{i-1}, \partial P_i, P_{i+1}, \dots, P_k)\\ \subset &\partial \mathcal{M}^{\text{main}}_{k+1}(\{J_\rho\}_\rho : \beta; \text{top}(\rho); P_1, \dots, P_k).\end{aligned}$$

Case (4). $\rho_{\alpha_0} = 0$.

$$(-1)^n \mathcal{M}^{\text{main}}_{k+1}(\beta; P_1, \dots, P_k; J_0) \subset \partial \mathcal{M}^{\text{main}}_{k+1}(\{J_\rho\}_\rho : \beta; \text{top}(\rho); P_1, \dots, P_k).$$

In Case (2), we have $(\beta(\alpha), k_\alpha) \neq (\beta_0, 1)$ and $(\beta', k-\ell+1) \neq (\beta_0, 1)$ because of the stability condition. Cases (3) and (4) may be regarded as the cases that $(\beta(\alpha), k_\alpha) = (\beta_0, 1)$ and $(\beta', k-\ell+1) = (\beta_0, 1)$, respectively, i.e., the classical boundary operators $\partial = (-1)^n \mathfrak{m}_{1,0}$ on L and L' appear in Case (3) and (4), respectively. In the proof of this proposition, we only need to note that

$$\begin{aligned}&(-1)^{n+1}\mathcal{M}^{\text{main}}_{k+1}(\beta; P_1, \dots, P_k; J_1)\\ \subset &\partial \mathcal{M}^{\text{main}}_{k+1}(\{J_\rho\}_\rho : \beta; \text{top}(\rho); P_1, \dots, P_k),\end{aligned}$$

and

$$\begin{aligned}&(-1)^{n}\mathcal{M}^{\text{main}}_{k+1}(\beta; P_1, \dots, P_k; J_0)\\ \subset &\partial \mathcal{M}^{\text{main}}_{k+1}(\{J_\rho\}_\rho : \beta; \text{top}(\rho); P_1, \dots, P_k),\end{aligned}$$

see (52.1) and Convention 45.1(1). Then we can show Proposition 52.4 in a similar way to Proposition 48.1.

We put

$$\mathfrak{f}_{k,\beta}(P_1, \dots , P_k) = \Big(\mathcal{M}^{\text{main}}_{k+1}(M', L', \{J_\rho\}_\rho : \beta; \text{top}(\rho); P_1, \dots , P_k), ev_0\Big),$$

for $(k, \beta) \neq (1, \beta_0)$ and

$$\mathfrak{f}_{1,\beta_0}(P) = P.$$

It is clear that $\mathfrak{f}_{1,\beta_0} \circ \mathfrak{m}_{1,\beta_0} = \mathfrak{m}_{1,\beta_0} \circ \mathfrak{f}_{1,\beta_0}$.

The strata of Case (1) correspond to

$$(-1)^{n+1}(\mathfrak{m}_{\ell,\beta'} \circ \widehat{\mathfrak{f}}_{\beta''})(P_1, \dots, P_k)$$

with $\beta = \beta' + \beta''$, $(\beta', \ell) \neq (\beta_0, 1)$. Here, recall that

$$\widehat{\mathfrak{f}}_{\beta''} = \sum \mathfrak{f}_{i_1,\beta^{(1)}} \otimes \cdots \otimes \mathfrak{f}_{i_\ell,\beta^{(\ell)}}$$

such that $\beta'' = \beta^{(1)} + \cdots + \beta^{(\ell)}$, see (7.28). The strata of Case (2) correspond to

$$(-1)^n(\mathfrak{f}_{k-\ell+1,\beta'} \circ \widehat{\mathfrak{m}}_{\ell,\beta''})(P_1, \ldots, P_k)$$

with $(\beta', k - \ell + 1) \neq (\beta_0, 1)$ and $(\beta'', \ell) \neq (\beta_0, 1)$. Here, recall that $\widehat{\mathfrak{m}}_{\ell,\beta''}$ is the extension of $\mathfrak{m}_{\ell,\beta''}$ as a graded coderivation, see (7.15). The strata of Case (3) correspond to

$$(-1)^n(\mathfrak{f}_{k,\beta} \circ \widehat{\mathfrak{m}}_{1,\beta_0})(P_1, \ldots, P_k).$$

The strata of Case (4) correspond to

$$(-1)^n(\mathfrak{f}_{1,\beta_0} \circ \mathfrak{m}_{k,\beta})(P_1, \ldots, P_k).$$

On the other hand, $\partial\mathcal{M}^{\text{main}}_{k+1}(\{J_\rho\}_\rho : \beta; \text{top}(\rho); P_1, \ldots, P_k)$ corresponds to

$$(-1)^n(\mathfrak{m}_{1,\beta_0} \circ \mathfrak{f}_{k,\beta})(P_1, \ldots, P_k).$$

Hence, taking $\mathfrak{f}_{1,\beta_0} \circ \mathfrak{m}_{1,\beta_0} = \mathfrak{m}_{1,\beta_0} \circ \mathfrak{f}_{1,\beta_0}$ into account, we find that

$$\mathfrak{m} \circ \widehat{\mathfrak{f}} = \mathfrak{f} \circ \widehat{d}.$$

Finally, we define orientations on the moduli spaces involving the time-wise-product, which are used in previous chapters. For the moduli space

$$\mathcal{M}^{\text{main}}_{k+1}(M', L', \{J_{1,s}\}_s : \beta; \text{twp}(s)) = \bigcup_{s \in (-\epsilon, 1+\epsilon)} \{s\} \times \mathcal{M}^{\text{main}}_{k+1}(L', J_{1,s}), \beta),$$

we give the orientation by putting the parameter $s \in (-\epsilon, 1 + \epsilon)$ before the moduli space $\mathcal{M}^{\text{main}}_{k+1}(L', J_{1,s}), \beta)$. Similarly, for the moduli space

$$\begin{aligned}&\mathcal{M}^{\text{main}}_{k+1}(M', L', \{J_{\rho,s}\}_{\rho,s} : \beta; \text{top}(\rho), \text{twp}(s))\\ &= \bigcup_{s \in (-\epsilon, 1+\epsilon)} \{s\} \times \mathcal{M}^{\text{main}}_{k+1}(M', L', \{J_{\rho,s}\}_\rho : \beta; \text{top}(\rho)),\end{aligned}$$

we give the orientation by putting the parameter $s \in (-\epsilon, 1 + \epsilon)$ before the moduli space $\mathcal{M}^{\text{main}}_{k+1}(M', L', \{J_{\rho,s}\}_\rho : \beta; \text{top}(\rho))$.

§53. Homotopy unit, operators $\mathfrak{p}, \mathfrak{q}$ and continuous family of perturbations.

53.1. Homotopy unit.

We first discuss the orientation of the moduli space $\mathcal{M}^{\text{main}}_{k+1}(\beta; \vec{P}^{+}) \times [0,1]^{|\vec{a}|}$ used in §31. Recall that we start with $P_1, \cdots, P_{k-|a|}$ and insert copies of L at a_j-th positions, where $\vec{a} = (a_1, \ldots, a_{|\vec{a}|})$, to obtain $\vec{P}^{+}$. Write $\vec{P}^{+} = (P_1^{+}, \ldots, P_k^{+})$. Then we put the orientation to the moduli space in Proposition 31.10, by

$$(-1)^{\epsilon}[0,1]^{|\vec{a}|} \times \mathcal{M}^{\text{main}}_{k+1}(\beta; \vec{P}^{+}),$$

where $\epsilon = \sum_{j=1}^{|\vec{a}|} \sum_{i=1}^{a_j-1} (\deg P_i^{+} + 1)$. It is straightforward to check that this orientation convention is compatible with the homotopy unit formulae.

