MIRROR SYMMETRY OF ABELIAN VARIETIES AND MULTI-THETA FUNCTIONS

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Introduction.

In this paper, we study mirror symmetry of complex and symplectic tori as an example of homological mirror symmetry conjecture of Kontsevich [K1], [K2] between symplectic and complex manifolds. We discussed mirror symmetry of tori in [Fu5] emphasizing its noncommutative generalization. In this paper, we concentrate on the case of a commutative (usual) torus. Our result is a generalization of one by Kontsevich [K2] and Polishchuk-Zaslow [PZ], Polishchuk [Pl], who studied the case of elliptic curve. The main results of this paper prove a dictionary of mirror symmetry between symplectic geometry and complex geometry partially in the case of tori of arbitrary dimension. Namely we prove it in the case of mutually transversal affine Lagrangian submanifolds and semi homogeneous sheaves (under transversality assumptions.) The argument presented in this paper suggests a possibility of its generalization. However there are various serious difficulties for the generalization, some of which we mention in this paper. We will discuss the general case more in [Fu7].

In this paper, we will define a new family of theta functions on complex tori, which we call multi theta function. It is a generating function of the numbers obtained by counting holomorphic polygons in tori and describe various product structures (Yoneda and Massey Yoneda products) of the sheaf cohomology group on its mirror. We recall that one famous consequence [COGP],[Gi],[LLY],[BDPP] of mirror symmetry is a coincidense of Gromov-Witten potential, the generating function of the number counting rational curves in a Calabi-Yau manifold, with the Yukawa coupling, a product structure of sheaf cohomology of its mirror. In the case of complex tori, there is no rational curve. Hence the statement above is void. However, if we include Lagrangian submanifolds in symplectic side and coherent sheaves in complex side, we can derive many nontrivial consequences of mirror symmetry. Exploring them is the purpose of this paper. Namely we find relations between counting problem of holomorphic polygons (0 loop correlation function of topological open string) and product structures of sheaf cohomology of its mirror. We remark that including Lagrangian submanifolds and coherent sheaves corresponds to including branes. So it is naturally related to the recent progress of string theory. (See for example [Po].)

To describe the main result of this paper, we discuss homological mirror symmetry conjecture briefly here. The scope of homological mirror symmetry conjecture is huge. So it seems yet impossible to state it precisely and rigorously. We mention only a small part of it, which we can prove partially in the case of tori. See [FKOOO],[Fu6],[Fu7] for some of the other parts of the story.

We consider a symplectic manifolds (M,ω), together with B-field B, (that is a closed 2 form). We put $\Omega = \omega + \sqrt{-1}B$. First of all, the (homological) mirror symmetry conjecture predicts the existence of a complex manifolds $(M,\Omega)^\vee$ which is called a mirror, to some of the pairs (M,Ω). Here are two remarks.

Remark 0.1. Usually mirror symmetry conjecuture predicts the existence of pairs of Calabi-Yau manifolds M and $M^\vee$, the mirror pairs. Calabi-Yau manifold is a Kähler manifold and is both symplectic and complex. (Symplectic structure is determined by the Kähler form.) It is also conjectured that the complex structure of $M^\vee$ depends on the symplectic structure of M (together with B field), and the complex structure of M depends on the symplectic structure of $M^\vee$. Here we only take symplectic structure of M and complex structure of $M^\vee$ and forget the other half of the structures.

Remark 0.2. It seems too much to expect that every symplectic manifold has a mirror. The author has no candidate of a good condition for a symplectic manifold to have a mirror. (One sufficent condition might be an existence of (singular) fibration $M \to B$ such that the (general) fiber is a Lagrangian torus.)

Also, even in case when a mirror exists, it may not be unique. We have such an example already in the case of symplectic torus. (See §1.)

Suppose we have a mirror pair (M, Ω), $(M, \Omega)^\vee$. It is conjectured (and is proved in some cases) that the Gromov-Witten invariant of (M, Ω) coincides with Yukawa-coupling that is a product structure of sheaf cohomology of $(M, \Omega)^\vee$. We do not discuss this point here since it is discussed by many other authors.

The first part of homological mirror symmetry conjecture is stated as follows.

Conjecture A. For each pair $(L, \mathcal{L})$ of unobstructed Lagrangian submanifold L and a flat line bundle $\mathcal{L}$, we can associate an object $\mathcal{E}(L, \mathcal{L})$ of derived category of coherent sheaves on $(M, \Omega)^\vee$.

Here "unobstructed" means the vanishing of the obstruction for the existence of Lagrangian intersection Floer homology, which we introduced in [FKOOO] . We do not discuss the obstrution theory (to Lagrangian intersection Floer homology) and its relation to mirror symmetry here, since it is the thema of [FKOOO]. For the purpose of this paper, it is enough to remark that L is unobstructed if $H_*(L; \mathbb{Q}) \to H_*(M; \mathbb{Q})$ is injective. (The case we are mainly concern with in this paper is one when L is an affine Lagrangian submanifold of symplectic torus M. In that case this condition is certainly satisfied.)

To state the second part of homological mirror symmetry conjecture, we remark that, if $(L_i, \mathcal{L}_i)$ are unobstructed, we can define a Floer homology $HF((L_1, \mathcal{L}_1), (L_2, \mathcal{L}_2))$ between them. (We review in §2, the definition of Floer homology in the case we use.)

Remark 0.3. In fact, Floer homology $HF((L_1, \mathcal{L}_1), (L_2, \mathcal{L}_2))$, in the general situation, depends not only on $(L_1, \mathcal{L}_1)$, $(L_2, \mathcal{L}_2)$ but also on other parameters. (See [FKOOO].) We do not discuss this point here since it is unnecessary in our case of affine Lagrangian submanifold in symplectic torus, (where $\pi_2(T^{2n}, L) = 0$.)

Remark 0.4. In the general situation, we can define Floer homology only over an appropriate Novikov ring [N] (which is a kind of formal power series ring). In fact, the boundary operator is defined as a formal power series whose convergence question is yet open. However the operations (the boundary operator and the product structures $\mathfrak{m}_k$ we will introduce later) always converge in the case of affine Lagrangian submanifolds in a symplectic torus, as we will see in this paper (Proposition 10.5). As a consequence, we can and will use $\mathbb{C}$ as a coefficient ring of Floer homology.

Conjecture B. There exists a canonical isomorphism

$$HF((L_1, \mathcal{L}_1), (L_2, \mathcal{L}_2)) \cong \operatorname{Ext}(\mathcal{E}(L_1, \mathcal{L}_1), \mathcal{E}(L_2, \mathcal{L}_2)), \tag{0.5}$$

between Floer homology and extension.

To state the third part of the conjecture, we need to review the product structure of Floer homology introduced in [Fu1] based on the ideas due to Donaldson [Do2] and Segal. Namely, in case $(L_i, \mathcal{L}_i)$ are unobstructed for $i = 1, 2, 3$, we have

$$\begin{aligned}\mathfrak{m}_2 : HF((L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2)) \otimes HF((L_2,\mathcal{L}_2),(L_3,\mathcal{L}_3)) \\ \to HF((L_1,\mathcal{L}_1),(L_3,\mathcal{L}_3)).\end{aligned} \tag{0.6}$$

The outline of the construction of $\mathfrak{m}_2$ was given in [Fu1]. Kontsevitch [K1],[K2] gave a modification necessary to apply it to homological mirror symmetry conjecture. In fact, to define $\mathfrak{m}_2$ rigorously in the general situation, we need to introduce various correction terms (similar to ones in [FKOOO]) to the definition of [Fu1]. The rigorous definition in the general case is not written up yet.

Fortunately, in our case of affine Lagrangian submanifold of symplectic torus, we do not need those correction terms. So we do not discuss the modification we need to define $\mathfrak{m}_2$ rigorously in the general situation. We will give a rigorous definition of $\mathfrak{m}_2$ in our case later in this paper. (§10.)

We remark that Yoneda product defines a map

$$\mathfrak{m}_2 : \mathrm{Ext}(\mathcal{E}_1,\mathcal{E}_2) \otimes \mathrm{Ext}(\mathcal{E}_2,\mathcal{E}_3) \to \mathrm{Ext}(\mathcal{E}_1,\mathcal{E}_3). \tag{0.7}$$

Here $\mathcal{E}_i$ are objects of derived category of coherent sheaves on $M^\vee$.

Conjecture C. Let $\mathcal{E}_i = \mathcal{E}(L_i, \mathcal{L}_i)$. Then the map (0.7) coincides with the map (0.6) through the isomorphism (0.5). In other words, the following diagram commutes.

Diagram 1

To go further, we need to review the notion of A_∞ category.

Definition 0.8. A_∞ category $\mathcal{C}$ is a correction of a set $Ob(\mathcal{C})$, (the set of objects), chain complex $\mathcal{C}(c_1, c_2)$ for each $c_1, c_2 \in Ob(\mathcal{C})$ (the set of morphisms), and maps

$$\mathfrak{m}_k : \mathcal{C}(c_0,c_1) \otimes \cdots \otimes \mathcal{C}(c_{k-1},c_k) \to \mathcal{C}(c_0,c_k)$$

((higher) compositions), such that $\mathfrak{m}_1 : \mathcal{C}(c_0,c_1) \to \mathcal{C}(c_0,c_1)$ is the boundary operator and

$$\begin{aligned}\sum_{0<\ell\le\ell'\le k} \pm\mathfrak{m}_{k-\ell'+\ell}(x_1 \otimes \cdots \otimes x_{\ell-1} \\ \otimes\, \mathfrak{m}_{\ell'-\ell+1}(x_\ell \otimes \cdots \otimes x_{\ell'}) \otimes x_{\ell'+1} \otimes \cdots \otimes x_k) = 0.\end{aligned} \tag{0.9}$$

We do not discuss the sign $\pm$ here. (See §11.) We can rewrite (0.9) as follows. We define bar complex $B\mathcal{C}(c,c')$ by :

$$B\mathcal{C}(c,c') = \bigoplus_{k=1}^{\infty} \bigoplus_{c_0=c,c_1,\cdots,c_{k-1},c_k=c'} \mathcal{C}(c_0,c_1) \otimes \cdots \otimes \mathcal{C}(c_{k-1},c_k) \to \mathcal{C}(c_0,c_k).$$

$\mathfrak{m}_k$ then defines a homomorphism $\hat{\mathfrak{m}}_k : B\mathcal{C}(c,c') \to B\mathcal{C}(c,c')$ by

$$\hat{\mathfrak{m}}_k(x_1 \otimes \cdots \otimes x_m) = \sum_i \pm x_1 \otimes \cdots \otimes \mathfrak{m}_k(x_i,\cdots,x_{i+k-1}) \otimes \cdots \otimes x_m.$$

(0.9) then is equivalent to

$$\sum_{k_1+k_2=k+1} \pm\hat{\mathfrak{m}}_{k_1} \circ \hat{\mathfrak{m}}_{k_2} = 0. \tag{0.10}$$

(See [FKOOO] Chapter 4.) We write $\mathfrak{m}_k$ in place of $\hat{\mathfrak{m}}_k$ from now on.

Remark 0.11. Usually existence of identity morphism (the unit) is assumed in the definition of category. There is a similar notion in the case of A^∞ category. However, in the case of A_∞ category, it is a bit complicated to prove the existence of the unit, especially in the case of our main example $\mathcal{LAG}$ we introduce below. (See [Fu4] §13 and [FKOOO] §20.) In the case of $\mathcal{LAG}$, the construction of the unit is related to the transversality of diagonal to itself and hence is rather an essential point. This point was left open in [Fu4] §13. However the difficulty there is solved in [FKOOO] §20. We will discuss it elsewhere.

The main construction of [Fu1] (together with various modifications required) will give an A_∞ category $\mathcal{LAG} = \mathcal{LAG}(M,\Omega)$ to each symplectic manifold (M,Ω) (+ B-field) such that :

(0.12.1) The object is a pair $(L,\mathcal{L})$ of unobstructed Lagrangian submanifold and a flat line bundle on it.
(0.12.2) The homology of $\mathcal{LAG}((L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2))$ is canonically isomorphic to the Floer homology $HF((L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2))$.
(0.12.3) $\mathfrak{m}_2$ in Definition 0.9 induces the map (0.6) in the homolopgy.

The detail of the rigorous construction of A_∞ category, $\mathcal{LAG}(M,\Omega)$ is not written up yet. (However [FKOOO] provides many of the necessary techiniques to do so.)

One of the main result of this paper gives a part of the construction of subcategory $\mathcal{LAG}(T^{2n},\Omega)$. Namely the full subcategory whose object is a pair $(L,\mathcal{L})$ where L is an affine Lagragian submanifold. In this case, we have more. Namely the operation $\mathfrak{m}_k$ in this case is defined in a constructive way. (In other words, we provide an algorism to calculate it.) In the general situation, $\mathfrak{m}_k$, by definition, is a generating function defined by the number counting appropriate pseudoholomorphic disks. Hence computing $\mathfrak{m}_k$ directly from the definition is difficult.

Let us next consider the complex side. Let $M^\vee$ be a complex manifold. Let $\mathcal{O}$ be the set of all chain homotopy equivalence classes of chain complexes of $\mathcal{O}_{M^\vee}$ module sheaves on $M^\vee$ with coherent homology sheaves. For each $\mathcal{F} \in \mathcal{O}$ we fix a represemtatove $C(\mathcal{F})$ of it such that $C^k(\mathcal{F})$ is locally projective and is flabby. We define an A_∞ category $\mathcal{SH} = \mathcal{SH}(M^\vee)$ as follows.

(0.13.1) The set of objects of $\mathcal{SH}$ is $\mathcal{O}$.
(0.13.2) If $\mathcal{F}_1,\mathcal{F}_2 \in \mathcal{O}$, then $\mathcal{SH}^k(\mathcal{F}_1,\mathcal{F}_2) = \bigoplus_\ell \operatorname{Hom}(C^\ell(\mathcal{F}_1), C^{\ell+k}(\mathcal{F}_2))$.
(0.13.2) $\mathfrak{m}_2$ is the usual composition of homomorphisms. $\mathfrak{m}_3$ and higher are all 0.

Remark 0.14. The definition above is a bit adhoc. For example, from the definition, it is not so clear in which sense the construction is independent of the choices of representatives $C(\mathcal{F})$. Actually $\mathcal{SH}(M^\vee)$ is independent of representatives up to homotopy equivalence. The definition of homotopy equivalence of A_∞ category was given in [Fu4].

To state the next conjecture, we need the notion of A_∞ functor. We omit the definition here. (See [Fu4] and §11 of present paper.)

Conjecture D. There exists an A_∞ functor $\mathfrak{F} : \mathcal{LAG}(M,\Omega) \to \mathcal{SH}((M,\Omega)^\vee)$ such that :

(0.15.1) The quasi isomorphism class of $\mathfrak{F}(L,\mathcal{L})$ is $\mathcal{E}(L,\mathcal{L})$, where $\mathcal{E}(L,\mathcal{L})$ is as in Conjecture A.

(0.15.2) The homomorphism

$$\mathfrak{F}_{((L_1,\mathcal{L}_1);(L_2,\mathcal{L}_2))} : \mathcal{LAG}((L_1,\mathcal{L}_1);(L_2,\mathcal{L}_2)) \to \mathcal{SH}(\mathcal{E}(L_1,\mathcal{L}_1);\mathcal{E}(L_2,\mathcal{L}_2)))$$

induces the canonical isomorphism in Conjecture B.

We remark that Conjecture D implies Conjecture C. Moreover it implies the coincidense of (higher) massey (Yoneda) products.

Remark 0.16. It seems too much to expect the A_∞ functor $\mathfrak{F}$ to be a homotopy equivalence, since there seems to be more coherent sheaves than Lagrangian submanifolds. So the image of $\mathfrak{F}$ is expected to be homotopy equivalent to a full subcategory of $\mathcal{SH}$. ((0.15.2) implies that $\mathfrak{F}$ is "injective" up to homotopy equivalence.) Which "singular" Lagrangian submanifolds etc. are to be included in order to make $\mathfrak{F}$ to be a homotopy equivalence, is an interesting but is very hard question.

Now, let us describe the results of this paper. Let V be a $2n$ dimensional vector space and ω be a homogeneous symplectic form on V. Let B be a homogeneous closed 2 form on V. We put $\Omega = \omega + \sqrt{-1}B$. Let $\Gamma \cong \mathbb{Z}^{2n}$ be a lattice on V. We study a torus $T^{2n} = V/\Gamma$ together with a form induced by Ω. (We denote it by Ω also.) We regard $V = \mathbb{C}^n$ and assume that $\Omega \in \Lambda^{1,1}(V)$.

In §1, we explain an idea to construct a complex manifold which is a moduli space of Lagrangian submanifolds (plus flat line bundles on it) of a symplectic manifolds (M,ω), together with B-field B. ($\Omega = \omega + \sqrt{-1}B$.) According to Conjecture A, these complex manifolds are expected to be components of the moduli space of coherent sheaves (more precisely objects of the derived category of coherent sheaves) of the mirror $(M,\Omega)^\vee$. A component of this moduli space which is to correspond to the moduli space of the skyscraper sheaves is the mirror manifold $(M,\Omega)^\vee$ itself. This is an idea due to Strominger-Yau-Zaslow [SYZ]. There are various troubles to make this construction rigorous in the general situation. In the case of a torus, we can make it rigorous and define a mirror torus in this way.

In §2, we review Floer homology of Lagrangian submanifolds especially in the case of affine Lagrangian submanifolds in a symplectic torus. To make the exposition selfcontained, we avoid using pseudoholomorphic disks etc. to define Floer homology in our case. Instead we just take the result of the calculation as a definition. We include a discussion of the degree of Floer homology in §2. (That is a discussion of Maslov index.) (We also refer [Sei] for this poit.)

In §3,4 we discuss Conjecture A in the case of affine Lagrangian submanifolds in symplectic tori. Our construction is motivated by a general idea which is expected to work in the general situation. (Namely the idea, which is based on the discussion with M.Kontsevich, to use family of Floer homologies to construct $\mathcal{E}(L,\mathcal{L})$ in Conjecture A.) This idea will be explained in more detail in [Fu7]. Since there are several troubles which prevent the author to realize this idea rigorously, so in this paper we concentrate on the case of affine Lagrangian submanifolds of tori and gives a rigorous construction in that case. Namely we construct a coherent sheaf $\mathcal{E}(L,\mathcal{L})$ on $(T^{2n},\Omega)^\vee$ to each pair $(L,\mathcal{L})$ of an affine Lagrangian submanifold L of (T^{2n},Ω) and a flat line bundle $\mathcal{L}$ on L.

In §5 we will present a family version of the construction in §3,4 and construct a universal object on $\mathcal{E} \to \mathcal{M} \times (T^{2n},\Omega)^\vee$. Here $\mathcal{M}$ is a (finite) covering space of a component of the moduli space of $(L,\mathcal{L})$ introduced in §1, which is a complex manifold (torus). Namely the restriction of $\mathcal{E}$ to $\{(L,\mathcal{L})\} \times (M,\Omega)^\vee$ is identified with $\mathcal{E}(L,\mathcal{L})$. The holomorphic structure on universal family $\mathcal{E}$ introduced in this section is essential for the discussion of Chapters 2,3.

Chapter 2 is devoted to the study of $\mathfrak{m}_k$ in $\mathcal{LAG}(T^{2n},\Omega)$. In §6 we give a brief summary of the definition of it in the general case.

The operator $\mathfrak{m}_k$ in the case of affine Lagrangian submanifold in the symplectic torus will be a multi theta function. Multi theta function is a generating function of the counting problem of holomorphic polygons in $\mathbb{C}^n$ with affine boundary conditions.

Let us now describe the problem more precisely. We take a compatible complex structure on V and regard $V=\mathbb{C}^n$. Let $\tilde{L}_i$ be a Lagrangian linear subspace of V. Let $v_i \in V/\tilde{L}_i$. We put $\hat{L}_i(v_i)=\tilde{L}_i+v_i \subset V$. $\hat{L}_i(v_i)$ is an affine Lagrangian submanifold of $\mathbb{C}^n$. We assume that $\hat{L}_i(v_i)$ is transversal to $\hat{L}_j(v_j)$. Let $p_{i,j} \in \hat{L}_i(v_i)\cap \hat{L}_j(v_j)$. We concern with holomorphic maps $\varphi : D^2 \to V$ together with points $z_i \in \partial D^2$ with the following properties.

(0.17.1) $z_i \in \partial D^2$, $(z_1,\cdots,z_{k+1})$ respects the cyclic order of ∂D^2.
(0.17.2) $\varphi(z_i)=p_{i,i+1}$.
(0.17.3) $\varphi(\partial_i D^2)\subseteq \hat{L}_i(v_i)$, where $\partial_i D^2$ is the part of ∂D^2 between $p_{i-1,i}$ and $p_{i,i+1}$.

Figure 1

We consider the quotient space of set of all φ by the obvious $PSL(2;\mathbb{R})=\mathrm{Aut}(D^2)$ action and denote it by $\mathcal{M}(\hat{L}_1(v_2),\cdots,\hat{L}_{k+1}(v_{k+1}))$. The problem studied in §§7,8,9 is :

Problem 0.18. Count the order (with sign) of the set $\mathcal{M}(\hat{L}_1(v_1),\cdots,\hat{L}_{k+1}(v_{k+1}))$ in case its virtual dimension is 0.

The author would like to thank M. Gromov who introduced the problem counting holomorphic polygons in V to the author.

The calculation of $\mathfrak{m}_k$ in the case of affine Lagrangian torus in (T^{2n},Ω) will be reduced to Problem 0.18 by taking the universal cover.

The number in Problem 0.18 is the simplest nontrivial case of open string analogue of Gromov-Witten invariant. By the same reason as Gromov-Witten invariant, there is a transversality problem to define this number rigorously.

In our case of open string version, the problem is more serious. Namely the methods developed to define Gromov-Witten invariants rigorously (see for example [BF],[FOn1],[FOn2],[LiT],[LuT],[Ru],[Sie]) are not enough to establish its "open string analogue". In fact, in the most naive sense, this number is ill-defined. Oh [Oh] discovered this trouble in a related context of Floer homology theory of Lagrangian intersection.

The basic reason of it is similar to the wall crossing problem discovered by Donaldson [Do1] to define Donaldson invariant of 4-manifolds with $b_2^+=1$. In our case, this problem is related to the fact that Massey product is well-defined only as an element of some quotient group. Donaldson introduced a chamber structure to study the ill-definedness of Donaldson invariant. For our problem of counting holomorphic polygons, we need also to study a chamber structure.

In our case, the wall (that is the boundary of the chamber) may also be ill-defined. Namely the point where the number of holomorphic polygons jumps may also depend on the perturbation. (This problem is pointed out in [Fu5] §5.) So only some kind of family version of the number in Problem 0.18 is well-defined, up to some kind of homotopy. We are going to make precise what we mean by it.

See [FKOOO] Chapter 4 for the problem of wall-crossing of Floer homology in the general situation. In this paper, we concentrate on the case of affine Lagrangian submanifolds in symplectic tori and will obtain more explicite result on the chamber structure, wall crossing etc. The result proved in this paper provides a good example to see what happens in the general case.

In this paper, we do not try to attach the analytic side of the transversality problem and do not try to define the number in Problem 0.18 rigorously. Instead, we list up the properties which those numbers are supposed to satisfy as axioms and prove that there exists such system of numbers and they are unique up to some kind of homotopy equivalence.

It implies that if we can find some perturbation so that transversality is satisfied and the numbers obtained by that perturbation satisfy the axiom listed below, then it is homotopy equivalent to one given in this paper. (The proof of the existence of such perturbation seems to be similar to [FKOOO] Chapter 6, but we do not try to present it in this paper since we do not need it for the purpose of present paper.)

Let us now describe the axiom. We first review the virtual dimension of $\mathcal{M}(\hat L_1(v_1), \cdots, \hat L_{k+1}(v_{k+1}))$. The virtual dimension is independent of v_i and depends only on $\tilde L_i$. We denote it by $-\deg(\tilde L_1, \cdots, \tilde L_{k+1})$. It related to Kashiwara-Maslov index $\eta(\tilde L_1, \cdots, \tilde L_{k+1})$ [KS] by the formula

$$\deg(\tilde L_1, \cdots, \tilde L_{k+1}) = \eta(\tilde L_1, \cdots, \tilde L_{k+1}) + 2 - k, \tag{0.19}$$

and satisfies

$$\deg(\tilde L_1, \cdots, \tilde L_{k+1}) = \deg(\tilde L_1, \cdots, \tilde L_i, \tilde L_j, \cdots, \tilde L_{k+1}) + \deg(\tilde L_i, \cdots, \tilde L_j) + 1, \tag{0.20}$$

(§§6,7.) To give a good axioms, we need to include also the case when the virtual dimension $-\deg(\tilde L_1, \cdots, \tilde L_{k+1})$ is not 0 but is negative. Off course, in such a case, we cannot count the order of the set $\mathcal{M}(\hat L_1(v_1), \cdots, \hat L_{k+1}(v_{k+1}))$. Instead we consider a family version and proceed as follows. We put

$$\tilde L(\tilde L_1, \cdots, \tilde L_{k+1}) = \prod_{i=1}^{k+1} V/\tilde L_i$$

We embed V to $\tilde L(\tilde L_1, \cdots, \tilde L_{k+1})$ by $v \mapsto ([v], \cdots, [v])$ and put

$$L(\tilde L_1, \cdots, \tilde L_{k+1}) = \frac{\tilde L(\tilde L_1, \cdots, \tilde L_{k+1})}{V}.$$

We define

$$\mathcal{M}(\tilde L_1, \cdots, \tilde L_{k+1}) = \bigcup_{[v_1, \cdots, v_{k+1}] \in L(\tilde L_1, \cdots, \tilde L_{k+1})} \mathcal{M}(\hat L_1(v_1), \cdots, \hat L_{k+1}(v_{k+1})) \times \{[v_1, \cdots, v_{k+1}]\}.$$

Let $\mathcal{M}(\tilde L_1, \cdots, \tilde L_{k+1}) \to L(\tilde L_1, \cdots, \tilde L_{k+1})$ be the natural projection.

If the actual dimension of $\mathcal{M}(\tilde L_1,\cdots,\tilde L_{k+1})$ is equal to its virtual dimension, then $\mathcal{M}(\tilde L_1,\cdots,\tilde L_{k+1})$ defines a codimension $\deg(\tilde L_1,\cdots,\tilde L_{k+1})$ chain on $L(\tilde L_1,\cdots,\tilde L_{k+1})$. We regard it as degree $\deg(\tilde L_1,\cdots,\tilde L_{k+1})$ (integral) current and denote it by $\mathfrak{C}_{hol}(\tilde L_1,\cdots,\tilde L_{k+1})$. (More precisely we do not try to construct it rigorously as the fundamental chain of moduli space but only construct such current satisfying expected properties describe below.)

We are going to list up the properties which $\mathfrak{C}_{hol}(\tilde L_1,\cdots,\tilde L_{k+1})$ is supposed to satisfy. We need more notations for it.

For $a \neq b$, we can identify

$$L(\tilde L_1,\cdots,\tilde L_{k+1}) \simeq \prod_{i\neq a,b} \mathbb{C}^n/\tilde L_i,$$

(since $\tilde L_i$ are transverse each other.) Therefore, we have a canonical isomorphism

$$L(\tilde L_1,\cdots,\tilde L_{k+1}) \simeq L(\tilde L_1,\cdots,\tilde L_a,\tilde L_b,\cdots,\tilde L_{k+1}) \times L(\tilde L_a,\cdots,\tilde L_b). \tag{0.21}$$

Let us next consider $\varphi : D^2 \to V$ satisfying (0.17.1), (0.17.2), (0.17.3). (However we do not assume φ to be holomorphic here.) We put

$$Q(v_1,\cdots,v_{k+1}) = \int_{D^2} \varphi^*\Omega. \tag{0.22}$$

Using Stokes' theorem, we can easily verify that (0.22) is independent of φ and depends only on v_i. Furthermore, since the right hand side is independent of the translation : $v_i \mapsto v_i + v$ by $v \in \mathbb{C}^n$, Q induces a map from $L(\tilde L_1,\cdots,\tilde L_{k+1})$. One can then prove that its real part $\Re Q$ is a non degenerate quadratic form on $L(\tilde L_1,\cdots,\tilde L_{k+1})$ and

$$\operatorname{Index} \Re Q = \eta(\tilde L_1,\cdots,\tilde L_{k+1}) \tag{0.23}$$

(See §7. Lemma 7.3.)

Now we are ready to describe axioms.

Axiom I. $\mathfrak{C}(\tilde L_1,\cdots,\tilde L_{k+1})$ is an integral current on $L(\tilde L_1,\cdots,\tilde L_{k+1})$ with the following properties.

(I.1) Degree of $\mathfrak{C}(\tilde L_1,\cdots,\tilde L_{k+1})$ is $\deg(\tilde L_1,\cdots,\tilde L_{k+1})$.
(I.2) The current $\overline{\mathfrak{C}}(\tilde L_1,\cdots,\tilde L_{k+1})$ is invariant of the $\mathbb{R}^+$ action, $r\cdot[v_1,\cdots,v_{k+1}] = [rv_1,\cdots,rv_{k+1}]$.
(I.3) Let $\|\|$ be a norm induced by an inner product on $L(\tilde L_1,\cdots,\tilde L_{k+1})$. Then there exists $\delta > 0$ such that for each $[v_1,\cdots,v_{k+1}] \in L(\tilde L_1,\cdots,\tilde L_{k+1})$ contained in the support of $\mathfrak{C}(\tilde L_1,\cdots,\tilde L_{k+1})$ we have

$$\Re Q([v_1,\cdots,v_{k+1}]) > \delta\|[v_1,\cdots,v_{k+1}]\|^2.$$

In particular the quadratic form $\Re Q$ is positive definite on the support of $\mathfrak{C}(\tilde L_1,\cdots,\tilde L_{k+1})$.

Axiom II. We have

$$d\mathfrak{C}(\tilde L_1,\cdots,\tilde L_{k+1}) = \sum_{1\le \ell<m\le k+1} \pm\mathfrak{C}(\tilde L_\ell,\cdots,\tilde L_m)\times \mathfrak{C}(\tilde L_1,\cdots,\tilde L_\ell,\tilde L_m,\cdots,\tilde L_{k+1}).$$

Here we use (0.21) to see that the left hand side is defined in the same space as the right hand side. We use (0.20) to see that the degree of the right hand side coincides with the degree of left hand side. The sign $\pm$ is specified later in §7 (Definition 7.26).

The third axiom concerns with the case when $k+1=3$, $\mathfrak{C}(\tilde L_1,\tilde L_2,\tilde L_3)$. Axiom II implies that $d\mathfrak{C}(\tilde L_1,\tilde L_2,\tilde L_3)=0$. Axiom I then implies that $\mathfrak{C}(\tilde L_1,\tilde L_2,\tilde L_3)$ determines a cohomology class

$$[\mathfrak{C}(\tilde L_1,\tilde L_2,\tilde L_3)\cap S(\tilde L_1,\tilde L_2,\tilde L_3)] \in H^{\deg(\tilde L_1,\tilde L_2,\tilde L_3)}(S(\tilde L_1,\tilde L_2,\tilde L_3);\mathbb{Z})$$

where

$$S(\tilde L_1,\tilde L_2,\tilde L_3) = \{[v_1,\cdots,v_{k+1}] | Q([v_1,\cdots,v_{k+1}])>0, \|[v_1,\cdots,v_{k+1}]\|=1\}.$$

Axiom III. $[\mathfrak{C}(\tilde L_1,\tilde L_2,\tilde L_3)\cap S(\tilde L_1,\tilde L_2,\tilde L_3)]$ is a generator of $H^{\deg(\tilde L_1,\tilde L_2,\tilde L_3)}(S(\tilde L_1,\tilde L_2,\tilde L_3);\mathbb{Z})\cong\mathbb{Z}$.

We can show the isomorphism $H^{\deg(\tilde L_1,\tilde L_2,\tilde L_3)}(S(\tilde L_1,\tilde L_2,\tilde L_3);\mathbb{Z})\cong\mathbb{Z}$ by using (0.23). (Corollary 6.14.) Now the main result of Chapter 2 is the following :

Theorem α. *There exists a system of currents $\mathfrak{C}(\tilde L_1,\cdots,\tilde L_{k+1})$ satisfying Axioms I,II,III.*

We can also prove that such $\mathfrak{C}(\tilde L_1,\cdots,\tilde L_{k+1})$ is unique up to homotopy.

Theorem β. *Let $\mathfrak{C}(\tilde L_1,\cdots,\tilde L_{k+1})$ and $\mathfrak{C}'(\tilde L_1,\cdots,\tilde L_{k+1})$ be system of currents satisfying Axiom I,II. We also assume*

$$[\mathfrak{C}(\tilde L_1,\tilde L_2,\tilde L_3)\cap S(\tilde L_1,\tilde L_2,\tilde L_3)] = [\mathfrak{C}'(\tilde L_1,\tilde L_2,\tilde L_3)\cap S(\tilde L_1,\tilde L_2,\tilde L_3)].$$

Then $\mathfrak{C}(\tilde L_1,\cdots,\tilde L_{k+1})$ is homotopy equivalent to $\mathfrak{C}'(\tilde L_1,\cdots,\tilde L_{k+1})$.

We will define homotopy equivalence in §8. Axiom III determines the homology class $[\overline{\mathfrak{C}}(\tilde L_1,\tilde L_2,\tilde L_3)\cap S(\tilde L_1,\tilde L_2,\tilde L_3)]$ up to sign. We explain the way to fix the sign (the orientation) in §7.

Using the current $\mathfrak{C}(\tilde L_1,\cdots,\tilde L_{k+1})$, obtained in Theorem α, we define multi theta current. We give its definition in §10. Here we discuss the case when $\deg(\tilde L_1,\cdots,\tilde L_{k+1})=0$. In this case, $\mathfrak{C}(\tilde L_1,\cdots,\tilde L_{k+1})$ is a degree zero (integral) current. In other words, we may regard it as an integer valued function, which we write $\mathfrak{c}_k : L(1,\cdots,k+1)\to\mathbb{Z}$. Now we define a multi theta series by

$$\begin{aligned}\sum \mathfrak{c}_k[v_1+\gamma_1,\cdots,v_{k+1}+\gamma_{k+1}] \exp\big(&-2\pi Q(v_1+\gamma_1,\cdots,v_{k+1}+\gamma_{k+1}) \\ &+2\pi\sqrt{-1}\sum \alpha_i(p_{i,i+1}-p_{i-1,i})\big).\end{aligned} \tag{0.24}$$

Here v_i parametrize the affine Lagrangian submanifold in V parallel to $\tilde{L}_i$. The sum is taken over $(\gamma_1, \cdots, \gamma_{k+1})$ which is in certain lattice in $\mathbb{R}^{n(k-2)}$. α_i is an element of the dual space $\tilde{L}_i^*$, which may be identified to a flat connection on $\tilde{L}_i$. $p_{i,j}$ is the point where two affine Lagrangian submanifolds (i-th and j-th) intersect. (See §10 for precise definition.)

We will prove that (0.24) converges in §10. (0.24) gives a usual theta function in the case $k = 2$. (In case $n = 1$, this fact was observed by Kontsevich in [K2].) In case $k = 3$, (0.23) is an indefinite theta series which looks similar to those used by Götche-Zagier [GZ] to study Donaldson's invariant of 4-manifolds with $b_2^+ = 1$. In case $k \geq 4$, it seems that (0.23) is a new family of theta series.

Using these multi theta functions as matrix elements, we obtain maps

$$\begin{aligned}\mathfrak{m}_k : HF((L_1, \alpha_1), (L_2, \alpha_2)) \otimes \cdots \otimes HF((L_k, \alpha_k), (L_{k+1}, \alpha_{k+1})) \\ \to HF((L_1, \alpha_1), (L_{k+1}, \alpha_{k+1})),\end{aligned} \tag{0.25}$$

of degree $2-k$. If we move L_i, α_i then $\mathfrak{m}_k$ move. Thus, we may regard $v_1, \cdots, v_{k+1}$, $\alpha_1, \cdots, \alpha_{k+1}$ as variables of $\mathfrak{m}_k$ also. (See §10 for precise formulation.)