53.2. Operators $\mathfrak{p}$, $\mathfrak{q}$.

In the definition of the operators $\mathfrak{p}, \mathfrak{q}$, etc., we consider the moduli spaces of holomorphic discs with interior marked points on the domain.

Firstly, we consider the case where the zero-th marked point is a boundary marked point. We put

$$\mathcal{M}_{(1,k),\ell}(\beta) = \mathcal{M}_{k+1,\ell}(\beta),$$

that is a moduli space of pseudo-holomorphic discs of the class β with $k+1$ boundary marked points and ℓ interior marked points. We define its orientation by the following equalities;

$$\begin{aligned}\widetilde{\mathcal{M}}_{(1,k),\ell}(\beta) &:= \widetilde{\mathcal{M}}(\beta) \times \partial D_0^2 \times D_1^2 \times \cdots \times D_\ell^2 \times \partial D_{\ell+1}^2 \times \cdots \times \partial D_{\ell+k}^2 \\ \mathcal{M}_{(1,k),\ell}(\beta) &:= \widetilde{\mathcal{M}}_{(1,k),\ell}(\beta)/PSL(2;\mathbb{R}).\end{aligned}$$

Definition 53.1. For smooth singular simplexes $f_i : P_i \to L$ in L and $g_j : Q_j \to M$ in M, we define

$$\begin{aligned}&\mathcal{M}^{\text{main}}_{(1,k),\ell}(\beta; Q_1, \ldots, Q_\ell; P_1, \ldots, P_k) \\ :=&(-1)^{\epsilon}\mathcal{M}^{\text{main}}_{(1,k),\ell}(L;\beta)_{(ev_1^{\text{int}},\ldots,ev_\ell^{\text{int}},ev_1,\ldots ev_k)} \\ &\times_{g_1\times\cdots\times g_\ell\times f_1\times\cdots\times f_k} \left(\prod_{i=1}^{\ell} Q_i \times \prod_{j=1}^{k} P_j\right),\end{aligned}$$

where

$$\epsilon = (n+1)\sum_{j=1}^{k-1}\sum_{i=1}^{j}\deg P_i + ((k+1)(n+1)+1)\sum_{i=1}^{\ell}\deg Q_i.$$

Note that when $\ell = 0$, the moduli space $\mathcal{M}_{(1,k),0}(\beta)$ is nothing but $\mathcal{M}_{k+1}(\beta)$ and the orientation above is the same as in Definition 47.1. In Definition 53.1, we only deal with the main component of the moduli spaces. We also adopt the same orientation convention for other moduli spaces, which will be used in the proof of Proposition 53.4 below. When the zero-th marked point is an interior marked point, we define

$$\begin{aligned}&\widetilde{\mathcal{M}}_{k,(1,\ell)}(\beta)\\ :=&\widetilde{\mathcal{M}}(\beta)\times D_0^2\times D_1^2\times\cdots\times D_\ell^2\times\partial D_{\ell+1}^2\times\cdots\times\partial D_{\ell+k}^2\\ &\mathcal{M}_{k,(1,\ell)}(\beta) := \widetilde{\mathcal{M}}_{k,(1,\ell)}(\beta)/PSL(2;\mathbb{R}).\end{aligned}$$

Definition 53.2. For smooth singular simplexes $f_i : P_i \to L$ in L and $g_j : Q_j \to M$ in M, we define

$$\begin{aligned}&\mathcal{M}^{\text{main}}_{k,(1,\ell)}(\beta; Q_1,\dots,Q_\ell; P_1,\dots,P_k)\\ :=&(-1)^\epsilon\mathcal{M}^{\text{main}}_{k,(1,\ell)}(L;\beta)_{(ev_1^{\text{int}},\dots,ev_\ell^{\text{int}},ev_1,\dots ev_k)}\times_{g_1\times\cdots\times g_\ell\times f_1\times\cdots\times f_k}(\prod_{i=1}^{\ell}Q_i\times\prod_{j=1}^{k}P_j),\end{aligned}$$

where

$$\epsilon = (n+1)\sum_{j=1}^{k-1}\sum_{i=1}^{j}\deg P_i + ((k+1)(n+1)+1)\sum_{i=1}^{\ell}\deg Q_i.$$

When $PSL(2;\mathbb{R})$ is oriented so that the embedding

$$g\in PSL(2;\mathbb{R})\mapsto(g\cdot 1, g\cdot\sqrt{-1}, g\cdot(-1))\in\partial D_0^2\times\partial D_1^2\times\partial D_2^2$$

preserves the orientations, then the embedding $g \in PSL(2;\mathbb{R}) \mapsto (g\cdot O, g\cdot z_1) \in D_0^2\times\partial D_1^2$ preserves the orientations. Here O is the origin of the disc D_0^2 and D_0^2 is oriented by the complex structure. Since we take the quotient by the right action, we adopt the opposite orientation to the one above (see Convention 46.1). This implies that $\mathfrak{p}_{1,0} \equiv i_!$ mod $\Lambda_{0,nov}$ as in (13.10.1).

We may identify

$$\mathcal{M}^{\text{main}}_{(1,k),\ell}(\beta; Q_1,\dots,Q_\ell; P_1,\dots,P_k)$$

and

$$\mathcal{M}^{\text{main}}_{1,(1,0)}(\beta_0;\mathcal{M}^{\text{main}}_{(1,k),\ell}(\beta; Q_1,\dots,Q_\ell; P_1,\dots,P_k)),$$

where β_0 is the class represented by constant maps from D^2 to L. Under this identification, it is easy to see that

$$\begin{aligned}&\mathcal{M}^{\text{main}}_{(1,k),\ell}(\beta; Q_1, \dots, Q_\ell; P_1, \dots, P_k)\\ \subset&(-1)^n \partial \mathcal{M}^{\text{main}}_{k,(1,\ell)}(\beta; Q_1, \dots, Q_\ell; P_1, \dots, P_k).\end{aligned}$$

Here we note the following graded symmetry for the operator $\mathfrak{p}$. Since each inner marked point carries 2-dimensional freedom and M is of even dimension, it is clear that, for $i \neq j$,

$$\begin{aligned}&\mathcal{M}^{\text{main}}_{k,(1,\ell)}(\beta; Q_1, \dots, Q_i, Q_{i+1}, \dots, Q_\ell; P_1, \dots, P_k)\\ =&(-1)^{\deg\, Q_i \cdot \deg\, Q_{i+1}} \mathcal{M}^{\text{main}}_{k,(1,\ell)}(\beta; Q_1, \dots, Q_{i+1}, Q_i, \dots, Q_\ell; P_1, \dots, P_k),\end{aligned}$$

where $\deg Q = 2n - \dim Q$. It is straightforward to see that we can use these orientation conventions to define the operators $\mathfrak{p}$, $\mathfrak{q}$, etc. such that the cyclic symmetry condition for $Q_i's$ in Theorem 13.32, etc. holds *with sign.*

Note that

$$\dim \mathcal{M}^{\text{reg}}_{m_1,(1,\ell_1)}(\beta') = \dim \mathcal{M}^{\text{reg}}_{(1,m_1),\ell_1}(\beta') + 1 \equiv n + m_1 + 1$$

modulo 2. Taking $+1$ in the right hand side into account, we can show an analog of Proposition 46.3 as follows. (This $+1$ appears in the exponent of (-1) in the right hand side of the equality in Proposition 53.3.)

Proposition 53.3. *We have an isomorphism*

$$\begin{aligned}&\partial \mathcal{M}^{\text{reg}}_{m,(1,\ell)}(\beta' + \beta'')\\ =&\bigcup (-1)^{(m_1-1)(m_2-1)+(n+m_1)+1} \mathcal{M}^{\text{reg}}_{m_1,(1,\ell_1)}(\beta')_{ev_1^{\beta'}} \times_{ev_0^{\beta''}} \mathcal{M}^{\text{reg}}_{(1,m_2),\ell_2}(\beta'')\end{aligned}$$

as oriented spaces with Kuranishi structures. Here the union is taken over $\beta = \beta_1 + \beta_2$, $\ell = \ell_1 + \ell_2$ and $m = m_1 + m_2 - 1$.

Proposition 53.4. (1) *For $\beta = \beta_1 + \beta_2$, we have*

$$\begin{aligned}&\mathcal{M}^{\text{main}}_{k-k_2+1,(1,\ell_1)}\Big(\beta_1; Q_1, \dots, Q_{\ell_1}; P_1, \dots, P_{i-1},\\ &\qquad \mathcal{M}^{\text{main}}_{(1,k_2),\ell_2}(\beta_2; Q_{\ell_1+1}, \dots, Q_{\ell_1+\ell_2}; P_i, \dots, P_{i+k_2-1}), P_{i+k_2}, \dots, P_k\Big)\\ \subset\ &(-1)^{\epsilon_1} \partial \mathcal{M}^{\text{main}}_{k,(1,\ell_1+\ell_2)}(\beta; Q_1, \dots, Q_{\ell_1}, Q_{\ell_1+1}, \dots, Q_{\ell_1+\ell_2};\\ &\qquad P_1, \dots, P_{i-1}, P_i, \dots, P_{i+k_2-1}, P_{i+k_2}, \dots, P_k),\end{aligned}$$

where

$$\epsilon_1 = n + \sum_{j=1}^{i-1}(\deg P_j + 1) + \sum_{j=1}^{\ell_1} \deg Q_j + \sum_{j=1}^{i-1}(\deg P_j + 1)(\sum_{i=\ell_1+1}^{\ell_1+\ell_2} \deg Q_i).$$

(2) *We have*

$$\begin{aligned}&\mathcal{M}^{\mathrm{main}}_{k,(1,\ell)}(\beta; Q_1,\dots,Q_\ell; P_1,\dots,\partial P_r,\dots,P_k)\\ \subset\ &(-1)^{\epsilon_2}\partial\mathcal{M}^{\mathrm{main}}_{k,(1,\ell)}(\beta; Q_1,\dots,Q_\ell; P_1,\dots,P_r,\dots,P_k),\end{aligned}$$

where

$$\epsilon_2 = \sum_{j=1}^{r-1}(\deg P_j + 1) + \sum_{j=1}^{\ell}\deg Q_j.$$

(3) *We have*

$$\begin{aligned}&\mathcal{M}^{\mathrm{main}}_{k,(1,\ell)}(\beta; Q_1,\dots,\partial Q_h,\dots,Q_\ell; P_1,\dots,P_k)\\ \subset\ &(-1)^{\epsilon_3}\partial\mathcal{M}^{\mathrm{main}}_{k,(1,\ell)}(\beta; Q_1,\dots,Q_h,\dots,Q_\ell; P_1,\dots,P_k),\end{aligned}$$

where

$$\epsilon_3 = 1 + \sum_{j=1}^{h-1}\deg Q_j.$$

Proof. The proof is a modification of the proof of Proposition 48.1. The proof of (1) is divided into several steps.
Step 1. As in the proof of Proposition 48.1, we will use the components other than main components at the intermediate stage of the calculation. Note that

$$\begin{aligned}&\deg\mathcal{M}^{\mathrm{main}}_{(1,k_2),\ell_2}(\beta_2; Q_{\ell_1+1},\dots,Q_{\ell_1+\ell_2}; P_i,\dots,P_{i+k_2-1})\\ \equiv &\sum_{j=i}^{i+k_2-1}(\deg P_j+1) + \sum_{i=\ell_1+1}^{\ell_1+\ell_2}\deg Q_i \pmod 2.\end{aligned}$$

Then by using an analog of Lemma 47.3 repeatedly, we find

$$\begin{aligned}&\mathcal{M}^{\mathrm{main}}_{k-k_2+1,(1,\ell_1)}(\beta_1; Q_1,\dots,Q_{\ell_1}; P_1,\dots,P_{i-1},\\ &\qquad\mathcal{M}^{\mathrm{main}}_{(1,k_2),\ell_2}(\beta_2; Q_{\ell_1+1},\dots,Q_{\ell_1+\ell_2}; P_i,\dots,P_{i+k_2-1}), P_{i+k_2},\dots,P_k)\\ \subseteq&(-1)^{\sigma_1}\mathcal{M}_{k-k_2+1,(1,\ell_1)}(\beta_1; Q_1,\dots,Q_{\ell_1};\\ &\qquad\mathcal{M}^{\mathrm{main}}_{(1,k_2),\ell_2}(\beta_2; Q_{\ell_1+1}\dots,Q_{\ell_1+\ell_2}; P_i,\dots,P_{i+k_2-1}),\\ &\qquad P_1,\dots,P_{i-1},P_{i+k_2},\dots,P_k),\end{aligned}$$

where

$$\sigma_1 = \left(1 + \sum_{j=i}^{i+k_2-1}(\deg P_j+1) + \sum_{i=\ell_1+1}^{\ell_1+\ell_2}\deg Q_i\right)\left(\sum_{j=1}^{i-1}(\deg P_j+1)\right).$$

Step 2. Next, we compare the orientation on

$$\begin{aligned}&\mathcal{M}_{k-k_2+1,(1,\ell_1)}(\beta_1;Q_1,\dots,Q_{\ell_1};\\&\qquad\mathcal{M}^{\text{main}}_{(1,k_2),\ell_2}(\beta_2;Q_{\ell_1+1},\dots,Q_{\ell_1+\ell_2};P_i,\dots,P_{i+k_2-1}),\\&\qquad P_1,\dots,P_{i-1},P_{i+k_2},\dots,P_k)\end{aligned}$$

with that on

$$\begin{aligned}&\partial\mathcal{M}_{k,(1,\ell_1+\ell_2)}(\beta;Q_1,\dots,Q_{\ell_1},Q_{\ell_1+1},\dots,Q_{\ell_1+\ell_2};\\&\qquad P_i,\dots,P_{i+k_2-1},P_1,\dots,P_{i-1},P_{i+k_2},\dots,P_k).\end{aligned}$$

By Definition 53.1, we have

$$\begin{aligned}&\mathcal{M}_{k,(1,\ell_1+\ell_2)}(\beta;Q_1,\dots,Q_{\ell_1},Q_{\ell_1+1},\dots,Q_{\ell_1+\ell_2};\\&\qquad P_i,\dots,P_{i+k_2-1},P_1,\dots,P_{i-1},P_{i+k_2},\dots,P_k)\\&=(-1)^{\gamma_1}\mathcal{M}_{k,(1,\ell_1+\ell_2)}(\beta)_{(ev_1^{\text{int}},\dots,ev_\ell^{\text{int}},ev_1,\dots,ev_k)}\times\\&\qquad\left(\prod_{i=1}^{\ell_1+\ell_2}Q_i\times\prod_{j=i}^{i+k_2-1}P_j\times\prod_{j=1}^{i-1}P_j\times\prod_{j=i+k_2}^{k}P_j\right),\end{aligned}$$

where

$$\begin{aligned}\gamma_1=&(n+1)\Bigg(\sum_{j=i}^{i+k_2-1}\sum_{\ell=i}^{j}\deg P_\ell+\sum_{j=1}^{i-1}\Big(\sum_{m=i}^{i+k_2-1}\deg P_m+\sum_{\ell=1}^{j}\deg P_\ell\Big)\\&+\sum_{j=i+k_2}^{k-1}\Big(\sum_{m=i}^{i+k_2-1}\deg P_m+\sum_{m=1}^{i-1}\deg P_m+\sum_{\ell=i+k_2}^{j}\deg P_\ell\Big)\Bigg)\\&+\Big((k+1)(n+1)+1\Big)\sum_{i=1}^{\ell_1+\ell_2}\deg Q_i.\end{aligned}$$