We remark that $\mathfrak{c}_k[v_1, \cdots, v_{k+1}]$ is an integer valued function and jumps at the support of the current $d\mathfrak{C}(\tilde{L}_1, \cdots, \tilde{L}_{k+1})$.

We call the support of $d\mathfrak{C}(\tilde{L}_1, \cdots, \tilde{L}_{k+1})$ the *wall*.

The value of the series (0.25) is not defined at the point $[v_1, \cdots, v_{k+1}]$ such that $[v_1+\gamma_1, \cdots, v_{k+1}+\gamma_{k+1}]$ lies on the wall for some $(\gamma_1, \cdots, \gamma_{k+1})$. The set of all such bad $[v_1, \cdots, v_{k+1}]$ is dense but has measure zero. Thus $\mathfrak{m}_k$ is well-defined only when $L_1, \cdots, L_{k+1}$ is generic. Moreover the wall does depend on the particular choice of $\mathfrak{C}(\tilde{L}_1, \cdots, \tilde{L}_{k+1})$ and hence is not invariant. We can prove that it is invariant "up to homotopy" by Theorem β. (Theorem 11.12).

We can use the current $\mathfrak{C}(\tilde{L}_1, \cdots, \tilde{L}_{k+1})$ in the case when $\deg(\tilde{L}_1, \cdots, \tilde{L}_{k+1}) > 0$ in a similar way to (0.24) and obtain a current version $\mathfrak{m}_k^{(d)}$ of $\mathfrak{m}_k$. We consider its $(0, d)$ part $\mathfrak{m}_k^{(0,d)}$. It is a current of $(0, d)$ type with respect to $v_1, \cdots, v_{k+1}, \alpha_1, \cdots$, α_{k+1} and is a linear map in the same sense as (0.25). (See §10 for precise formulation.)

The following result is a consequence of the axioms (especially Axiom II) and is a generalization of A_∞ formulae.

Theorem γ.

$$\bar{\partial}\mathfrak{m}_k^{(0,d)} + \sum_{\substack{d_1+d_2=d+1 \\ k_1+k_2=k+1}} \pm \mathfrak{m}_{k_1}^{(0,d_1)} \circ \mathfrak{m}_{k_2}^{(0,d_2)} = 0 \tag{0.26}$$

Here $\bar{\partial}$ is the Dolbaut operator. We use the complex structure introduced in §5 to define $\bar{\partial}$. The sign is discussed in §11. The proof of Theorem γ is in §11.

In case when $\ell = -1$, the first term of (0.26) is zero. Hence we have

$$\sum_{k_1+k_2=k+1} \pm \mathfrak{m}_{k_1}^{(0,0)} \circ \mathfrak{m}_{k_2}^{(0,0)} = 0.$$

This is the A_∞ formula (0.10) in Definition 0.8.

(0.26) is a version of Maurer-Cartan equation (or Batalin-Vilkovisky master equation). They appear in many literatures recently, in related context. (See [ASKZ],[BK],[K3],[Ma],[Sch],[St2] etc.) The L_∞ version appears mainly in those literatures. (0.26) is an A_∞ version. (Here L stands for Lie and A for associative.)

In §11, we also show that if we change the choice of $\mathfrak{C}(\tilde L_1, \cdots, \tilde L_{k+1})$ then Theorem β will imply that the resulting A_∞ structures are homotopy equivalent. In particular, all the (higher) massey products coincide.

Using the result we thus described, we are going to prove partially Conjectures A,B,C,D in our case, in Chapter 3.

Let $\mathcal{E}(L,\mathcal{L})$ be a coherent sheaf of on $(T^{2n},\Omega)^\vee$ associated to the pair $(L,\mathcal{L})$ of an affine Lagrangian submanifold L of (T^{2n},Ω) and a flat line bundle $\mathcal{L}$ on L. In this case, we have the following result which partially prove Conjectuer B.

Theorem δ.

$$\mathrm{Ext}^k(\mathcal{E}(L_1,\mathcal{L}_1),\mathcal{E}(L_2,\mathcal{L}_2)) \cong HF^k((L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2)). \tag{0.27}$$

We remark that we do not assume that L_1 is transversal to L_2 here. (We assume that they are affine Lagrangian submanifolds in the torus.) Theorem δ is proved in §§12,13,15. However the proof there is unsatisfactory. Namely we only prove there that the ranks of the vector spaces in (0.27) coincide to each other and does not provide any canonical isomorphism. (Hence Theorem δ is not enough to prove Conjecture B.)

In §14 we provide a canonical isomorphism (0.27) using operators $\mathfrak{m}_k^{(0,d)}$ constructed in Chapter 2. However in §14 we study only the case when L_1 is transversal to L_2, (since we assumed transversality in Chapter 2.) It seems that by modifying the argument of Chapter 2 and §15 we may construct canonical isomorphism in the general case also. We leave it for future research.

The proof of Theorem δ in §§12,13,15 proceeds roughly as follows.

In §12 we prove Theorem δ in the case when $\mathcal{E}(L_1,\mathcal{L}_1)$ is the structure sheaf and $\mathcal{E}(L_2,\mathcal{L}_2)$ is a line bundle. In that case, the group

$$\mathrm{Ext}^k(\mathcal{E}(L_1,\mathcal{L}_1),\mathcal{E}(L_2,\mathcal{L}_2)) = H^k((T^{2n},\Omega)^\vee;\mathcal{E}(L_2,\mathcal{L}_2))$$

can be calculated if we know the first Chern class of $\mathcal{E}(L_2,\mathcal{L}_2)$. We will calculate the Chern class of $\mathcal{E}(L_2,\mathcal{L}_2)$ by looking the explicite description of it.

In §13, we study the case when $\mathcal{E}(L_i,\mathcal{L}_i)$ are vector bundles. We use isogeny trick to reduce this case to the case proved in §12. The argument here is somewhat similar to one by Polishchuk-Zaslow [PZ]. In §15, we complete the proof of Theorem δ. The argument of §15 is rather technical and is mainly a careful check that Maslov index in symplectic side coincides with the degree where extension is nonzero in complex side.

§14 is the heart of Chapter 3. There we construct the isomorphism in Theorem δ in a canonical way. Moreover we will prove Conjecture C in this case. Namely we prove :

Theorem ϵ. *We assume that L_1,L_2,L_3 are mutually transversal. We assume that $\mathcal{E}(L_i,\mathcal{L}_i)$ are vector bundles. Then we have a canonical isomorhpism (0.27) such that the Diagram 1 commutes.*

In fact, we can prove more. Namely we can prove the coincidence of various secondary operations. This should follow from Conjecture D. Since we exclude the case when L_1 is not transversal to L_2 (especially the case $L_1 = L_2$), we do not prove Conjecture D for affine Lagrangian submanifolds. However as far as mutually transversal affine Lagrangian submannifolds concern, we can prove everything which would follow from Conjecture D. To be specific, we will prove the coincidence of Massey product (Theorem 14.12.).

Up to Chapter 3, we consider only affine Lagrangian submanifold. Chapter 4 is devoted to some preliminary study of the case of Lagrangian submanifold which is not affine. A method to construct non affine Lagrangian submanifold in symplectic torus is by Lagrangian surgery. In [FKOOO] Chapter 7, we discuss some examples to show how the Floer homology behave by Lagrangian surgery. In Chapter 4, we present a construction in complex geometry side which are conjectured to correspond to Lagrangian surgery in the symplectic geometry side.

Let L_i be mutually transversal Lagrangian submanifolds. We assume that $\mathcal{E}(L_i, \mathcal{L}_i)$ are vector bundles. Let $x_{i,j} \in HF((L_i, \mathcal{L}_i), (L_j, \mathcal{L}_j))$. (To be precise, we need to specify the integers determing degree shift. See §16.) We consider equations of the form :

$$\sum_k \sum_{i_0=i, i_1, \cdots, i_{k-1}, i_k=j} \pm \mathfrak{m}_k \left(x_{i_0,i_1}, \cdots, x_{i_{k-1},i_k} \right) = 0 \tag{0.28}$$

for each i, j. (See §16 for precise notation and sign.) We prove :

Theorem ϕ. *There exists a family of the objects of derived category of coherent sheaves on $(T^{2n}, \Omega)^\vee$ parametrized by the solution of (0.28).*

We call the system $\mathbb{L} = (\cdots, (L_i, \mathcal{L}_i), \cdots; \cdots, x_{i,j}, \cdots)$ satisfying (0.28) a Lagrangian resolution. We denote by $\mathcal{E}(\mathbb{L})$ the object of derived category constructed in Theorem ϕ from the Lagrangian resolution $\mathbb{L}$.

Note that (0.28) is a polynomial of x_{ij} and its coefficients are special values of multi theta functions. Roughly speaking, the object in Theorem ϕ is the cohomology sheaf of the modified Dolbeault operator $\bar\partial^\wedge = \bar\partial + \sum \pm \mathfrak{m}_k^{(0,d)}(\bullet, x \cdots x)$. Theorem ϕ seems to be related to the monad or quiver description of the moduli space of stable sheaves. (See Examples 16.32,16.33,16.35.)

As we will explain in [Fu7], a Lagrangian resolution $\mathbb{L}$ gives an A_∞ functor $\mathfrak{F}_{\mathbb{L}} : \mathcal{LAG} \to \mathcal{CH}$, where $\mathcal{CH}$ is the category of all chain complexes. (See [Fu4] §10 for the definition of A_∞ functor and $\mathcal{CH}$.) As we mentioned before it seeems that there are more coherent sheaves than Lagrangian submanifolds. So, to obtain an A_∞ category homotopy equivalent to $\mathcal{SH}$, we need to increase the number of objects in $\mathcal{LAG}$. A_∞ functor $\mathcal{LAG} \to \mathcal{CH}$ can be throught as a generalization of the objects of $\mathcal{LAG}$, (since an object of $\mathcal{LAG}$ gives a (representable) A_∞ functor $\mathcal{LAG} \to \mathcal{CH}$).

The next result describes the extension between objects obtained in Theorem ϕ. Let $\mathbb{L}^{(k)} = (\cdots, (L_i^{(k)}, \mathcal{L}_i^{(k)}), \cdots; \cdots, x_{i,j}^{(k)}, \cdots)$, $(k = 1, 2)$ be Lagrangian resolutions. We assume that $L_i^{(1)}$ is transversal to $L_j^{(2)}$.

We put

$$C(\mathbb{L}^{(1)}, \mathbb{L}^{(2)}) \cong \bigoplus_{i,j} HF((L_i^{(1)}, \mathcal{L}_i^{(1)}), (L_j^{(2)}, \mathcal{L}_j^{(2)})).$$

For $i' \le i$, $j \le j'$, we define

$$\partial_{i',i;j,j'} : HF((L_i^{(1)}, \mathcal{L}_i^{(1)}), (L_j^{(2)}, \mathcal{L}_j^{(2)})) \to HF((L_{i'}^{(1)}, \mathcal{L}_{i'}^{(1)}), (L_{j'}^{(2)}, \mathcal{L}_{j'}^{(2)}))$$

by

$$\partial_{i',i;j,j'}(s_{i,j}) = \sum_{k,\ell} \sum_{i_1=i',\cdots,i_k=i} \sum_{j_1=j,\cdots,j_\ell=j'} \pm \mathfrak{m}_{k+\ell+1}\left(x^{(1)}_{i_1,i_2},\cdots,x^{(1)}_{i_{k-1},i_k},s_{i,j},x^{(2)}_{j_1,j_2},\cdots,x^{(2)}_{j_{\ell-1},j_\ell}\right). \tag{0.29}$$

(See §16 for the sign convention.) We put

$$\partial^\vee = \sum \partial_{i',i;j,j'}.$$

We will prove $\partial^\vee\partial^\vee = 0$ in §16. (Lemma 16.13.)

Theorem γ. *The group* $\operatorname{Ext}(\mathcal{E}(\mathbb{L}^{(1)}),\mathcal{E}(\mathbb{L}^{(2)}))$ *is isomorphic to* $\operatorname{Ker}\partial^\vee/\operatorname{Im}\partial^\vee$.

Note that the set $\mathcal{C}(\mathfrak{F}_{\mathbb{L}^{(1)}},\mathfrak{F}_{\mathbb{L}^{(2)}})$ of all pre A_∞ transformations from $\mathfrak{F}_{\mathbb{L}^{(1)}}$ to $\mathfrak{F}(\mathbb{L}^{(2)})$ is a chain complex. ([Fu4] §10.) We can prove that the cohomology of $\mathcal{C}(\mathfrak{F}_{\mathbb{L}^{(1)}},\mathfrak{F}_{\mathbb{L}^{(2)}})$ is identified to the cohomology of $\partial^\vee$. Thus, Theorem γ can be regarded as a proof of Conjecture B in this case. We do not discuss A_∞ category systematically in this paper. So the result of §16 should be regarded as a preliminary one.

We also explain in Chapter 4 of present paper and Chapter 7 of [FKO3], that a mirror of the system satisfying (0.28), is expected to be a smooth Lagrangian submanifold obtained from affine Lagrangian submanifolds L_i by Lagrangian surgery (see (16.40) for some evidence). In this way, we can include non affine Lagrangian submanifold in the story also. Namely, in various cases, a mirror of a sheaves which is not semi-homogeneous, is a Lagrangian submanifold which is not affine. In the case Lagrangian submanifold $\mathfrak{L}(\mathbb{L}^{(i)})$ obtained from Lagrangian resolution $\mathbb{L}^{(i)}$ by surgery, the cohomology of $\partial^\vee$ is expected to coincide with the Floer homology $HF(\mathfrak{L}(\mathbb{L}^{(1)}),\mathfrak{L}(\mathbb{L}^{(2)}))$. Thus Theorem γ may provide a way to calculate Floer homology between Lagrangian submanifolds in tori systematically, when the program mentioned here will be sucessfully completed.

One of the main ideas of this paper, that is to use family of Floer homology to construct mirror sheaf, is based on a discussion between the author and M.Kontsevich during the author's stay I.H.E.S in August 1997. The author thanks I.H.E.S. for hospitality during his stay there.

Chapter 1. Construction of Mirror torus and of Mirror bundles.

§1. Moduli space of Lagrangian submanifolds.

In this section, we construct a mirror torus of a given (flat) symplectic torus (T^{2n},Ω) such as $T^{2n}=\mathbb{C}^n/(\mathbb{Z}\oplus\sqrt{-1}\mathbb{Z})^n$, $\Omega\in\Lambda^{1,1}(T^{2n})$. (Note that the complex structure of the torus T^{2n} is not important here. We use it only to set the condition $\Omega\in\Lambda^{1,1}(T^{2n})$.) We first give an idea which the author expects to work in more general situations. (See [Fu6],[Fu7] for more detail on general case.) We then will make it rigorous in the case of a torus.

Let (M,Ω) be a symplectic manifold (M,ω) together with a closed 2 form B on M. Here we put $\Omega=\omega+\sqrt{-1}B$. (Note $-B+\sqrt{-1}\omega$ is used in many of the literatures.)

Definition 1.1. $\mathcal{LAG}^{\sim+}(M,\Omega)$ is the set of all pairs $(L,\mathcal{L})$ with the following properties :

(1.2.1) L is a Lagrangian submanifold of (M,ω).
(1.2.2) $\mathcal{L}\to L$ is a line bundle together with a connection $\nabla^{\mathcal{L}}$ such that $F_{\nabla^{\mathcal{L}}}=2\pi\sqrt{-1}B|_L$.

We put the C^∞ topology on $\mathcal{LAG}^{\sim+}(M,\Omega)$. This space is of infinite dimension. We will divide it by the group of Hamiltonian diffeomorphisms. The quotient space is a finite dimensional "manifold". To be precise, we proceed as follows.

Let $f:M\times[0,1]\to\mathbb{R}$ be a smooth function and we put $f_t(x)=f(x,t)$. Let $X_{f,t}$ be the Hamiltonian vector field associated to f_t. It induces a one parameter family of symplectic diffeomorphisms $\varphi:M\times[0,1]\to M$ by:

$$\varphi(x,0)=x,\quad \frac{\partial}{\partial t}\varphi(x,t)=X_{f,t}(\varphi(x,t)). \tag{1.3}$$

We put $\varphi_t(x)=\varphi(x,t)$. The diffeomorphism $\varphi_1(x)$ is called a *Hamiltonian diffeomorphism.*

Definition 1.4. Let $(L,\mathcal{L}),(L',\mathcal{L}')\in\mathcal{LAG}^{\sim+}(M,\Omega)$. We say that $(L,\mathcal{L})$ is *Hamiltonian equivalent* to $(L',\mathcal{L}')$ if the following holds. There exists $f:M\times[0,1]\to\mathbb{R}$ such that the map $\varphi:M\times[0,1]\to M$ solving (1.3) satisfies $\varphi_1(L)=L'$. Also there exists a connection ∇ on $L\times[0,1]$ with the following properties.

(1.5.1) $F_\nabla=2\pi\sqrt{-1}\varphi^*B|$,
(1.5.2) $\nabla|_{L\times\{0\}}=\nabla^{\mathcal{L}}$.
(1.5.3) There exists an isomorphism $(L,\nabla|_{L\times\{1\}})\cong(L',\nabla^{\mathcal{L}'})$ coverging φ_1.

It is easy to see that Hamiltonian equivalence defines an equivalence relation on $\mathcal{LAG}^{\sim+}(M,\Omega)$. Let $\mathcal{LAG}^{+}(M,\Omega)$ denote the quotient space with quotient topology. $\mathcal{LAG}^{+}(M,\Omega)$ is the moduli space of Lagrangian submanifold we use in this paper.

In [SYZ], Strominger-Yau-Zaslow proposed closely related but a bit different moduli space. Namely they proposed the moduli space $\mathcal{LAG}_{\rm sp}$ which consists of the pairs $(L,\mathcal{L})\in\mathcal{LAG}^{\sim+}(M,\Omega)$ of special Lagrangian submanifolds L and flat line bundles $\mathcal{L}$ on it. It seems that, by taking a special Lagrangian submanifold, we take a representative of Hamiltonian equivalence. However we need to study some open questions to clarify the relation between these two moduli spaces. Especially we need to prove the following :

Conjecture 1.6. The induced map $\mathcal{LAG}_{\rm sp} \to \mathcal{LAG}^+(M,\Omega)$ is injective.

It seems unlikely that the map $\mathcal{LAG}_{\rm sp} \to \mathcal{LAG}^+(M,\Omega)$ is surjective. So to determine the image of it is another important question.

A complex structure on $\mathcal{LAG}_{\rm sp}$ is defined in [SYZ]. In a similar way, our moduli space $\mathcal{LAG}^+(M,\Omega)$ has a "complex structure" as we will soon define. (In our case, we do not need to assume that M is a Calabi-Yau manifold and can start with a general symplectic manifold.) However, in fact, we do not know whether $\mathcal{LAG}^+(M,\Omega)$ is a manifold or not, since we do not know whether it is Hausdorff or not. So we consider Hausdoff part (or stable part) of it only.

Let $(L,\mathcal{L}) \in \mathcal{LAG}^{\sim +}(M,\Omega)$. By Darbout-Weinstein theorem, a neighborhood U of L in M is symplectically diffeomorphic to a neighborhood of the zero section of the cotangent bundle T^*L. We denote it by $\psi : U \to T^*L$. Let ω' be the standard symplectic form on T^*L and B' be a closed 2 form on T^*L which coincides with $\psi^{-1*}B$ in a neighborhood of zero section.

Condition 1.7. ψ induces a homeomorphism from a closed neighborhood $\mathcal{U}$ of $[L,\mathcal{L}] \in \mathcal{LAG}^+(M,\Omega)$ to a closed neighborhoods of $[L,\mathcal{L}] \in \mathcal{LAG}^+(T^*L,\omega' + \sqrt{-1}B')$.

We can show easily that if $[L,\mathcal{L}] \in \mathcal{LAG}^+(M,\Omega)$ satisfies Condition 1.7, $[L',\mathcal{L}'] \in \mathcal{LAG}^+(M,\Omega)$, and if $[L,\mathcal{L}] \ne [L',\mathcal{L}']$, then they have disjoint neighborhoods. (Note that we assume $\mathcal{U}$ in Condition 1.7 to be closed.)

Condition 1.7 also implies the following : For each $\epsilon > 0$, there exists $\mathcal{U}_\epsilon$ a neighborhood of $(L,\mathcal{L})$ in $\mathcal{LAG}^{\sim +}(M,\Omega)$, such that if $(L_1,\mathcal{L}_1),(L_2,\mathcal{L}_2) \in \mathcal{U}_\epsilon$ are Hamiltonian equivalent to each other, then the function f in Definition 1.4 can be chosen so that its C^1 norm is smaller than ϵ.

The reader who is familiar with symplectic geometry may find that Condition 1.7 is closely related to the flux conjecture. (See [LMP].)

We let $\mathcal{LAG}(M,\Omega)$ be the subset of $\mathcal{LAG}^+(M,\Omega)$ consisting the equivalece class of the pairs $[L,\mathcal{L}]$ satisfying Condition 1.7.

Proposition 1.8. *Let $K \subset \mathcal{LAG}^+(M,\Omega)$. We assume that Condition 1.7 is satisfied for each $[L,\mathcal{L}] \in K$. Then a neighborhood of K in $\mathcal{LAG}^+(M,\Omega)$ has a structure of complex manifold.*

Proof. Let $[L,\mathcal{L}] \in K$. We are going to construct a chart on its neighborhood. Let $\ell_1,\cdots,\ell_b$ be loops representing a basis of $H_1(L;\mathbb{Z})$ and $[L',\mathcal{L}']$ be in a neighborhood of $[L,\mathcal{L}]$. By Condition 1.7, we may assume that L' is C^1 close to L. Hence we may assume that it is a graph of a closed one form u on L. We define $\phi_i : S^1 \times [0,1] \to T^*L$ by $\phi_i(s,t) = su(\ell_i(t))$. We put

$$h_i(L',\mathcal{L}') = h_{\phi_i(\cdot,1)}(\mathcal{L}') \exp\left(-2\pi \int \phi_i^*\Omega\right). \tag{1.9}$$

Here $h_{\phi_i(\cdot,1)}(\mathcal{L}')$ is the holonomy $\in U(1) \subseteq \mathbb{C}$ of the flat connection $\mathcal{L}'$ along the loop $\phi_i(\cdot,1)$.

Lemma 1.10. *h_i defines a map from a neighborhood of $[L,\mathcal{L}]$ in $\mathcal{LAG}^+(M,\Omega)$ to $\mathbb{C}$.*

Proof. Suppose $[L',\mathcal{L}'] = [L'',\mathcal{L}'']$. We need to prove $h_i(L',\mathcal{L}') = h_i(L'',\mathcal{L}'')$. To save notation, we assume $(L,\mathcal{L}) = (L'',\mathcal{L}'')$. We may assume also that $L' =$

L_u = graph of u. Using Condition 1.7, we can prove that u is exact and that the function f in Definition 1.4 can be chosen to be independent of t. Moreover $u = df$. Therefore $\exp(-\int \phi_i^*\omega) = 1$. We put $\nabla^{\mathcal{L}} = d/ds + \alpha ds$, $\nabla^{\mathcal{L}'} = d/ds + \beta ds$ where $s \in [0, 2\pi]$ is the coordinate of S^1 and α, β are $u(1) = \sqrt{-1}\mathbb{R}$ valued functions on S^1. Then (1.5.2) and (1.5.3) imply

$$\int_0^{2\pi} \beta ds - \int_0^{2\pi} \alpha ds = \int_0^{2\pi} ds \int_0^1 dt\, F_{\nabla}.$$

Therefore we have

$$h_{\phi_i(\cdot,1)}(\mathfrak{L}') \exp\left(-2\pi\sqrt{-1}\int \phi_i^* B\right) = h_{\phi_i(\cdot,0)}(\mathfrak{L}).$$

Lemma 1.10 follows. □

By Lemma 1.10, $h = (h_1, \cdots, h_b)$ is a map from a neighborhood of $[L, \mathcal{L}]$ in $\mathcal{LAG}^+(M, \Omega)$ to $\mathbb{C}^b$. Then again Condition 1.7 implies that h is injective there. We take h as a coordinate around $[L, \mathcal{L}]$. It is straightforward to verify that the coordinate change is biholomorphic. We thus proved Proposition 1.8. □

We now restrict ourselves to the case of affine Lagrangian submanifold of symplectic torus (T^{2n}, Ω) (equipped with B field). We put $V = \tilde{T}^{2n}$, the universal cover of T^{2n}. V is a $2n$-dimensional real vector space. We put $\Gamma = \pi_1(T^{2n}) \cong \mathbb{Z}^{2n}$. In this paper, we study the case when the mirror of (T^{2n}, Ω) exists as a usual manifold. According to the discussion of [Fu5], it means that we assume the following Assumption 1.11. (Otherwise the mirror is a kind of noncommutative torus.)

Assumption 1.11. There exist n-dimensional linear subspaces $\tilde{L}_{\rm pt}$, $\tilde{L}_{\rm st}$ of V such that $\Omega|_{\tilde{L}_{\rm pt}} = \Omega|_{\tilde{L}_{\rm st}} = 0$ and that $(\tilde{L}_{\rm pt} \cap \Gamma) \oplus (\tilde{L}_{\rm st} \cap \Gamma) = \Gamma$.

As we remarked in [Fu5], Assumption 1.11 is satisfied if $T^{2n} = \mathbb{C}^n/(\mathbb{Z} + \sqrt{-1}\mathbb{Z}^n)$, and if Ω is of 1-1 type. Namely we may take $\tilde{L}_{\rm pt} = \mathbb{R}^n$, $\tilde{L}_{\rm st} = \sqrt{-1}\mathbb{R}^n$. We may restrict ourselves to this case without loosing generality.

Let $\tilde{L}$ be an n-dimensional linear subspace of V such that $\Omega|_{\tilde{L}} = 0$ and $\Omega|_{\tilde{L}} \cap \Gamma \cong \mathbb{Z}^n$. We put $L(0) = \tilde{L}/(\Omega|_{\tilde{L}} \cap \Gamma) \cong T^n \subset T^{2n}$, and

$$\mathcal{M}(\tilde{L}) = \{[L, \mathcal{L}] \mid L \text{ is a flat Lagrangian submanifold of } T^{2n} \text{ parallel to } L(0)\}.$$

Lemma 1.12. *Elements of $\mathcal{M}(\tilde{L})$ satisfy Condition 1.7. $\mathcal{M}(\tilde{L})$ is a connected component of $\mathcal{LAG}^+(M, \Omega)$.*

Proof. We can prove easily that two flat Lagrangian submanifolds of T^{2n} parallel to $L(0)$ are Hamiltonian equivalent to each other if and only if they coincide.

By the proof of Proposition 1.8, we find that a neighborhood of $\mathcal{M}(\tilde{L})$ in $\mathcal{LAG}^+(M, \Omega)$ is an n dimensional complex manifold. On the other hand, $\mathcal{M}(\tilde{L})$ is a compact n dimensional complex manifold. Hence $\mathcal{M}(\tilde{L})$ is an open subset of $\mathcal{LAG}^+(M, \Omega)$. We are going to show that $\mathcal{M}(\tilde{L})$ is closed.

Let $[L_i, \mathcal{L}_i] \in \mathcal{M}(\tilde{L})$. We assume that $(L_i, \mathcal{L}_i)$ converges to $(L', \mathcal{L}') \in \mathcal{LAG}^{\sim+}(M, \Omega)$ in C^∞ topology. Then there is a sequence of closed 1-forms u_i on L' such that L_i is a graph of u_i. We consider $[u_i] \in H^1(L'; \mathbb{R})$. Let $(L_i', \mathcal{L}_i')$ be an affine Lagrangian submanifold of T^{2n} parallel to $\hat{L}(0)$ such that $[L_i, \mathcal{L}_i] = [L'_i, \mathcal{L}'_i]$. We take one form u'_i on L'_i such that $-[u'_i] = [u_i]$ in $H^1(L';\mathbb{R}) \simeq H^1(L'_i;\mathbb{R})$. We may choose u'_i so that the graph of u'_i is again an affine Lagrangian submanifold. Let L''_i be the graph of u'_i. Proposition 1.16 below implies that L''_i is Hamiltonian equivalent to L'. Hence $[L', \mathcal{L}'] \in \mathcal{M}(\tilde{L})$. □

We will prove Proposition 1.16 in a bit more general situation than torus. We need some notations. Let M be a symplectic manifold and L_t be a smooth family of Lagrangian submanifolds. We put $L = L_0$. Let $\varphi_t : L \to L_t$ be a smooth family of diffeomorphisms. If $s-t$ is small then L_s is contained in a small neighborhood of L_t. Hence we may regards $L_s \subset T^*(L_t)$. Then, for small $s-t$, L_s may be identified to a graph of close one form $v_{s,t}$ on L_t. Let $[v_{s,t}] \in H^1(L_t;\mathbb{R})$ be its De-Rham cohomology class. We put

$$u_t = \varphi_t^* \left.\frac{d[v_{s,t}]}{ds}\right|_{s=t} \in H^1(L;\mathbb{R}). \tag{1.13}$$

We define the *flux* of our family of Lagrangian submanifold L_t by

$$\mathrm{Flux}(L_t) = \int u_t dt \in H^1(L;\mathbb{R}). \tag{1.14}$$

It is easy to see that $\mathrm{Flux}(L_t)$ is an analogue of flux homomorphism (Calabi invariant) of symplectic isotopy. (See [MS2] 10.3.) We remark that $\mathrm{Flux}(L_t)$ is invariant of Hamiltonian isotopy. Namely if ψ_t is a family of Hamiltonian diffeomorphisms with $\psi_0 = identity$, then $\mathrm{Flux}(L_t) = \mathrm{Flux}(\psi_t(L_t))$.

The following lemma is easy to show.

Lemma 1.15. *If $u_t \equiv 0$, then there exists an exact symplectic diffeomorphism ψ such that $\psi(L) = L_1$.*

Now we prove the following :

Proposition 1.16. *Let L_t, L'_t be families of Lagrangian submanifolds with $L_0 = L'_0 = L$. We assume $\mathrm{Flux}(L_t) = \mathrm{Flux}(L'_t)$ and that $H^1(M;\mathbb{Q}) \to H^1(L;\mathbb{Q})$ is surjective. Then there exists a Hamiltonian diffeomorphism ψ such that $\psi(L_1) = L'_1$.*

Proof. We define u_t, u'_t by (1.13). Let α_t and α'_t be closed one forms on M whose restrictions to L represents De-Rham cohomology class of u_t and u'_t respectively. We may assume $[\alpha_1] = [\alpha'_1]$. Let X_t, X'_t be vector fields on M such that $i_{X_t}\omega = \alpha_t$, $i_{X'_t}\omega = \alpha'_t$. Let ϕ_t, ϕ'_t be family of selfdiffeomorphisms of M such that $d\phi_t/dt = X_t$, $d\phi'_t/dt = X'_t$ and $\phi_0 = \phi'_0 = id$. The maps ϕ_t, ϕ'_t are symplectic diffeomorphisms and the flux of ϕ_t, ϕ'_t are $[\alpha_t]$ and $[\alpha'_t]$ respectively. Since $[\alpha_1] = [\alpha'_1]$, [MS2] Theorem 10.12 implies that $\phi_1^{-1} \circ \phi'_1$ is a Hamiltonian diffeomorphism.

We consider the family $\phi_t^{-1}(L_t)$ of Lagrangian submanifolds. We can apply Lemma 1.15 to this family and find a Hamiltonian diffeomorphism ϕ such that $\phi(L) = \phi_1^{-1}(L_1)$. Similary we obtain ϕ' such that $\phi'(L) = \phi_1'^{-1}(L'_1)$.

We put

$$\psi = \phi'_1 \circ \phi' \circ \phi^{-1} \circ \phi_1^{-1}.$$

Since the group of exact symplectic diffeomorphisms is normal in the group of all symplectic diffeomorphisms, ψ is an exact symplectic diffeomorphism. It satisfies the required properties. □

Problem 1.17. Remove the condition that $H^1(M;\mathbb{Q}) \to H^1(L;\mathbb{Q})$ is surjective. Under which condition, we can still prove the conclusion of Proposiiton 1.16 ?

There is an exampe, due to Chekanov [Ch], which shows that the conclusion of Proposition 1.17 does not hold for arbitrary L. (The author would like to thank Prof. Y.Eliashberg who pointed it out to him.)

We now go back to the case of symplectic torus and describe explicitely the complex structure on $\mathcal{M}(\tilde{L})$ we defined by Proposition 1.8. We put $V^* = Hom_{\mathbb{R}}(V,\mathbb{R})$. Let $x \in V$. We define $I_x : V \oplus V^* \to \mathbb{C}$ by

$$I_x(v,\sigma) = \Omega(x,v) + \sqrt{-1}\sigma(x). \tag{1.18}$$

It is easy to see that there exists a unique complex structure on $V \oplus V^*$ such that I_x is complex linear for each $x \in V$.

Let $\tilde{L}$ be a Lagrangian linear subspace of (V,Ω). Then there exists a natural $\mathbb{R}$-linear surjection : $V \oplus V^* \to V/\tilde{L} \oplus \tilde{L}^*$, where $\tilde{L}^* = Hom_{\mathbb{R}}(\tilde{L},\mathbb{R})$. It is also easy to see that there exist a unique complex structure on $V/\tilde{L} \oplus \tilde{L}^*$ such that the map : $V \oplus V^* \to V/\tilde{L} \oplus \tilde{L}^*$ is complex linear.

Let $(v,\sigma) \in V/\tilde{L} \oplus \tilde{L}^*$. We obtain an affine subspace $\hat{L}(v) = \tilde{L} + v$ and its quotient $L(v) \subset T^{2n}$. On the other hand, σ is regarded as a flat connection ∇_σ of the trivial bundle on $L(v)$, by the isomorphism $\mathbb{R} \simeq u(1)$, $\sigma \mapsto 2\pi\sqrt{-1}\sigma$. Let $\mathcal{L}(\sigma)$ denote the pair of the trivial line bundle and the connection ∇_σ. Hence $(L(v),\mathcal{L}(\sigma))$ is an element of $\mathcal{LAG}^{\sim +}(T^{2n},\Omega)$. We put

$$\left(\Gamma \cap \tilde{L}_{\rm pt}\right)^\vee = \left\{\mu \in \tilde{L}_{\rm pt}^* \;\middle|\; \forall\gamma \in \Gamma \cap \tilde{L}_{\rm pt}, \quad \mu(\gamma) \in \mathbb{Z}\right\} \tag{1.19}$$

It is easy to see that $(L(v_1),\mathcal{L}(\sigma_1))$ is Hamiltonian equivalent to $(L(v_2),\mathcal{L}(\sigma_2))$ if and only if $v_1 - v_2 \in \Gamma/(\Gamma \cap \tilde{L}_{\rm pt})$ and $\sigma_1 - \sigma_2 \in (\Gamma \cap \tilde{L}_{\rm pt})^\vee$. We define :

Definition 1.20.

$$\mathcal{M}(\tilde{L}) = \frac{V/\tilde{L} \oplus \tilde{L}^*}{(\Gamma/\Gamma \cap \tilde{L}) \oplus (\Gamma \cap \tilde{L})^\vee.}$$

It is easy to see that the complex structure in Definition 1.4 coincides with one in Definition 1.20 in this case. Now we use Strominger-Yau-Zaslow's idea to define :

Definition 1.21. A mirror $(T^{2n},\Omega)^\vee$ of (T^{2n},Ω) is $\mathcal{M}(\tilde{L}_{\rm pt})$.

We remark that $\mathcal{M}(\tilde{L}_{\rm pt})$ may depend on the choice of $\tilde{L}_{\rm pt}$. Hence there are many different mirrors of (T^{2n},Ω). Note that $L_{\rm st}$ does not play a role in Definition 1.20. $L_{\rm st}$ is not used untill §3.

We also remark that the moduli space $\mathcal{M}(\tilde{L})$ also coincides with a component of $\mathcal{LAG}_{\rm sp}$ since flat Lagrangian submanifold is a special Lagrangian submanifold. It is also easy to see that special Lagrangian submanifold which is Hamiltonian equivalent to affine Lagrangian submanifold is affine.

§2. Floere homology of affine Lagrangian submanifolds in symplectic torus.

We first review briefly Floer homology of Lagrangian submanifolds. See [Fl],[Oh1],[FKOOO],[Fu6],[Fu7] for more detail. In this paper, we study affine Lagrangian submanifold in symplex torus. For this purpose, it is enough to consider the case studied by Floer [Fl] himself. Namely we consier a Lagrangian submanifold L_i of (M,ω) such that :

(2.1) $\pi_2(M, L_i) = 1.$

Under Condition (2.1), Floer [Fl] defined a graded $\mathbb{Z}_2$ module, the Floer homology $HF(L_1, L_2)$ of the pair (L_1, L_2). Floer used a $\mathbb{Z}_2$ coefficient to go around the problem of orientation of the moduli space of pseudoholomorphic disks. We can define a Floer homology over integer coefficient under additional assumption :

(2.2) L_i is spin.

Namely in case (2.1),(2.2), $HF(L_1, L_2)$ is a graded $\mathbb{Z}$ module. (See [FKO3] Chapter 6.)

The calculation of Floer homology in general is very hard. One method we can use to do so is Bott-Morse theory ([Fu3],[Pz],[PSS],[RT],[FKOOO]) and a spectral sequence.

Definition 2.3. We say that a pair (L_1, L_2) cleanly intersects to each other in M, if $L_1 \cap L_2$ is a smooth submanifold and if $(N_{L_1}M)_p \cap (N_{L_2}M)_p = T_p(L_1 \cap L_2)$ for $p \in L_1 \cap L_2$. Here $N_{L_i}M$ is a normal bundle.

We put $L_1 \cap L_2 = \cup R_i$ where R_i is connected. The following result is proved in [Pz],[FKOOO].

Theorem 2.4. *Let (L_1, L_2) be a pair of Lagrangian submanifolds of M satisfying* (2.1), (2.2). *We assume that they cleanly intersect to each other in M. Then, for each connected component R_i of $L_1 \cap L_2$, we can define a Maslov index $\eta(R_i) \in \mathbb{Z}$ and a local system $\pi_1(R_i) \to \{\pm 1\}$ with* $\operatorname{Ker}\rho \subseteq \operatorname{Ker}(\pi_1(R_i) \to \pi_1(M))$. *There exists a spectral sequence E_*^* with the following properties.*

(2.4.1) $E_2^k \cong \oplus_i H^{k-\eta(R_i)}(R_i; \mathbb{Z}^\rho).$
(2.4.2) $E_\infty^* \cong HF(L_1, L_2).$
(2.4.3) *If $H_*(L_1 \cap L_2; \mathbb{Z}) \to H_*(M; \mathbb{Z})$ is injective, then the spectral sequence degenerates at E_2 stage and ρ is trivial.*

In fact, in [FKOOO] Chapters 2,5, Theorem 2.4 is proved under a milder assumption.

Now we consider the case of affine Lagrangian submanifolds. Let $\tilde L_1$, $\tilde L_2$ be Lagrangian linear subspaces of $V = \tilde T^{2n}$, such that $\tilde L_i \cap \Gamma \simeq \mathbb{Z}^n$, where $\Gamma = \pi_1(T^{2n})$. Let $v_i \in V/\tilde L_1$. We obtain affine Lagrangian submanifolds $L_i(v_i) = (\tilde L_i + v_i)/(\Gamma \cap \tilde L_i) \subset T^{2n}$. In this section (but not in later sections) we write $L_i = L_i(v_i)$ for simplicity. They cleanly intersect each other. Moreover $L_1 \cap L_2$ is again an affine subtorus. Hence $H(L_1 \cap L_2; \mathbb{Z}) \to H(M; \mathbb{Z})$ is injective, in case $L_1 \cap L_2$ is connected.

Therefore if $L_1 \cap L_2$ is connected Theorem 2.4 implies

$$HF^k(L_1, L_2) \simeq H^{k-\eta(L_1,L_2)}(L_1 \cap L_2; \mathbb{Z}). \tag{2.5}$$

Here $\eta(L_1, L_2)$ is the Maslov index which we will define later in this section.

In fact, we can prove that the spectral sequence degenerates also in case when $L_1 \cap L_2$ is disconnected. (By studying the pseudoholomorphic disks in the universal cover.) In this paper, to make the exposition selfcontained, we use the right hand side of (2.5) as a *definition* of Floer homology.

In this paper, we need to include flat line bundle on L_i. The way to include it in the Floer homology (in the general situation) is described in [Fu6],[Fu7]. In this paper, we use the following Definition 2.6 (which in fact is a calculation of Floer homology in our case.) Let $\mathcal{L}_i$ be flat line bundles on L_i :

Definition 2.6.

$$HF^k((L_1, \mathcal{L}_1), (L_2, \mathcal{L}_2)) \simeq H^{k-\eta(L_1,L_2)}(L_1 \cap L_2; Hom(\mathcal{L}_1, \mathcal{L}_2)).$$

Here the right hand side is the cohomology with local coefficient.

We now define Maslov index η in the case of affine Lagrangian submanifolds. We first take a Lagrangian subspace $\tilde{L}_0$ of V. We put $\overline{V} = V/\tilde{L}_0$ and let $\pi : V \to \overline{V}$ be the projection. We may identify $V = T^*\overline{V}$ so that π is the projection $T^*\overline{V} \to \overline{V}$. Let $\tilde{L}_i$, $i = 1, 2, 3$ be Lagrangian subspace of V transversal to $\tilde{L}_0$. There exists a quadratic function $f_{\tilde{L}_i;\tilde{L}_0}$ such that $\tilde{L}_i \subset T^*\overline{V} = V$ is the graph of $df_{\tilde{L}_i;\tilde{L}_0}$. We put

$$f_{\tilde{L}_i,\tilde{L}_j;\tilde{L}_0} = f_{\tilde{L}_i;\tilde{L}_0} - f_{\tilde{L}_j;\tilde{L}_0}.$$

Definition 2.7. Let $\bar{\eta}_{\tilde{L}_0}(\tilde{L}_i, \tilde{L}_j)$ be the index (the sum of multiplicities of strictly negative eigenvalues) of the quadratic function $f_{\tilde{L}_i,\tilde{L}_j;\tilde{L}_0}$.

We put

$$\eta_{\tilde{L}_0}(\tilde{L}_1, \tilde{L}_2, \tilde{L}_3) = 2n - \left(\bar{\eta}_{\tilde{L}_0}(\tilde{L}_1, \tilde{L}_2) + \bar{\eta}_{\tilde{L}_0}(\tilde{L}_2, \tilde{L}_3) + \bar{\eta}_{\tilde{L}_0}(\tilde{L}_3, \tilde{L}_1)\right).$$

Lemma 2.8. *$\eta_{\tilde{L}_0}(\tilde{L}_1, \tilde{L}_2, \tilde{L}_3)$ is independent of the choices of $\tilde{L}_0$.*

Proof. Let $\tilde{L}_0'$ be the other choice. There are one parameter family of Lagrangian subspaces $\tilde{L}_0^s$ such that $\tilde{L}_0^0 = \tilde{L}_0$ and $\tilde{L}_0^1 = \tilde{L}_0'$. It is easy to see from definition that $\eta_{\tilde{L}_0^s}(\tilde{L}_1, \tilde{L}_2, \tilde{L}_3)$ is independent of s as far as $\tilde{L}_0^s$ is transversal to $\tilde{L}_i$, $i = 1, 2, 3$. By perturbing the family $\tilde{L}_0^s$ a bit, we may assume that $\tilde{L}_0^s$ is transversal to $\tilde{L}_i$, $i = 1, 2, 3$ if $s \neq s_1, \cdots, s_k$ and also that $\tilde{L}_0^{s_i}$ is transversal to only one of $\tilde{L}_i$, $i = 1, 2, 3$ for $s = s_1, \cdots, s_k$.

It suffices to consider the case when $k = 1$. (Namely the case when $\tilde{L}_0^s$ is not transversal to one of $\tilde{L}_i$, $i = 1, 2, 3$ only once.) Furthermore it suffices to consider the case when $\dim \tilde{L}_0^{s_1} \cap \tilde{L}_1 = 1$ and $\tilde{L}_0^{s_1}$ is transversal to $\tilde{L}_2$ and $\tilde{L}_3$.

We consider $f_{\tilde{L}_1;\tilde{L}_0^s}$. Let $\lambda_j(s)$, $j = 1, \cdots, n$ be the eigenvalues of $f_{\tilde{L}_1;\tilde{L}_0^s}$ such that $\lambda_j(s) \le \lambda_{j+1}(s)$. We may assume one of the following happens, (by perturbing our family $\tilde{L}_0^s$ if necessary.)

(2.9.1) $\lim_{s\uparrow s_1} \lambda_1(s) = -\infty$, $\lim_{s\downarrow s_1} \lambda_n(s) = +\infty$, $\lim_{s\uparrow s_1} \lambda_j(s) = \lim_{s\downarrow s_1} \lambda_{j-1}(s)$ for $j = 2, \cdots, n$.

(2.9.2) $\lim_{s\uparrow s_1} \lambda_n(s) = +\infty$, $\lim_{s\downarrow s_1} \lambda_1(s) = -\infty$, $\lim_{s\uparrow s_1} \lambda_j(s) = \lim_{s\downarrow s_1} \lambda_{j+1}(s)$ for $j = 1, \cdots, n-1$.

In case (2.9.1) happens we have

$$\lim_{s\uparrow s_1} \bar\eta_{\tilde L_0^s}(\tilde L_1, \tilde L_2) = \lim_{s\downarrow s_1} \bar\eta_{\tilde L_0^s}(\tilde L_1, \tilde L_2) + 1,$$
$$\lim_{s\uparrow s_1} \bar\eta_{\tilde L_0^s}(\tilde L_3, \tilde L_1) + 1 = \lim_{s\downarrow s_1} \bar\eta_{\tilde L_0^s}(\tilde L_3, \tilde L_1).$$

Hence $\eta_{\tilde L_0^s}(\tilde L_1, \tilde L_2, \tilde L_3)$ does not change. Similarly $\eta_{\tilde L_0^s}(\tilde L_1, \tilde L_2, \tilde L_3)$ does not change in case (2.9.2) either. The proof of Lemma 2.8 is complete. □

From now on, we write $\eta(\tilde L_1, \tilde L_2, \tilde L_3)$ in place of $\eta_{\tilde L_0}(\tilde L_1, \tilde L_2, \tilde L_3)$. We remark that $\eta(\tilde L_1, \tilde L_2, \tilde L_3)$ is well defined for arbitrary mutually transversal triple $\tilde L_1, \tilde L_2, \tilde L_3$ of Lagrangian subspaces, since there exists always a fourth Lagrangian subspace $\tilde L_0$ transversal to $\tilde L_i$.

Lemma 6.13 proved in §6 implies

$$\eta(\tilde L_1, \tilde L_2, \tilde L_3) \in \{0, 1, 2, \cdots, n\}. \tag{2.10}$$

The following lemma is easy to prove.

Lemma 2.11.

$$\eta_{\tilde L_0}(\tilde L_1, \tilde L_2) + \eta_{\tilde L_0}(\tilde L_2, \tilde L_1) = n - \dim \tilde L_1 \cap \tilde L_2.$$

We next turn to the definition of $\eta(\tilde L_1, \tilde L_2)$. We put $k_i = \dim \tilde L_{\rm pt} \cap \tilde L_i$, $k_{ij} = \dim \tilde L_i \cap \tilde L_j$.

Lemma-Definition 2.12. *There exists unique $\eta(\tilde L_i, \tilde L_j)$ such that*

(2.13.1) $\eta(\tilde L_1, \tilde L_2) + \eta(\tilde L_2, \tilde L_3) = \eta(\tilde L_1, \tilde L_3) + \eta(\tilde L_1, \tilde L_2, \tilde L_3) - k_{12} - k_{23}.$
(2.13.2) $\eta(\tilde L_1, \tilde L_2) + \eta(\tilde L_2, \tilde L_1) = n - k_{12}.$
(2.13.3) $\eta(\tilde L, \tilde L_{\rm pt}) = 0.$

Remark 2.14. Using (2.13.2), we can rewrite (2.13.1) to the following more symmetric form :

$$\eta(\tilde L_1, \tilde L_2) + \eta(\tilde L_2, \tilde L_3) + \eta(\tilde L_3, \tilde L_1) = \eta(\tilde L_1, \tilde L_2, \tilde L_3) + n - k_{12} - k_{23} - k_{31}. \tag{2.15}$$

Proof. Putting $\tilde L_3 = \tilde L_{\rm pt}$, (2.13.3) and (2.13.1) impliy

$$\eta(\tilde L_i, \tilde L_j) = \eta(\tilde L_i, \tilde L_j, \tilde L_{\rm pt}) - k_{ij} - k_j. \tag{2.16}$$

The uniqueness follows.

To show the existence, we define $\eta(\tilde L_i, \tilde L_j)$ by the right hand side of (2.16) and will check (2.13).

We take $\tilde L_0$ and let $\eta_{\tilde L_0}$ be as before. We calculate

$$\begin{aligned} &\eta(\tilde L_1, \tilde L_2) + \eta(\tilde L_2, \tilde L_1) + k_1 + k_2 + 2k_{12} \\ &\quad = 4n - (\eta_{\tilde L_0}(\tilde L_1, \tilde L_2) + \eta_{\tilde L_0}(\tilde L_2, \tilde L_{\rm pt}) + \eta_{\tilde L_0}(\tilde L_{\rm pt}, \tilde L_1)) \\ &\qquad - (\eta_{\tilde L_0}(\tilde L_2, \tilde L_1) + \eta_{\tilde L_0}(\tilde L_1, \tilde L_{\rm pt}) + \eta_{\tilde L_0}(\tilde L_{\rm pt}, \tilde L_2)). \end{aligned}$$

(2.13.2) then follows from Lemma 2.11.

By Lemma 2.11, $\eta_{\tilde{L}_0}(\tilde{L}_{\rm pt}, \tilde{L}_{\rm pt}) = 0$ and definition, we have

$$\begin{aligned}\eta(\tilde{L}_1, \tilde{L}_{\rm pt}) &= \eta(\tilde{L}_1, \tilde{L}_{\rm pt}, \tilde{L}_{\rm pt}) - n - k_1 \\ &= 2n - (\eta_{\tilde{L}_0}(\tilde{L}_1, \tilde{L}_{\rm pt}) + \eta_{\tilde{L}_0}(\tilde{L}_{\rm pt}, \tilde{L}_{\rm pt}) + \eta_{\tilde{L}_0}(\tilde{L}_{\rm pt}, \tilde{L}_1)) - k_1 - n = 0.\end{aligned}$$

(2.13.3) follows.

Finally we calculate

$$\begin{aligned}\eta(\tilde{L}_1, \tilde{L}_2) + \eta(\tilde{L}_2, \tilde{L}_3) + \eta(\tilde{L}_3, \tilde{L}_1) + (k_{12} + k_{23} + k_{31}) + (k_1 + k_2 + k_3) \\ = 6n - (\eta_{\tilde{L}_0}(\tilde{L}_1, \tilde{L}_2) + \eta_{\tilde{L}_0}(\tilde{L}_2, \tilde{L}_{\rm pt}) + \eta_{\tilde{L}_0}(\tilde{L}_{\rm pt}, \tilde{L}_1)) \\ - (\eta_{\tilde{L}_0}(\tilde{L}_2, \tilde{L}_3) + \eta_{\tilde{L}_0}(\tilde{L}_3, \tilde{L}_{\rm pt}) + \eta_{\tilde{L}_0}(\tilde{L}_{\rm pt}, \tilde{L}_2)) \\ - (\eta_{\tilde{L}_0}(\tilde{L}_3, \tilde{L}_1) + \eta_{\tilde{L}_0}(\tilde{L}_1, \tilde{L}_{\rm pt}) + \eta_{\tilde{L}_0}(\tilde{L}_{\rm pt}, \tilde{L}_3)) \\ = n + (k_1 + k_2 + k_3) \\ + 2n - (\eta_{\tilde{L}_0}(\tilde{L}_1, \tilde{L}_2) + \eta_{\tilde{L}_0}(\tilde{L}_2, \tilde{L}_3) + \eta_{\tilde{L}_0}(\tilde{L}_3, \tilde{L}_1)).\end{aligned}$$

(2.15) follows. □

Remark 2.17. Theorem δ implies $\eta(\tilde{L}_i, \tilde{L}_j) \in \{1, \cdots, n\}$.

Remark 2.18. In the general situation, the Maslov index $\eta(L_1, L_2, L_3)$ for triple of Lagrangian submanifolds is well-defined for each homotopy class of maps from disk to M with boundary condition determined by L_1, L_2, L_3. However, the Maslov index $\eta(L_1, L_2)$ of a pair of Lagrangian submanifolds is determined only up to overall constant. In this section, we use the choice of $\tilde{L}_{\rm pt}$ to fix this ambiguity. (Compare [Sei] for this point.)

Our definition of η is designed so that it coincides with the degree of Extension in the mirror.

§3. Construction of coherent sheaf (transversal case).

We next construct a sheaf from an affine Lagrangian submanifold. In the general situation, family of Floer homologies seems to give a way to do so systematically. However there are several troubles in doing so. We will discuss the general case in [Fu7]. In this paper, we concentarate on the case of flat symplectic tori and affine Lagrangian submanifolds.

Let $\tilde L_{\rm st}$ be an n-dimensional linear subspace of V such that $\Omega|_{L_{\rm st}} = 0$, $\Gamma \cap \tilde L_{\rm st} \cong \mathbb{Z}^n$ and that $L_{\rm st} \cap L_{\rm pt}(0)$ is one point. Hence $L_{\rm st} = \tilde L_{\rm st}/\tilde L_{\rm st} \cap \Gamma$.

Let $\tilde L \subseteq V$ be another n-dimensional linear subspace such that $\Omega|_{\tilde L} = 0$. We assume also that $\Gamma \cap \tilde L \cong \mathbb{Z}^n$. Let $w \in V/\tilde L$. We take an affine space $\hat L(w) = \tilde L + w$ parallel to $\tilde L$ and put $L(w) = \hat L(w)/\Gamma \cap \tilde L$. L is a closed Lagrangian submanifold of T^{2n}. Let $\alpha \in Hom(\tilde L, \mathbb{R})$ and we regard it as a connection of a trivial bundle on $L(w)$. Hence $(L(w), \alpha)$ is regarded as an element of $\mathcal{LAG}(T^{2n}, \Omega)$. (From now on, we write $(L(w), \alpha)$ in place of $(L(w), \mathcal{L}(\alpha))$.) In this section, we assume that $\tilde L$ is transversal to $\tilde L_{\rm pt}$. We first construct a smooth complex vector bundle on $(T^{2n}, \Omega)^\vee$, and then define a holomorphic structure on it.

We will define a $\left(\Gamma/\Gamma \cap \tilde L_{\rm pt}\right) \oplus \left(\Gamma \cap \tilde L_{\rm pt}\right)^\vee$ action on the trivial bundle $\tilde{\mathcal{E}}(L, \alpha)$ on $V/\tilde L_{\rm pt} \oplus \tilde L^*_{\rm pt}$. Let $(v, \sigma) \in V/\tilde L_{\rm pt} \oplus \tilde L^*_{\rm pt}$. We put $\hat L_{\rm pt}(v) = \tilde L_{\rm pt} + v$ and let $L_{\rm pt}(v) \subseteq T^{2n}$ be its quotient. We put

$$\tilde{\mathcal{E}}(L(w), \alpha)_{(v,\sigma)} = \bigoplus_{p \in L(w) \cap L_{\rm pt}(v)} \mathbb{C}[p]. \tag{3.1}$$

Let $\gamma \in (\Gamma/\Gamma \cap \tilde L_{\rm pt})$. It is easy to see that $L_{\rm pt}(v) = L_{\rm pt}(v + \gamma)$. Therefore, by definition, $\tilde{\mathcal{E}}(L(w), \alpha)_{(v,\sigma)}$ coincides with $\tilde{\mathcal{E}}(L(w), \alpha)_{(\gamma+v,\sigma)}$. Thus we defined an action of $\Gamma/\Gamma \cap \tilde L_{\rm pt}$ on $\tilde{\mathcal{E}}(L(w), \sigma)$.

We next define an action of $(\Gamma \cap \tilde L_{\rm pt})^\vee$. Let $\mu \in (\Gamma \cap \tilde L_{\rm pt})^\vee$. μ is a homomorphism from $\tilde L_{\rm pt}$ to $\mathbb{R}$. We regard it as a gauge transformation on $\tilde L_{\rm pt}(v)$ as follows. We take the (unique) point $x_0(v) \in \hat L_{\rm pt}(v) \cap \tilde L_{\rm st}$. (If we identify $V/\tilde L_{\rm pt} = \tilde L_{\rm st}$ then $x_0(v) = v$.) For $x \in \hat L_{\rm pt}(v)$, we put :

$$g_{\mu,v}(x) = \exp\left(2\pi\sqrt{-1}\mu(x - x_0(v))\right). \tag{3.2.1}$$

$g_{\mu,v}$ is a $U(1)$ valued map and hence is a gauge transformation. Since $\mu(\gamma) \in \mathbb{Z}$ for $\gamma \in \Gamma \cap \tilde L_{\rm pt}$, it follows that $g_{\mu,v}$ induces a map $L_{\rm pt}(v) \to U(1)$. We denote it by the same symbol. Then we define

$$\mu(c\,[p]) = g_{\mu,v}(p)\; c\,[p], \tag{3.2.2}$$

where $p \in L(w) \cap L_0(v)$. Here we remark that we regard the right hand side as an element of $\tilde{\mathcal{E}}(L, \beta)_{(v,\mu+\sigma)}$.

Lemma 3.3. *The actions of $\gamma \in (\Gamma/\Gamma \cap \tilde L_{\rm pt})$ and $\mu \in (\Gamma \cap \tilde L_{\rm pt})^\vee$ on $\tilde{\mathcal{E}}(L, \alpha)_{(v,\sigma)}$ we defined above commute to each other.*

Proof. We recall $(\Gamma \cap \tilde L_{\rm pt}) \oplus (\Gamma \cap \tilde L_{\rm st}) = \Gamma$. Hence we may regards $\gamma \in \Gamma \cap \tilde L_{\rm st}$. Then, by definition, we have $g_{\mu,v+\gamma}(x + \gamma) = g_{\mu,v}(x)$. Lemma 3.3 follows from the definition. □

Thus we defined an action of $(\Gamma/\Gamma\cap\tilde L_{\rm pt})\oplus(\Gamma\cap\tilde L_{\rm pt})^\vee$ on $\tilde{\mathcal E}(L(w),\alpha)$. Let $\mathcal E(L(w),\alpha)\to(T^{2n},\Omega)^\vee$ be the quotient bundle.

We next are going to construct a holomorphic structure on $\mathcal E(L(w),\alpha)$. It suffices to construct its local (holomorphic) frame. We use a term of theta series for this purpose as follows. Let $(v,\sigma)\in V/\tilde L_{\rm pt}\oplus\tilde L^*_{\rm pt}$. We take $p\in L(w)\cap L_0(v)$. We will define a frame $\mathbf e_{\tilde p}$ whose value at (v,σ) is $[p]$. Here $\tilde p$ is a lift of p to $\hat L_{\rm pt}(v)$. (Then, $\tilde p\in\hat L(w+\gamma_0)$, $\gamma_0\in\Gamma$.) Let $(v',\sigma')\in V/\tilde L_{pt}\oplus\tilde L^*_{pt}$ be in a small neighborhood of (v,σ). We find $p'\in L\cap L_0(v')$ and its lift $\tilde p'$ which lie in a small neighborhood of p and $\tilde p$ respectively. We define

$$\begin{aligned}e_{\tilde p,\sigma}(v',\sigma') = \exp\Bigg(2\pi\int_{D(\tilde p,x_0(v),x_0(v'),\tilde p')}\Omega \\ + 2\pi\sqrt{-1}\left(\sigma(x_0(v)-\tilde p)+\sigma'(\tilde p'-x_0(v'))+\alpha(\tilde p-\tilde p')\right)\Bigg).\end{aligned}\tag{3.4}$$

Figure 2

Here $D(\tilde p,x_0(v),x_0(v'),\tilde p')$ in (3.4) is the union of two triangles $\Delta_{\tilde p\;x_0(v)\;x_0(v')}$ and $\Delta_{x_0(v')\;\tilde p'\;\tilde p}$. Hereafter we write

$$Q(a,b,c,d)=\int_{D(a,b,c,d)}\Omega.\tag{3.5}$$

Using Stokes' theorem, we can prove $Q(a,b,c,d)=Q(b,c,d,a)$. We put

$$\mathbf e_{\tilde p,\sigma}(v',\sigma')=e_{\tilde p,\sigma}(v',\sigma')[p'].\tag{3.6}$$

Lemma 3.7 below implies that $\mathbf e_{\tilde p,\sigma}$ is a section of $\mathcal E(L,\alpha)$ in a neighborhood of $(v,\sigma)\in(T^{2n},\Omega)^\vee$. If we take $\tilde p$ for each $p\in L\cap L_0(v)$, then $\mathbf e_{\tilde p,\sigma}$, $p\in L\cap L_0(v)$ is a local frame of the bundle $\mathcal E(L,\alpha)$.

Lemma 3.7. *If $\gamma\in(L\cap\tilde L_{\rm pt})$ and $\mu\in(L\cap\tilde L_{\rm pt})^\vee$, then there exists a holomorphic function $g(v',\sigma')$ such that*

$$\mathbf e_{\tilde p,\sigma}(v',\sigma')=g(v',\sigma')\mathbf e_{\tilde p+\gamma,\sigma+\mu}(v',\sigma'+\mu).$$

Proof. We put

$$g(v',\sigma')=e_{\tilde p,\sigma}(v',\sigma')/e_{\tilde p+\gamma,\sigma}(v',\sigma').$$

By (3.4), we have

$$\log e_{\tilde p+\gamma}(v',\sigma')-\log e_{\tilde p}(v',\sigma')=2\pi I_\gamma(v'-v,\sigma'-\sigma).\tag{3.8}$$

Here I_γ is as in (1.18).

Figure 3

By the construction of complex structure, (3.8) implies that $g(v',\sigma')$ is a holomorphic function of (v',σ'). On the other hand, we have

$$\begin{aligned} e_{\tilde p,\sigma+\mu}(v',\sigma'+\mu)/e_{\tilde p,\sigma+\mu}(v,\sigma+\mu) &= \exp\left(2\pi\sqrt{-1}\mu(\tilde p' - x_0(v') - \tilde p + x_0(v))\right) \\ &= g_{\mu,v}(\tilde p')g_{\mu,v}(\tilde p)^{-1}. \end{aligned}$$

Hence

$$\mu\left(\mathbf{e}_{\tilde p,\sigma}(v',\sigma')\right) = g_{\mu,v}(p)\mathbf{e}_{\tilde p,\sigma+\mu}(v',\sigma'+\mu).$$

The proof of Lemma 3.7 is now complete. □

Lemma 3.7 implies that there exists a unique holomorphic structure on $\mathcal{E}(L(w),\beta) \to (T^{2n},\Omega)^\vee$ such that $\mathbf{e}_{\tilde p}$ is a local holomorphic section. We thus constructed a holomorphic vector bundle $\mathcal{E}(L(w),\beta) \to (T^{2n},\Omega)^\vee$.

Proposition 3.9. *If $(L(w),\alpha)$ is Hamiltonian equivalent to $(L'(w'),\alpha')$ then the bundle $\mathcal{E}(L(w),\alpha)$ is isomorphic to $\mathcal{E}(L'(w'),\alpha')$.*

Proof. By Lemma 1.12 there exists $\xi \in \Gamma/(\Gamma\cap\tilde L)$, $\zeta \in (\Gamma\cap\tilde L)^\vee$ such that $w' = w+\xi$, $\alpha' = \alpha+\zeta$. Since $L(w) = L(w+\xi)$ it follows that $\mathcal{E}(L(w),\alpha) \cong \mathcal{E}(L(w+\xi),\alpha)$. Choose and fix $y \in L(w)$. Let $p \in L(w)\cap L_{\rm pt}(v)$, $\tilde p \in \hat L(w)\cap \hat L_{\rm pt}(v)$. We let $\tilde y$ be a lift of y in $\hat L(w)$. We define

$$\tilde\Psi([p],(v,\sigma)) = \left(\exp\left(2\pi\sqrt{-1}\zeta(\tilde p - \tilde y)\right)[p],(v,\sigma)\right). \tag{3.10}$$

Here $([p],(v,\sigma)) \in \tilde{\mathcal{E}}(L(w),\alpha)_{v,\sigma}$. It is straightforward to see that (3.10) is compatible with the actions of $\Gamma/(\Gamma\cap\tilde L_{\rm pt})\oplus(\Gamma\cap\tilde L_{\rm pt})^\vee$ and is independent of the lift $\tilde y$. We can also verify easily that $\tilde\Psi$ define an isomorphism $\Psi : \mathcal{E}(L(w),\alpha) \to \mathcal{E}(L(w),\alpha+\zeta)$. Hence Proposition 3.9. □

We will prove a converse of Proposition 3.9 in §13 (Proposition 13.26).

Before going further we add several remarks on our construction. First the way we constructed the bundle $\mathcal{E}(L(w),\alpha)$ is a consequence of the dictionary between symplectic and complex geometry itself. To see this, we recall that, by the construction of Strominger-Yau-Zaslow, the pair $(L_{\rm pt}(v),\sigma)$ corresponds to the skyscraper sheaf at the point $(L_{\rm pt}(v),\sigma) \in (T^{2n},\Omega)^\vee$. (We write it $\mathcal{E}(L_{\rm pt}(v),\sigma)$.) Namely an n-brane $(L_{\rm pt}(v),\sigma)$ in (T^{2n},Σ) corresponds to a 0-brane in $(T^{2n},\Sigma)^\vee$. Let $(L(w),\alpha) \in \mathcal{LAG}(T^{2n},\Sigma)$ be another element. Suppose that it corresponds to a sheaf $\mathcal{E}(L(w),\alpha)$ on (T^{2n},Σ). Then Conjecture B in this case is :

$$HF((L(w),\alpha),(L_{\rm pt}(v),\sigma)) \cong \operatorname{Ext}(\mathcal{E}(L(w),\alpha),\mathcal{E}(L_{\rm pt}(v),\sigma)). \tag{3.11}$$

In our case, $\mathcal{E}(L(w),\alpha)$ is a vector bundle. Hence we have

$$\operatorname{Ext}^k(\mathcal{E}(L(w),\alpha),\mathcal{E}(L_{\rm pt}(v),\sigma)) = \begin{cases} 0 & k\neq 0 \\ \mathcal{E}(L,\alpha)^*_{[v,\sigma]} & k=0. \end{cases} \tag{3.12}$$

On the other hand, Definition 2.6, (2.13.1), (2.13.2) imply

$$HF^k((L,\alpha),(L_{\rm pt}(v),\sigma)) = \begin{cases} 0 & k\neq 0 \\ \displaystyle\bigoplus_{p\in L(w)\cap L_{\rm pt}(v)} \mathbb{C}[p] & k=0. \end{cases} \tag{3.13}$$

Thus (3.1) is consistent with homological mirror symmetry conjecture.

We next explain the reason why we need to fix $\tilde{L}_{\rm st}$ to define $\mathcal{E}(L(w),\alpha) \to (T^{2n},\Omega)^\vee$. Our purpose is to construct a functor:

$$\mathcal{LAG}(T^{2n},\Omega) \to \mathbb{D}((T^{2n},\Omega)^\vee), \tag{3.14}$$

where $\mathbb{D}((T^{2n},\Omega)^\vee)$ is the derived category of the category of coherent sheaves on $(T^{2n},\Omega)^\vee$. We suppose that (3.14) sends Lagrangian submanifolds parallel to $\tilde{L}_{\rm pt}$ to the skyscraper sheaves. Note that the automorphism group of the category $\mathbb{D}((T^{2n},\Omega)^\vee)$ is rather big. Mukai constructed ([Mu2], [Mu3]) a symmetry, called Fourier-Mukai transformation. In fact, we can see such a symmetry from mirror symmetry itself. Namely the mirror of a Fourier-Mukai transformation (or S-duality) is realized by a symplectic diffeomorphism of (T^{2n},Ω). This phenomenon, that is S-duality will become easier duality in the mirror, is observed by physicists in more general situations and is called the duality of duality.

So there can be many possible ways to construct the functor (3.14). The ambiguity is described by Fourier-Mukai transformation which sends skyscraper sheaves to skyscraper sheaves. If we see such transformation in the mirror (T^{2n},Ω) they are (linear) symplectic diffeomorphisms which preserve $\tilde{L}_{\rm pt}$. For example, if we consider the case when $n=1$, then the group of linear symplectic diffeomorphisms of T^2 is an extension of T^2 by $SL(2;\mathbb{Z})$. This group $SL(2;\mathbb{Z})$ will become the S-duality group of the mirror, $T^{2\vee}$. The element of $SL(2;\mathbb{Z})$ which preserves $\mathbb{R}\subset\mathbb{C}$ is a matrix of the form

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

To kill this symmetry we need to fix a direction transversal to $\mathbb{R}$. This is equivalent to fix the Lagrangian submanifold which becomes the structure sheaf of the mirror torus.

§4 Construction of coherent sheaf (general case).

In this section, we consider the case when Lagrangian subspace $\tilde L$ does not intersect transversally to $\tilde L_{\rm pt}$. In this case, we obtain a coherent sheaf which is not a vector bundle. We consider $(\tilde L + \tilde L_{\rm pt})/\tilde L_{\rm pt}$ and $(\tilde L \cap \tilde L_{\rm pt})^{\perp} = \{\sigma \in \tilde L_{\rm pt}^* \mid \sigma|_{\tilde L\cap \tilde L_{\rm pt}} = 0\}$. The sum $(\tilde L + \tilde L_{\rm pt})/\tilde L_{\rm pt} \oplus (\tilde L \cap \tilde L_{\rm pt})^{\perp}$ is a subspace of the universal cover $V/\tilde L_{\rm pt} \oplus \tilde L_{\rm pt}^*$ of $(T^{2n},\Omega)^{\vee}$. It is easy to see that this subspace is complex linear. The sheaf $\mathcal{E}(L(w),\alpha)$, we are going to construct has a support on a subtorus parallel to $(\tilde L + \tilde L_{\rm pt})/\tilde L_{\rm pt} \oplus (\tilde L \cap \tilde L_{\rm pt})^{\perp}$. To explain the reason, we first recall the following calculation of Floer homology. By Definition 3.7, we have:

$$\begin{aligned} &HF^k((L_1(w),\alpha),(L_2(v),\beta)) \\ &\qquad = H^{k-\eta(\tilde L_1,\tilde L_2)}(L_1(w)\cap L_2(v), \beta|_{L_1(w)\cap L_2(v)} - \alpha|_{L_1(w)\cap L_2(v)}) \end{aligned} \tag{4.1}$$

We remark that the cohomology in the right hand side is trivial unless the flat connection $\beta|_{L_1(w)\cap L_2(v)} - \alpha|_{L_1(w)\cap L_2(v)}$ is trivial and the intersection $L_1(w)\cap L_2(v)$ is nonempty. Hence we have the following:

Lemma 4.2. *If* $w - v \in \tilde L + \tilde L_{\rm pt} \mod \Gamma$ *and* $\beta|_{L(w)\cap L_{\rm pt}(v)} - \sigma|_{L(w)\cap L_{\rm pt}(v)} = \mu_{L(w)\cap L_{\rm pt}(v)}$ *for some* $\mu \in (\tilde L_{\rm pt}\cap \Gamma)^{\vee}$*, then*

$$HF^k((L(w),\alpha),(L_{\rm pt}(v),\sigma)) = H^{k-\eta(\tilde L,\tilde L_{\rm pt})}(T^m;\mathbb{C}),$$

where $m = \dim L_1(w)\cap L_{\rm pt}(v)$*. Otherwise* $HF^k((L(w),\alpha),(L_{\rm pt}(v),\sigma)) = 0$*.*

We are going to define $\mathcal{E}(L(w),\alpha)$ such that

$$\operatorname{Ext}^k(\mathcal{E}(L(w),\alpha),\mathcal{E}(L_{\rm pt}(v),\sigma)) \cong HF^k((L(w),\alpha),(L_{\rm pt}(v),\sigma)). \tag{4.3}$$

We put

$$\begin{aligned} T(L(w),\alpha) = &\Big\{[v,\sigma] \mid w - v \in \tilde L + \tilde L_{\rm pt},\ \alpha|_{\tilde L\cap\tilde L_{\rm pt}} - \sigma|_{\tilde L\cap \tilde L_{\rm pt}} = 0\Big\} \\ &\subseteq (T^{2n},\Omega)^{\vee}. \end{aligned}$$

Recall that $\mathcal{E}(L_{\rm pt}(v),\sigma)$ will become the skyscraper sheaf at $[v,\sigma] \in (T^{2n},\Omega)^{\vee}$. Therefore, in order (4.3) to be satisfied, the support of $\mathcal{E}(L(w),\alpha)$ should be contained in $T(L(w),\alpha)$.