Using the iteration formula Lemma 45.3 (3), we have

$$\begin{aligned}&\mathcal{M}_{k,(1,\ell_1+\ell_2)}(\beta)_{(ev_1^{\text{int}},\dots,ev_{\ell_1+\ell_2}^{\text{int}},ev_1,\dots,ev_k)}\times\left(\prod_{i=1}^{\ell_1+\ell_2}Q_i\times\prod_{j=i}^{i+k_2-1}P_j\times\prod_{j=1}^{i-1}P_j\times\prod_{j=i+k_2}^{k}P_j\right)\\&=(-1)^{\gamma_2}\left(\mathcal{M}_{k,(1,\ell_1+\ell_2)}(\beta)_{(ev_{\ell_1+1}^{\text{int}},\dots,ev_{\ell_1+\ell_2}^{\text{int}},ev_1,\dots,ev_{k_2})}\times\left(\prod_{i=\ell_1+1}^{\ell_1+\ell_2}Q_i\times\prod_{j=i}^{i+k_2-1}P_j\right)\right)\\&\qquad_{(ev_1^{\text{int}},\dots,ev_{\ell_1}^{\text{int}},ev_{k_2+1},\dots,ev_k)}\times\left(\prod_{i=1}^{\ell_1}Q_i\times\prod_{j=1}^{i-1}P_j\times\prod_{j=i+k_2}^{k}P_j\right),\end{aligned}$$

where

$$\begin{aligned}\gamma_2 =&(k-k_2)n\Big(k_2n+\sum_{j=i}^{i+k_2-1}(n-\deg P_j)+\sum_{i=\ell_1+1}^{\ell_1+\ell_2}(2n-\deg Q_i)\Big)\\&+\Big(\sum_{i=1}^{\ell_1}(2n-\deg Q_i)\Big)\Big(\sum_{j=i}^{i+k_2-1}(n-\deg P_j)\Big)\\&+\Big(\sum_{i=1}^{\ell_1}(2n-\deg Q_i)\Big)\Big(\sum_{i=\ell_1+1}^{\ell_1+\ell_2}(2n-\deg Q_i)\Big),\end{aligned}$$

which is congruent to

$$(k-k_2)n\sum_{j=i}^{i+k_2-1}\deg P_j+(k-k_2)n\sum_{i=\ell_1+1}^{\ell_1+\ell_2}\deg Q_i+\Big(\sum_{i=1}^{\ell_1}\deg Q_i\Big)\Big(\sum_{j=i}^{i+k_2-1}(n-\deg P_j)\Big)$$
$$+\Big(\sum_{i=1}^{\ell_1}\deg Q_i\Big)\Big(\sum_{i=\ell_1+1}^{\ell_1+\ell_2}\deg Q_i\Big)\qquad \pmod 2.$$

By Proposition 53.3, we find that

$$\begin{aligned}&\partial\left(\mathcal{M}_{k,(1,\ell_1+\ell_2)}(\beta)_{(ev^{\rm int}_{\ell_1+1},\dots,ev^{\rm int}_{\ell_1+\ell_2},ev_1,\dots,ev_{k_2})}\times(\prod_{i=\ell_1+1}^{\ell_1+\ell_2}Q_i\times\prod_{j=i}^{i+k_2-1}P_j)\right)\\&\qquad\qquad{}_{(ev^{\rm int}_1,\dots,ev^{\rm int}_{\ell_1},ev_{k_2+1},\dots,ev_k)}\times(\prod_{i=1}^{\ell_1}Q_i\times\prod_{j=1}^{i-1}P_j\times\prod_{j=i+k_2}^{k}P_j)\\&\supset(-1)^{\gamma_3}\Bigg(\Big(\mathcal{M}_{k-k_2+1,(1,\ell_1)}(\beta_1)_{ev_1^{\beta_1}}\times_{ev_0^{\beta_2}}\mathcal{M}_{(1,k_2),\ell_2}(\beta_2)\Big)\\&\qquad\qquad{}_{(ev^{\beta,\rm int}_{\ell_1+1},\dots,ev^{\beta,\rm int}_{\ell_1+\ell_2},ev^{\beta}_1,\dots,ev^{\beta}_{k_2})}\times\Big(\prod_{i=\ell_1+1}^{\ell_1+\ell_2}Q_i\times\prod_{j=i}^{i+k_2-1}P_j\Big)\Bigg)\\&\qquad\qquad{}_{(ev^{\beta,\rm int}_1,\dots,ev^{\beta,\rm int}_{\ell_1},ev^{\beta}_{k_2+1},\dots,ev^{\beta}_k)}\times\left(\prod_{i=1}^{\ell_1}Q_i\times\prod_{j=1}^{i-1}P_j\times\prod_{j=i+k_2}^{k}P_j\right),\end{aligned}$$

where

$$\gamma_3=(k-k_2)(k_2-1)+n+k-k_2\equiv n+kk_2+k_2\pmod 2.$$

By the associativity property Lemma 45.3 (2) and Definition 53.1, we have

$$\begin{aligned}
&\Bigg(\Big(\mathcal{M}_{k-k_2+1,(1,\ell_1)}(\beta_1)_{ev_1^{\beta_1}} \times_{ev_0^{\beta_2}} \mathcal{M}_{(1,k_2),\ell_2}(\beta_2)\Big)_{(ev^{\beta,\mathrm{int}}_{\ell_1+1},\dots,ev^{\beta,\mathrm{int}}_{\ell_1+\ell_2},ev^{\beta}_1,\dots,ev^{\beta}_{k_2})}\\
&\times\Big(\prod_{i=\ell_1+1}^{\ell_1+\ell_2} Q_i \times \prod_{j=i}^{i+k_2-1} P_j\Big)\Bigg)\\
&\qquad {}_{(ev^{\beta,\mathrm{int}}_1,\dots,ev^{\beta,\mathrm{int}}_{\ell_1},ev^{\beta}_{k_2+1},\dots,ev^{\beta}_{k})} \times \left(\prod_{i=1}^{\ell_1} Q_i \times \prod_{j=1}^{i-1} P_j \times \prod_{j=i+k_2}^{k} P_j\right)\\
=&\Bigg(\mathcal{M}_{k-k_2+1,(1,\ell_1)}(\beta_1)_{ev_1^{\beta_1}} \times_{ev_0^{\beta_2}} \Big(\mathcal{M}_{(1,k_2),\ell_2}(\beta_2)_{(ev^{\beta_2,\mathrm{int}}_{1},\dots,ev^{\beta_2,\mathrm{int}}_{\ell_2},ev^{\beta_2}_1,\dots,ev^{\beta_2}_{k_2})}\\
&\quad\times(\prod_{i=\ell_1+1}^{\ell_1+\ell_2} Q_i \times \prod_{j=i}^{i+k_2-1} P_j)\Big)\Bigg)\\
&\qquad {}_{(ev^{\beta,\mathrm{int}}_1,\dots,ev^{\beta,\mathrm{int}}_{\ell_1},ev^{\beta}_{k_2+1},\dots,ev^{\beta}_{k})} \times \left(\prod_{i=1}^{\ell_1} Q_i \times \prod_{j=1}^{i-1} P_j \times \prod_{j=i+k_2}^{k} P_j\right)\\
=&(-1)^{\gamma_4}\Bigg(\mathcal{M}_{k-k_2+1,(1,\ell_1)}(\beta_1)_{ev_1^{\beta_1}} \times_{ev_0^{\beta_2}} \mathcal{M}_{(1,k_2),\ell_2}(\beta_2; Q_{\ell_1+1},\dots,Q_{\ell_1+\ell_2};\\
&P_i,\dots,P_{i+k_2-1})\Bigg)\\
&\qquad {}_{(ev^{\beta,\mathrm{int}}_1,\dots,ev^{\beta,\mathrm{int}}_{\ell_1},ev^{\beta}_{k_2+1},\dots,ev^{\beta}_{k})} \times \left(\prod_{i=1}^{\ell_1} Q_i \times \prod_{j=1}^{i-1} P_j \times \prod_{j=i+k_2}^{k} P_j\right),
\end{aligned}$$