Let us now study $T(L(w),\alpha)$.

Lemma 4.4.
(4.4.1) *$T(L(w),\alpha)$ depends only on $[w] \in (V/\tilde L)/(\Gamma/(\tilde L\cap\Gamma))$ and $[\alpha] \in \tilde L^*/(\tilde L\cap \Gamma)^{\vee}$.*
(4.4.2) *$T(L(w),\alpha)$ is a complex subtorus of $(T^{2n},\Omega)^{\vee}$.*

Proof. The proof of (4.4.1) is easy and is omitted. To prove (4.4.2) remark that

$$I_x(v,\sigma) = \Omega(x,v) + \sqrt{-1}\sigma(x).$$

is complex linear : $V \oplus V^* \to \mathbb{C}$ for each x. By definition, we find that

$$\bigcap_{x\in\tilde L\cap\tilde L_{\rm pt}} \operatorname{Ker} I_V$$

is the universal cover of $T(L(w), \alpha)$. The lemma follows. $\square$

We put

$$V' = \left(\tilde{L} + \tilde{L}_{\rm pt}\right)\Big/\left(\tilde{L} \cap \tilde{L}_{\rm pt}\right), \quad \Gamma' = \left(\Gamma + \tilde{L} \cap \tilde{L}_{\rm pt}\right)\Big/\left(\tilde{L} \cap \tilde{L}_{\rm pt}\right), \tag{4.5.1}$$

In fact, V' is a (linear) symplectic reduction of V with respect to $\tilde{L} \cap \tilde{L}_{pt}$. (See [MS2] Chapter 2.) We define :

$$\tilde{L}' = \tilde{L}\Big/\left(\tilde{L} \cap \tilde{L}_{\rm pt}\right), \quad \tilde{L}'_{\rm pt} = \tilde{L}_{\rm pt}\Big/\left(\tilde{L} \cap \tilde{L}_{\rm pt}\right), \tag{4.5.2}$$

$$\tilde{L}'_{\rm st} = \left(\tilde{L}_{\rm st} \cap \left(\tilde{L} + \tilde{L}_{\rm pt}\right)\right)\Big/\left(\tilde{L}_{\rm st} \cap \tilde{L} \cap \tilde{L}_{\rm pt}\right) \cong \tilde{L}_{\rm st} \cap \left(\tilde{L} + \tilde{L}_{\rm pt}\right). \tag{4.5.3}$$

Since $\tilde{L}, \tilde{L}_{\rm pt}$ are both Lagrangian linear subspaces, it follows that V' has a symplectic structure and $\tilde{L}'$, $\tilde{L}'_{\rm pt}$, $\tilde{L}'_{\rm st}$ are Lagrangian subspaces of V' ([MS2] Lemma 2.7). The B field B on V induces B' on V'. Γ' is a lattice of V'. Thus we obtain a mirror torus $(V'/\Gamma', \Omega')^\vee$ using $\tilde{L}'_{\rm pt}$. It is easy to see

$$V'/\tilde{L}'_{pt} \cong \tilde{L}/(\tilde{L} \cap \tilde{L}_{\rm pt}) \subseteq V/\tilde{L}_{\rm pt}, \tag{4.6.1}$$

$$(\tilde{L}'_{\rm pt})^* \cong (\tilde{L} \cap \tilde{L}_{\rm pt})^\perp \subseteq \tilde{L}^*_{\rm pt}. \tag{4.6.2}$$

Hence we may regard $(V'/\Gamma', \Omega')^\vee$ as a subgroup of $(T^{2n}, \Omega)^\vee$. It is easy to see that $T(L(w), \alpha)$ is an orbit of $(V'/\Gamma', \Omega')^\vee$. We fix $(v_0, \sigma_0) \in T(L(w), \alpha)$ and define an isomorphism $I_{(v_0,\sigma_0)} : (V'/\Gamma', \Omega')^\vee \to T(L(w), \alpha)$ by $I_{(v_0,\sigma_0)}(g) = g(v_0, \sigma_0)$.

We next associate an affine Lagrangian subspace $L'(\overline{w}; v_0)$ on $(V'/\Gamma', \Omega')$ to each element $(v_0, \sigma_0) \in T(L(w), \alpha)$. We fix a lift $\tilde{v}_0$ of v_0.

Let $[u, \sigma'] \in T(L(w), \alpha)$, $p \in L_{\rm pt}(u) \cap L(w)$. We can find a lift $\tilde{p} \in V$ of p such that

$$\tilde{p} - \tilde{v}_0 \in \tilde{L} + \tilde{L}_{\rm pt}. \tag{4.7}$$

Moreover the lift $\tilde{p} \in V$ satisfying (4.7) is unique modulo the action of $(\tilde{L}_{\rm pt} + \tilde{L}) \cap \Gamma$. Hence

$$f_{w,v_0}(p) := [\tilde{p} - \tilde{v}_0] \in V'/\Gamma'$$

depends only on p, w, v_0. We put

$$L'(\overline{w}(v_0)) = \{f_{w,v_0}(p) | \exists u \; p \in L_{\rm pt}(u) \cap L(w)\}. \tag{4.8}$$

In fact it is easy to see that the right hand side of (4.8) is parallel to L'.

Using the splitting $V = \tilde{L}_{\rm st} \oplus \tilde{L}_{\rm pt}$, we have a projection $\pi_{\tilde{L}_{\rm st}} : V \to \tilde{L}_{\rm pt}$. We put

$$\bar{\alpha}_{\sigma_0} = \alpha - \pi^*_{\tilde{L}_{\rm pt}}(\sigma_0) \in \tilde{L}'^* \subseteq \tilde{L}^*. \tag{4.9}$$

We remark that $\tilde{L}'$ is transversal to $\tilde{L}'_{\rm pt}$. Hence by the construction of §4, we obtain a holomorphic vector bundle $\mathcal{E}(L'(\overline{w}(v_0)), \bar{\alpha}_{\sigma_0})$ on $(V'/\Gamma', \Omega')^\vee$.

Lemma 4.10. *The holomorphic vector bundle* $I_{[v_0,\sigma_0]*}\mathcal{E}(L'(\overline{w}(v_0)),\bar{\alpha}_{\sigma_0})$ *on* $T(L(w),\alpha)$ *is independent of the choice of* $(v_0,\sigma_0) \in T(L(w),\alpha)$.

Proof. If $(v_0',\sigma_0') \in T(L(w),\alpha)$ be another choice. We put $v_0' - v_0 = t, \sigma_0' - \sigma_0 = \rho$. $[t,\rho] \in (V'/\Gamma',\Omega')^{\vee}$. We remark

$$L'(\overline{w}(v_0')) = t \cdot L'(\overline{w}(v_0)), \tag{4.11.1}$$

$$\bar{\alpha}_{\sigma_0'} = \bar{\alpha}_{\sigma_0} + \pi^*_{\tilde{L}_{\rm pt}}(\rho). \tag{4.11.2}$$

On the other hand, by definition we have $[t,\rho]\cdot I_{(v_0',\sigma_0')}(g) = I_{(v_0,\sigma_0)}(g)$. Lemma 4.10 follows easily from (4.11). □

Definition 4.12. $\mathcal{E}(L(w),\alpha) = \oplus I_* I_{[v_0,\sigma_0]*}\mathcal{E}(L'(\overline{w}(v_0)),\bar{\alpha}_{\sigma_0})$. Here $I : T(L(w),\alpha) \to (T^{2n},\Omega)^{\vee}$ is the inclusion and the direct product $\oplus$ is taken over all connected components of $T(L(\omega),\alpha)$.

We can easily prove the following

Lemma 4.13.

$$\mathrm{Ext}^k(\mathcal{E}(L(w),\alpha),\mathcal{E}(L_{\rm pt}(v),\sigma)) \cong H^{k-\eta(\tilde{L},\tilde{L}_{\rm pt})}(L(w)\cap L_{\rm pt})(v);\mathbb{C}).$$

Lemma 4.13 is consistent with Lemma 4.2 and hence justifies our definition.

§5. Universal bundles.

Let $\tilde L$ be a Lagrangian linear subspace of (V,Ω) such that $\tilde L\cap\Gamma\cong\mathbb{Z}^n$ and that $\tilde L\cap\tilde L_{\rm pt}=\{0\}$. We constructed a complex manifold (torus) $\mathcal{M}(\tilde L)$ in §2. On the other hand, for each element $[L(w),\alpha]\in\mathcal{M}(\tilde L)$, we constructed a holomorphic vector bundle $\mathcal{E}(L(w),\alpha)$ on $(T^{2n},\Omega)^\vee$. In this section, we construct a universal family of vector bundles on $\mathcal{M}(\tilde L)$. One delicate point in doing so is gauge fixing which we mentioned in §3. In fact, during the proof of Proposition 3.9, we need to choose a base point (denoted by y there) on $L(w)$. In other words, the isomorphism $\mathcal{E}(L(w),\alpha)\simeq\mathcal{E}(L(w),\alpha+\mu)$ for $\mu\in(\Gamma\cap\tilde L)^\vee$ depends on the choice of the base point on $L(w)$ and is not canonical. To choose a base point on $L(w)$ systematically, we need an additional data. Namely we fix another affine Lagrangian submanifold. (Which we will write M.) To do it precisely is the purpose of this section.

We remark that the bundles $\mathcal{E}(L(w),\alpha)$ has nontrivial automorphism. Hence defining universal object on the moduli space of such bundles is a delicate question. The problem discussd in this section is related to this point.

We start with the following situation.

Let $\tilde L_1$, $\tilde L_2$ and $\tilde M_1$, $\tilde M_2$ be Lagrangian linear subspaces of (V,Ω) such that

(5.1.1) $\tilde L_i\cap\Gamma\cong\mathbb{Z}^n,\quad \tilde M_i\cap\Gamma\cong\mathbb{Z}^n.$
(5.1.2) $\tilde L_i$ is transversal to $\tilde M_i$.
(5.1.3) $\tilde L_1$ is transversal to $\tilde L_2$.

By a modification of the argument of this section (combined with the construction of §4), we might remove Condition (5.1.3). However we are not trying to do so since the universal bundle is applied only in §10,11,14, where the transversal case is discussed.

Definition 5.2. $\mathcal{M}(\tilde L_i;M_i)$ is the set of pairs $([L_i(v_i),\alpha_i],x_i)\in\mathcal{M}(\tilde L_i)\times T^{2n}$ such that $x_i\in M_i\cap L_i(w_i)$.

$([L_i(w_i),\alpha_i],x_i)\mapsto[L_i(w_i),\alpha_i]$ is a $M_i\bullet L_i(w_i)$ hold covering. Hence the complex structure on $\mathcal{M}(\tilde L_i)$ induces one on $\mathcal{M}(\tilde L_i;M_i)$. We are going to define a holomorphic vector bundle $\mathcal{P}(\tilde L_1,\tilde L_2;M_1,M_2)\to\mathcal{M}(\tilde L_1;M_1)\times\mathcal{M}(\tilde L_2;M_2)$ such that its fiber at $([L_1(w_1),\alpha_1],[L_2(w_2),\alpha_2])$ is identified with the Floer homology $HF((L_1(w_1),\alpha_1),(L_2(w_2),\alpha_2))$.

Let

$$(5.3)\quad ((w_1,\alpha_1),(w_2,\alpha_2))\in\left(V\Big/\tilde L_1\times\tilde L_1^*\right)\times\left(V\Big/\tilde L_2\times\tilde L_2^*\right),\quad x_i\in M_i\cap L_i(v_i).$$

Let $V(\tilde L_1,\tilde L_2;M_1,M_2)$ be the totality of all $((w_1,\alpha_1),(w_2,\alpha_2);x_1,x_2)$ satisfying (5.3). We put

$$(5.4)\qquad \tilde{\mathcal{P}}(\tilde L_1,\tilde L_2;M_1,M_2)_{((w_1,\alpha_1),(w_2,\alpha_2);x_1,x_2)}=\bigoplus_{p\in L_1(w_1)\cap L_2(w_2)}\mathbb{C}[p].$$

$\tilde{\mathcal{P}}(\tilde L_1,\tilde L_2;M_1,M_2)$ is a complex vector bundle on $V(\tilde L_1,\tilde L_2;M_1,M_2)$. We put

$$\Gamma(\tilde L_1,\tilde L_2)=\left(\Gamma/\Gamma\cap\tilde L_1\right)\times\left(\Gamma\cap\tilde L_1\right)^\vee\times\left(\Gamma/\Gamma\cap\tilde L_2\right)\times\left(\Gamma\cap\tilde L_2\right)^\vee.$$

It acts on $V(\tilde L_1,\tilde L_2;M_1,M_2)$ by

$$\begin{aligned}(\gamma_1,\mu_1,\gamma_2,\mu_2)\cdot((w_1,\alpha_1),(w_2,\alpha_2);x_1,x_2)\\ =(w_1+\gamma_1,\alpha_1+\mu_1,w_2+\gamma_2,\alpha_2+\mu_2;x_1,x_2).\end{aligned}$$

The quotient space is $\mathcal{M}(\tilde L_1;M_1)\times\mathcal{M}(\tilde L_2;M_2)$.

We define an action of $\Gamma(\tilde L_1,\tilde L_2)$ on $\tilde{\mathcal{P}}(\tilde L_1,\tilde L_2;\tilde M_1,\tilde M_2)$ as follows. Let

$$((w_1,\alpha_1),(w_2,\alpha_2);x_1,x_2)\in V(\tilde L_1,\tilde L_2;M_1,M_2), \tag{5.5.1}$$

$$p\in L_1(w_1)\cap L_2(w_2), \tag{5.5.2}$$

$$[p]\in\tilde{\mathcal{P}}(\tilde L_1,\tilde L_2;\tilde M_1,\tilde M_2)_{((w_1,\alpha_1),(w_2,\alpha_2);x_1,x_2)}, \tag{5.5.3}$$

$$(\gamma_1,\mu_1,\gamma_2,\mu_2)\in\Gamma(\tilde L_1,\tilde L_2). \tag{5.5.4}$$

We have $\lambda_1\in\Gamma/\Gamma\cap\tilde L_1$, $\lambda_2\in\Gamma/\Gamma\cap\tilde L_2$ such that $p=\pi(\tilde p)$, $\{\tilde p\}=\hat L_1(w_1+\lambda_1)\cap\hat L_2(w_2+\lambda_2)$. Let $\tilde x_i\in\hat L_i(w_i+\lambda_i)$ be the lift of x_i. We then put

$$\begin{aligned}&(\gamma_1,\mu_1,\gamma_2,\mu_2)\cdot[p]\\ &=\exp\left(2\pi\sqrt{-1}\mu_1\left(\tilde p-\tilde x_1\right)+2\pi\sqrt{-1}\mu_2\left(\tilde x_2-\tilde p\right)\right)[p]\\ &\in\tilde P(\tilde L_1,\tilde L_2;M_1,M_2)_{((w_1+\gamma_1,\alpha_1+\mu_1),(w_2+\gamma_2,\alpha_2+\mu_2);x_1,x_2)}.\end{aligned} \tag{5.6}$$

Lemma 5.7. (5.6) *defines an action of* $\Gamma(\tilde L_1,\tilde L_2)$ *on* $\tilde{\mathcal{P}}(\tilde L_1,\tilde L_2;\tilde M_1,\tilde M_2)$.

Proof. It is easy to see that the right hand side of (5.5) is independent of λ_i.

To complete the proof of Lemma 5.6, we need to show that

$$\begin{aligned}(\gamma_1,&\mu_1,\gamma_2,\mu_2)\cdot(\gamma_1',\mu_1',\gamma_2',\mu_2')\cdot[p]\\ &=(\gamma_1+\gamma_1',\mu_1+\mu_1',\gamma_2+\gamma_2',\mu_2+\mu_2')\cdot[p].\end{aligned}$$

This equality follows from the fact that

$$\exp\left(-2\pi\sqrt{-1}\mu_1\left(\tilde p-\tilde x_1\right)+2\pi\sqrt{-1}\mu_2\left(\tilde x_2-\tilde p\right)\right)$$

depends only on x_1,x_2,p and is independent of its lifts. The proof of Lemma 5.6 is complete. □

Definition 5.8. $\mathcal{P}(\tilde L_1,\tilde L_2;M_1,M_2)\to\mathcal{M}(\tilde L_1;M_1)\times\mathcal{M}(\tilde L_2;M_2)$ is the quotient bundle of $\tilde{\mathcal{P}}(\tilde L_1,\tilde L_2;M_1,M_2)/\Gamma(\tilde L_1,\tilde L_2)$.

Remark 5.9. We put $\tilde L_1=\tilde L_{\rm pt}$, $\tilde L_2=\tilde L$, $\tilde M_1=\tilde L_{\rm st}$. $\tilde M_2$ is an arbitrary Lagrangian submanifold satisfying the assumption. We have $\mathcal{M}(\tilde L_{\rm pt},L_{\rm st})=(T^{2n},\Omega)^\vee$. We restrict $\mathcal{P}(\tilde L_{\rm pt},\tilde L;L_{\rm st},M_2)\to(T^{2n},\Omega)^\vee\times\mathcal{M}(\tilde L;M_2)$ to $(T^{2n},\Omega)^\vee\times\{[w,\alpha]\}$. Then, the bundle we obtain is $\mathcal{E}(L(w),\alpha)$. In fact, $\tilde x_1$ in (5.5) will be $x_0(v)$ in (3.2.1). M_2 does not play a role here since $[w,\alpha]$ is fixed. The right hand side of (5.5) will become

$$\exp(2\pi\sqrt{-1}\mu_1(\tilde p-x_0(v)))$$

and coincides with (3.2)

We next construct a holomorphic structure on $\mathcal{P}(\tilde L_1, \tilde L_2; M_1, M_2)$. We again construct a holomorphic local frame. We use the same notation as in (5.5). Let $((w_1', \alpha_1'), (w_2', \alpha_2'); x_1', x_2') \in V(\tilde L_1, \tilde L_2; M_1, M_2)$ be in a neighborhood of $((w_1, \alpha_1), (w_2, \alpha_2); x_1, x_2)$. There exists a point $\tilde p' \in \hat L_1(w_1' + \lambda_1) \cap \hat L_2(w_2' + \lambda_2)$ in a neighborhood of $\tilde p$. (See Figure 4.)

Figure 4

We define:

$$\begin{aligned}
&e_{((w_1,\alpha_1),(w_2,\alpha_2),\tilde x_1,\tilde x_2,\tilde p)}\left((w_1', \alpha_1'), (w_2', \alpha_2')\right) \\
&\quad = \exp\big(-2\pi Q\left(\tilde p, \tilde x_1, \tilde x_1', \tilde p', \tilde x_2', \tilde x_2\right) \\
&\qquad +2\pi\sqrt{-1}(\alpha_1(\tilde x_1 - \tilde p) + \alpha_1'(\tilde p' - \tilde x') + \alpha_2'(\tilde x_2' - \tilde p') + \alpha_2(\tilde p - \tilde x_2))\big),
\end{aligned} \tag{5.10.1}$$

where $Q\left(\tilde p, \tilde x_1, \tilde x_1', \tilde p', \tilde x_2', \tilde x_2\right)$ is a integration of Ω over union of 4 triangles $\Delta_{\tilde p\, \tilde x_1 \tilde x_1'}$, $\Delta_{\tilde p\, \tilde x_1, \tilde p'}$, $\Delta_{\tilde p\, \tilde p', \tilde x_2}$, $\Delta_{\tilde p\, \tilde x_2', \tilde x_2}$. Then the frame is :

$$\begin{aligned}
&\mathbf{e}_{((w_1,\alpha_1),(w_2,\alpha_2),\tilde x_1,\tilde x_2,\tilde p)}\left((w_1', \alpha_1'), (w_2', \alpha_2')\right) \\
&\quad = e_{((w_1,\alpha_1),(w_2,\alpha_2),\tilde x_1,\tilde x_2,\tilde p)}\left((w_1', \alpha_1'), (w_2', \alpha_2')\right)[p'].
\end{aligned} \tag{5.10.2}$$

Lemma 5.11. *There exists a unique holomorphic structure on $\mathcal{P}(\tilde L_1, \tilde L_2; \tilde M_1, \tilde M_2)$ such that $\mathbf{e}_{((w_1,\alpha_1),(w_2,\alpha_2),\tilde x_1,\tilde x_2,\tilde p)}$ is a local holomorphic section.*

The proof is a straight forward analogue of the proof of Lemma 3.7 and hence is omitted. We remark that we did not need to assume $\tilde M_i$ to be a Lagrangian submanifold to prove Lemma 5.6 and to define $\mathcal{P}(\tilde L_1, \tilde L_2; \tilde M_1, \tilde M_2)$ as a smooth complex vector bundle. However to show Lemma 5.10 we do need to assume $\tilde M_i$ to be a Lagrangian submanifold.

We thus constructed a holomorphic vector bundle

$$\mathcal{P}(\tilde L_1, \tilde L_2; \tilde M_1, \tilde M_2) \to \mathcal{M}(\tilde L_1; M_1) \times \mathcal{M}(\tilde L_2; M_2)$$

in case (5.1.1),(5.1.2),(5.1.3) are satisfied.

We assume $\tilde L \cap \tilde L_{\rm pt} = 0$ and put $\tilde L_1 = \tilde L_{\rm pt}$, $\tilde L_2 = \tilde L$, $\tilde M_1 = \tilde L_{\rm st}$, $\tilde M_2 = \tilde L_{\rm pt}$. We obtain

$$\mathcal{P}(\tilde L_{\rm pt}, \tilde L; L_{\rm st}, L_{\rm pt}) \to (T^{2n}, \Omega)^\vee \times \mathcal{M}(\tilde L, L_{\rm pt}). \tag{5.12}$$

Let us consider the group $G = L_{\rm pt} \cap L(0)$. It is easy to see $\mathcal{M}(\tilde L, L_{\rm pt})/G = \mathcal{M}(\tilde L)$. For

$\circ\ p \in L_{\rm pt}(w_1) \cap L(w_2)$,

$\circ\ [p] \in \tilde{\mathcal{P}}(\tilde L_{\rm pt}, \tilde L; L_{\rm st}, L_{\rm pt})_{((w_1,\alpha_1),(w_2,\alpha_2);x_1,x_2)}$,

$\circ\ g \in G = L_{\rm pt} \cap L(0)$,

We put

$$g \cdot [p] = [g + p] \in \tilde{\mathcal{P}}(\tilde L_{\rm pt}, \tilde L; L_{\rm st}, L_{\rm pt})_{((w_1,\alpha_1),(w_2,\alpha_2);x_1,x_2+g)}. \tag{5.13}$$

It is easy to see that (5.13) defines a G action on $\mathcal{P}(\tilde L_{\rm pt}, \tilde L; L_{\rm st}, L_{\rm pt})$ such that (5.12) is G equivalent. Thus we divide (5.12) by G and obtain

$$\mathcal{P}(\tilde L_{\rm pt}, \tilde L) \to (T^{2n}, \Omega)^\vee \times \mathcal{M}(\tilde L). \tag{5.14}$$

It is easy to verify the following :

Proposition 5.15. *Let* $(L,\alpha) \in \mathcal{M}(\tilde{L})$. *The restriction of* $\mathcal{P}(\tilde{L}_{\rm pt}, \tilde{L})$ *to* $pr^{-1}(L,\alpha)$ *is isomorphic to* $\mathcal{E}(L,\alpha)$.

We prove a converse of Proposition 5.15 in §13. (Proposition 13.26.) Proposition 5.15 and Proposition 13.26 imply that we may regard (5.14) as the universal bundle. We next put $\tilde{L}_1 = \tilde{L}_{\rm pt}$, $\tilde{L}_2 = \tilde{L}_{\rm st} = \tilde{M}_1$, $\tilde{M}_2 = \tilde{L}_{\rm pt}$ and obtain

$$\mathcal{P} \to (T^{2n}, \Omega)^{\vee} \times \mathcal{M}(\tilde{L}_{\rm st}). \tag{5.16}$$

Since $L_{\rm st} \bullet L_{\rm pt} = 1$ it follows that $\mathcal{M}(\tilde{L}_{\rm st}, L_{\rm pt}) = \mathcal{M}(\tilde{L}_{\rm st})$

Proposition 5.17. $\mathcal{M}(\tilde{L}_{\rm st})$ *is the dual torus* $(T^{2n}, \Omega)^{\vee\wedge}$. *The bundle* (5.16) *is the Poincaré bundle.*

The proof is easy and is omitted, (since we do not use it in later sections).

Chapter 2. Product structure in Floer homology.

§6. Explanation of Axioms I,II,III.

Product structures of Lagrangian intersection Floer homology was introduced by the author in [Fu1], based on the idea due to Donaldson [Do2] and Segal. The rough idea is to count the number of pseudoholomorphic polygons to define matrix element. Kontsevich [K1],[K2] introduced its modification by putting a weight, which is the exponential of the symplectic area of the pseudoholomorphic polygons. It leads a family of operators :

$$\begin{aligned} \mathfrak{m}_k : HF^{d_{1,2}}(L_1, L_2) \otimes \cdots \otimes HF^{d_{k-1,k}}(L_k, L_{k+1}) \\ \to HF^{\sum d_{i,i+1}+2-k}(L_1, L_{k+1}), \end{aligned} \tag{6.1}$$

of degree $2-k$. (We can include flat line bundles on L_i.) In the case when $L = L_1 = \cdots = L_{k+1}$, this operator coincides with one defined (rigorously) in [FKOOO] Chapter 3. A more detailed discussion on the general case is in [FKOOO],[Fu6],[Fu7]. The full detail of the proof of the construction of operations (6.1), in the general case, will appear elesewhere.

In this paper, we concentrate on the case of affine Lagrangian submanifold. Also, in this chapter, we consider only the case when L_i are transversal to each other. In that case, Floer homology is

$$HF^d((L_1(v_1), \mathcal{L}(\alpha_1)), (L_2(v_2), \mathcal{L}(\alpha_2)) \simeq \begin{cases} \mathbb{C}^{\sharp(L_1(v_1)\cap L_2(v_2))} & \text{if } \eta(\tilde L_1, \tilde L_2) = d, \\ 0 & \text{otherwise,} \end{cases}$$

according to Definition 2.6. Therefore, the operator $\mathfrak{m}_k$ can be nonzero only if

$$\sum_{i=1}^{k} \eta(\tilde L_i, \tilde L_{i+1}) + 2 - k = \eta(\tilde L_1, \tilde L_{k+1}). \tag{6.2}$$

The Condition (6.2) is related to the Kashiwara-Maslov index [KM], which we define now.

Definition 6.3.

$$\eta(\tilde L_1, \cdots, \tilde L_{k+1}) = \sum_{i=1}^{k} \eta(\tilde L_i, \tilde L_{i+1}) - \eta(\tilde L_1, \tilde L_{k+1}).$$

Note, in case $k = 2$, Definition 6.3 is consisitent with the previous one, because of (2.13.1). (Note $k_{12} = k_{23} = 0$ in our case.)

We are going to study the operator (6.1) in the case when $\eta(\tilde L_1, \cdots, \tilde L_{k+1}) = k-2$. It is useful however to include the case when $\eta(\tilde L_1, \cdots, \tilde L_{k+1}) \ne k-2$ by considering a family version.

To study it, we work in the universal cover V and "count" pseudoholomorphic disks whose bondary is $\hat L_1(v_1) \cup \cdots \cup \hat L_{k+1}(v_{k+1})$. We will not present a rigorous definition of the order of this moduli space. Here we consider the properties they are supposed to satisfy and prove that the operators satisfying them exists and is unique upto homotopy equivalence. Those properties are the axioms we stated in the introduction.

In this section, we define some of the notations used in the introduction and also gives some explanation of the axiom.

Let $v_i \in V/\tilde L_i$. We recall $\hat L_i(v_i) = \tilde L_i + v_i$. The moduli space of pseudoholomorphic polygons we concern with is the following. We put $\{p_{i,j}(\vec v)\} = \hat L_i(v_i) \cap \hat L_j(v_j)$.

$$\begin{aligned}&\tilde{\mathcal M}(\hat L_1(v_1), \cdots, \hat L_{k+1}(v_{k+1}))\\ &= \left\{ (\varphi; z_1, \cdots, z_{k+1}) \left| \begin{array}{l} \varphi : D^2 \to V \text{ is holomorphic.} \\ z_i \in \partial D^2, (z_1, \cdots, z_{k+1}) \text{ respects the} \\ \quad \text{cyclic order of } \partial D^2. \\ \varphi(z_i) = p_{i,i+1}(\vec v), \quad \varphi(\partial_i D^2) \subseteq \hat L_i(v_i). \end{array} \right. \right\}\end{aligned} \tag{6.4}$$

Here $\partial_i D^2$ is the part of ∂D^2 between $p_{i-1,i}$ and $p_{i,i+1}$. The group $PSL(2;\mathbb R) = \mathrm{Aut}(D^2)$ acts on $\tilde{\mathcal M}(\hat L_1(v_1), \cdots, \hat L_{k+1}(v_{k+1}))$ by

$$g \cdot (\varphi; z_1, \cdots, z_{k+1}) = (\varphi \circ g^{-1}; g(z_1), \cdots, g(z_{k+1})).$$

Let $\mathcal M(\hat L_1(v_1), \cdots, \hat L_{k+1}(v_{k+1}))$ be the quotient space. One can prove

$$\mathrm{Virdim}\, \mathcal M(\hat L_1(v_1), \cdots, \hat L_{k+1}(v_{k+1})) = k - 2 - \eta(\tilde L_1, \cdots, \tilde L_{k+1}). \tag{6.5}$$

We do not prove (6.5) in this paper since we use it only to motivate axioms for the chains (integral current) $\mathfrak C(\tilde L_1, \cdots, \tilde L_{k+1})$ we gave in the introduction. Here the right hand side is the virtual dimension. We study also the case when this virtual dimension is negative.

In that case, we need to study a family version. We recall

$$\tilde L(\tilde L_1, \cdots, \tilde L_{k+1}) = \prod_{i=1}^{k+1} V/\tilde L_i. \tag{6.6.1}$$

$$L(\tilde L_1, \cdots, \tilde L_{k+1}) = \frac{\tilde L(\tilde L_1, \cdots, \tilde L_{k+1})}{V}. \tag{6.6.2}$$

Here we regard $V \subset \tilde L(\tilde L_1, \cdots, \tilde L_{k+1})$ by

$$v \mapsto (v \mod \tilde L_1, v \mod \tilde L_2, \cdots, v \mod \tilde L_k).$$

We write $\vec v = (v_1, \cdots, v_{k+1})$. The chain $\mathfrak C_{hol}(\tilde L_1, \cdots, \tilde L_{k+1})$ is defined as follows :

$$\begin{aligned}\mathfrak C_{hol}(\tilde L_1, \cdots, \tilde L_{k+1}) = \{[\vec v] \in L(1, \cdots, k+1) \mid \\ \tilde{\mathcal M}(\hat L_1(v_1), \cdots, \hat L_{k+1}(v_{k+1})) \ne \emptyset\Big\}.\end{aligned} \tag{6.7}$$

The chain (integral current) $\mathfrak C(\tilde L_1, \cdots, \tilde L_{k+1})$ in the introduction is one which are supposed to coincides with this. We will prove elesewhere that $\mathfrak C_{hol}(\tilde L_1, \cdots, \tilde L_{k+1})$ satisfies the axioms in the introduction. We only give here a brief discussion why those axioms are expected.

We first define a function Q in the introduction. We put

$$\{p_{i,j}(\vec v)\} = \hat L_i(v_i) \cap \hat L_j(v_j).$$

Let $\Delta_{i,j,k}(\vec v)$ be the triangle whose vertices are $p_{i,j}(\vec v)$, $p_{j,k}(\vec v)$, $p_{k,i}(\vec v)$.

Definition 6.8. The function $Q_{j_1,\cdots,j_\ell} : L(\tilde{L}_{j_1}, \cdots, \tilde{L}_{j_\ell}) \to \mathbb{C}$ is defined by :

$$Q_{i,j,k}(v_i, v_j, v_k) = \int_{\Delta_{i,j,k}(\vec{v})} \Omega,$$

$$Q_{j_1,\cdots,j_\ell}(\vec{v}) = \sum_{i=1}^{\ell-2} Q_{j_i,j_{i+1},j_{i+2}}(v_{j_i}, v_{j_{i+1}}, v_{j_{i+2}}).$$

$Q_{j_1,\cdots,j_\ell}$ induces a map : $L(\tilde{L}_{j_1}, \cdots, \tilde{L}_{j_\ell}) \to \mathbb{C}$ also, which we denote by the same symbol.

(In Definition 6.8 we write $Q_{j_1,\cdots,j_\ell}(\vec{v})$ in place of $Q_{j_1,\cdots,j_\ell}(v_{j_1}, \cdots, v_{j_\ell})$, for simplicity.)

Lemma 6.9. *$Q_{j_1,\cdots,j_\ell}$ is a quadratic form. It satisfies the following :*

(6.9.1) $\quad Q_{j_1,\cdots,j_\ell}(v_{j_1}, \cdots, v_{j_\ell}) = Q_{j_\ell,j_1,\cdots,j_{\ell-1}}(v_{j_\ell}, v_{j_1}, \cdots, v_{j_{\ell-1}}).$

(6.9.2) $\quad Q_{j_1,\cdots,j_\ell}(\vec{v}) = Q_{j_1,\cdots,j_{a-1},j_a,j_b,j_{b+1},\cdots,j_\ell}(\vec{v}) + Q_{j_a,\cdots,j_b}(\vec{v}).$

The lemma can be proved easily by using Stokes theorem. We can prove also the following.

Lemma 6.10. *If $(v_{j_1}, \cdots, v_{j_\ell}) \in \mathfrak{C}_{hol}(\tilde{L}_{j_1}, \cdots, \tilde{L}_{j_\ell})$ then*

$$\Re\left(Q_{j_1,\cdots,j_\ell}(v_{j_1}, \cdots, v_{j_\ell})\right) \geq 0.$$

If moreover $(\varphi; z_{j_1}, \cdots, z_{j_\ell}) \in \mathcal{M}(\hat{L}_{j_1}(v_{j_1}), \cdots, \hat{L}_{j_\ell}(v_{j_\ell}))$ and φ is nonconstant, then

$$\Re\left(Q_{j_1,\cdots,j_\ell}(v_{j_1}, \cdots, v_{j_\ell})\right) > 0.$$

Proof. Let $(\varphi; z_{j_1}, \cdots, z_{j_\ell}) \in \mathcal{M}(\hat{L}_{j_1}(v_{j_1}), \cdots, \hat{L}_{j_\ell}(v_{j_\ell}))$. It is easy to see from definition and Stokes' theorem that

$$\int_{D^2} \varphi^*\omega = \Re\left(Q_{j_1,\cdots,j_\ell}(v_{j_1}, \cdots, v_{j_\ell})\right).$$

On the other hand, $\int_{D^2} \varphi^*\omega \geq 0$ since φ is pseudoholomorphic. Moreover if φ is nonconstant then $\int_{D^2} \varphi^*\omega > 0$. The lemma follows. □

Lemma 6.10 is a motivation of Axiom I (I.3).