where

$$\gamma_4 = (n+1)\sum_{j=i}^{i+k_2-2}\sum_{\ell=i}^{j}\deg P_\ell + \Big((k_2+1)(n+1)+1\Big)\sum_{i=\ell_1+1}^{\ell_1+\ell_2}\deg Q_i.$$

Again by using the iteration formula and Definition 53.2, we find that

$$\Big(\mathcal{M}_{k-k_2+1,(1,\ell_1)}(\beta_1)_{ev_1^{\beta_1}} \times_{ev_0^{\beta_2}} \mathcal{M}_{(1,k_2),\ell_2}(\beta_2; Q_{\ell_1+1}, \dots, Q_{\ell_1+\ell_2}; P_i, \dots, P_{i+k_2-1})\Big)$$

$$_{(ev_1^{\beta,\mathrm{int}},\dots,ev_{\ell_1}^{\beta,\mathrm{int}},ev_{k_2+1}^{\beta},\dots,ev_k^{\beta})} \times \left(\prod_{i=1}^{\ell_1} Q_i \times \prod_{j=1}^{i-1} P_j \times \prod_{j=i+k_2}^{k} P_j\right)$$

$$=(-1)^{\gamma_5}\mathcal{M}_{k-k_2+1,(1,\ell_1)}(\beta_1)_{(ev_1^{\beta_1,\mathrm{int}},\dots,ev_{\ell_1}^{\beta_1,\mathrm{int}},ev_1^{\beta_1},\dots,ev_{k-k_2+1}^{\beta_1})} \times$$

$$\Big(\mathcal{M}_{(1,k_2),\ell_2}(\beta_2; Q_{\ell_1+1}, \dots, Q_{\ell_1+\ell_2}; P_i, \dots, P_{i+k_2-1})$$

$$\times (\prod_{i=1}^{\ell_1} Q_i \times \prod_{j=1}^{i-1} P_j \times \prod_{j=i+k_2}^{k} P_j)\Big)$$

$$=(-1)^{\gamma_5+\gamma_6}\mathcal{M}_{k-k_2+1,(1,\ell_1)}(\beta; Q_1, \dots, Q_{\ell_1}; \mathcal{M}_{(1,k_2),\ell_2}(\beta_2; Q_{\ell_1+1}, \dots, Q_{\ell_1+\ell_2};$$

$$P_i, \dots, P_{i+k_2-1}), P_1, \dots, P_{i-1}, P_{i+k_2}, \dots, P_k),$$

where

$$\begin{aligned}
\gamma_5 &= n(k-k_2)\Big(2n - \sum_{j=i}^{i+k_2-1}(\deg P_j + 1) - \sum_{i=\ell_1+1}^{\ell_1+\ell_2} \deg Q_i\Big) \\
&\equiv n(k-k_2)\Big(k_2 + \sum_{j=i}^{i+k_2-1} \deg P_j - \sum_{i=\ell_1+1}^{\ell_1+\ell_2} \deg Q_i\Big), \\
\gamma_6 &= (n+1)\Bigg(\deg \mathcal{M}_{(1,k_2),\ell_2}(\beta_2; Q_{\ell_1}, \dots, Q_{\ell_1+\ell_2}; P_i, \dots, P_{i+k_2-1}) \\
&\quad + \sum_{j=1}^{i-1}\Big(\deg \mathcal{M}_{(1,k_2),\ell_2}(\beta_2; Q_{\ell_1}, \dots, Q_{\ell_1+\ell_2}; P_i, \dots, P_{i+k_2-1}) + \sum_{\ell=1}^{j} \deg P_\ell\Big) \\
&\quad + \sum_{j=i+k_2}^{k-1}\Big(\deg \mathcal{M}_{(1,k_2),\ell_2}(\beta_2; Q_{\ell_1+1}, \dots, Q_{\ell_1+\ell_2}; P_i, \dots, P_{i+k_2-1}) + \sum_{m=1}^{i-1} \deg P_m \\
&\quad + \sum_{\ell=i+k_2}^{j} \deg P_\ell\Big)\Bigg) + \Big((k-k_2+2)(n+1)+1\Big)\sum_{i=1}^{\ell_1} \deg Q_i \\
&\quad + \Big(\sum_{i=1}^{\ell_1}(2n - \deg Q_i)\Big)\Big(\dim \mathcal{M}_{(1,k_2),\ell_2}(\beta_2; Q_{\ell_1+1}, \dots, Q_{\ell_1+\ell_2}; P_i, \dots, P_{i+k_2-1})\Big) \\
&\equiv (n+1)\Bigg((k-k_2)\Big(k_2 + \sum_{m=i}^{i+k_2-1} \deg P_m + \sum_{i=\ell_1+1}^{\ell_1+\ell_2} \deg Q_i\Big) + (k-i-k_2)\Big(\sum_{m=1}^{i-1} \deg P_m\Big) \\
&\quad + \sum_{j=1}^{i-1}\sum_{\ell=1}^{j} \deg P_\ell + \sum_{j=i+k_2}^{k-1}\sum_{\ell=i+k_2}^{j} \deg P_\ell\Bigg) + \Big((k-k_2)(n+1)+1\Big)\sum_{i=1}^{\ell_1} \deg Q_i \\
&\quad + \Big(\sum_{i=1}^{\ell_1} \deg Q_i\Big)\Big(n - (k_2 + \sum_{m=i}^{i+k_2-1} \deg P_m + \sum_{i=\ell_1+1}^{\ell_1+\ell_2} \deg Q_i)\Big).
\end{aligned}$$

Then an elementary calculation shows that

$$\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6 \equiv n + \sum_{i=1}^{\ell_1} \deg Q_i \quad (\bmod\ 2)\ .$$

Hence we have found that

$$
\begin{aligned}
&\mathcal{M}_{k-k_2+1,(1,\ell_1)}(\beta_1; Q_1 \dots, Q_{\ell_1};\\
&\qquad \mathcal{M}^{\text{main}}_{(1,k_2),\ell_2}(\beta_2; Q_{\ell_1+1}, \dots, Q_{\ell_1+\ell_2} Q_i; P_i, \dots, P_{i+k_2-1}),\\
&\qquad P_1, \dots, P_{i-1}, P_{i+k_2}, \dots, P_k)\\
\subset&(-1)^{n+\sum_{i=1}^{\ell_1} \deg Q_i} \partial \mathcal{M}_{k,(1,\ell_1+\ell_2)}(\beta; Q_1, \dots, Q_{\ell_1+\ell_2};\\
&\qquad P_i, \dots, P_{i+k_2-1}, P_1, \dots, P_{i-1}, P_{i+k_2}, \dots, P_k).
\end{aligned}
$$

Step 3. On the other hand, by using Lemma 47.3 again, we can see that

$$
\begin{aligned}
&\mathcal{M}_{k,(1,\ell_1+\ell_2)}(\beta; Q_1, \dots, Q_{\ell_1+\ell_2};\\
&\qquad P_i, \dots, P_{i+k_2-1}, P_1, \dots, P_{i-1}, P_{i+k_2}, \dots, P_k)\\
=&(-1)^{\sigma_2} \mathcal{M}^{\text{main}}_{k,(1,\ell_1+\ell_2)}(\beta; Q_1, \dots, Q_{\ell_1+\ell_2};\\
&\qquad P_1, \dots, P_{i-1}, P_i, \dots, P_{i+k_2-1}, P_{i+k_2}, \dots, P_k),
\end{aligned}
$$

where

$$
\sigma_2 = \left(\sum_{j=i}^{i+k_2-1} (\deg P_j + 1) \right) \left(\sum_{j=1}^{i-1} (\deg P_j + 1) \right).
$$

Therefore we have

$$
\begin{aligned}
&\sigma_1 + n + \sum_{i=1}^{\ell_1} \deg Q_i + \sigma_2\\
\equiv& n + \sum_{i=1}^{\ell_1} \deg Q_i + \sum_{j=1}^{i-1} (\deg P_j + 1) + \sum_{j=1}^{i-1} (\deg P_j + 1) \Big(\sum_{i=\ell_1+1}^{\ell_1+\ell_2} \deg Q_i \Big) \pmod 2,
\end{aligned}
$$

which proves Proposition 53.4 (1).

Next, we prove (2) and (3). We recall that

$$
\begin{aligned}
&\mathcal{M}^{\text{main}}_{k,(1,\ell)}(\beta; Q_1, \dots, Q_\ell; P_1, \dots, P_i, \dots, P_k)\\
=&(-1)^{\delta_1} \mathcal{M}^{\text{main}}_{k+1}(\beta)_{(ev_1^{\text{int}}, \dots, ev_\ell^{\text{int}}, ev_1, \dots, ev_k)} \times_{g_1 \times \cdots g_\ell \times f_1 \times \cdots \times f_k} \left(\prod_{i=1}^{\ell} Q_i \times \prod_{j=1}^{k} P_j \right)
\end{aligned}
$$

with

$$
\delta_1 = (n+1) \sum_{j=1}^{k-1} \sum_{i=1}^{j} \deg P_i + \Big((k+1)(n+1) + 1 \Big) \sum_{i=1}^{\ell} \deg Q_i.
$$

By Lemma 45.3 (1), we find that

$$\begin{aligned}
&\partial\left(\mathcal{M}^{\rm main}_{k,(1,\ell)}(\beta)_{(ev^{\rm int}_1,\dots,ev^{\rm int}_\ell,ev_1,\dots,ev_k)}\times_{g_1\times\cdots g_\ell\times f_1\times\cdots\times f_k}\left(\prod_{i=1}^{\ell}Q_i\times\prod_{j=1}^{k}P_j\right)\right)\\
&=\left(\partial\mathcal{M}^{\rm main}_{k,(1,\ell)}(\beta)\right)_{(ev^{\rm int}_1,\dots,ev^{\rm int}_\ell,ev_1,\dots,ev_k)}\times_{g_1\times\cdots g_\ell\times f_1\times\cdots\times f_k}\left(\prod_{i=1}^{\ell}Q_i\times\prod_{j=1}^{k}P_j\right)\\
&\qquad\bigsqcup(-1)^{n+k+1+nk}\mathcal{M}^{\rm main}_{k,(1,\ell)}(\beta)_{(ev^{\rm int}_1,\dots,ev^{\rm int}_\ell,ev_1,\dots,ev_k)}\times_{g_1\times\cdots g_\ell\times f_1\times\cdots\times f_k}\\
&\qquad\qquad\partial\left(\prod_{i=1}^{\ell}Q_i\times\prod_{j=1}^{k}P_j\right),
\end{aligned}$$

because $\dim\mathcal{M}^{\rm main}_{k,(1,\ell)}(\beta)=n+\mu(\beta)-3+k+2(\ell+1)\equiv n+k+1$ and M is even dimensional. Note also that

$$\begin{aligned}
&\partial\left(\prod_{i=1}^{\ell}Q_i\times\prod_{j=1}^{k}P_j\right)\\
&=\bigsqcup_{h=1}^{\ell}(-1)^{\sum_{i=1}^{h-1}\dim Q_i}Q_1\times\cdots\times\partial Q_h\times\cdots Q_\ell\times\prod_{j=1}^{k}P_j\\
&\bigsqcup\bigsqcup_{r=1}^{k}(-1)^{\sum_{i=1}^{\ell}\dim Q_i+\sum_{j=1}^{r-1}\dim P_j}\prod_{i=1}^{\ell}Q_i\times P_1\times\cdots\times\partial P_r\times\cdots\times P_k.
\end{aligned}$$

On the other hand, Definition 53.2 yields that

$$\begin{aligned}
&\mathcal{M}^{\rm main}_{k,(1,\ell)}(\beta;Q_1,\dots,Q_\ell;P_1,\dots,\partial P_r,\dots,P_k)\\
=&(-1)^{\delta_2}\mathcal{M}^{\rm main}_{k,(1,\ell)}(\beta)_{(ev^{\rm int}_1,\dots,ev^{\rm int}_\ell,ev_1,\dots,ev_k)}\times_{g_1\times\cdots\times g_\ell\times f_1\times\cdots\times f_k}\\
&\qquad\Big(\prod_{i=1}^{\ell}Q_i\times\prod_{j=1}^{r-1}P_j\times\partial P_r\times\prod_{j=r+1}^{k}P_j\Big)
\end{aligned}$$

with

$$\delta_2=(n+1)\sum_{j=1}^{k-1}\sum_{\ell=1}^{j}(\deg P_\ell)'+\Big((k+1)(n+1)+1\Big)\sum_{j=1}^{\ell}\deg Q_j,$$

where

$$(\deg P_\ell)'=\begin{cases}\deg P_\ell & \text{for } \ell\neq r\\ \deg P_\ell+1 & \text{for } \ell=r.\end{cases}$$

Therefore we can see that

$$\begin{aligned}&\delta_1 + n + k + 1 + nk + \sum_{j=1}^{r-1}(n - \deg P_j) + \sum_{i=1}^{\ell}(2n - \deg Q_i) + \delta_2 \\ \equiv &\sum_{j=1}^{r-1}(\deg P_j + 1) + \sum_{i=1}^{\ell} \deg Q_i \pmod 2,\end{aligned}$$

which proves Proposition 53.4 (2).