The other part of Axiom I is rather an obivious properties of $\mathfrak{C}_{hol}(\tilde{L}_1, \cdots, \tilde{L}_{k+1})$. For eample (I.2) is implilied by the fact that

$$\mathcal{M}(\hat{L}_1(v_1), \cdots, \hat{L}_{k+1}(v_{k+1})) = \mathcal{M}(\hat{L}_1(rv_1), \cdots, \hat{L}_{k+1}(rv_{k+1})),$$

for $r > 0$. To explain the motivation of Axiom II, we consider the following moduli space.

$$\mathcal{M}(\tilde{L}_1, \cdots, \tilde{L}_{k+1}) = \bigcup_{[\vec{v}] \in L(\tilde{L}_1, \cdots, \tilde{L}_{k+1})} \mathcal{M}(\hat{L}_1(v_1), \cdots, \hat{L}_{k+1}(v_{k+1})) \times \{[\vec{v}]\}.$$

Note that the projection of $\mathcal{M}(\tilde{L}_1, \cdots, \tilde{L}_{k+1}))$ to $L(\tilde{L}_1, \cdots, \tilde{L}_{k+1})$ defines the current $\mathfrak{C}_{hol}(\tilde{L}_1, \cdots, \tilde{L}_{k+1})$. The compactification of this moduli space is obtained in a way similar to the compactification of the moduli space of pseudoholomorphic disks bounding a (single) Lagrangian submanifold L, which is discussed in [FKOOO] Chapter 6. Then in a way similar to the proof of [FKOOO] Chapter 6, we find

$$\begin{aligned} & d\mathcal{M}(\tilde L_1,\cdots,\tilde L_{k+1}) \\ & = \bigcup_{1\le a<b\le k+1} \pm\mathcal{M}(\tilde L_1,\cdots,\tilde L_a,\tilde L_b,\cdots,\tilde L_{k+1})\times\mathcal{M}(\tilde L_a,\cdots,\tilde L_b) \end{aligned} \tag{6.11}$$

as integral currents on $L(\tilde L_1,\cdots,\tilde L_{k+1})$. Let us explain a heuristic argument justfying (6.11). Let $[\varphi^j;z_1^j,\cdots,z_{k+1}^j]$ be a divergent sequence of elements of $\mathcal{M}(\tilde L_1,\cdots,\tilde L_{k+1})$. We may assume that $\lim_{j\to\infty} z_i^j = z_i^\infty$ converges. Then either

(6.12.1) $\lim_{j\to\infty}\varphi_j$ does not converge, or
(6.12.2) $z_i^\infty = z_i'^\infty$ for $i\ne i'$.

In the first case, there happens a bubble either in the interior of the disk or the boundary of the disk. However the first case can not happen since there is no holomorphic sphere or no holomorphic disk boundaing an affine Lagrangian submanifold.

In the second case, we see as a limit, the pseudoholomorphic curves which looks like Figure 5. This figure corresponds elements of $\mathcal{M}(\tilde L_1,\cdots,\tilde L_a,\tilde L_b,\cdots,\tilde L_{k+1})\times\mathcal{M}(\tilde L_a,\cdots,\tilde L_b)$. (6.11) will follow from this fact. (We remark that the sign $\pm$ is hard to determine from this argument. See [FKOOO] Chapter 6, where the sign (orientation) is determined in a closely related case. We explain a sign convention in §7, where we also justify it in a way way different from [FKOOO].)

Figure 5

We now turn to the explanation of the last Axiom III. We used the following Lemma 6.13 and Corollary 6.13 to state it in introduction.

Lemma 6.13.

$$\mathrm{Index}\,\Re Q_{1,2,3} = \eta(\tilde L_1,\tilde L_2,\tilde L_3).$$

Proof. We first remark that the right hand side of the equality does not change when we move $\tilde L_1,\tilde L_2,\tilde L_3$ without changing $\dim\tilde L_i\cap\tilde L_j$, $\dim\tilde L_1\cap\tilde L_2\cap\tilde L_3$.

We can prove also the same property for the left hand side. (We remark that $\eta(\tilde L_1,\tilde L_2)$ jumps when L_1 or L_2 will become tangent to $\tilde L_{\mathrm{pt}}$. But $\eta(\tilde L_1,\tilde L_2,\tilde L_3)$ does not jump. Lemma 2.8.)

Let us take $\tilde L_0$ transversal to $\tilde L_i$. We identify $V = T^*(V/\tilde L_0)$. It then suffices to consider the case when $\tilde L_3 = V/\tilde L_0$ (the zero section) and $\tilde L_1$ is equal to the graph of $Nd(x_1^2+\cdots+x_n^2)$, where N is large. Let $\tilde L_2$ be the graph of $df_{\tilde L_2}$. Then, by definition, $\eta_{\tilde L_0}(\tilde L_1,\tilde L_2)=0$, $\eta_{\tilde L_0}(\tilde L_2,\tilde L_3) = \mathrm{Index}\,f_{\tilde L_2}$, $\eta_{\tilde L_0}(\tilde L_3,\tilde L_1)=n$. Therefore, by Definition 2.7, we have

$$\eta(\tilde L_1,\tilde L_2,\tilde L_3) = n - \mathrm{Index}\,f_{\tilde L_2}.$$

Next we consider $\Re Q_{1,2,3}$. We perturb $\tilde L_1$ a bit and may take $\tilde L_1 = \tilde L_0 =$ the fiber of $T^*(V/\tilde L_0)$. We may identify $L(\tilde L_1,\tilde L_2,\tilde L_3)$ with $V/\tilde L_2$. Then we find

$$Q_{1,2,3}(0,v_2,0) = \frac{1}{2}\Omega(\pi_1(v_2),\pi_3(v_2)) = -f(\pi_3(v_2)).$$

Here $(\pi_1, \pi_3) : V \to \tilde L_1 \oplus \tilde L_3$ is the canonical isomorphism. (Here we regard Ω as a skew symmetric form on V.) Lemma 6.13 follows easily. □

As in introduction, we put :

$$S(Q; \tilde L_1, \tilde L_2, \tilde L_3) = \left\{ [v_1, v_2, v_3] \in L(\tilde L_1, \tilde L_2, \tilde L_3) \;\middle|\; Q(v_1, v_2, v_3) > 0,\ \Vert [v_1, v_2, v_3]\Vert = 1 \right\}.$$

Lemma 6.13 implies that $S(Q; \tilde L_1, \tilde L_2, \tilde L_3)$ is homotopy equivalent to $S^{n-\eta(\tilde L_1,\tilde L_2,\tilde L_3)-1}$. Hence by Poincaré duality we have :

Corollary 6.14.

$$H^{\eta(\tilde L_1,\tilde L_2,\tilde L_3)}(S(Q; \tilde L_1, \tilde L_2, \tilde L_3); \mathbb Z) \simeq \mathbb Z.$$

Now we turn to the discussion of Axiom III. The transversality problem is easy to handle in the case of the moduli space $\mathcal M(\hat L_1(v_1), L_2(v_2), \hat L_3(v_3))$ which we are discussing now, since there is no bubble and the moduli space is compact. (See [Fu5].) So we present the precise statement and a proof here. Let $S(L(\tilde L_1, \tilde L_2, \tilde L_3))$ be the unit sphere in $L(\tilde L_1, \tilde L_2, \tilde L_3)$. Let us consider the moduli space $\mathcal M(\tilde L_1, \tilde L_2, \tilde L_3)$ and the projection map $\pi : \mathcal M(\tilde L_1, \tilde L_2, \tilde L_3) \to L(\tilde L_1, \tilde L_2, \tilde L_3)$. First we have:

Proposition 6.15. *$\mathcal M(\tilde L_1, \tilde L_2, \tilde L_3)$ has an oriented Kuranishi structure without boundary. Its dimension is $n - \eta(\tilde L_1, \tilde L_2, \tilde L_3)$.*

Moreover $\mathcal M(\tilde L_1, \tilde L_2, \tilde L_3) \cap \pi^{-1} S(L(\tilde L_1, \tilde L_2, \tilde L_3))$ is compact.

The compactness is proved in [Fu5]. The construction of Kuranishi structure is similar to and is easier than one given in [FKOOO] Chapter 5 and is omitted. The calculation of (virtual) dimension follows from the proof of Theorem 6.16 blow.

We remark that the Kuranishi structure on $\mathcal M(\tilde L_1, \tilde L_2, \tilde L_3) \cap \pi^{-1} S(L(\tilde L_1, \tilde L_2, \tilde L_3))$ is one without boundary since there can be no bubble. We remark that the action of the group $\mathrm{Aut}(D^2)$ on $\mathcal M(\hat L_1(v_1), L_2(v_2), L_3(v_3))$ is free unless $\hat L_1(v_1) \cap L_2(v_2) \cap L_3(v_3) \ne \emptyset$. Hence automorphism group of elements of $\mathcal M(\tilde L_1, \tilde L_2, \tilde L_3) \cap \pi^{-1} S(L(\tilde L_1, \tilde L_2, \tilde L_3))$ is trivial. It follows from [FOn2], that we have a fundamental cycle

$$\begin{aligned}&\pi_*[\mathcal M(\tilde L_1, \tilde L_2, \tilde L_3) \cap \pi^{-1} S(L(\tilde L_1, \tilde L_2, \tilde L_3))] \\ &\qquad \in H_{n-\eta(\tilde L_1,\tilde L_2,\tilde L_3)-1}(S(Q; \tilde L_1, \tilde L_2, \tilde L_3); \mathbb Z).\end{aligned}$$

Theorem 6.16. *$\pi_*[\mathcal M(\tilde L_1, \tilde L_2, \tilde L_3) \cap \pi^{-1} S(L(\tilde L_1, \tilde L_2, \tilde L_3))]$ is a generator.*

Proof. We use Morse homotopy [Fu2] in a similar way to the proof of [Fu5] Theorem 7.18. Let $\tilde L_0$ be Lagrangian linear subspace transversal to $\tilde L_i$. We have $V \cong T^*(V/\tilde L_0)$. We may choose the isomorphism so that $\tilde L_1$ is the zero section. We may furthermore assume that $\tilde L_{\rm pt} = \sqrt{-1}\mathbb R^n$ is tranvsersal to $\tilde L_2$ and $\tilde L_3$. We hence may regard $\tilde L_2$ and $\tilde L_3$ as graphs of $df_{\tilde L_2}$ and $df_{\tilde L_3}$. Where $df_{\tilde L_2}$ and $df_{\tilde L_3}$ are quadratic functions on $\tilde L_{\rm st}$. We let $\tilde L_{2,\epsilon}$ and $\tilde L_{3,\epsilon}$ be the graphs of $\epsilon df_{\tilde L_2}$ and $\epsilon df_{\tilde L_3}$. We consider the isomorphism

$$I_\varepsilon : V/\tilde L_1 \times V/\tilde L_2 \times V/\tilde L_3 \cong V/\tilde L_1 \times V/\tilde L_2^\varepsilon \times V/\tilde L_3^\varepsilon$$

We have

$$Q(I_\epsilon(v_1), I_\epsilon(v_2), I_\epsilon(v_3)) = \epsilon^2 Q(v_1, v_2, v_3).$$

It follows from well established cobordism argument using the compactness in Proposition 6.15 (see [Fu5]), that the homology class in Theorem 6.16 does not change when we replace $\tilde L_1, \tilde L_2, \tilde L_3$ by $\tilde L_{1,\epsilon}, \tilde L_{2,\epsilon}, \tilde L_{3,\epsilon}$. So we consider the limit where $\epsilon \to 0$. By [FOh], this limit is described by Morse homotopy. Let us recall it here.

We remark that $L(\tilde L_1, \tilde L_2, \tilde L_3) \simeq V/\tilde L_2$. It is easy to see that there exsits a linear isomorphism $I : V/\tilde L_2 \to \tilde L_1^*$ such that $\hat L_2(v_2)$ is a graph of $df_{\tilde L_2} + I(v_2)$. Note $\tilde L_3 \cap \tilde L_1 = \{q_{3,1}\} = \{0\}$. Let $q_{1,2}(v_2)$, $q_{2,3}(v_2)$ be the intersection of $\hat L_2(v_2)$ with $\tilde L_1$, $\tilde L_3$ respectively. They are the unique critical points of $f_{\tilde L_2} + I(v_2)$ and $f_{\tilde L_3} - f_{\tilde L_2} + I(v_2)$, respectively. Let $U_{1,2}(v_2)$, $U_{2,3}(v_2)$, $U_{3,1}$ be the unstable manifolds of $\mathrm{grad}(f_{\tilde L_2} + I(v_2))$, $\mathrm{grad}(f_{\tilde L_3} - f_{\tilde L_2} + I(v_2))$, and $\mathrm{grad} f_{\tilde L_3}$, respectively. ($U_{1,2}(v_2)$, $U_{2,3}(v_2)$, $U_{3,1}$ are affine since $f_{\tilde L_2} + I(v_2)$ and $f_{\tilde L_3} - f_{\tilde L_2} + I(v_2)$ are quadratic.)

The main theorem of [FOh] says that, when $\epsilon \to 0$, the moduli spaces $\pi(\mathcal M(\tilde L_1, \tilde L_2, \tilde L_3))$ will converge to the space

$$\{v_2 \,|\, U_{1,2}(v_2) \cap U_{2,3}(v_2) \cap U_{3,1} \neq \emptyset\}. \tag{6.17}$$

It is easy to see that (6.17) is a linear subspace of $L(\tilde L_1, \tilde L_2, \tilde L_3) \simeq V/\tilde L_2$ and its codimension is $\eta(\tilde L_1, \tilde L_2, \tilde L_3) = 2n - (\dim U_{1,2} + \dim U_{2,3} + \dim U_{3,1})$. This implies Theorem 6.16. □

§7. Counting pseudoholomorphic polygons (existence).

In this section we prove Theorem α. Namely we prove the existence of the family of integral currents $\mathfrak{C}(\tilde L_{j_1},\cdots,\tilde L_{j_{k+1}})$ satisfying Axioms I,II,III.

We first define several notations. We put:

$$\deg(j_1,\cdots,j_{k+1}) = \eta(L_{j_1},\cdots,L_{j_{k+1}}) + 2 - k, \tag{7.1}$$

$$\Gamma(j_1,\cdots,j_{k+1}) = \prod_{i=1}^{k+1}\Gamma/(\Gamma\cap\tilde L_{j_i}). \tag{7.2}$$

Lemma 7.3. *The index of $\Re Q_{j_1,\cdots,j_{k+1}}$ is $\eta(L_{j_1},\cdots,L_{j_{k+1}}) = \deg(j_1,\cdots,j_{k+1}) + k-2$.*

Proof. The proof is by induction. The case when $k=2$ is Lemma 6.13. On the other hand, (6.9.2) implies that

$$\begin{aligned}\operatorname{Index}\Re Q_{j_1,\cdots,j_{k+1}} = \operatorname{Index}\Re Q_{j_1,\cdots,j_{a-1},j_a,\cdots,j_b,j_{b+1},\cdots,j_{k+1}}\\ +\operatorname{Index}\Re Q_{j_a,\cdots,j_b}.\end{aligned} \tag{7.4}$$

On the other hand by Definition 6.3, we have :

$$\begin{aligned}\eta(L_{j_1},\cdots,L_{j_{k+1}}) = \eta(L_{j_1},\cdots,L_{j_{a-1}},L_{j_a},L_{j_b},L_{j_{b+1}},\cdots,L_{j_{k+1}})\\ +\eta(L_{j_a},\cdots,L_{j_b}).\end{aligned} \tag{7.5}$$

Hence

$$\begin{aligned}\deg(j_1,\cdots,j_k) = \deg(j_1,\cdots,j_{a-1},j_a,\cdots,j_b,j_{b+1},\cdots,j_{k+1})\\ +\deg(j_a,\cdots,j_b)-1.\end{aligned} \tag{7.6}$$

By (7.4),(7.5),(7.6), we can prove the lemma by induction on k. □

Now we start the proof of Theorem α. We construct $\mathfrak{C}(\tilde L_{j_1},\cdots,\tilde L_{j_{k+1}})$ by induction on k. Hereafter we write $\mathfrak{C}(1,\cdots,k+1)$, $\deg(1,\cdots,k+1)$ etc. in place of $\mathfrak{C}(\tilde L_{j_1},\cdots,\tilde L_{j_{k+1}})$, $\deg(j_1,\cdots,j_k)$ etc. for simplicity.

In case when $k+1=3$, we need to find $\mathfrak{C}(1,2,3)$ satisfying Axiom III. Let $d=\deg(1,2,3)$. Lemma 6.12 implies $d\ge 0$.

In case $d=0$ we put $\mathfrak{C}(1,2,3) = L(1,2,3)$.

Let $d>0$. By Lemma 6.12, we can take a codimension d linear subspace $\mathfrak{C}(1,2,3)$ of $L(1,2,3)$ such that Q is positive on it. $\mathfrak{C}(1,2,3)$ thus defined satisfies Axioms III. We will define an orientation of $\mathfrak{C}(1,2,3)$ later during the proof of Lemma 7.9.

We next consider the case when $k+1=4$. Let $\deg(1,2,3,4)=d$. We consider the cycle :

$$\begin{aligned}(-1)^{\mu_1}\left[(\mathfrak{C}(1,2,3)\times\mathfrak{C}(1,3,4))\cap S(Q;1,2,3,4)\right]\\ +(-1)^{\mu_2}\left[(\mathfrak{C}(2,3,4)\times\mathfrak{C}(1,2,4))\cap S(Q;1,2,3,4)\right],\end{aligned} \tag{7.7}$$

where

$$\mu_1 = \deg(1,3,4)+\deg(1,3) \tag{7.8.1}$$

$$\begin{aligned}\mu_2 = \deg(1,2,3)+\deg(2,3,4)+\deg(1,2,4)\\ +\deg(1,2)\deg(2,3,4)+\deg(1,3)+1.\end{aligned} \tag{7.8.2}$$

(7.7) represents an element of $H^{d+1}(S(Q;j_1,j_2,j_3,j_4);\mathbb{Z})\cong\mathbb{Z}$. (Note we regard codimension $d+1$ chain and a current of degree $d+1$.)

Proposition 7.9. (7.7) *represents* 0 *in the homology group.*

Proof. Axiom III implies that the cycles $[(\mathfrak{C}(1,2,3)\times\mathfrak{C}(1,3,4))\cap S(Q;1,2,3,4)]$ and $[(\mathfrak{C}(2,3,4)\times\mathfrak{C}(1,2,4))\cap S(Q;1,2,3,4)]$ both represent the generator of the group $H^d(S(Q;j_1,j_2,j_3,j_4);\mathbb{Z})$ if it is nonempty. (6.9.2) and the definition of $\mathfrak{C}(i,j,k)$ implies that $\mathfrak{C}(2,3,4)\times\mathfrak{C}(1,2,4)$ is nonempty if and only if $\mathfrak{C}(1,2,3)\times\mathfrak{C}(1,3,4)$ is nonempty. Hence (7.7) holds over $\mathbb{Z}_2$.

We next check the sign and show that (7.7) over $\mathbb{Z}$.

For this purpose we give a coorientation on the moduli spaces $\mathfrak{C}(1,2,3)$, etc. The definition of coorientation is done by going to the Morse homotopy limit as in the proof of Theorem 6.16. We regards $\tilde L_i$ as the graph of df_i, where f_i is a quadratic form on $V/\tilde L_0$ and we identify $V=T^*(V/\tilde L_0)$. Let $S(a,b)$ be the stable manifold of $\mathrm{grad}(f_a-f_b)$, which is equal to the unstable manifold of $\mathrm{grad}(f_b-f_a)$. In other words, $S(a,b)$ is the sum of the eigen spaces of f_a-f_b belonging to the positive eigenvalues. We remark that we may regards $\mathfrak{C}(1,2,3)$ as a linear subspace of $V/\tilde L_0$. We will define a co-orientation on $\mathfrak{C}(1,2,3)$ etc. In general, co-orientation of a submanifold $N\subset M$ means a orientation of the normal bundle. We want to regard $\mathfrak{C}(1,2,3)$ as currents so we need to define its coorientation rather than orientation.

We first *fix* co-orientation to each $S(i,j)$. (The co-orientation of $\mathfrak{C}(i,j,k)$ will depend on them.) Hence $S(i,j)$ determines a current. Namely if $e_1^*,\cdots,e_\ell^*$ is an oriented base of the dual of the normal bundle of $S(i,j)$, we identify $S(i,j)$ with a current

$$\delta_{S(i,j)}e_1^*\wedge\cdots\wedge e_\ell^*.$$

Here $\delta_{S(i,j)}$ is a delta function supported at $S(i,j)$. (Delta function is a measure and hence is determined independent of the orientations.)

Note

$$S(1,2)\oplus S(2,3)=\mathfrak{C}(1,2,3)\oplus S(1,3).$$

Now we *define* the coorientation of $\mathfrak{C}(1,2,3)$ so that the following equality holds as currents.

$$S(1,2)\wedge S(2,3)=\mathfrak{C}(1,2,3)\wedge S(1,3). \tag{7.10}$$

Then we have

$$\begin{aligned}S(1,2)\wedge S(2,3)\wedge S(3,4)&=\mathfrak{C}(1,2,3)\wedge S(1,3)\wedge S(3,4)\\&=\mathfrak{C}(1,2,3)\wedge\mathfrak{C}(1,3,4)\wedge S(1,4)\end{aligned}$$

$$\begin{aligned}S(1,2)\wedge S(2,3)\wedge S(3,4)&=S(1,2)\wedge\mathfrak{C}(2,3,4)\wedge S(2,4)\\&=(-1)^{\deg(2,3,4)\eta(1,2)}\mathfrak{C}(2,3,4)\wedge S(1,2)\wedge S(2,4)\\&=(-1)^{\deg(2,3,4)\eta(1,2)}\mathfrak{C}(2,3,4)\wedge\mathfrak{C}(1,2,4)\wedge S(1,4).\end{aligned}$$

Namely we have

$$\mathfrak{C}(1,2,3)\wedge\mathfrak{C}(1,3,4)=(-1)^{\deg(2,3,4)\eta(1,2)}\mathfrak{C}(2,3,4)\wedge\mathfrak{C}(1,2,4). \tag{7.11}$$

Note

$$\deg(2,3,4)+\deg(1,2,4)=\deg(1,2,3)+\deg(1,3,4). \tag{7.12}$$

Proposition 7.9 follows from (7.7),(7.8),(7.11),(7.12). □

By Proposition 7.9, we can choose a chain $\mathfrak{C}(1,2,3,4)$ which satisfies Axiom II.

Now the induction for the general k is as follows. We assume that we have constructed $\mathfrak{C}(1,\cdots,k')$ satisfying Axioms II for $k'\le k$. Let $\deg(j_1,\cdots,j_{k+1})=d$.

Lemma 7.13.

$$d\sum_{\ell,m}\pm\left(\mathfrak{C}(\ell,\cdots,m)\times\mathfrak{C}(1,\cdots,\ell-1,\ell,m,m+1,\cdots,k+1)\right)=0.$$

Proof. We first prove the lemma up to sign. (Namely over $\mathbb{Z}_2$ coefficient.) We will discuss sign later. We also fix the sign $\pm$ in the statement there. We remark that the left hand side is a sum of the terms of the form

$$\mathfrak{C}(a,\cdots,b)\times\mathfrak{C}(\ell,\cdots,a,b,\cdots,m)\times\mathfrak{C}(1,\cdots,\ell,m,\cdots,k+1) \tag{7.14}$$

for $1<\ell<a<b<m<k$. We put $\deg(a,\cdots,b)=d_1$, $\deg(\ell,\cdots,a,b,\cdots,m)=d_2$, $\deg(1,\cdots,\ell,m,\cdots,k+1)=d_3$. We may assume $d_1,d_2,d_3\geq 0$. We consider the following three cases.

Case 1: $d_1+d_2>0$, $d_2+d_3>0$. Since $\deg(\ell,\cdots,m)=d_1+d_2-1$, $\deg(1,\cdots,a,b,\cdots,k+1)=d_2+d_3-1$, it follows that both $\mathfrak{C}(a,\cdots,b)\times\mathfrak{C}(1,\cdots,a-1,a,b,b+1,\cdots,k+1)$ and $\mathfrak{C}(\ell,\cdots,m)\times\mathfrak{C}(1,\cdots,\ell-1,\ell,m,m+1,\cdots,k+1)$ appear in the left hand side of Lemma 7.13 and contains (7.14). Hence the term (7.14) cancels to each other (up to sign.)

Case 2: $d_1+d_2=0$. We apply the induction hypothesis (Axiom II) to $(\ell,\cdots,m)$. We then obtain

$$\sum_{a,b}\pm\mathfrak{C}(a,\cdots,b)\times\mathfrak{C}(\ell,\cdots,a,b,\cdots,m)=0.$$

Hence the sum of such terms in the left hand side of Lemma 7.13 vanishes.

Case 3: $d_2+d_3=0$. The same as Case 2.

Thus we proved Lemma 7.13 up to sign.

Now we are going to discuss the sign in A_∞ formulae. The sign is related to suppersymmetry and is in fact an important matter. We use a trick due to Getzler-Jones [GJ]. We consider a finite set of Lagrangian linear subspaces $\tilde{L}_j$ indexed by $j\in\mathfrak{J}$. We order $\mathfrak{J}$ and let $\mathrm{Ts}(\mathfrak{J})$ be the graded vector space spanned by the symbols $\left[e_{j_1,j_2}|\cdots|e_{j_k,j_{k+1}}\right]$, $j_1<\cdots<j_{k+1}$. (We write it sometimes $[e_{1,2}|\cdots|e_{k,k+1}]$ for simplicity.) We put

$$\begin{aligned}\deg e_{j_1,j_2}&=\eta(\tilde{L}_{j_1},\tilde{L}_{j_2})-1\equiv\deg(j_1,j_2)\mod 2,\\ \deg\left[e_{j_1,j_2}|\cdots|e_{j_k,j_{k+1}}\right]&=\sum\deg e_{j_i,j_{i+1}}=\sum\eta(j_i,j_{i+1})-k.\end{aligned}$$

(Here we shift the degree of $[e_{1,2}|\cdots|e_{k,k+1}]$ by $-k$. This construction, the suspension in the terminology of [GJ], is the main idea to simplify the sign.)

Suppose we have integers $\mathfrak{b}_{j_1,\cdots,j_{k+1}}(=\mathfrak{b}_{1,\cdots,k+1})$ for each $j_1,\cdots,j_{k+1}$ with $\deg(j_1,\cdots,j_{k+1})=0$. We use it to obtain a map

$$[e_{1,2}|\cdots|e_{k,k+1}]\mapsto\mathfrak{b}_{1,\cdots,k+1}[e_{1,k+1}],$$

of degree $+1$. (Note $\eta(1,2)+\cdots+\eta(k,k+1)-\eta(1,k+1)=\eta(1,\cdots,k+1)=k-2$. Here

$$\eta(1,2)+\cdots+\eta(k,k+1)-k+1+\eta(1,k+1)-1.$$

We extend it to a map $\mathfrak{b} : \mathrm{Ts}(\mathfrak{J}) \to \mathrm{Ts}(\mathfrak{J})$ of degree +1 by

$$\begin{aligned}\mathfrak{b}\left[e_{1,2}\left|\cdots\right|e_{k,k+1}\right] = &\sum_{\ell<m}(-1)^{\sum_{i=1}^{\ell-1}\deg e_{i,i+1}} \\ &\mathfrak{b}_{\ell,\cdots,m}\left[e_{1,2}\left|\cdots\right|e_{\ell-1,\ell}|e_{\ell,m}\left|e_{m,m+1}\right|\cdots\left|e_{k,k+1}\right]. \end{aligned} \tag{7.15}$$

Here the sum is taken for all ℓ, m such that $\deg(\ell,\cdots,m) = 0$. (Note that the sign in (7.15) is usual one since degree of $\mathfrak{b}$ is +1.)

We say that $\mathfrak{b}$ is a *derivative* if $\mathfrak{b}\circ\mathfrak{b} = 0$.

Next let $\mathrm{T}(\mathfrak{J})$ be the graded vector space spanned by $\hat e_{j_1,j_2}\otimes\cdots\hat e_{k,k+1}$. Here $\deg \hat e_{j_1,j_2}\otimes\cdots\otimes\hat e_{k,k+1} = \sum\eta(j_i,j_{i+1})$. (We do not shift the degree this time.) We define s by :

$$s\hat e_{1,2}\otimes\cdots\otimes s\hat e_{k,k+1} = \left[e_{1,2}\left|\cdots\right|e_{k,k+1}\right].$$

We then find:

$$(s\otimes\cdots\otimes s)\left(\hat e_{1,2}\otimes\cdots\otimes\hat e_{k,k+1},\right) = (-1)^{\mu(1,\cdots,k+1)}\left[e_{1,2}\left|\cdots\right|e_{k,k+1}\right].$$

where

$$\mu(1,\cdots,k+1) = (k-1)\eta(1,2) + (k-2)\eta(2,3) + \cdots + \eta(k-1,k) + k(k-1)/2.$$

(Note that the sign is determined by the fact that $\deg s = 1$.) We define

$$\mathfrak{c}_k = s^{-1}\circ\mathfrak{b}_k\circ(s\otimes\cdots\otimes s) \tag{7.16}$$

and

$$\mathfrak{c}_k(\hat e_{1,2}\otimes\cdots\otimes\hat e_{k,k+1}) = \mathfrak{c}_{1,\cdots,k+1}\hat e_{1,k+1}. \tag{7.17}$$

By definition $\mathfrak{c}_{1,\cdots,k+1} = (-1)^{\mu(1,\cdots,k+1)}\mathfrak{b}_{1,\cdots,k+1}$. The equation $\mathfrak{b}\circ\mathfrak{b} = 0$ is equivalent to an equation

$$(-1)^{\mu(1,\cdots,k+1;\ell,m)}\mathfrak{c}_{1,\cdots,\ell,m,\cdots,k+1}\wedge\mathfrak{c}_{\ell,\cdots,m} = 0.$$

Here the sum is taken for all ℓ, m such that $\deg(\ell,\cdots,m) = \deg(1,\cdots,\ell,m,\cdots,k+1) = 0$, and

$$\begin{aligned}\mu(1,\cdots,k+1;\ell,m) =&\mu(1,\cdots,k+1) + \mu(\ell,\cdots,m) \\ &+ \mu(1,\cdots,\ell,m,\cdots,k+1) + \sum_{i=1}^{\ell-1}\eta(i,i+1).\end{aligned} \tag{7.18}$$

The sign here is messy and complicated. But in fact we do not need to calculate it so much, since most of the calculation will be done by using $\mathfrak{b}$ in place of $\mathfrak{c}$. (The reason we introduced $\mathfrak{c}$ (and $\mathfrak{m}$) is that the degree coincides with natural one (in sheaf cohomology) for them.)

In order to fix a sign in Axiom II we need to generalize it more and discuss the case when the degree of $\mathfrak{b}$ is not necessary +1. (In other words, the case when the degree of $\mathfrak{c}$ is not necessary 0.) Let $\deg(1,\cdots,k+1) = d$. We consider integral current $\mathfrak{b}^{(d)}_{1,\cdots,k+1}[v_1,\cdots,v_{k+1}]$ of degree d on $L(1,\cdots,k+1)$. Let $\Lambda^{(\lambda)}$ [resp. $\Lambda^{(\lambda)}_{smooth}$] denote the vector space of all degree λ current [resp. smooth differential forms] on $L(1,\cdots,k+1)$. We define

$$\mathfrak{b}^{(d)} : \Lambda^{(\lambda)}_{smooth} \otimes \mathrm{Ts}(J) \to \Lambda^{(\lambda+d)} \otimes \mathrm{Ts}(J)$$

by

$$\begin{aligned} &\mathfrak{b}^{(d)}\,(u \otimes [e_{1\,2}\,|\cdots|e_{k\,k+1}\,]) \\ &= \sum_{\ell<m} (-1)^{\lambda+(d+1)\sum_{i=1}^{\ell-1}\deg(i,i+1)} \left(u \wedge \mathfrak{b}^{(d)}_{\ell,\cdots,m}\right) \otimes \\ &\qquad [e_{1,2}\,|\cdots\,|e_{\ell-1\ell}\,|e_{\ell,m}\,|e_{m,m+1}\,|\cdots\,|e_{k,k+1}\,]. \end{aligned} \tag{7.19}$$

(Note that $\mathrm{Ts}(J)$ degree of $\mathfrak{b}^{(d)}$ is $1-d$ and $\mathrm{Ts}(J)$ degree of $e_{i,j}$ is $\deg(i,j)-1$.) Since the current degree of $\mathfrak{b}^{(d)}$ is d, the total degree of $\mathfrak{b}^{(d)}$ is $+1$ and is odd. We got the sign

$$(-1)^{\lambda+\sum_{i=1}^{\ell-1}\deg(i,i+1)}$$

when we exchange $\mathfrak{b}^{(d)}$ with $u, e_{12},\cdots,e_{\ell-1,\ell}$. We got the sign

$$(-1)^{d\sum_{i=1}^{\ell-1}\deg(i,i+1)}$$

when we exchange $\mathfrak{b}^{(d)}_{1,\cdots,k+1}$ with $e_{1,2},\cdots,e_{\ell-1,\ell}$.

We consider the equation

$$d\mathfrak{b}^{(d)} + \sum_{d_1+d_2=d+1} (-1)^{d_2}\mathfrak{b}^{(d_2)}\circ\mathfrak{b}^{(d_1)} = 0. \tag{7.20}$$

Here d is the exterior derivative on $L(1,\cdots,k+1)$ and we put

$$d\mathfrak{b}^{(d)}(\alpha) = d(\mathfrak{b}^{(d)}(\alpha)) + \mathfrak{b}^{(d)}(d(\alpha)).$$

Lemma 7.21. *(7.20) is equivalent to*

$$\begin{aligned} d\mathfrak{b}^{(d)}_{1,\cdots,k+1} + \sum_{\substack{\ell<m \\ d_1+d_2=d+1}} &(-1)^{d_2+(d_1+1)\sum_{i=1}^{\ell-1}(\deg(i,i+1)+1)} \\ &\mathfrak{b}^{(d_1)}_{\ell,\cdots,m} \wedge \mathfrak{b}^{(d_2)}_{1,\cdots,\ell,m,\cdots,k+1} = 0. \end{aligned} \tag{7.22}$$

Proof. We apply $d\mathfrak{b}^{(d)} + \sum_{d_1+d_2=d+1}(-1)^{d_2}\mathfrak{b}^{(d_2)}\circ\mathfrak{b}^{(d_1)}$ to $[e_{1,2}\,|\cdots|e_{k,k+1}\,]\otimes 1$ and obtain (7.21). On the contrary, if we apply $d\mathfrak{b}^{(d)} + \sum_{d_1+d_2=d+1}(-1)^{d_2}\mathfrak{b}^{(d_2)}\circ\mathfrak{b}^{(d_1)}$ to general $u\otimes[e_{1,2}\,|\cdots|e_{h,h+1}\,]$ we obtain the terms

$$\begin{aligned} &\sum_{\substack{k\le\ell<m\le n \\ n-k>0}} \sum_{d} (-1)^{d(\deg(1,2)+\cdots+\deg(k-1,k))} \\ &\qquad \left(u \wedge \left(d\mathfrak{b}^{(d)}_{k,\cdots,n} + \sum_{\ell<m}(-1)^{d_2+(d_1+1)\sum_{i=k}^{\ell-1}\deg(i,i+1)}\right.\right. \\ &\qquad\qquad \left.\left.\mathfrak{b}^{(d_1)}_{\ell,\cdots,m}\wedge\mathfrak{b}^{(d_2)}_{k,\cdots,\ell,m,\cdots,n}\right)\right). \\ &\qquad \otimes [e_{1,2}\,|\cdots|\,e_{k,n}|\cdots|e_{h,h+1}\,] \end{aligned} \tag{7.23}$$

and

$$\pm\left[e_{\bullet\bullet}\middle|\cdots\middle|\mathfrak{b}^{(d_2)}(e_{\bullet\bullet})\middle|\cdots\middle|\mathfrak{b}^{(d_1)}(e_{\bullet\bullet})\middle|\cdots\middle|e_{\bullet\bullet}\right] \\ \pm\left[e_{\bullet\bullet}\middle|\cdots\middle|\mathfrak{b}^{(d_1)}(e_{\bullet\bullet})\middle|\cdots\middle|\mathfrak{b}^{(d_2)}(e_{\bullet\bullet})\middle|\cdots\middle|e_{\bullet\bullet}\right]. \tag{7.24}$$

(7.23) vanishes by (7.22). (7.24) cancels to each other since the total degree of $\mathfrak{b}^{(d)}$ is odd. The proof of Lemma 7.21 is complete. □

We now put

$$\mathfrak{c}_k^{(d)} = s^{-1}\circ\mathfrak{b}_k^{(d)}\circ(s\otimes\cdots\otimes s). \tag{7.25}$$

We regards our chain $\mathfrak{C}^{(d)}(j_1,\cdots,j_\ell)$ in Axiom III as a degree d integral current. (Note $\mathfrak{C}^{(d)}(j_1,\cdots,j_\ell)$ is defined on $L(j_i,\cdots,j_\ell)$. We pull it back to $L(1,\cdots,k)$. From now on, we write $\mathfrak{C}^{(d)}(j_1,\cdots,j_\ell)$ when we regard it as an integral current and will write $\mathfrak{c}^{(d)}_{j_1,\cdots,j_\ell}:\Lambda^{(\lambda)}\otimes\mathrm{T}(J)\to\Lambda^{(\lambda+d)}\otimes\mathrm{T}(J)$ when we regard it as an operation.