Recall that

$$\begin{aligned}&\mathcal{M}^{\text{main}}_{k,(1,\ell)}(\beta; Q_1, \dots, \partial Q_h, \dots, Q_\ell; P_1, \dots, P_k) \\ =&(-1)^{\delta_3}\mathcal{M}^{\text{main}}_{k,(1,\ell)}(\beta)_{(ev_1^{\text{int}},\dots,ev_\ell^{\text{int}},ev_1,\dots,ev_k)}\times_{g_1\times\cdots\times g_\ell\times f_1\times\cdots\times f_k} \\ &\qquad(\prod_{i=1}^{h=1} Q_i \times \partial Q_h \times \prod_{i=h+1}^{\ell} Q_i \times \prod_{j=1}^{k} P_j)\end{aligned}$$

with

$$\delta_3 = (n+1)\sum_{j=1}^{k-1}\sum_{i=1}^{j}(\deg P_i) + ((k+1)(n+1)+1)\,(\sum_{i=1}^{\ell}\deg Q_i - 1).$$

Therefore we can see that

$$\delta_1 + n + k + 1 + nk + \sum_{i=1}^{h-1}(2n - \deg Q_i) + \delta_3 \equiv \sum_{i=1}^{h-1}\deg Q_i + 1 \pmod 2,$$

which proves Proposition 53.4 (3). □

In a similar way, we find the following:

Proposition 53.5. (1) *For $\beta = \beta_1 + \beta_2$, we have*

$$\begin{aligned}&\mathcal{M}^{\text{main}}_{(1,k-k_2+1),\ell_1}(\beta_1; Q_1, \dots, Q_{\ell_1}; P_1, \dots, P_{i-1}, \\ &\qquad\qquad \mathcal{M}^{\text{main}}_{(1,k_2),\ell_2}(\beta_2; Q_{\ell_1+1}, \dots, Q_{\ell_1+\ell_2}; P_i, \dots, P_{i+k_2-1}), P_{i+k_2}, \dots, P_k) \\ \subset\ &(-1)^{\epsilon'_1}\partial\mathcal{M}^{\text{main}}_{(1,k),\ell_1+\ell_2}(\beta; Q_1, \dots, Q_{\ell_1}, Q_{\ell_1+1}, \dots, Q_{\ell_1+\ell_2}; \\ &\qquad\qquad P_1, \dots, P_{i-1}, P_i, \dots, P_{i+k_2-1}, P_{i+k_2}, \dots, P_k),\end{aligned}$$

where

$$\epsilon'_1 = n + 1 + \sum_{j=1}^{i-1}(\deg P_j + 1) + \sum_{j=1}^{\ell_1}\deg Q_j + \sum_{j=1}^{i-1}(\deg P_j + 1)(\sum_{i=\ell_1+1}^{\ell_1+\ell_2}\deg Q_i).$$

(2) *We have*

$$\begin{aligned}&\mathcal{M}^{\rm main}_{(1,k),\ell}(\beta;Q_1,\dots,Q_\ell;P_1,\dots,\partial P_r,\dots,P_k)\\ \subset\ &(-1)^{\epsilon'_2}\partial\mathcal{M}^{\rm main}_{(1,k),\ell}(\beta;Q_1,\dots,Q_\ell;P_1,\dots,P_r,\dots,P_k),\end{aligned}$$

where

$$\epsilon'_2=\sum_{j=1}^{r-1}(\deg P_j+1)+\sum_{j=1}^{\ell}\deg Q_j+1.$$

(3) *We have*

$$\begin{aligned}&\mathcal{M}^{\rm main}_{(1,k),\ell}(\beta;Q_1,\dots,\partial Q_h,\dots,Q_\ell;P_1,\dots,P_k)\\ \subset\ &(-1)^{\epsilon'_3}\partial\mathcal{M}^{\rm main}_{(1,k),\ell}(\beta;Q_1,\dots,Q_h,\dots,Q_\ell;P_1,\dots,P_k),\end{aligned}$$

where

$$\epsilon'_3=\sum_{j=1}^{h-1}\deg Q_j.$$

Next, in order to check the sign in (13.10.3), we need to study the orientation of $\mathcal{M}_{0,2}(M;\widetilde\beta)$ and $\mathcal{M}_{0,(1,0)}(L;\beta)$, where $\widetilde\beta\in H_2(M)$ and β is the image of $\widetilde\beta$ by $i_*:H_2(M)\to H_2(M,L)$, see §13.

Proposition 53.6. *The fiber product orientation on $\mathcal{M}^{\rm reg}_{0,2}(M;\widetilde\beta)_{ev_1}\times L$ coincides with the orientation as the boundary of $\mathcal{M}^{\rm reg}_{0,(1,0)}(L;\beta)$.*

Proof. Fixing $0,\infty\in\mathbb{C}P^1$ as the 0-th and first marked points, we write

$$\widetilde{\mathcal{M}}^{\rm reg}(M;\widetilde\beta)/\mathrm{Aut}(\mathbb{C}P^1;0,\infty)=\mathcal{M}^{\rm reg}(M;\widetilde\beta).$$

Similarly, fix $0,1$ on the closed unit disc as the 0-th interior and first boundary marked points on D^2. Then the moduli space $\mathcal{M}^{\rm reg}_{1,(1,0)}(L;0)$, resp. $\mathcal{M}^{\rm reg}_{1,(1,0)}(L;\beta)$, is identified with the space $\widetilde{\mathcal{M}}^{\rm reg}(L;0)$, resp. $\widetilde{\mathcal{M}}^{\rm reg}(L;\beta)$ of pseudo-holomorphic maps from D^2 in the class 0, resp. β.

In a similar way to Lemma 46.5, we find that the gluing map

$$\widetilde{\mathcal{M}}^{\rm reg}(M;\widetilde\beta)_{ev^{\rm int}_\infty}\times_{ev^{\rm int}_0}\widetilde{\mathcal{M}}^{\rm reg}(L;0)\to\widetilde{\mathcal{M}}^{\rm reg}(\beta)$$

is orientation preserving in the sense of Kuranishi structure. Therefore we have the identification (in the level of tangent spaces)

$$\begin{aligned}&\Big(\mathcal{M}^{\rm reg}_{0,2}(M;\widetilde\beta)\times\mathrm{Aut}(\mathbb{C}P^1;0,\infty)\Big)_{ev^{\rm int}_\infty}\times_{ev^{\rm int}_0}\mathcal{M}^{\rm reg}_{1,(1,0)}(L;0)\\ =&\mathcal{M}^{\rm reg}_{1,(1,0)}(\beta).\end{aligned}$$

Note that $\mathrm{Aut}(\mathbb{C}P^1; 0, \infty) \cong \mathbb{C}^\times \cong \mathbb{R}_{>0} \times S^1$. We give an orientation on $\mathbb{R}_{>0}$ so that the action on $\mathbb{C}P^1$ induces a flow from 0 to ∞. Then, as in the proof of Lemma 46.5, $\mathbb{R}_{>0}$ corresponds to the outer normal vector field. Note also that $ev_0^{\mathrm{int}} : \mathcal{M}^{\mathrm{reg}}_{1,(1,0)}(L; 0) \to L$ is orientation preserving. Hence we have

$$\begin{aligned} &\mathbb{R}_{out} \times \left(\mathcal{M}^{\mathrm{reg}}_{0,2}(M; \widetilde{\beta}) \times S^1\right)_{ev^{\mathrm{int}}_\infty} \times_L L \\ =&\mathcal{M}^{\mathrm{reg}}_{1,(1,0)}(\beta). \end{aligned}$$

Note also that, after gluing, the S^1-action moves the first boundary marked point in the counter-clockwise direction. Therefore we obtain

$$\begin{aligned} &\mathcal{M}^{\mathrm{reg}}_{0,2}(M; \widetilde{\beta})_{ev^{\mathrm{int}}_\infty} \times_L L \\ \subset&\partial\mathcal{M}^{\mathrm{reg}}_{0,(1,0)}(\beta). \end{aligned}$$

□

Finally, we prove the sign in the formula

$$\mathfrak{p}_{k,\beta}(P_1, \cdots, P_k) = (-1)^{\deg' P_k \times (\sum_{i=1}^{k-1} \deg' P_i)} \mathfrak{p}_{k,\beta}(P_k, P_1, \cdots, P_{k-1})$$

stated in Lemma 13.25. Here $\deg'$ is the shifted degree.