Let $\mathfrak{b}_k^{(d)}$ correspond to our $\mathfrak{C}^{(d)}(j_1,\cdots,j_\ell)=\mathfrak{c}^{(d)}_{j_1,\cdots,j_\ell}$ by (7.25).

Definition 7.26. We choose the sign in Lemma 7.13 so that it is equivalent to (7.20) or (7.22).

To check that the sign in the proof of Theorem α is correct, we proceed as follows. We construct $\mathfrak{b}_k^{(d)}$ by induction on k such that

$$d\mathfrak{b}_k^{(d)}+\sum_{\substack{d_1+d_2=\ell+1\\k_1+k_2=k+1}}(-1)^{d_2}\mathfrak{b}_{k_2}^{(d_2)}\circ\mathfrak{b}_{k_1}^{(d_1)}=0 \tag{7.27}$$

The proof of Theorem α for $k\le 3$ gives $\mathfrak{b}_2^{(d)}$, $\mathfrak{b}_3^{(d)}$. We calculate

$$\begin{aligned}
&d\sum_{\substack{d_1+d_2=\ell+1\\k_1+k_2=k+1}}(-1)^{d_2}\mathfrak{b}_{k_2}^{(d_2)}\circ\mathfrak{b}_{k_1}^{(d_1)}\\
&=\sum_{\substack{d_1+d_2=\ell+1\\k_1+k_2=k+1}}(-1)^{d_2}d\mathfrak{b}_{k_2}^{(d_2)}\circ\mathfrak{b}_{k_1}^{(d_1)}+\sum_{\substack{d_1+d_2=\ell+1\\k_1+k_2=k+1}}\mathfrak{b}_{k_2}^{(d_2)}\circ d\mathfrak{b}_{k_1}^{(d_1)}\\
&=-\sum_{\substack{d_1+d_2=\ell+1,d_3\\k_1+k_2+k_3=k+2}}(-1)^{d_2+d_3}\mathfrak{b}_{k_3}^{(d_3)}\circ\mathfrak{b}_{k_2}^{(d_2-d_3+1)}\circ\mathfrak{b}_{k_1}^{(d_1)}\\
&\qquad-\sum_{\substack{d_1+d_2=\ell+1,d_3\\k_1+k_2+k_3=k+2}}(-1)^{d_3}\mathfrak{b}_{k_2}^{(d_2)}\circ\mathfrak{b}_{k_3}^{(d_3)}\circ\mathfrak{b}_{k_1}^{(d_1-d_3+1)}\\
&=0.
\end{aligned}$$

Thus induction works.

We next check that the choice of sign above coincides with (7.7), (7.8). We have

$$\mathfrak{b}_2[x_{j_1,j_2}|x_{j_2,j_3}]=(-1)^{\eta(j_1,j_2)+1}s\mathfrak{c}_2(\hat{x}_{j_1,j_2}\otimes\hat{x}_{j_2,j_3})$$

(where $s\hat{x}_{j_1,j_2} = x_{j_1,j_2}$), by

$$\begin{aligned}\hat{x}_{j_1,j_2} \otimes \hat{x}_{j_2,j_3} &= (s^{-1} \otimes s^{-1})[x_{j_1,j_2}|x_{j_2,j_3}] \\ &= (-1)^{\eta(j_1,j_2)+1}(s \otimes s)^{-1}[x_{j_1,j_2}|x_{j_2,j_3}].\end{aligned}$$

We then obtain

$$\begin{aligned}&\sum_{d_1,d_2} (-1)^{d_2}\mathfrak{b}_2^{(d_2)} \circ \mathfrak{b}_2^{(d_1)} [x_{1,2}|\, x_{2,3}|\, x_{3,4}] \\ &= \sum (-1)^{\deg(1,3,4)}\mathfrak{b}_2^{(\deg(1,3,4))} \left[\mathfrak{b}_2^{(\deg(1,2,3))} [x_{1,2}|\, x_{2,3}]\ \ |x_{3,4}\right] \\ &\quad + (-1)^{(\deg(2,3,4)+1)(\eta(1,2)+1)+\deg(1,2,4)}\mathfrak{b}_2^{(\deg(1,2,4))} \left[x_{1,2}|\ \ \mathfrak{b}_2^{(\deg(2,3,4))} [x_{2,3}|\, x_{3,4}]\right] \\ &= (-1)^{\eta(1,2)+\mu_1} s\mathfrak{c}_2^{(\deg(1,3,4))}(\mathfrak{c}_2^{(\deg(1,2,3))}(\hat{x}_{1,2}, \hat{x}_{2,3}), \hat{x}_{3,4}) \\ &\quad + (-1)^{\eta(1,2)+\mu_2} s\mathfrak{c}_2^{(\deg(1,2,4))}(\hat{x}_{1,2}, \mathfrak{c}_2^{(\deg(2,3,4))}(\hat{x}_{2,3}, \hat{x}_{3,4})),\end{aligned}$$

where

$$\begin{aligned}\mu_1 &= \deg(1,3,4) + \eta(1,3) \\ \mu_2 &= (\deg(2,3,4)+1)(\deg(1,2)+1) + \deg(1,2,4) + \eta(2,3) \\ &= \eta(1,2)\deg(2,3,4) + \deg(1,2,3) \\ &\qquad + \deg(1,2,4) + \deg(2,3,4) + \eta(1,3) + 1.\end{aligned}$$

(We use $\eta(1,2) + \eta(2,3) = \deg(1,2,3) + \eta(1,3)$ to calculate μ_2.) Thus the sign coincide with one of (7.7), (7.8). The proof of Theorem α is complete. □

§8. Counting pseudoholomorphic polygons (uniqueness).

In this section, we show Theorem β. Namely we prove the uniqueness of the family of currents $\mathfrak{C}(\tilde L_{j_1},\cdots,\tilde L_{j_{k+1}})$ satisfying Axiom I,II,III.

We start with defining the notion that two system of chains $\mathfrak{C}^{(1)}(\tilde L_{j_1},\cdots,\tilde L_{j_{k+1}})$ $\mathfrak{C}^{(2)}(\tilde L_{j_1},\cdots,\tilde L_{j_{k+1}})$ to be homotopy equivalent. Let $\mathfrak{b}_k^{(1)}$ and $\mathfrak{b}_k^{(2)}$ be determined by $\mathfrak{C}^{(1)}(\tilde L_{j_1},\cdots,\tilde L_{j_{k+1}})$, $\mathfrak{C}^{(2)}(\tilde L_{j_1},\cdots,\tilde L_{j_{k+1}})$ in the same way as the last section.

Let $\mathfrak{f}^{(d)}_{j_1,\cdots,j_{k+1}}(=\mathfrak{f}_{\tilde L(j_1,\cdots,j_{k+1})})$ be integral currents of degree d on $L(j_1,\cdots,j_{k+1})$, where $d+1=\deg(j_1,\cdots,j_{k+1})$. We use it to define $\mathfrak{f}^{(d)}=\sum\mathfrak{f}^{(d)}_k:\Lambda^{(\lambda)}_{smooth}\otimes \mathrm{Ts}(J)\to\Lambda^{(\lambda+d)}\otimes\mathrm{Ts}(J)$ by

$$\begin{aligned}&\mathfrak{f}^{(d)}_k\left(u\otimes[e_{1,2}\,|\cdots|e_{k,k+1}]\right)\\ &=\sum_i\sum_{d_1+\cdots+d_{i-1}=d}(-1)^{\mu}\left(u\wedge\mathfrak{f}^{(d_1)}_{a(1),\cdots,a(2)}\wedge\cdots\wedge\mathfrak{f}^{(d_{i-1})}_{a(i-1),\cdots,a(i)}\right)\\ &\qquad\qquad\otimes\left[e_{a(1),a(2)}\,|\cdots\,\left|e_{a(i-1),a(i)}\right.\right],\end{aligned}\tag{8.1.1}$$

where $1=a(1)<a(2)<\cdots<a(i)=k+1$, and

$$\begin{aligned}\mu=&d_{i-1}\left(\eta(1,2)+\cdots+\eta(a(i-1)-1,a(i-1))\right)\\ &+\cdots\cdots+d_2\left(\eta(1,2)+\cdots+\eta(a(2)-1,a(2))\right).\end{aligned}\tag{8.1.2}$$

We note that $\mathrm{Ts}(J)$ degree of $\mathfrak{f}^{(d)}$ is $-d$ and current degree is d. Hence its total degree is 0 and is even.

Let $\mathfrak{f}^{(d)}_\ell$ be the sum of components of (8.1) ($d=\deg(j_1,\cdots,j_{k+1})-1$) such that $i+\ell=k+1$.

Definition 8.2. We say that $\mathfrak{C}^{(1)}(\tilde L_{j_1},\cdots,\tilde L_{j_{k+1}})$ is homotopy equivalent to $\mathfrak{C}^{(2)}(\tilde L_{j_1},\cdots,\tilde L_{j_{k+1}})$ if there exists $\mathfrak{f}^{(d)}_{j_1,\cdots,j_{k+1}}[v_1,\cdots,v_{k+1}]$ such that

(8.3.1) $\mathfrak{f}^{(0)}_{j_1,j_2}$ is the fundamental class.

(8.3.2) $\mathfrak{f}^{(d)}_{j_1,\cdots,j_{k+1}}$ is invariant of the $\mathbb{R}^+$ action $r\cdot[v_1,\cdots,v_{k+1}]=[rv_1,\cdots,rv_{k+1}]$.

(8.3.3) There exists $\delta>0$ such that if $[v_1,\cdots,v_{k+1}]$ is in the support of $\mathfrak{f}^{(d)}_{j_1,\cdots,j_{k+1}}$ then

$$\Re Q(v_1,\cdots,v_{k+1})>\delta\left\|[v_1,\cdots,v_{k+1}]\right\|^2.$$

(8.3.4) Let $d\mathfrak{f}^{(d)}_k=d\circ\mathfrak{f}^{(d)}_k-\mathfrak{f}^{(d)}_k\circ d$. Then, we have

$$d\mathfrak{f}^{(d)}_k+\sum_{\substack{d_1+d_2=d+1\\ k_1+k_2=k+1}}(-1)^{d_2}\mathfrak{f}^{(d_2)}_{k_2}\circ\mathfrak{b}^{1(d_1)}_{k_1}-\sum_{\substack{d_1+d_2=d+1\\ k_1+k_2=k+1}}(-1)^{d_2}\mathfrak{b}^{2(d_2)}_{k_2}\circ\mathfrak{f}^{(d_1)}_{k_1}=0.$$

We will explain in §11 consequences of two system of integral currents to be homotopy equivalent. In a way similar to Lemma 7.22, we can check that (8.34) is equivalent to

$$0 = d\mathfrak{f}^{(d)}_{1,\cdots,k+1} + \sum_{\substack{d_1+d_2=d+1\\ \ell<m}} (-1)^{(d_1+1)(\deg(1,2)+\cdots+\deg(\ell-1,\ell))} \mathfrak{b}^{(d_1)}_{\ell,\cdots,m} \wedge \mathfrak{f}^{(d_2)}_{1,\cdots,\ell,m,\cdots,k+1}$$
$$- \sum_{\substack{1=a(1)<\cdots<a(i)=k+1\\ \sum d_i+d'=d}} (-1)^{\mu} \mathfrak{f}^{(d_1)}_{a(1),\cdots,a(2)} \wedge \cdots \wedge \mathfrak{f}^{(d_{i-1})}_{a(i-1),\cdots,a(i)} \wedge \mathfrak{b}^{(d')}_{a(1)\cdots a(i)},$$

where μ is as in (8.1.2).

Now we start the proof of Theorem β. We solve the equation (8.3.4) by induction on k in the same way as the proof of Theorem α. (8.3.1) gives $\mathfrak{f}^{(0)}_{j_1,j_2}$, that is the case $k = 1$. Then (8.3.4) is automatically satisfied.

In case $k = 2$, (8.3.4) becomes

$$d\mathfrak{f}^{(d)}_2 + \mathfrak{b}^{1(d)}_2 - \mathfrak{b}^{2(d)}_2 = 0, \tag{8.4}$$

in view of (8.3.1).

$\mathfrak{b}^{1(d)}_2$ and $\mathfrak{b}^{2(d)}_2$ determine the same homology class in $S(\tilde L_{j_1}, \tilde L_{j_2}, \tilde L_{j_3})$ by assumption. Hence we have $\mathfrak{f}^{(d)}_2$ satisfying (8.4),(8.3.2),(8.3.3).

Let $k > 2$. We can check, using induction hypothesis, that

$$d\left(\sum_{d_1+d_2=\ell+1} (-1)^{d_2}\mathfrak{f}^{(d_2)}\circ\mathfrak{b}^{1(d_1)} - \sum_{d_1+d_2=\ell+1} (-1)^{d_2}\mathfrak{b}^{2(d_2)}\circ\mathfrak{f}^{(d_1)}\right) = 0.$$

Hence the current

$$\sum_{d_1+d_2=\ell+1} (-1)^{d_2}\mathfrak{f}^{(d_2)}\circ\mathfrak{b}^{1(d_1)} - \sum_{d_1+d_2=\ell+1} (-1)^{d_2}\mathfrak{b}^{2(d_2)}\circ\mathfrak{f}^{(d_1)}$$

defines a De-Rham cohomology class on

$$S(\tilde L_{j_1}, \cdots, \tilde L_{j_{k+1}}) = \{[v_1, \cdots, v_{k+1}] | Q([v_1, \cdots, v_{k+1}]) > 0, \|[v_1, \cdots, v_{k+1}]\| = 1\}.$$

We can check the degree of it using Lemma 6.13, and show the De-Rham cohomology group of $S(\tilde L_{j_1}, \cdots, \tilde L_{j_{k+1}})$ of degree $d = \deg(j_1, \cdots, j_{k+1})$ vanishes. Therefore, by induction, we find $\mathfrak{f}^d_k$. The proof of Theorem β is now complete. □

§9. Counting pseudoholomorphic polygons (chamber structure).

In this section, we show by an example how Axioms I,II,III can be used to find the chain $\mathfrak{C}(1,\cdots,k+1)$ in the case when $\deg(1,\cdots,k+1)=0$ and $k+1=4,5,6$.

We first consider the case $k+1=4$ and $\deg(1,2,3,4)=0$. (For example $\eta(1,2)=1$, $\eta(i,j)=0$ for other $i<j$.) We have

$$d\mathfrak{C}(1,2,3,4)=\pm(\mathfrak{C}(1,3,4)\times\mathfrak{C}(1,2,3))\pm(\mathfrak{C}(1,2,4)\times\mathfrak{C}(2,3,4))$$

Note $\deg(1,3,4)+\deg(1,2,3)=1$. In case $\deg(1,3,4)=1$, we may choose $\mathfrak{C}(1,3,4)$ as a codimension 1 linear subspace of $L(1,3,4)$. Hence $\mathfrak{C}(1,3,4)\times\mathfrak{C}(1,2,3)$ is a codimension 1 linear subspace of $L(1,2,3,4)$. The other case and other term $\mathfrak{C}(1,2,4)\times\mathfrak{C}(2,3,4)$ can also be chosen to be a codimension 1 linear subspace. On the other hand, the index of $\Re Q$ on $L(1,2,3,4)$ also is 1 by Lemma 7.3. Hence we have the following Figure 6.

Figure 6

Here $\Re Q<0$ on A. We have

$$\mathfrak{C}(1,2,3,4)=\pm(B-C).$$

In the case when $k+1=5$, there are several possibilities according to the Maslov index. We first consider the case $\eta(1,2)=2$, $\eta(i,j)=0$ for other $i<j$. (Note then $\deg(1,2,3,4,5)=0$.) We find that $\overline{\mathfrak{C}}(i,j,k,\ell)$ is of negative virtual codimension (and hence is empty), except $\mathfrak{C}(1,2,3,4)$, $\mathfrak{C}(1,2,3,5)$, $\mathfrak{C}(1,2,4,5)$. Hence

$$\begin{aligned} d\mathfrak{C}(1,2,3,4,5)=&\pm(\mathfrak{C}(1,2,3,4)\times\mathfrak{C}(1,4,5))\pm(\mathfrak{C}(1,2,3,5)\times\mathfrak{C}(3,4,5))\\ &\pm(\mathfrak{C}(1,2,4,5)\times\mathfrak{C}(2,3,4)).\end{aligned}$$

We remark also that index of $\Re Q$ is 2 on $L(1,2,3,4,5)$ in this case. Therefore combinatorially $\mathfrak{C}(1,2,4,5)$ looks like as the following Figure 7.

Figure 7

Note

$$\mathfrak{C}(1,2,5)\times\mathfrak{C}(2,4,5)\times\mathfrak{C}(2,3,4)=\mathfrak{C}(1,2,5)\times\mathfrak{C}(2,3,5)\times\mathfrak{C}(3,4,5)$$

holds because $\mathfrak{C}(2,4,5)=L(2,4,5)$, $\mathfrak{C}(2,3,4)=L(2,3,4)$, $\mathfrak{C}(2,3,5)=L(2,3,5)$, $\mathfrak{C}(3,4,5)=L(3,4,5)$.

We remark however that Figure 7 is combinatorial or topological picture. Namely faces $\mathfrak{C}(1,2,3,4)\times\mathfrak{C}(1,4,5)$ etc. are not linear in this case. In fact, let us consider the case of $n=2$. Then $L(1,2,3,4)=\mathbb{R}^4$ and $\mathfrak{C}(1,3,4)\times\mathfrak{C}(1,2,3)\cong\mathbb{R}^2$. Hence $\partial\mathfrak{C}(1,2,3,4)$ is a union of two $\mathbb{R}^2$'s in $\mathbb{R}^4$. It is impossible to find a chain $\mathfrak{C}(1,2,3,4)$ contained in a single (flat) hyperplain. (We can take it as a union of two flat 3 dimensional sectors.)

Let us consider other cases of $k+1=5$. Here we take the negative eigenspace $L(-)$ of $\Re Q$ and draw the figure of the intersection of $\mathfrak{C}(1,2,3,4,5)$ with a 2 dimensional plain parallel to $L(-)$.

If $\eta(1,2)=\eta(2,3)=1$ and $\eta(i,j)=0$ for other $i<j$, then 4 of $\mathfrak{C}(j_1,j_2,j_3,j_4)$'s can appear in $\partial\mathfrak{C}(1,2,3,4,5)$ and we find Figure 8-1.

If $\eta(1,2) = \eta(3,4) = 1$ and $\eta(i,j) = 0$ for other $i < j$ we have Figure 8-2.

Figure 8

Let us next study the case when $k+1 = 6$. In this case, index of $\Re Q$ on $L(1,2,3,4,5,6)$ is 3. We take a 3 dimensional subspace $L(-)$ and are going to draw the figures of the intersection of $\mathfrak{C}(1,2,3,4,5,6)$ with a 3 dimensional plain parallel to $L(-)$.

Let us consider first the case $\eta(1,2) = 3$, $\eta(i,j) = 0$ for other $i < j$. Then $\partial\mathfrak{C}(1,2,3,4,5,6)$ is a union of 4 faces and looks like

Figure 9

Next we consider the case $\eta(1,2) = \eta(3,4) = \eta(5,6) = 1$, $\eta(i,j) = 0$ for other $i < j$. The we find that $\partial\mathfrak{C}(1,2,3,4,5,6)$ looks like

Figure 10

This is a cell Stasheff introduced to study A_∞-structure in [St1].

Finally we consider the case $\eta(1,2) = \eta(3,4) = \eta(4,5) = 1$, $\eta(i,j) = 0$ for other $i < j$. We then find that the following figure :

Figure 11

Here the shaded region can belong any one of $\mathfrak{C}(1,2,3,6)\times\mathfrak{C}(3,4,5,6)$, $\mathfrak{C}(1,2,3,4,5)\times \mathfrak{C}(5,6,1)$, $\mathfrak{C}(1,2,3,4)\times\mathfrak{C}(4,5,6,1)$. Note that we need to use the formula

$$\pm[\mathfrak{C}(1,5,6)\times\mathfrak{C}(1,2,3,5)] \pm [\mathfrak{C}(2,3,5)\times\mathfrak{C}(1,2,5,6)] \pm [\mathfrak{C}(3,5,6)\times\mathfrak{C}(1,2,3,6)]$$

to draw Figure 11.

§10. Multi theta function.

In §§6,7,8,9, we have been working on the universal cover $V = \tilde T^{2n}$ of the torus. We start working on the symplectic torus (T^{2n}, Ω) itself in this section.

Let $\tilde L_i \subset V$ be Lagrangian linear subspaces. We assume that they are mutually transversal and $\tilde L_i \cap \Gamma \cong \mathbb{Z}^n$. We also choose $\tilde M_i$ which is transversal to $\tilde L_i$ and which satisfies $\tilde M_i \cap \Gamma \cong \mathbb{Z}^n$. We consider the moduli space $\mathcal{M}(\tilde L_i; M_i)$ constructed in §5 and its direct product $\prod \mathcal{M}(\tilde L_i; M_i)$. Let $\pi_{i,j} : \prod \mathcal{M}(\tilde L_i; M_i) \to \mathcal{M}(\tilde L_i; M_i) \times \mathcal{M}(\tilde L_j; M_j)$ be the projection.

In §5 we constructed universal bundles $\mathcal{P}(\tilde L_i, \tilde L_j; M_i, M_j)$. We pull it back to $\prod \mathcal{M}(\tilde L_i; M_i)$. We are going to construct a (distribution valued) homomorphism

$$\begin{aligned}\mathfrak{m}_k^{(0,d)} : \pi_{1,2}^*\mathcal{P}(\tilde L_1, \tilde L_2; M_1, M_2) \otimes \cdots \otimes \pi_{k,k+1}^*\mathcal{P}(\tilde L_k, \tilde L_{k+1}; M_k, M_{k+1}) \\ \to \Lambda^{(0,d)}(\mathcal{M}(\tilde L_1; M_1) \times \cdots \times \mathcal{M}(\tilde L_{k+1}; M_{k+1})) \\ \otimes \pi_{1,k+1}^*\mathcal{P}(\tilde L_1, \tilde L_{k+1}; M_1, M_{k+1}).\end{aligned} \tag{10.1}$$

Namely it will be a $(0, d)$ current section of appropriate hom bundle on $\mathcal{M}(\tilde L_1; M_1) \times \mathcal{M}(\tilde L_{k+1}; M_{k+1})$.

Let $\mathbf{v}_i \in \mathcal{M}(\tilde L_i; M_i)$. $\mathbf{v} = (\mathbf{v}_1, \cdots, \mathbf{v}_{k+1})$. Let (v_i, α_i) be a representative of $\mathbf{v}_i$. Set $\vec v = (v_1, \cdots, v_{k+1})$. We put $p_{i,j}(\vec v) = \hat L_i(v_i) \cap \hat L_j(v_j)$.

Definition 10.2.

$$H(\alpha_1, \cdots, \alpha_{k+1}; v_1, \cdots, v_{k+1}) = \sum_{i=1}^{k+1} \alpha_i(p_{i,i+1}(\vec v) - (p_{i-1,i}(\vec v)))$$

Let $q_{i,j} \in L_i(v_i) \cap L_j(v_j)$. We put

$$V(q_{1,2} \cdots q_{k,k+1}, q_{k+1,1})(\mathbf{v}) = \left\{ \vec v = (v_1', \cdots, v_{k+1}') \;\middle|\; \begin{array}{r} [p_{1,2}(\vec v)] = q_{1,2}, \\ \cdots, \\ [p_{k,k+1}(\vec v)] = q_{k,k+1}, \\ [p_{k+1,1}(\vec v)] = q_{k+1,1}. \end{array} \right\}$$

There exists a subgroup $\Gamma(\tilde L_1, \cdots, \tilde L_{k+1})$ of $\prod \Gamma/(\Gamma \cap \tilde L_i)$ such that

$$\left[\prod \Gamma/(\Gamma \cap \tilde L_i) : \Gamma(\tilde L_1, \cdots, \tilde L_{k+1})\right] = \prod L_i \bullet L_{i+1},$$

and that $\Gamma(\tilde L_1, \cdots, \tilde L_{k+1})$ acts transitively on $V(q_{1,2} \cdots q_{k,k+1} q_{k+1,1})$. We embed $V \subset \tilde L(\tilde L_1, \cdots, \tilde L_{k+1})$ and put $\overline\Gamma(\tilde L_1, \cdots, \tilde L_{k+1}) = V \cap \Gamma(\tilde L_1, \cdots, \tilde L_{k+1})$. Let

$$\overline V(\tilde L_1, \cdots, \tilde L_{k+1}) = \frac{V(q_{1,2} \cdots q_{k,k+1} q_{k+1,1})}{\overline\Gamma(\tilde L_1, \cdots, \tilde L_{k+1})}.$$

We then define :

Definition 10.3.

$$\begin{aligned}\Theta_k\left(\mathbf{v};\Omega\right)_{q_{1,2}\cdots q_{k,k+1}q_{k+1,1}} \\ = \sum_{[\vec v]\in \overline{V}(q_{1,2}\cdots q_{k,k+1}q_{k+1,1})(\mathbf{v})} \mathfrak{c}_k^{(d)}[v_1,\cdots,v_{k+1}] \\ \exp(-2\pi Q(v_1,\cdots,v_{k+1}) \\ + 2\pi\sqrt{-1}H(\alpha_1,\cdots,\alpha_{k+1};v_1,\cdots,v_{k+1})).\end{aligned} \tag{10.4}$$

Let us clarify the notation $\mathfrak{c}_k^{(d)}(v_1,\cdots,v_{k+1})$ we used in (10.4).

We fix $\mathbf{v}_i^0 \in \mathcal{M}(\tilde L_i;M_i)$ and consider a small neighborhood $\mathcal{U}_i$ of it only. We fix a lift (v_i^0,α_i^0) of $\mathbf{v}_i^0$. We can find a representative $(v_i(\mathbf{v}_i),\alpha_i(\mathbf{v}_i))$ of $\mathbf{v}_i$ which is close to (v_i^0,α_i^0) for $\mathbf{v}_i \in \mathcal{U}_i$. Then, for each $(\gamma_1,\cdots,\gamma_{k+1}) \in \Gamma(\tilde L_1,\cdots,\tilde L_{k+1})$, the map

$$(\mathbf{v}_1,\cdots,\mathbf{v}_{k+1}) \mapsto (\gamma_1,\cdots,\gamma_{k+1})\cdot[v_1(\mathbf{v}_1),\cdots,v_{k+1}(\mathbf{v}_{k+1})]$$

is a smooth enbedding

$$\prod \mathcal{U}_i \to L(\tilde L_1,\cdots,\tilde L_{k+1}).$$

We pull back the (integral) current $\mathfrak{C}(\tilde L_1,\cdots,\tilde L_{k+1})$ by this map and denote it by $\mathfrak{c}_k^{(0,d)}[v_1,\cdots,v_{k+1}]$ where

$$[v_1,\cdots,v_{k+1}] = (\gamma_1,\cdots,\gamma_{k+1})\cdot[v_1(\mathbf{v}_1),\cdots,v_{k+1}(\mathbf{v}_{k+1})].$$

Thus each term of (10.4) makes sense as a current on $\prod \mathcal{U}_i$.

We next discuss the convergence of (10.4). Let us fix a positive definite quadratic form on $L(\tilde L_1,\cdots,\tilde L_{k+1})$ and $B_D(\tilde L_1,\cdots,\tilde L_{k+1})$ denote the metric ball of radius D centered at 0 of $L(\tilde L_1,\cdots,\tilde L_{k+1})$.

We now prove :

Proposition 10.5.

$$\begin{aligned}\lim_{D\to\infty} \sum_{\vec v\in V(q_{1,2}\cdots q_{k,k+1}q_{k+1,1})(\mathbf{v})\cap B_D(\tilde L_1,\cdots,\tilde L_{k+1})} \mathfrak{c}_k^{(d)}[v_1,\cdots,v_{k+1}] \\ \exp(-2\pi Q(v_1,\cdots,v_{k+1}) + 2\pi\sqrt{-1}H(\alpha_1,\cdots,\alpha_{k+1};v_1,\cdots,v_{k+1})).\end{aligned}$$

converges as current on $\prod \mathcal{M}(\tilde L_i;M_i)$.

Proof. Let α be a smooth form on $\prod \mathcal{M}(\tilde L_i;M_i)$ of codegree d. Then by Axiom (I.2), we have

$$\int_{B_D(\tilde L_1,\cdots,\tilde L_{k+1})} \mathfrak{C}_k^{(d)}] \wedge \pi^*\alpha < CD^{n(k-1)},$$

where $\pi : V(q_{1,2}\cdots q_{k,k+1}q_{k+1,1}) \to \bar V(q_{1,2}\cdots q_{k,k+1}q_{k+1,1})$ is the projection. On the other hand, if $[v_1,\cdots,v_{k+1}] \notin B_D(\tilde L_1,\cdots,\tilde L_{k+1})$ then the C^k norm of

$$\exp\left(-2\pi Q(v_1,\cdots,v_{k+1}) + 2\pi\sqrt{-1}H(\alpha_1,\cdots,\alpha_{k+1};v_1,\cdots,v_{k+1})\right)$$

is smaller than $C_k\exp(-D/C_k)$ for some C_k independent of D. (This fact follows from Axiom (I.3).) The proposition follows immediately. □

The following function equality is obvious from definition.

Lemma 10.6.

$$\begin{aligned}\Theta_k(\mathbf{v}+(\gamma_1,\cdots,\gamma_n;\mu_1,\cdots,\mu_n);\Omega)_{q_{1,2}\cdots q_{k,k+1}q_{k+1,1}} \\ = \prod_i \exp(2\pi\sqrt{-1}\mu_i(p_{i,i+1}(\mathbf{v})-(p_{i-1,i}(\mathbf{v})))) \cdot \Theta_k\left(\mathbf{v};\Omega\right)_{q_{1,2}\cdots q_{k,k+1}q_{k+1,1}}.\end{aligned}$$

Lemma 10.6 and (5.5) implies the well definedness of the following :

Definition 10.7.

$$\mathfrak{m}_k^{(d)}([q_{1,2}]\otimes\cdots\otimes[q_{k-1,k}]) = \sum_{q_{k+1,1}} \Theta_k\left(\mathbf{v};\Omega\right)_{q_{1,2}\cdots q_{k,k+1}q_{k+1,1}}[q_{1,k+1}]$$

$\mathfrak{m}_k^{(d)}$ gives a distribution map (10.1). We let $\mathfrak{m}_k^{(0,d)}$ be the type $(0,d)$ part of $\mathfrak{m}_k^{(0,d)}$.

§11. Maurer-Cartan equation.

The main purpose of this section is to prove Theorem γ. To clarify the idea we first study the simplest case $k+1=3$, $d=0$. Then, $\mathfrak{m}_2^{(0)}$ is a homomorphism :

$$\mathfrak{m}_2^{(0)} : \pi_{1,2}^*\mathcal{P}(\tilde L_1,\tilde L_2;M_1,M_2)\otimes\pi_{2,3}^*\mathcal{P}(\tilde L_2,\tilde L_3;M_2,M_3)\to\pi_{1,3}^*\mathcal{P}(\tilde L_1,\tilde L_3;M_1,M_3).$$

In this case, Theorem γ means that $\mathfrak{m}_2^{(0)}$ is holomorphic. Let us prove it first.

By Axiom III

$$\mathfrak{c}_2^{(0)}(v_1,v_2,v_3)\equiv 1.$$

Hence

$$\Theta_2(\mathbf{v};\Omega)_{q_{1,2}q_{2,3}q_{3,1}} = \sum_{[\vec v]\in\overline{V}(q_{1,2}q_{2,3}q_{3,1})}\exp(-2\pi Q(v_1,v_2,v_3)+ 2\pi\sqrt{-1}H(\alpha_1,\alpha_2,\alpha_3;v_1,v_2,v_3)). \tag{11.1}$$

To simplify the notation, we fix (v_1,α_1), (v_3,α_3) and prove the holomorphicity with respect to the second factor (v_2,α_2) only. Let $x=q_{1,2}$, $y=q_{2,3}$, $z=q_{3,1}$, and $\tilde x,\tilde y,\tilde z$ be their lifts. If we move v_2 to v_2', then $\tilde x,\tilde y$ will move to $\tilde x',\tilde y'$ respectively. (See Figure 12.)

Figure 12

We define $f(v_2',\alpha_2')$ and $h(v_2',\alpha_2')$ by :

$$\begin{aligned}\mathfrak{m}_2([x'],[y']) &= \sum f(v_2',\alpha_2')[z]\\ h(v_2',\alpha_2') &= f(v_2',\alpha_2')\times e_{((v_1,\alpha_1),(v_2,\alpha_2),\tilde p_1,\tilde p_2,\tilde x)}((v_1,\alpha_1),(v_2',\alpha_2'))^{-1}\\ &\qquad\times e_{((v_2,\alpha_2),(v_3,\alpha_3),\tilde p_2,\tilde p_3,\tilde z)}((v_2',\alpha_2'),(v_3,\alpha_3)).\end{aligned}$$

It suffices to show that $h(v_2',\alpha_2')$ is holomorphic with respect to (v_2',α_2'). Let $\tilde x'(\gamma)=\tilde L_2(v_2'+\gamma)\cap\tilde L_1(v_1)$, $\tilde y'(\gamma)=\tilde L_2(v_2'+\gamma)\cap\tilde L_3(v_3)$. We put

$$\begin{aligned}f_\gamma(v_2',\alpha_2') =& -2\pi Q(\tilde x'(\gamma),\tilde y'(\gamma),\tilde z)\\ &+2\pi\sqrt{-1}\left(\alpha_2'(\tilde y'(\gamma)-\tilde x'(\gamma))+\alpha_3(\tilde z-\tilde y'(\gamma))+\tilde\alpha_1(\tilde x'(\gamma)-\tilde z)\right).\end{aligned}$$

Then

$$f(v_2',\alpha_2')=\sum_\gamma\exp f_\gamma(v_2',\alpha_2'). \tag{11.2}$$

On the other hand,

$$\begin{aligned}&\log e_{((v_1,\alpha_1),(v_2,\alpha_2),\tilde p_1,\tilde p_2,\tilde x)}((v_1,\alpha_1),(v_2',\alpha_2'))\\ &=-2\pi Q(\tilde p_2,\tilde x,\tilde x',\tilde p_2')+2\pi\sqrt{-1}\left(\alpha_2(\tilde x-\tilde p_2)+\alpha_1(\tilde x'-\tilde x)+\alpha_2'(\tilde p_2'-\tilde x')\right),\end{aligned} \tag{11.3}$$

$$\begin{aligned}&\log e_{((v_2,\alpha_2),(v_3,\alpha_3),\tilde p_2,\tilde p_3,\tilde z)}((v_2',\alpha_2'),(v_3,\alpha_3))\\ &=-2\pi Q(\tilde y,\tilde p_2,\tilde p_2',\tilde y')+2\pi\sqrt{-1}\left(\alpha_2(\tilde y-\tilde p_2)+\alpha_2'(\tilde p_2'-\tilde y')+\alpha_3(\tilde y'-\tilde y)\right).\end{aligned} \tag{11.4}$$

Then we have

$$f_\gamma(v_2', \alpha_2') + (11.3) - (11.4) = 2\pi I_{c(\gamma)}(v_2' - v_2),$$

where

$$c(\gamma) = (\tilde{y}'(\gamma) - \tilde{x}'(\gamma)) - (\tilde{y}' - \tilde{x}').$$

Figure 13

Therefore, by (11.2), $h(v_2', \alpha_2')$ is holomorphic. $\square$

Proof of Theorem γ. We use the same notation as we introduced just after Definition 10.3. We put $(v_1, \cdots, v_{k+1}) = (\gamma_1, \cdots, \gamma_{k+1}) \cdot (v_1(\mathbf{v}_1), \cdots, v_{k+1}(\mathbf{v}_{k+1}))$. where $\mathbf{v}_i \in \mathcal{U}_i$.

Lemma 11.5. $\exp(-2\pi Q(v_1, \cdots, v_{k+1}) + 2\pi\sqrt{-1}H(\alpha_1, \cdots, \alpha_{k+1}; v_1, \cdots, v_{k+1}))$ *is a holomorhpic function of* $\mathbf{v}_i \in \mathcal{U}_i$.

The proof of Lemma 11.5 is similar to the proof of Theorem γ in the case $k = 2$, $d = 0$ we discussed above. Hence we omit it.

Now by Axiom II we have

$$\begin{aligned} &d\mathfrak{c}_k^{(d)}(v_1, \cdots, v_{k+1}) \\ &= \sum_{1 \le a < b \le k+1} \pm \mathfrak{c}_{b-a}^{(d_1)}(v_a, \cdots, v_b) \wedge \mathfrak{c}_{k-b+a+1}^{(d_2)}(v_1, \cdots, v_a, v_b, \cdots, v_{k+1}) \end{aligned}$$

(Here the sign is as in §7.) We take its $(0, d+1)$ part and obtains

$$\begin{aligned} &\overline{\partial}\mathfrak{c}_k^{(0d+1)}(v_1, \cdots, v_{k+1}) \\ &= \sum_{1 \le a < b \le k+1} \pm \mathfrak{c}_{b-a}^{(0,d_1)}(v_a, \cdots, v_b) \wedge \mathfrak{c}_{k-b+a+1}^{(0,d_2)}(v_1, \cdots, v_a, v_b, \cdots, v_{k+1}). \end{aligned} \tag{11.6}$$

Theorem γ follows from Lemma 11.5 and (11.6) immediately. $\square$

We next show that the system of maps $\mathfrak{m}_*$ changes by homotopy equivalence when we change the choice of the currents $\mathfrak{C}(\tilde{L}_{j_1}, \cdots, \tilde{L}_{j_{k+1}})$.

We begin with defining homotopy equivalence between two A_∞ structures. Let $\mathfrak{m}_k^{(0,d)}$, $\mathfrak{m}_k'^{(0,d)}$ be families of operations both of which satisfy the Maurer-Cartan equations (0.26). We consider maps

$$\begin{aligned} &\mathfrak{h}_k^{(0,d)} : \pi_{1,2}^*\mathcal{P}(\tilde{L}_1, \tilde{L}_2; M_1, M_2) \otimes \cdots \otimes \pi_{k,k+1}^*\mathcal{P}(\tilde{L}_k, \tilde{L}_{k+1}; M_k, M_{k+1}) \to \\ &\quad \Lambda^{(0,d)}(\mathcal{M}(\tilde{L}_1; M_1) \times \mathcal{M}(\tilde{L}_{k+1}; M_{k+1})) \otimes \pi_{1,k+1}^*\mathcal{P}(\tilde{L}_1, \tilde{L}_{k+1}; M_1, M_{k+1}). \end{aligned} \tag{11.7}$$

Definition 11.8. We say $\mathfrak{h}_k^{(0,d)}$ is a *homotopy* between $\mathfrak{m}_k^{(0,d)}$ and $\mathfrak{m}_k'^{(0,d)}$ if

$$\sum_{\substack{d_1+d_2=d\\k_1+k_2=k}} (\mathfrak{m}_k^{(0,d_1)}\circ\mathfrak{h}_k^{(0,d_2)}\pm\mathfrak{h}_k^{(0,d_1)}\circ\mathfrak{m}_k'^{(0,d_2)})\pm\left(\overline{\partial}\circ\mathfrak{h}_k^{(0,d-1)}\pm\mathfrak{h}_k^{(0,d-1)}\circ\overline{\partial}\right)=0. \tag{11.9}$$

Here the sign $\pm$ is determined as follows. We first move from $\mathfrak{c}$ to $\mathfrak{b}$ as in §7. We then define $\tilde{\mathfrak{m}}$ using $\mathfrak{b}$ instead of $\mathfrak{c}$. (In other words, we define

$$s^{-1}\circ\tilde{\mathfrak{m}}_k\circ(s\otimes\cdots\otimes s)=\mathfrak{m}_k.) \tag{11.10}$$

In the same way we replace $\mathfrak{h}$ by $\tilde{\mathfrak{h}}$. We next consider the Bar complex

$$\mathfrak{B}=\bigoplus_{j_*,k}\pi_{j_1j_2}^*\mathcal{P}(\tilde{L}_{j_1},\tilde{L}_{j_2};M_{j_1},M_{j_2})\otimes\cdots\otimes\pi_{j_kj_{k+1}}^*\mathcal{P}(\tilde{L}_{j_k},\tilde{L}_{j_{k+1}};M_{j_k},M_{j_{k+1}}).$$

Then $\overline{\partial}$, $\tilde{\mathfrak{m}}$, $\tilde{\mathfrak{m}}'$, $\tilde{\mathfrak{h}}$ defines maps

$$\bigoplus_d\Gamma(\Lambda^{(0,d)})\otimes\mathfrak{B}\to\bigoplus_d\Gamma(\Lambda^{(0,d)})\otimes\mathfrak{B}.$$

Let $\tilde{\partial}$, $\tilde{\partial}'$, $\tilde{\mathfrak{H}}$ be obtained from $\tilde{\mathfrak{m}}$, $\tilde{\mathfrak{h}}$ respectively. Namely

$$\begin{aligned}\tilde{\partial}(u\otimes x_1\otimes\cdots\otimes x_k)=&\sum_{1\le\ell<\ell+2\le\ell+m\le k}(-1)^{\deg x_1+\cdots+\deg x_{\ell-1}+\deg u}\\&\quad u\otimes x_1\otimes\cdots\otimes\tilde{\mathfrak{m}}_m^{(0,d)}(x_\ell,\cdots,x_{\ell+m-1})\otimes x_m\otimes\cdots\otimes x_k\\&+\overline{\partial}u\otimes x_1\otimes\cdots\otimes x_k,\end{aligned}$$

$$\begin{aligned}\tilde{\mathfrak{H}}(u\otimes x_1\otimes\cdots\otimes x_k)=&\sum_{1\le\ell<\ell+1\le\ell+m\le k}(-1)^{\deg x_1+\cdots+\deg x_{\ell-1}+\deg u}\\&\quad u\otimes x_1\otimes\cdots\otimes\tilde{\mathfrak{h}}_m^{(0,d-1)}(x_\ell,\cdots,x_{\ell+m-1})\otimes x_m\otimes\cdots\otimes x_k,\end{aligned}$$

where $d=\deg(x_{j_\ell,j_{\ell+1}})+\cdots+\deg(x_{j_{\ell+m-2},j_{\ell+m-1}})-\deg(x_{j_\ell,j_{\ell+m-1}})+2-m$. We remark that $\tilde{\partial}\circ\tilde{\partial}=0$ is the Maurer-Cartan equation (0.25). Moreover (11.9) is equivalent to

$$\tilde{\partial}\circ\tilde{\mathfrak{H}}=\tilde{\mathfrak{H}}\circ\tilde{\partial}', \tag{11.11}$$

up to sign. So we choose the sign in (11.9) so that it will become (11.11) together with the sign.

Now we prove the following :

Theorem 11.12. *Let $\mathfrak{C}(\tilde{L}_{j_1},\cdots,\tilde{L}_{j_{k+1}})$ $\mathfrak{C}'(\tilde{L}_{j_1},\cdots,\tilde{L}_{j_{k+1}})$ be families of integral currents satisfying Axioms I,II,III. Let they determine $\mathfrak{m}_k^{(0,d)}$, $\mathfrak{m}_k'^{(0,d)}$. We assume that $\mathfrak{C}(\tilde{L}_{j_1},\cdots,\tilde{L}_{j_{k+1}})$ is homotopy equivalent to $\mathfrak{C}'(\tilde{L}_{j_1},\cdots,\tilde{L}_{j_{k+1}})$ in the sense of Definition 8.2. Then $\mathfrak{m}_k^{(0,d)}$ is homotopy equivalent to $\mathfrak{m}_k'^{(0,d)}$ in the sense of Definition 11.8.*

Proof. The proof is quite similar to the proof of Theorem γ. Let $\mathfrak{f}_{j_1,\cdots,j_{k+1}}^{(d)}[v_1,\cdots,v_{k+1}]$ be as in Definition 11.8. We use it in the same was as we used $\mathfrak{c}_{\tilde{L}_{j_1},\cdots,\tilde{L}_{j_{k+1}}}$

in Definition 10.7 and obtain $\mathfrak{h}_k^{(0,d)}$. In fact the convergence of the corresponding power series follows from (8.3.4). Then we use (8.3.1) and (8.3.5) in the same way as the proof of Theorem γ to verify (11.9). The proof of Theorem 11.12 is complete. □

We remark that in our case $\mathfrak{m}_1$ is zero. Hence the Floer homology group does not change when we replace $\mathfrak{C}$ by $\mathfrak{C}'$. However the operator $\mathfrak{m}_2$ may change. The following is an easy consequence of Theorem 11.12.

Corollary 11.13. $\mathfrak{m}_2$ *and* $\mathfrak{m}_2'$ *gives the same map on cohomology.*

Proof. (11.9) implies

$$\mathfrak{m}_2 - \mathfrak{m}_2' = \pm\overline{\partial} \circ \mathfrak{h}_2 \pm \mathfrak{h}_2 \circ \overline{\partial}.$$

(Note $\mathfrak{h}_2 = id$.) Therefore $\mathfrak{m}_2$ is chain homotopic to $\mathfrak{m}_2'$. □

Furthermore Theorem 11.2 implies coincidense of various secondary operators. We consider the case $k = 3$. We assume

$$\mathfrak{m}_2(x_{12}, x_{23}) = \mathfrak{m}_2(x_{23}, x_{34}) = 0.$$

This is the situation where we can define Massey triple product. In our case, it is represented by $\mathfrak{m}_3(x_{12}, x_{23}, x_{34})$ or $\mathfrak{m}_3'(x_{12}, x_{23}, x_{34})$. Then, Theorem 11.12 and (11.9) imply

$$\begin{aligned} &\mathfrak{m}_3(x_{12}, x_{23}, x_{34}) - \mathfrak{m}_3'(x_{12}, x_{23}, x_{34}) \\ &\qquad = \pm\, \mathfrak{m}_2(x_{12}, \mathfrak{h}_2(x_{23}, x_{34})) \pm \mathfrak{m}_2(\mathfrak{h}_2(x_{12}, x_{23}), x_{34}) \\ &\qquad\qquad + \bar{\partial}(\mathfrak{m}_3(x_{12}, x_{23}, x_{34})). \end{aligned} \tag{11.14}$$

It follows that $\mathfrak{m}_3(x_{12}, x_{23}, x_{34})$ coincides with $\mathfrak{m}_3'(x_{12}, x_{23}, x_{34})$ modulo elements of the form $\mathfrak{m}_2(x_{12}, \bullet) + \mathfrak{m}_2(\bullet, x_{34}) + \bar{\partial}(\bullet)$. This is consistent with the usual definition of Massey triple product.

Chapter 3. Floer homology and Extension.

§12. Calculation of cohomology (the case of line bundle).

In this section, we proof a special case of Theorem δ. Namely we show

Proposition 12.1. *If L is transversal to $L_{\rm pt}$ and if $|L \bullet L_{\rm pt}| = 1$. Then, we have :*

$$HF^k((L_{\rm st},0),(L,\alpha)) \cong H^k((T^{2n},\Omega)^\vee, \mathcal{E}(L,\alpha))$$

We remark that the condition $|L \bullet L_{\rm pt}| = 1$ is equivalent to the condition that $\mathcal{E}(L,\alpha)$ is a line bundle.

As we mentioned in the introduction, the proof we give in this section is not satisfactory, since here we only check the coincidence of the rank of the vector spaces in Proposition 12.1 and does not provide a canonical isomorphism.

To show Proposition 12.1, we calculate the first Chern class of the line bundle $\mathcal{E}(L,\alpha)$. To state it, we need some notations. We assumed that $\tilde L$ is transversal to $\tilde L_{\rm pt}$. Hence $\tilde L$ may be regarded as a graph of a linear isomorphism : $\tilde L_{\rm st} \to \tilde L_{\rm pt}$. We write it as $\phi_L : \tilde L_{\rm st} \to \tilde L_{\rm pt}$. In other words, $x + \phi_L(x) \in \tilde L$ for any $x \in \tilde L_{\rm st}$.

We next remark that there exists an isomorphism $V/\tilde L_{\rm pt} \cong \tilde L_{\rm st}$. We have

$$\tilde L_{\rm st} \oplus \tilde L_{\rm pt}^* \cong V/\tilde L_{\rm pt} \oplus \tilde L_{\rm pt}^*, \qquad \Gamma/(\Gamma\cap \tilde L_{\rm pt}) \cong \Gamma \cap \tilde L_{\rm st} \subseteq \tilde L_{\rm st}.$$

Definition 12.2. Let $\gamma,\gamma' \in \Gamma/(\Gamma\cap\tilde L_{\rm pt}) \cong \Gamma\cap\tilde L_{\rm st}$. $\mu,\mu' \in \left(\Gamma\cap\tilde L_{\rm pt}\right)^\vee$. We define :

$$E_L((\gamma,\mu),(\gamma',\mu')) = \mu(\phi_L(\gamma')) - \mu'(\phi_L(\gamma)).$$

Since $|L \bullet L_{\rm pt}| = 1$, it follows that $\phi_L\left(\Gamma/\Gamma\cap\tilde L_{\rm pt}\right) = \phi_L(\Gamma\cap\tilde L_{\rm st}) \subseteq \Gamma\cap\tilde L_{\rm pt}$. Therefore E_L is integer valued. By definition, E_L is anti-symmetric. We can extend E_L to an $\mathbb{R}$-bilinear anti-symmetric form on $\tilde L_{\rm st}\oplus\tilde L_{\rm pt}^* \cong V/\tilde L_{\rm pt}\oplus\tilde L_{\rm pt}^*$, we denote it by the same symbol.

Lemma 12.3. *$E_L(J_\Omega x, J_\Omega y) = E_L(x,y)$, where J_Ω is the complex structure on $\tilde L_{\rm st}\oplus\tilde L_{\rm pt}^*$ introduced in §1.*

We will prove Lemma 12.3 later in this section.

Theorem 12.4. *$c_1(\mathcal{E}(L,\alpha)) = E_L$. Here we regard an anti-symmetric bilinear map E_L on $\left(\Gamma/\Gamma\cap\tilde L_{pt}\right)\oplus\left(\Gamma\cap\tilde L_{pt}\right) = \pi_1((T^{2n},\Omega)^\vee)$ as an element of $H^2((T^{2n},\Omega)^\vee;\mathbb{Z})$.*

Note that Lemma 12.3 implies that $E_L \in H^{1,1}((T^{2n},\Omega)^\vee)$. We will prove Theorem 12.4 later in this section.

We next show that Theorem 12.4 implies Proposition 12.1. For this purpose, we need to recall some standard results on the cohomology of line bundles on complex tori. We define a quadratic form $H_{L,\Omega}$ on $V/\tilde L_{\rm pt}\oplus\tilde L_{\rm pt}^*$ by

$$H_{L,\Omega}(x,y) = E_L(J_\Omega x, y) + \sqrt{-1}E_L(x,y) \tag{12.5}$$

Lemma 5.6 implies that $H_{L,\Omega}$ is hermitian. Theorem 12.4 and a classical result (see [Mum] §16, [LB] Theorem 5.5.) implies the following :

Corollary 12.6. *Suppose that $H_{L,\Omega}$ is nondegenerate, then we have*

$$H^k((T^{2n},\Omega)^\vee,\mathcal{E}(L,\alpha)) = \begin{cases} 0 & k \ne \operatorname{Index} H_{L,\Omega}, \\ \mathbb{C}^{\mathrm{Pf}E_L} & k = \operatorname{Index} H_{L,\Omega}. \end{cases}$$

Here $\mathrm{Pf}E_L$ is the Pfaffian of the anti-symmetric form E_L and $\operatorname{Index} H_{L,\Omega}$ is the number of negative eigenvalues of the hermitian form $H_{L,\Omega}$. Usng $|L \bullet L_{\rm pt}| = 1$ it is easy to see

$$\mathrm{Pf}\, E_L = \sharp \frac{\Gamma \cap \tilde L_{\rm pt}}{\phi_L\left(\Gamma/(\Gamma\cap \tilde L_{\rm pt})\right)} = \sharp\frac{\Gamma}{(\Gamma\cap\tilde L) + (\Gamma\cap \tilde L_{\rm st})} = |L\bullet L_{\rm st}|\,. \tag{12.7}$$

We next need the following :

Lemma 12.8. $\operatorname{Index} H_{L,\Omega} = \eta(\tilde L_{\rm st},\tilde L)$, *where η is defined by Definition 2.12.*

The proof will be given later in this section.

Theorem 12.1 in the case when $H_{L,\Omega}$ is nondegenerate follows from Corollary 12.6, Lemma 12.8 and (12.7).

Let us consider the case when $H_{L,\Omega}$ is degenerate. In other words, we consider the case when L is not transversal to $L_{\rm st}$. We consider the kernel $\operatorname{Ker} H_{L,\Omega}$ of $H_{L,\Omega}$. It is easy to see

$$\dim \operatorname{Ker} H_{L,\Omega} = \dim L_{\rm st}\cap L. \tag{12.9}$$

We also have

$$\operatorname{Ker} H_{L,\Omega} \cap \pi_1((T^{2n},\Omega)^\vee) \cong \mathbb{Z}^{\dim \operatorname{Ker} H_{L,\Omega}}.$$

Hence $T_0 = \operatorname{Ker} H_{L,\Omega}/(\operatorname{Ker} H_{L,\Omega}\cap \pi_1((T^{2n},\Omega)^\vee))$ is a subtorus of $(T^{2n},\Omega)^\vee$. Let $T/T_0 = \bar T$. There exists a bundle $\bar{\mathcal{E}}$ on $\bar T$ such that $\mathcal{E}(L,\alpha)$ is isomorphic to the pull back of the bundle $\bar{\mathcal{E}}$. Therefore by Künnet formula we have

$$H^k((T^{2n},\Omega)^\vee,\mathcal{E}(L,\alpha)) \cong \bigoplus_{\ell} H^\ell(\bar T,\bar{\mathcal{E}})\otimes H^{k-\ell}(T_0;\mathbb{C}). \tag{12.10}$$

On the other hand, in a similar way to (12.7) we have

$$\mathrm{Pf}\, c^1(\bar{\mathcal{E}}) = \sharp \pi_0(L\cap L_{\rm st}). \tag{12.11}$$

Moreover if we define $\bar H(x,y) = c^1(\bar{\mathcal{E}})(Jx,y) + \sqrt{-1}c^1(\bar{\mathcal{E}})(x,y)$, then we have

$$\operatorname{Index} \bar H = \eta(\tilde L_{\rm st},\tilde L). \tag{12.12}$$

Since $c^1(\bar{\mathcal{E}})$ is nondegenerate we can apply Theorem 12.6 to $\bar{\mathcal{E}}$. Hence (2.9),(2.10), (2.11),(2.12) and Definition 2.6, imply Proposition 12.1. □

We now prove Lemmata 12.3, 12.8. We first prove Lemma 12.3. (In fact, Lemma 12.3 follows from Theorem 12.4 proven later. We prove Lemma 12.3 as a check of the calculation of the proof of Theorem 12.4.) We put

$$U = \left(\Gamma\cap\tilde L_{\rm pt}\right)\oplus\left(\Gamma/(\Gamma\cap\tilde L_{\rm pt})\right)^\vee, \quad U_1 = \Gamma\cap\tilde L_{\rm pt}, \quad U_2 = \left(\Gamma/\Gamma\cap\tilde L_{\rm pt}\right)^\vee.$$

We first calculate the complex structure J_Ω. The symplectic form ω defines an isomorphism $I_\omega : \tilde{L}_{\rm st} \to \tilde{L}^*_{\rm pt}$ such that $I_\omega(v)(x) = \omega(x,v)$. Similarly the closed 2 form B determines $I_B : \tilde{L}_{\rm st} \to \tilde{L}^*_{\rm pt}$ by $I_B(v)(x) = B(x,v)$. We then find, from the definition, that

$$v + \sigma \mapsto I_\omega v + \sqrt{-1}\,(I_B v + \sigma)$$

is a complex linear isomorphism : $\tilde{L}_{\rm st} \oplus \tilde{L}^*_{\rm pt} \to \tilde{L}^*_{\rm st} \otimes_{\mathbb{R}} \mathbb{C}$, where $v, v' \in \tilde{L}_{\rm st}$, $\sigma, \sigma' \in \tilde{L}^*_{\rm pt}$. Hence

$$\begin{aligned} J_\Omega \begin{pmatrix} v \\ \sigma \end{pmatrix} &= \begin{pmatrix} I_\omega & 0 \\ I_B & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} I_\omega & 0 \\ I_B & 1 \end{pmatrix} \begin{pmatrix} v \\ \sigma \end{pmatrix} \\ &= \begin{pmatrix} -I_\omega^{-1} I_B & -I_\omega^{-1} \\ I_\omega + I_B I_\omega^{-1} I_B & I_B I_\omega^{-1} \end{pmatrix} \begin{pmatrix} v \\ \sigma \end{pmatrix}. \end{aligned} \tag{12.13}$$

Therefore, for $v, v' \in \tilde{L}_{\rm st}$, we have :

$$\begin{aligned} &E_L(J_\Omega v, J_\Omega v') \\ &\quad = E_L(I_\omega(v) + I_B I_\omega^{-1} I_B(v), -I_\omega^{-1} I_B(v')) \\ &\qquad + E_L(-I_\omega^{-1} I_B(v), I_\omega(v') + I_B I_\omega^{-1} I_B(v')) \\ &\quad = -\omega\left(\phi_L\left(I_\omega^{-1} I_B(v')\right), v\right) - B\left(\phi_L\left(I_\omega^{-1} I_B(v')\right), I_\omega^{-1} I_B(v)\right) \\ &\qquad + \omega\left(\phi_L\left(I_\omega^{-1} I_B(v)\right), v'\right) + B\left(\phi_L\left(I_\omega^{-1} I_B(v)\right), I_\omega^{-1} I_B(v')\right). \end{aligned} \tag{12.14}$$

On the other hand, since $\omega|_{\tilde{L}} = B|_{\tilde{L}} = 0$ if follows that

$$\omega\left(v, \phi_L\left(v'\right)\right) = -\omega\left(\phi_L\left(v\right), v'\right), \quad B\left(v, \phi_L\left(v'\right)\right) = -B\left(\phi_L\left(v\right), v'\right). \tag{12.15}$$

Therefore

$$B\left(\phi_L\left(I_\omega^{-1} I_B(v')\right), I_\omega^{-1} I_B(v)\right) - B\left(\phi_L\left(I_\omega^{-1} I_B(v)\right), I_\omega^{-1} I_B(v')\right) = 0.$$

Moreover we have

$$\begin{aligned} &\omega\left(\phi_L\left(I_\omega^{-1} I_B(v)\right), v'\right) - \omega\left(\phi_L\left(I_\omega^{-1} I_B(v')\right), v\right) \\ &\qquad = \omega\left(\phi_L\left(v'\right), I_\omega^{-1} I_B(v)\right) - \omega\left(\phi_L\left(v\right), I_\omega^{-1} I_B(v')\right) \\ &\qquad = B\left(\phi_L\left(v'\right), v\right) - B\left(\phi_L\left(v\right), v'\right) = 0. \end{aligned}$$

Hence

$$E_L(J_\Omega v, J_\Omega v') = 0 = E_L(v, v'). \tag{12.16}$$

We can prove

$$E_L(J_\Omega \sigma, J_\Omega \sigma') = 0 = E_L(\sigma, \sigma') \tag{12.17}$$

for $\sigma, \sigma' \in \tilde{L}^*_{\rm pt}$, in a similar way. We next calculate using (12.13), (12.15) :

$$\begin{aligned} &E_L(J_\Omega v, J_\Omega \sigma) \\ &\quad = E_L(I_\omega(v) + I_B I_\omega^{-1} I_B(v), -I_\omega^{-1}(\sigma)) + E_L(-I_\omega^{-1} I_B(v), I_B I_\omega^{-1}(\sigma)) \\ &\quad = -\omega\left(\phi_L I_\omega^{-1}(\sigma), v\right) - B\left(\phi_L I_\omega^{-1}(\sigma), I_\omega^{-1} I_B(v)\right) \\ &\qquad + B\left(\phi_L I_\omega^{-1} I_B(v), I_\omega^{-1}(\sigma)\right) \\ &\quad = -\omega\left(\phi_L(v), I_\omega^{-1}(\sigma)\right) = -\sigma\left(\phi_L(v)\right) = E_L(v, \sigma). \end{aligned} \tag{12.18}$$

Lemma 12.3 follows from (12.16) , (12.17) and (12.18). □

We turn to the proof of Lemma 12.8. We put $\Omega_s = \omega + s\sqrt{-1}B$. We remark that the J_{Ω_s} hermitian form H_{L,Ω_s} is well-defined and is nondegenerate for each s. Hence its index is independent of s. So to prove Lemma 12.8, we may assume $B = 0$. Then we have

$$H_{L,\Omega_0}(v,v) = E_L\left(J_{\Omega_0}(v),v\right) = E_L\left(I_\omega(v),v\right) = \omega(\phi_L(v),v) \tag{12.19}$$

for $v \in \tilde{L}_{\rm st}$. Lemma 12.8 follows from (12.19) and the definition of η in the same way as the proof of Lemma 6.13. □

The rest of this section is devoted to the proof of Theorem 12.4. We remark that the first Chern class of $\mathcal{E}(L,\alpha)$ is independent of α. Hence we put $\alpha = 0$ for simplicity. We are going find a 1-cocycle

$$e_u(z) : U \times \left(\tilde{L}_{\rm st} \oplus \tilde{L}^*_{\rm pt}\right) \to \mathbb{C}\backslash\{0\}$$

representing our line bundle $\mathcal{E}(L,\alpha)$. For this purpose, we will find a holomorphic trivialization of the pull back bundle $\tilde{\mathcal{E}}(L,\alpha)$ on $V/\tilde{L}_{\rm pt} \oplus \tilde{L}^*_{\rm pt}$.

Note that there is an obvious choice of trivialization $\tilde{\mathcal{E}}(L,0) \cong \left(V/\tilde{L}_{\rm pt} \oplus \tilde{L}^*_{\rm pt}\right) \times \mathbb{C}$. Namely, by choosing a lift $\hat{L} \cong \mathbb{R}^n$ of L, we define a global frame s' of $\tilde{\mathcal{E}}(L,0)$ by $s'(v,\sigma) = [\tilde{p}(v)]$ for $(v,\sigma) \in \tilde{L}_{\rm st} \oplus \tilde{L}^*_{\rm pt}$. Here $\{\tilde{p}(v)\} = \hat{L} \cap \hat{L}_{\rm pt}(v)$. We recall $\tilde{\mathcal{E}}(L,0)_{(v,\sigma)} = \mathbb{C}[\tilde{p}(v)]$ since we assumed that $\tilde{\mathcal{E}}(L,0)$ is a line bundle.

However the frame s' does not respect the holomorphic structure introduced in §2. A holomorphic global frame of $\tilde{\mathcal{E}}(L,0)$ is obtained by

$$s(v,\sigma) = \exp\left(2\pi Q\left(\tilde{p}(0),0,v,\tilde{p}(v)\right) - 2\pi\sqrt{-1}\sigma(\tilde{p}(v)-v)\right)\ s'(v,\sigma). \tag{12.20}$$

Note $v = \hat{L}_{\rm st} \cap \hat{L}_{\rm pt}(v)$. (Compare (3.4). $v, \tilde{p}(v)$ in (12.12) corresponds $x_0(v), \tilde{p}$ in (3.4).) For $u_1 \in U_1$, $u_2 \in U_2$ we have :

$$(u_1 \cdot s') \cdot (v + u_1, \sigma) = s'(v,\sigma), \tag{12.21.1}$$

$$(u_2 \cdot s') \cdot (v, \sigma + u_2) = \exp\left(2\pi\sqrt{-1}u_2(\tilde{p}(v)-v)\right) s'(v,\sigma). \tag{12.21.2}$$

((12.12.2) follows from (3.2.1),(3.2.2).) By (12.21) we obtain the following formulae.

$$\begin{aligned}(u_1 \cdot s)(v,\sigma) &\\ = \exp &\left(2\pi Q\left(\tilde{p}(0),0,u_1+v,\tilde{p}(u_1+v)\right)\right.\\ &- 2\pi\sqrt{-1}\sigma\left((u_1+v)-\tilde{p}(u_1+v)\right)\\ &\left.-2\pi Q\left(\tilde{p}(0),0,v,\tilde{p}(v)\right) + 2\pi\sqrt{-1}\sigma\left(v-\tilde{p}(v)\right)\right) s(u_1+v,\sigma).\end{aligned} \tag{12.22.1}$$

$$(u_2 \cdot s)(v,\sigma) = s(v, u_2 + \sigma). \tag{12.22.2}$$

By the definition of ϕ_L we have

$$\tilde{p}(v) - v = \tilde{p}(0) + \phi_L(v). \tag{12.23}$$

Figure 14

Here we regards $v \in V/\tilde{L}_{\rm pt} \cong \tilde{L}_{\rm st}$. Therefore a 1 cocycle $e_u(z)$ defining $\mathcal{E}(L,0)$ is :

$$e_{u_1}(v,\sigma) = \exp\left(2\pi Q\left(\tilde{p}(v), v, v+u_1, \tilde{p}(v+u_1)\right) - 2\pi\sqrt{-1}\sigma(\phi_L(u_1))\right), \tag{12.24.1}$$

$$e_{u_2}(v,\sigma) = 1. \tag{12.24.2}$$

We put

$$f_{u_1}(v,\sigma) = -\sqrt{-1}Q\left(\tilde{p}(v), v, v+u_1, \tilde{p}(v+u_1)\right) - \sigma(\phi_L(u_1)), \tag{12.25.1}$$

$$f_{u_2}(v,\sigma) = 0. \tag{12.25.2}$$

Then, by a standard result (see Proposition in page 18 of [Mum]), we find that the first Chern class of $\mathcal{E}(L,0)$ is represented by :

$$E(u,u') = f_u(z+u') + f_{u'}(z) - f_{u'}(z+u) - f_u(z). \tag{12.26}$$

Let $u_1, u_1' \in U_1$, $u_2, u_2' \in U_2$.

$$E(u_2, u_2') = 0 \tag{12.27}$$

by (12.25.2). We next put $f^2_{u_1}(v,\sigma) = -\sigma(\phi_L(u_1))$, $f^1_{u_1}(v,\sigma) = f_{u_1}(v,\sigma) - f^2_{u_1}(v,\sigma)$. By Figure 15, we have

$$f^1_{u_1}(z+u_1') + f^1_{u_1'}(z) = f^1_{u_1+u_1'}(z) = f^1_{u_1'}(z+u_1) + f^1_{u_1}(z).$$

Figure 15

Moreover $f^2_{u_1}(v,\sigma)$ is independent of v. Therefore

$$E(u_1, u_1') = 0. \tag{12.28}$$

We finally calculate

$$E(u_1,u_2)(v,\sigma) = f_{u_1}(v,\sigma+u_2) - f_{u_1}(v,\sigma) = -u_2(\phi_L(u_1)) = E_L(u_1,u_2). \tag{17.29}$$

Theorem 12.4 follows from (17.27),(17.28),(17.29). □

§13. Isogenie.

In this section, we generalize the result of §12 and prove Theorem γ in the case when $L_1(w_1)$ and $L_2(w_2)$ are transversal to each other. We first generalize Proposition 12.1 a bit. Let $L_1(w_1)$, $L_2(w_2)$ be affine subtorus of T^{2n} such that $\Omega|_{L_i} = 0$. In this section, we assume that $\tilde{L}_i$ is transversal to $\tilde{L}_{\mathrm{pt}}$.

Lemma 13.1. *We assume furthermore that $\mathcal{E}(L_i(w_i), \alpha_i)$ $(i = 1, 2)$ are line bundles. Then*

$$\mathrm{Ext}^k(\mathcal{E}(L_1(w_1), \alpha_1), \mathcal{E}(L_2(w_2), \alpha_2)) \cong HF^k((L_1(w_1), \alpha_1), (L_2(w_2), \alpha_2)) \tag{13.2}$$

Proof. We have

$$\begin{aligned} &\mathrm{Ext}^k(\mathcal{E}(L_1(w_1), \alpha_1), \mathcal{E}(L_2(w_2), \alpha_2)) \\ &\qquad \cong H^k((T^{2n}, \Omega)^\vee, \mathcal{E}(L_1(w_1), \alpha_1)^* \otimes \mathcal{E}(L_2(w_2), \alpha_2)) \end{aligned} \tag{13.3}$$

in this case. By Theorem 12.4, we have

$$c^1\left(\mathcal{E}(L_1(w_1), \alpha_1)^* \otimes \mathcal{E}(L_2(w_2), \alpha_2)\right) = -E_{L_1} + E_{L_2}$$

Let ϕ_{L_i} be as in §12. Then we find a Lagrangian linear subspace $\tilde{L}_3$ such that $-\phi_{L_1} + \phi_{L_2} = \phi_{L_3}$. Then, putting $L_3 = L_3(v_3)$, we have

$$\mathrm{Pf}\, E_{L_3} = \sharp\left(L_{\mathrm{st}} \cap L_3\right) = \sharp\left(L_1 \cap L_2\right). \tag{13.4}$$

Hence Theorem 12.6, (13.3) and (13.4) imply Lemma 13.1. □

Now the main result of this section is :

Proposition 13.5. *The isomorphism (13.2) holds if L_1, L_2 are affine subtori of T^{2n} such that $\Omega|_{L_i} = 0$ and if $\tilde{L}_i$ is transversal to $\tilde{L}_{\mathrm{pt}}$.*

Proof. Here we follow Polishchuk and Zaslow [PZ] §5.3 to reduce the proof of Theorem 13.5 to the case of line bundles. We remark that the rank of the vector bundle $\mathcal{E}(L(w), \alpha)$ is $L(w) \bullet L_{\mathrm{pt}}(v)$. Hence there exists a finite group $G(L) \subseteq L_{\mathrm{pt}}(0) \subseteq T^{2n}$ with the following property :

(13.6.1) The order of $G(L)$ is $L(w) \bullet L_{\mathrm{pt}}(v)$.
(13.6.2) $L(w)$ is $G(L)$ invariant.
(13.6.3) $G(L)$ acts transitively on $L(w) \cap L_{\mathrm{pt}}(v)$.