The following is a straightforward generalization of Lemma 47.3.

Lemma 53.7. *Let σ be the transposition element $(i, i+1)$ in the k-th symmetric group $\mathfrak{S}_k$. $(i = 1, \ldots, k-1)$. Then the action of σ on*

$$\mathcal{M}_{k,(1,\ell)}(\beta; Q_1, \ldots, Q_\ell; P_1, \ldots, P_i, P_{i+1}, \ldots, P_k)$$

by changing the order of marked points is described by the following.

$$\begin{aligned} &\sigma(\mathcal{M}_{k,(1,\ell)}(\beta; Q_1, \ldots, Q_\ell; P_1, \ldots, P_i, P_{i+1}, \ldots, P_k)) \\ =&(-1)^{(\deg P_i + 1)(\deg P_{i+1}+1)} \mathcal{M}_{k,(1,\ell)}(\beta; Q_1, \ldots, Q_\ell; P_1, \ldots, P_{i+1}, P_i, \ldots, P_k). \end{aligned}$$

Proof. Recall Definition 53.2. Then the lemma can be proved in the same way as that of Lemma 47.3. □

Then Lemma 53.7 for $\ell = 0$ immediately implies the desired sign in Lemma 13.25.

53.3. Continuous family of perturbations.

We finally discuss orientation of the moduli spaces we used in §33 to define an A_K homomorphism $\mathfrak{f}_\ell : B(\mathbb{R}\mathcal{X}_L) \to \Omega(L)$. In §33, we defined the moduli space

$$U_{\ell+1}(\mathfrak{f}, \vec{P}') = (L \times_{L^{\ell+1}} (L \times P_1' \times \cdots \times P_\ell')) \times \mathcal{N}_{\ell+1}'$$

and used it to define $\mathfrak{X}_\ell(\vec{P}')$, by Formula (33.48) :

$$\mathfrak{X}_\ell(\vec{P}') = \{(\mathbf{p}, w) \in U_{\ell+1}(\mathfrak{f}, \vec{P}') \times W_\ell \mid \mathfrak{t}^{\vec{P}'}_{\mathfrak{f},\ell+1}(\mathbf{p}, w) = 0\}. \tag{33.48}$$

Here $\mathfrak{t}^{\vec{P}'}_{\mathfrak{f},\ell+1}$ is a section of the obstruction bundle on $U_{\ell+1}(\mathfrak{f}, \vec{P}') \times W_\ell$. We have a projection $U_{\ell+1}(\mathfrak{f}, \vec{P}') \to W_\ell$.

We used a top form ω_ℓ on W_ℓ and defined

$$\mathfrak{f}_\ell(\vec{P}') = ev_!(\pi^*_{W_\ell}(\omega_\ell)) \in \Omega(L). \tag{33.49}$$

Here $\pi : \mathfrak{X}_\ell(\vec{P}') \to L$ is the evaluation map at the 0-th marked point.

The orientation of W_ℓ is given by ω_ℓ. We need to find an orientation of $U_{\ell+1}(\mathfrak{f}, \vec{P}')$ which induces an orientation of $\mathfrak{X}_\ell(\vec{P}')$ and hence the sign of Formula (33.49) above. We will reduce this problem to ones on the moduli space used to define filtered A_∞ homomorphisms. This later problem is discussed already in §52.

Let $d|\omega_\ell|$ be the smooth *measure* on W_ℓ such that

$$\int_{W_\ell} f\, \omega_\ell = \int_{W_\ell} f d|\omega_\ell|$$

for any positive function f. By a standard transversality theorem, we can find a subset W_ℓ^0 of full $d|\omega_\ell|$ measure such that for each $w \in W_\ell^0$

$$\pi^{-1}_{W_\ell}(w) \subset \mathfrak{X}_\ell(\vec{P}')$$

is a smooth submanifold. If we have an orientation on $\pi^{-1}_{W_\ell}(w)$, then

$$\pi(\pi^{-1}_{W_\ell}(w)) \subset L$$

defines a current on L. Moreover it is easy to prove that

$$ev_!(\pi^*_{W_\ell}(\omega_\ell)) = \int_{w \in W_\ell^0} \pi(\pi^{-1}_{W_\ell}(w)) d|\omega_\ell|(w) \tag{53.8}$$

holds as an equality among smooth forms, up to sign. Hence the problem to find an appropriate sign for (33.49) reduces to the problem to find an appropriate orientation of $\pi^{-1}_{W_\ell}(w)$.

We next recall that W_ℓ is the space parameterizing the family of perturbations (sections) of the Kuranishi structure on $U_{\ell+1}(\mathfrak{f}, \vec{P}')$.

The set

$$\pi_{W_\ell}^{-1}(w) = \mathfrak{X}_\ell(\vec{P}') \cap (U_{\ell+1}(\mathfrak{f}, \vec{P}') \times \{w\})$$

is the zero set of the section

$$\mathbf{p} \mapsto \mathfrak{t}_{\mathfrak{f},\ell+1,w}^{\vec{P}'} := \mathfrak{t}_{\mathfrak{f},\ell+1}^{\vec{P}'}(\mathbf{p}, w).$$

If we regard $\mathfrak{t}_{\mathfrak{f},\ell+1}^{\vec{P}'}$ as a W_ℓ parameterized family $\{\mathfrak{t}_{\mathfrak{f},\ell+1,w}^{\vec{P}'}\}_{w\in W_\ell}$ of sections on $U_{\ell+1}(\mathfrak{f}, \vec{P}')$, then $\pi_{W_\ell}^{-1}(w)$ is its zero set.

We consider the space

$$U_{\ell+1}(\mathfrak{f}, \vec{P}')^+ = (L \times_{L^{\ell+1}} (P_1' \times \cdots \times P_\ell' \times L)) \times \mathcal{N}_{\ell+1}$$

with Kuranishi structure. (We remark that there is no prime for the notation $\mathcal{N}_{\ell+1}$ in the right hand side.)

Note $U_{\ell+1}(\mathfrak{f}, \vec{P}')^+$ coincides (together with Kuranishi structure) with the space

$$\mathcal{M}_{\ell+1}^{\mathrm{main}}(M, L, \{J_\rho\}_\rho : \beta_0; \mathrm{top}(\rho)) \otimes_{L^\ell} (P_1' \times \cdots \times P_\ell'). \tag{53.9}$$

We used (53.9) to construct an A_∞ homomorphism $\mathfrak{f} : (\mathbb{R}\mathcal{X}_L, \overline{\mathfrak{m}}) \to (\mathbb{R}\mathcal{X}_L', \overline{\mathfrak{m}}')$ between two A_∞ algebras (established by Theorem 9.8) which are obtained by two different choices of perturbation and two different choices of the sets of singular simplexes $\mathcal{X}_L$, $\mathcal{X}_L'$. Thus we already defined a coherent orientation of the space $U_{\ell+1}(\mathfrak{f}, \vec{P}')^+$ in §52.

On the other hand, by Theorem 33.63, there exists a smooth surjective map

$$\mathcal{N}_{\ell+1} \to \mathcal{N}_{\ell+1}',$$

which is a diffeomorphism outside the boundary. Therefore, orientation of $U_{\ell+1}(\mathfrak{f}, \vec{P}')^+$ induces orientation of $U_{\ell+1}(\mathfrak{f}, \vec{P}')$. It determines a sign of (33.49). □