Let G be a subgroup of $G(L)$. We put $(T^{2n}, \Omega)/G = (\bar{T}^{2n}, \bar{\Omega})$. We use $\tilde{L}_{\mathrm{st}} \subseteq V =$ the universal cover of $\bar{T}^{2n}$ to define a mirror $(\bar{T}^{2n}, \bar{\Omega})^\vee$. Let $G^\vee = Hom(G, U(1))$ be the dual group.

Lemma 13.7. *$G^\vee$ acts on $(\bar{T}^{2n}, \bar{\Omega})^\vee$ such that $(\bar{T}^{2n}, \bar{\Omega})^\vee / G^\vee = (T^{2n}, \Omega)^\vee$.*

Proof. The universal cover of $(\bar{T}^{2n}, \bar{\Omega})$ is identified to the universal cover $V/\tilde{L}_{\mathrm{pt}} \oplus \tilde{L}^*_{\mathrm{pt}}$ of (T^{2n}, Ω). We remark that $\Gamma' = \pi_1(\bar{T}^{2n}, \bar{\Omega})$ contains $\Gamma = \pi_1(T^{2n}, \Omega)$ as an index $\sharp G$ subgroup. It is easy to see

$$\Gamma'/(\Gamma' \cap \tilde{L}_{\mathrm{pt}}) \cong \Gamma/(\Gamma \cap \tilde{L}_{\mathrm{pt}}), \qquad (\Gamma' \cap \tilde{L}_{\mathrm{pt}})/(\Gamma \cap \tilde{L}_{\mathrm{pt}}) \cong G.$$

Hence

$$\left(\Gamma \cap \tilde{L}_{\rm pt}\right)^{\vee} \Big/ \left(\Gamma' \cap \tilde{L}_{\rm pt}\right)^{\vee} = G^{\vee}.$$

Lemma 13.7 follows. □

By (13.6.2), there exists a Lagrangian submanifold $\bar{L}(\bar{w}) = L(w)/G$ of $(\bar{T}^{2n}, \bar{\Omega})$. There is a flat connection $\bar{\alpha}$ on $\bar{L}(\bar{w})$ such that $\pi^*\bar{\alpha} = \alpha$. Here $\pi : L(w) \to \bar{L}(\bar{w})$ is the covering map. (13.6.3) implies that $\left|\bar{L}(\bar{w}) \bullet \bar{L}_{\rm pt}\right| = \sharp G(L)/G$. Hence $\operatorname{rank}\mathcal{E}(\bar{L}(\bar{w}), \bar{\alpha}) = \sharp G(L)/G$. Let $\pi : (\bar{T}^{2n}, \bar{\Omega})^{\vee} \to (T^{2n}, \Omega)^{\vee}$ be the $G^{\vee}$ covering constructed by Lemma 13.7.

Lemma 13.8. $\pi_*\left(\mathcal{E}(\bar{L}(\bar{w}), \bar{\alpha})\right) \cong \mathcal{E}(L(w), \alpha)$, *where* π_* *is the push forward of the bundle.*

Proof. We put

$$A(\sigma) = \left\{\sigma + \mu \;\middle|\; \mu \in \left(\Gamma \cap \tilde{L}_{\rm pt}\right)^{\vee}\right\} \tag{13.9}$$

$$B(v) = \left\{p \in \hat{L}_{\rm pt}(v) \;\middle|\; \pi(p) = L_{\rm pt}(v) \cap L\right\} \tag{13.10}$$

Let $\hat{E}(\sigma, v)$ be the vector space consisting of all maps $u(\lambda, p) : A(\alpha) \times B(v) \to \mathbb{C}$. For $(\gamma, \mu) \in \Gamma'/(\Gamma' \cap \tilde{L}) \times \left(\Gamma \cap \tilde{L}_{\rm pt}\right)^{\vee}$, we put

$$\left((\gamma, \mu)u\right)(\lambda + \mu, \gamma + p) = \exp\left(2\pi\sqrt{-1}\mu(p - x_0(v))\right) u(\lambda, p). \tag{13.11}$$

(13.11) defines actions of $\Gamma'/(\Gamma' \cap \tilde{L}) \times (\Gamma' \cap \tilde{L}_{\rm pt})^{\vee}$ and of $\Gamma/(\Gamma \cap \tilde{L}) \times (\Gamma \cap \tilde{L}_{\rm pt})^{\vee}$. We remark however that (13.11) does *not* define an action of $\Gamma'/(\Gamma' \cap \tilde{L}) \times (\Gamma \cap \tilde{L})^{\vee}$ since the action of the two factors $\Gamma'/(\Gamma' \cap \tilde{L})$, $(\Gamma \cap \tilde{L})^{\vee}$ do not commute each other.

The definition of $\mathcal{E}(L(w), \alpha)$ (see (3.2)) implies the following :

$$\begin{aligned} &\pi_*\left(\mathcal{E}(\bar{L}(\bar{w}), \bar{\alpha})\right)_{(L_{\rm pt}(v),\sigma)} \\ &\cong \left\{ u \in \hat{E}(\sigma, v) \;\middle|\; \forall(\gamma, \mu) \in \Gamma'/(\Gamma' \cap \tilde{L}) \times (\Gamma' \cap \tilde{L}_{\rm pt})^{\vee},\ (\gamma, \mu)u = u \right\}. \end{aligned} \tag{13.12}$$

$$\begin{aligned} &\mathcal{E}(L(w), \alpha)_{(L_{\rm pt}(v),\sigma)} \\ &\cong \left\{ u \in \hat{E}(\sigma, v) \;\middle|\; \forall(\gamma, \mu) \in \Gamma/(\Gamma \cap \tilde{L}) \times (\Gamma \cap \tilde{L}_{\rm pt})^{\vee},\ (\gamma, \mu)u = u\right\}. \end{aligned} \tag{13.13}$$

We are going to construct an isomorphism between (13.12) and (13.13) by "Fourier transformation". Let γ_i be representatives of

$$\frac{\Gamma'/(\Gamma' \cap \tilde{L}_{\rm pt})}{\Gamma/(\Gamma \cap \tilde{L}_{\rm pt})}$$

and μ_j be representatives of $(\Gamma \cap \tilde{L}_{\rm pt})^{\vee}/(\Gamma' \cap \tilde{L}_{\rm pt})^{\vee}$. For $u \in \mathcal{E}(L(w), \alpha)_{(L_{\rm pt}(v),\sigma)}$, $u' \in \pi_*\left(\mathcal{E}(\bar{L}(\bar{w}), \bar{\alpha})\right)_{(L_{\rm pt}(v),\sigma)}$ we put :

$$\mathfrak{F}(u)(\lambda, p) = \sum_i u(\lambda, \gamma_i + p) \tag{13.14}$$

$$\mathfrak{F}'(u')(\lambda, p) = \sum_j \exp\left(-2\pi\sqrt{-1}\mu_j(p - x_0(v))\right) u'(\lambda + \mu_j, p) \tag{13.15}$$

Sublemma 13.16. *$\mathfrak{F}(u) \in \pi_* \left(\mathcal{E}(\bar{L}(\bar{w}), \bar{\alpha})\right)_{(L_{\rm pt}(v),\sigma)}$. $\mathfrak{F}'(u') \in \mathcal{E}(L(w), \alpha)_{(\mathcal{L}_{\rm pt}(v),\sigma)}$, $\mathfrak{F}'\mathfrak{F}(u) = u$, $\mathfrak{F}\mathfrak{F}'(u) = u$.*

Proof. It is easy to see $\mathfrak{F}(u)(\lambda, \gamma'+p) = \mathfrak{F}(u)(\lambda, p)$ for $\gamma' \in \Gamma'$. Let $\mu' \in \left(\Gamma' \cap \tilde{L}_{\rm pt}\right)^{\vee}$. We have

$$\begin{aligned}\mathfrak{F}(u)(\mu' + \lambda, p) &= \sum_i u(\mu' + \lambda, \gamma_i + p) \\ &= \sum_i \exp\left(2\pi\sqrt{-1}\mu'(\gamma_i + p - x_0(v))\right) u(\lambda, \gamma_i + p) \\ &= \exp\left(2\pi\sqrt{-1}\mu'(p - x_0(v))\right) \mathfrak{F}(u)(\lambda, p).\end{aligned}$$

Therefore $\mathfrak{F}(u) \in \pi_* \left(\mathcal{E}(\bar{L}, \bar{\alpha})\right)_{(L_{\rm pt}(v),\sigma)}$. The proof of $\mathfrak{F}'(u') \in \mathcal{E}(L, \alpha)_{(L_{\rm pt}(v),\sigma)}$ is similar.

On the other hand, we have

$$\begin{aligned}\mathfrak{F}'\mathfrak{F}(u)(\lambda, p) &= \sum_{i,j} \exp\left(-2\pi\sqrt{-1}\mu_j(p - x_0(v))\right) u(\lambda + \mu_j, p+\gamma_i) \\ &= \sum_{i,j} \exp\left(2\pi\sqrt{-1}\mu_j(\gamma_i)\right) u(\lambda, p+\gamma_i)\end{aligned} \tag{13.17}$$

We may choose γ_i and μ_i so that $\gamma_1 = 0$, $\mu_1 = 0$ and

$$\sum_j \exp\left(2\pi\sqrt{-1}\mu_j(\gamma_i)\right) = 0 \text{ unless } i = 1, \tag{13.18.1}$$

$$\sum_i \exp\left(2\pi\sqrt{-1}\mu_j(\gamma_i)\right) = 0 \text{ unless } j = 1. \tag{13.18.2}$$

(13.18) implies $\mathfrak{F}'\mathfrak{F}(u) = u$. The proof of $\mathfrak{F}\mathfrak{F}'(u) = u$ is similar. □

Now it is easy to see that $\mathfrak{F}$, $\mathfrak{F}'$ give isomorphisms asserted in Lemma 13.8. □

The following is an immediate consequence of Proposition 13.8.

Corollary 13.19. *If $L(w)$ is transversal to $L_{\rm pt}$ then*

$$H^*((\bar{T}^{2n}, \bar{\Omega})^{\vee}, \mathcal{E}(\bar{L}, \bar{\alpha})) \cong H^*((T^{2n}, \Omega)^{\vee}, \mathcal{E}(L(w), \alpha)).$$

Proof. Corollary 13.19 follows from Lemma 13.8 and Leray spectral sequence since the higher direct image sheaf $(R\pi_*)_k\mathcal{E}(L(w), \alpha)$ is zero for $k > 0$. □

Note $L_{\rm st} \cap L(w) \cong \bar{L}_{\rm st} \cap \bar{L}(\bar{w})$. Hence Corollary 13.19 and Proposition 12.1 implies Theorem 13.5 in the case when $L_1 = L_{\rm pt}$. To prove Proposition 13.15 in the general case, we further study isogenie. Let $G(L_1)$, $G(L_2)$ be as above. Let $G \subseteq L_{\rm pt}$ be a finite subgroup. We define $(\bar{T}^{2n}, \bar{\Omega})$ as above. We study the following three cases sparately.

Case 1. $G \subseteq G(L_1) \cap G(L_2)$:

Let $\bar{\alpha}_i$ be a flat connection on $\bar{L}_1$ such that $\pi^*\bar{\alpha}_i = \alpha_i$. We remark that $\pi^*(\bar{\alpha}_i + \mu) = \alpha_i$ for $\mu \in G$.

Lemma 13.20.

$$\pi^*\mathcal{E}(L_i(w_i),\alpha_i) \cong \bigoplus_{\mu\in G^\vee} \mathcal{E}(\bar{L}_i(\bar{w}_i),\bar{\alpha}_i+\mu).$$

Proof. By Lemma 13.8, we have $\pi_*\left(\mathcal{E}(\bar{L}_i(\bar{w}_i),\bar{\alpha}_i)\right) \cong \mathcal{E}(L_i(w_i),\alpha_i)$. Note that $G^\vee$ is the deck transformation group of the covering $\pi : (\bar{T}^{2n},\bar{\Omega})^\vee \to (T^{2n},\Omega)^\vee$. Hence

$$\pi^*\mathcal{E}(L_i(w_i),\alpha_i) = \bigoplus_{\mu\in G^\vee} \mu^*\mathcal{E}(\bar{L}_i(\bar{w}_i),\bar{\alpha}_i).$$

Here we regard $\mu : (\bar{T}^{2n},\bar{\Omega})^\vee \to (\bar{T}^{2n},\bar{\Omega})^\vee$. It is easy to see $\mu^*\mathcal{E}(\bar{L}_i,\bar{\alpha}_i) = \mu^*\mathcal{E}(\bar{L}_i,\bar{\alpha}_i+\mu)$. Lemma 13.20 follows. □

Lemma 13.21.

$$\mathrm{Ext}^k(\mathcal{E}(L_1(w_1),\alpha_1),\mathcal{E}(L_2(w_2),\alpha_2)) \cong \bigoplus_{\mu\in G^\vee} \mathrm{Ext}^k(\mathcal{E}(\bar{L}_1(\bar{w}_1),\bar{\alpha}_1),\mathcal{E}(\bar{L}_2(\bar{w}_2),\bar{\alpha}_2+\mu)).$$

Proof. Lemma 13.8 and Lemma 13.20 imply

$$\begin{aligned}\mathrm{Ext}^k(\mathcal{E}(L_1(w_1),\alpha_1),\mathcal{E}(L_2(w_2),\alpha_2)) &\cong \mathrm{Ext}^k(\pi_*(\mathcal{E}(\bar{L}_1(\bar{w}_1),\bar{\alpha}_1)),\mathcal{E}(L_2(\bar{w}_2),\alpha_2))\\ &\cong \bigoplus_{\mu\in G^\vee} \mathrm{Ext}^k(\mathcal{E}(\bar{L}_1(\bar{w}_1),\bar{\alpha}_1),\mathcal{E}(\bar{L}_2(\bar{w}_2),\bar{\alpha}_2+\mu)).\end{aligned}$$

(Here we use the fact that $(R\pi_{k*})(\mathcal{E}(\bar{L}_1(\bar{w}_1),\bar{\alpha}_1)) = 0$ for $k > 0$.) □

We next consider Floer cohomology. The assumption $G \subseteq G(L_1)\cap G(L_2)$ implies that $L_1(w_1)$ and $L_2(w_2)$ are both G invariant. Hence G acts on $L_1(w_1)\cap L_2(w_2)$ freely. Therefore we have

$$\begin{aligned}&HF^k((L_1(w_1),\alpha_1),(L_2(w_2),\alpha_2))\\ &\qquad\cong HF^k((\bar{L}_1(\bar{w}_1),\bar{\alpha}_1),(\bar{L}_2(\bar{w}_2),\bar{\alpha}_2))\otimes R(G).\end{aligned} \tag{13.22}$$

Lemma 13.21 and (13.22) imply that if (13.2) holds for $(\bar{L}_1(\bar{w}_1),\bar{\alpha}_1),(\bar{L}_2(\bar{w}_2),\bar{\alpha}_2)$ then it holds for $(L_1(w_1),\alpha_1),(L_2(w_2),\alpha_2)$.

Case 2. $G \subseteq G(L_1)$, $G\cap G(L_2) = \{1\}$:

We put $G = \{\gamma_1,\cdots,\gamma_g\}$. By assumption $G\cap G(L_2) = \{1\}$, we have $(\gamma_1 + L_1(w_1))\cap(\gamma_2+L_2(w_2)) = \emptyset$. Let $\bar{L}_2(\bar{w}_2) = \pi(L_2(w_2))$. π induces an isomorphism $L_2(w_2)\cong\bar{L}_2(\bar{w}_2)$. Using this isomorphism we define $\bar{\alpha}_2$ on $\bar{L}_2(\bar{w}_2)$.

Lemma 13.23. $\mathcal{E}(\bar{L}_2(\bar{w}_2),\bar{\alpha}_2) \cong \pi^*\mathcal{E}(L_2(w_2),\alpha_2)$.

Proof. Let $(L_{\mathrm{pt}}(v),\sigma)\in(T^{2n},\Omega)^\vee$. Lemma 13.23 follows from

$$\begin{aligned}\mathcal{E}(L_2(w_2),\alpha_2)_{(L_{\mathrm{pt}}(v),\sigma)} &= \bigoplus_{p\in L_2(w_2)\cap L_{\mathrm{pt}}(v)} \mathbb{C}[p]\\ &\cong \bigoplus_{\bar{p}\in\bar{L}_2(\bar{w}_2)\cap\bar{L}_{\mathrm{pt}}(v)} \mathbb{C}[\bar{p}] = \mathcal{E}(\bar{L}_2(\bar{w}_2),\bar{\alpha}_2)_{(\bar{L}_{\mathrm{pt}}(v),\sigma)}.\end{aligned}$$

□

Lemma 13.24.

$$\mathrm{Ext}^k(\mathcal{E}(L_1(w_1),\alpha_1),\mathcal{E}(L_2(w_2),\alpha_2)) \cong \mathrm{Ext}^k(\mathcal{E}(\bar L_1(\bar w_1),\bar\alpha_1),\mathcal{E}(\bar L_2(\bar w_2),\bar\alpha_2)).$$

Proof. Lemma 13.23 and Proposition 13.8 imply :

$$\begin{aligned}\mathrm{Ext}^k(\mathcal{E}(L_1(w_1),\alpha_1),\mathcal{E}(L_2(w_2),\alpha_2)) &\cong \mathrm{Ext}^k(\pi_*\mathcal{E}(\bar L_1(\bar w_1),\bar\alpha_1),\mathcal{E}(L_2(w_2),\alpha_2)) \\ &\cong \mathrm{Ext}^k(\mathcal{E}(\bar L_1(\bar w_1),\bar\alpha_1),\pi^*\mathcal{E}(L_2(w_2),\alpha_2)) \\ &\cong \mathrm{Ext}^k(\mathcal{E}(\bar L_1(\bar w_1),\bar\alpha_1),\mathcal{E}(\bar L_2(\bar w_2),\bar\alpha_2)).\end{aligned}$$

□

On the other hand, π induces an isomorphism $L_1(w_1)\cap L_2(w_2) \cong \bar L_1(\bar w_1)\cap \bar L_2(\bar w_2)$. Hence

$$HF^k((L_1(w_1),\alpha_1),(L_2(w_2),\alpha_2)) \cong HF^k((\bar L_1(\bar w_1),\bar\alpha_1),(\bar L_2(\bar w_2),\bar\alpha_2)). \tag{13.25}$$

Lemma 13.24 and (13.25) imply that if (13.2) holds for $(\bar L_1(\bar w_1),\bar\alpha_1),(\bar L_2(\bar w_2),\bar\alpha_2)$ then it holds for $(L_1(w_1),\alpha_1),(L_2(w_2),\alpha_2)$.

Case 3. $G\subseteq G(L_2)$, $G\cap G(L_1)=\{1\}$:

We have $\mathcal{E}(\bar L_1(\bar w_1),\bar\alpha_1)\cong \pi^*\mathcal{E}(L_1(w_1),\alpha_1)$ and $\mathcal{E}(L_2(w_2),\alpha_2)\cong \pi_*\mathcal{E}(\bar L_2(\bar w_2),\bar\alpha_2)$. Hence

$$\begin{aligned}\mathrm{Ext}^k(\mathcal{E}(L_1(w_1),\alpha_1),\mathcal{E}(L_2(w_2),\alpha_2)) &\cong \mathrm{Ext}^k(\mathcal{E}(\bar L_1(\bar w_1),\bar\alpha_1),\pi_*\mathcal{E}(L_2(w_2),\alpha_2)) \\ &\cong \mathrm{Ext}^k(\pi^*\mathcal{E}(\bar L_1(\bar w_1),\bar\alpha_1),\mathcal{E}(L_2(w_2),\alpha_2)) \\ &\cong \mathrm{Ext}^k(\mathcal{E}(\bar L_1(\bar w_1),\bar\alpha_1),\mathcal{E}(\bar L_2(\bar w_2),\bar\alpha_2)).\end{aligned}$$

On the other hand, (13.25) holds also in this case. Therefore if (13.2) holds for $(\bar L_1(\bar w_1),\bar\alpha_1),(\bar L_2(\bar w_2),\bar\alpha_2)$ then it holds for $(L_1(w_1),\alpha_1),(L_2(w_2),\alpha_2)$.

Combining these three cases and Lemma 13.1, the proof of Theorem 13.5 is complete. □

We generalize Theorem 13.5 and prove Theorem δ in §15.

We finally prove the following converse to Proposition 3.9. We use the same notation as Proposition 3.9

Proposition 13.26. *If $\mathcal{E}(L(w),\alpha)$ is isomorphic to $\mathcal{E}(L'(w'),\alpha')$ then $[L(w),\alpha]=[L'(w'),\alpha']$. (Namely the affine Lagrangian submanifold $L(w)$ coincides with $L'(w')$ and the flat bundle determined by α is isomorphic to one determined by α'.)*

Proof. It is easy to see from Theorem 12.4 that $L(w)$ is parallel to $L'(w')$, if $\mathcal{E}(L(w),\alpha)$ is isomorphic to $\mathcal{E}(L'(w'),\alpha')$. Then Definition 2.6 implies $HF((L(w),\alpha),(L'(w'),\alpha'))=0$ unless $[L(w),\alpha]=[L'(w'),\alpha']$. On the other hand, if $\mathcal{E}(L(w),\alpha)$ is isomorphic to $\mathcal{E}(L'(w'),\alpha')$ then $\mathrm{Ext}^0(\mathcal{E}(L(w),\alpha),\mathcal{E}(L'(w'),\alpha'))$ is nontrivial. Proposition 13.26 then follows from Proposition 13.5. □

§14. Construction of Isomorphisms.

In this section, we construct a canonical isomorphism

$$\mathrm{Ext}^k(\mathcal{E}(L_1(w_1),\alpha_1),\mathcal{E}(L_2(w_2),\alpha_2)) \cong HF^k((L_1(w_1),\alpha_1),(L_2(w_2),\alpha_2))$$

under the assumption of Theorem ϵ. We use the operators $\mathfrak{m}_2^{(0,k)}$ we constructed in Chapter 2 for this purpose.

Let $s \in HF^k((L_1(w_1),\alpha_1),(L_2(w_2),\alpha_2))$. We are going to define

$$[\Phi(s)] \in \mathrm{Ext}^k(\mathcal{E}(L_1(w_1),\alpha_1),\mathcal{E}(L_2(w_2),\alpha_2)).$$

In our case, where we work under the assumption of Theorem ϵ, the sheaf $\mathcal{E}(L_i,\alpha_i)$ is a vector bundle. So we use Dolbault cohomology to calculate Ext. We are going to construct

$$\Phi(s) \in \mathfrak{D}'\left((T^{2n},\Omega)^\vee; Hom(\mathcal{E}(L_1(w_1),\alpha_1),\Lambda^{(0,k)}\otimes\mathcal{E}(L_2(w_2),\alpha_2))\right).$$

Here $\mathfrak{D}'$ denotes the space of distribution valued sections. In other words, $\Phi(s)$ is a $Hom(\mathcal{E}(L_1(w_1),\alpha_1),\mathcal{E}(L_2(w_2),\alpha_2))$ valued $(0,k)$ current.

We choose M_1, M_2 as in §5. Then we have a (distribution valued) homomorphism :

$$\begin{aligned}\mathfrak{m}_2^{(0,k)} : \pi_{0,1}^*\mathcal{P}(\tilde L_{\mathrm{pt}},\tilde L_1;L_{\mathrm{st}},M_1)\otimes\pi_{1,2}^*\mathcal{P}(\tilde L_1,\tilde L_2;M_1,M_2)\\ \to \Lambda^{(0,k)}\otimes\pi_{0,2}^*(\mathcal{P}(\tilde L_{\mathrm{pt}},\tilde L_2;L_{\mathrm{st}},M_2)).\end{aligned} \tag{14.1}$$

We lifts $(L_1(w_1),\alpha_1)\in\mathcal{M}(\tilde L_1)$, $(L_2(w_2),\alpha_2)\in\mathcal{M}(\tilde L_2)$, to $\mathcal{M}(\tilde L_1;M_1)$, $\mathcal{M}(\tilde L_2;M_2)$ respectively, and denote them by the same symbol. We recall $\mathcal{M}(\tilde L_{\mathrm{pt}};L_{\mathrm{st}}) = (T^{2n},\Omega)^\vee$, by definition.

We consider a submanifold $\mathcal{M}(\tilde L_{\mathrm{pt}};L_{\mathrm{st}})\times\{(L_1(w_1),\alpha_1)\}\times\{(L_2(w_2),\alpha_2)\}$ of $\mathcal{M}(\tilde L_{\mathrm{pt}};L_{\mathrm{st}})\times\mathcal{M}(\tilde L_1;M_1)\times\mathcal{M}(\tilde L_2;M_2)$. The operator $\mathfrak{m}_2^{(0,k)}$ in (14.1) is a disctribution valued section on $\mathcal{M}(\tilde L_{\mathrm{pt}};L_{\mathrm{st}})\times\mathcal{M}(\tilde L_1;M_1)\times\mathcal{M}(\tilde L_2;M_2)$.

We first need :

Lemma 14.2. *We can restrict $\mathfrak{m}_2^{(0,k)}$ in (14.1) to $\mathcal{M}(\tilde L_{\mathrm{pt}};L_{\mathrm{st}})\times\{(L_1(w_1),\alpha_1)\}\times\{(L_2(w_2),\alpha_2)\}$.*

Note that the restriction of distribution is not always possible. The notion of restriction of distribution is defined, for example, in [Hö].

Proof. We recall that $\bar{\mathfrak{C}}(\tilde L_{\mathrm{pt}},\tilde L_1,\tilde L_2)$ can be taken to be the negative eigenspace of Q on $L(\tilde L_{\mathrm{pt}},\tilde L_1,\tilde L_2)$. (See §7,8.) Hence its pull back $\mathfrak{C}(\tilde L_{\mathrm{pt}},\tilde L_1,\tilde L_2)$ to $\tilde L(\tilde L_{\mathrm{pt}},\tilde L_1,\tilde L_2) = V/\tilde L_{\mathrm{pt}}\times V/\tilde L_1\times V/\tilde L_2$ is transversal to $V/\tilde L_{\mathrm{pt}}\times 0\times 0$. Therefore, by definition (see [Hö]), the wave front set of the integral current $\mathfrak{C}(\tilde L_{\mathrm{pt}},\tilde L_1,\tilde L_2)$ is contained in the union of the conrmal bundles $T^*(\mathfrak{C}(\tilde L_{\mathrm{pt}},\tilde L_1,\tilde L_2)+w)$ over w. (Here $\mathfrak{C}(\tilde L_{\mathrm{pt}},\tilde L_1,\tilde L_2)+w$ is an affince spaces parallel to $\mathfrak{C}(\tilde L_{\mathrm{pt}},\tilde L_1,\tilde L_2)$.) We then find that the wave front set of the current $\mathfrak{m}_2^{(k)}$ is contained in the push out of this union of conormal bundles. Hence by [Hö] Theorem 2.5.11, we can restrict it to $\mathcal{M}(\tilde L_{\mathrm{pt}};L_{\mathrm{st}})\times\{(L_1(w_1),\alpha_1)\}\times\{(L_2(w_2),\alpha_2)\}$. □

Now we consider the restriction given by Lemma 14.2. The restriction of the bundle $\pi_{0i}^*\mathcal{P}(\tilde L_{\rm pt},\tilde L_i;L_{\rm st},M_1)$ to $\mathcal{M}(\tilde L_{\rm pt};L_{\rm st})\times\{(L_1,\alpha_1)\}\times\{(L_2,\alpha_2)\}\cong(T^{2n},\Omega)^\vee$ is $\mathcal{E}(L_i,\alpha_i)$. The restriction of $\pi_{12}^*\mathcal{P}(\tilde L_1,\tilde L_2;M_1,M_2)$ to $\mathcal{M}(\tilde L_{\rm pt};L_{\rm st})\times\{(L_1(w_1),\alpha_1)\}\times\{(L_2(w_2),\alpha_2)\}\cong(T^{2n},\Omega)^\vee$ is the product bundle $(T^{2n},\Omega)^\vee\times HF^k((L_1(w_1),\alpha_1);(L_2(w_2),\alpha_2))$. Hence the restriction gives a distribution valued homomorphism :

$$\begin{aligned}\mathfrak{m}_2^{(0,k)}:\mathcal{E}(L_1(w_1),\alpha_1)\otimes HF^k((L_1(w_1),\alpha_1);(L_2(w_2),\alpha_2))\\ \to\Lambda^{(0,k)}((T^{2n},\Omega)^\vee)\otimes\mathcal{E}(L_2(w_2),\alpha_2).\end{aligned}$$

Definition 14.3. Let v be a section of $\mathcal{E}(L_1(w_1),\alpha_1)$. We then put :

$$\Phi(s)(v)=\mathfrak{m}_2^{(0,k)}(v\otimes s).$$

(Note we do not assume v to be holomorphic.)

Lemma 14.4. $\overline\partial(\Phi(s))=0$.

Proof. By Theorem γ and $\mathfrak{m}_1=0$, we have

$$\overline\partial(\Phi(s)(v))=\overline\partial(\mathfrak{m}_2^{(0,k)}(v\otimes s))=(-1)^k\mathfrak{m}_2^{(0,k)}(\overline\partial v\otimes s)=(-1)^k\Phi(s)(\overline\partial v).$$

Lemma 14.4 follows. □

We thus obtained a homomorphism

$$\Phi_{12}:HF^k((L_1,\alpha_1),(L_2,\alpha_2))\to{\rm Ext}^k(\mathcal{E}(L_1,\alpha_1),\mathcal{E}(L_2,\alpha_2)).\tag{14.5}$$

Before proving that Φ is an isomorphism, we check the commutativity of Diagram 1 in introduction. We assume that $\tilde L_i$, $i=1,2,3$ are mutually transversal and are transversal to $L_{\rm pt}$. We assume $\deg(\tilde L_1,\tilde L_2,\tilde L_3)=0$. Then

$$\begin{aligned}\mathfrak{m}_2:HF((L_1(w_1),\alpha_1),(L_2(w_2),\alpha_2))\otimes HF((L_2(w_2),\alpha_2),(L_3(w_3),\alpha_3))\\ \to HF((L_1(w_1),\alpha_1),(L_3(w_3),\alpha_3))\end{aligned}$$

is defined for every v_i,α_i. (In other words, in this case, for the map (14.1), we have $k=0$ and the map is smooth.)

Theorem 14.6. *Let $s_{i,j}\in HF((L_i(w_i),\alpha_i),(L_j(w_j),\alpha_j))$. Then we have :*

$$\Phi_{1,3}(\mathfrak{m}_2(s_{1,2},s_{23,}))=\pm\Phi_{1,2}(s_{1,2})\circ\Phi_{2,3}(s_{2,3})$$

Here $\circ$ is the Yoneda product.

Proof. Let v be a section of $\mathcal{E}(L_1(w_1),\alpha_1)$. Then, by Theorem γ, we have

$$\begin{aligned}\Phi_{1,3}(\mathfrak{m}_2(s_{1,2},s_{2,3}))(v)&=\mathfrak{m}_2(v,\mathfrak{m}_2(s_{1,2},s_{2,3}))\\ &=\pm\mathfrak{m}_2(\mathfrak{m}_2(v,s_{1,2}),s_{2,3})\pm\overline\partial(\mathfrak{m}_3(v,s_{1,2},s_{2,3}))\pm\mathfrak{m}_3(\overline\partial v,s_{1,2},s_{2,3}).\end{aligned}$$

On the other hand, by definition

$$\mathfrak{m}_2(\mathfrak{m}_2(v,s_{1,2}),s_{2,3})=\pm(\Phi_{1,2}(s_{1,2})\circ\Phi_{2,3}(s_{2,3}))(v).$$

Theorem 14.6 follows. □

In the same way we can prove the coincidence of Massey products. We first recall its definition. We first define Massey-Yoneda product, in the case we use in this section. Let $\mathcal{E}_i$ be holomorphic vector bundles on $(T^{2n},\Omega)^\vee$. We assume that $\mathrm{Ext}^k(\mathcal{E}_i,\mathcal{E}_j)$ is zero for $k \ne \eta(\tilde L_i,\tilde L_j)$. Hereafter we will write $\eta(i,j)$ in place of $\eta(\tilde L_i,\tilde L_j)$. We assume furthermore

$$\eta(1,2)+\eta(2,3)+\eta(3,4)=\eta(1,4)+1, \tag{14.7}$$

$$\eta(1,2)+\eta(2,3)\ne\eta(1,3), \tag{14.8.1}$$

$$\eta(2,3)+\eta(3,4)\ne\eta(2,4). \tag{14.8.2}$$

Let $[u_{i,j}]\in \mathrm{Ext}^{\eta(i,j)}(\mathcal{E}_i,\mathcal{E}_j)$. We represent $[u_{i,j}]$ by a smooth section $u_{i,j}$ of $\Lambda^{(0,\eta(i,j))}\otimes Hom(\mathcal{E}_i,\mathcal{E}_j)$ satisfying $\overline\partial u_{i,j}=0$.

By (14.8), there exists $v_{1,3},v_{2,4}$ such that

$$\overline\partial v_{1,3}=u_{1,2}\wedge u_{2,3}, \tag{14.9.1}$$

$$\overline\partial v_{2,4}=u_{2,3}\wedge u_{3,4}. \tag{14.9.2}$$

We put

$$w_{1,2,3,4}=v_{1,3}\wedge u_{3,4}-(-1)^{\eta(1,2)}u_{1,2}\wedge v_{2,4}.$$

It is easy to check $\overline\partial w_{1,2,3,4}=0$.

Definition 14.10. $[w_{1,2,3,4}]\in \mathrm{Ext}^{\eta(1,4)}(\mathcal{E}_1,\mathcal{E}_4)$ is called the *Massey(-Yoneda) triple product* $\langle u_{1,2},u_{2,3},u_{3,4}\rangle$.

In general, Massey product is well-defined only as an element of a coset space. However in our case using the assumption $\mathrm{Ext}^d(\mathcal{E}_1,\mathcal{E}_4)=0$ for $d\ne\deg(1,4)$, the Massey triple product is defined as an element of $\mathrm{Ext}^{\eta(1,4)}(\mathcal{E}_1,\mathcal{E}_4)$. Namely we can check that it is independent of the choice of representatives $u_{1,2}$,$u_{2,3}$,$u_{3,4}$.

Next we turn to the Floer homology side. We assume that $\tilde L_i$, $i=1,2,3,4$ are mutually transversal and are transversal to $L_{\rm pt}$. We also assume $\deg(i,j)$ satisfy (14.7) and (14.8). We consider

$$\begin{aligned}\mathfrak{m}_3:\pi_{12}^*\mathcal{P}(\tilde L_1,\tilde L_2;M_1,M_2)&\otimes\pi_{23}^*\mathcal{P}(\tilde L_2,\tilde L_3;M_2,M_3)\\ &\otimes\pi_{34}^*\mathcal{P}(\tilde L_3,\tilde L_4;M_3,M_4)\to\pi_{14}^*\mathcal{P}(\tilde L_1,\tilde L_4;M_1,M_4).\end{aligned}$$

(We remark that $\deg(1,2,3,4)=0$ in this case.) $\mathfrak{m}_3$ is a distribution valued homomorphism and hence is not everywhere well-defined. However it is well-defined in a Bair subset of $\mathcal{M}(\tilde L_1;M_1)\times\mathcal{M}(\tilde L_2;M_2)\times\mathcal{M}(\tilde L_3;M_3)\times\mathcal{M}(\tilde L_4;M_4)$. Let $([w_1,\alpha_1],[w_2,\alpha_2],[w_3,\alpha_3],[w_4,\alpha_4])$ be in this set. Then $\mathfrak{m}_3$ defines a map

$$\begin{aligned}\mathfrak{m}_3:HF((L_1(w_1),\alpha_1),(L_2(w_2),\alpha_2))&\otimes HF((L_2(w_2),\alpha_2),(L_3(w_3),\alpha_3))\\ \otimes HF((L_3(w_3),\alpha_3),(L_4(w_4),\alpha_4))&\to HF((L_1(w_1),\alpha_1),(L_4(w_4),\alpha_4)).\end{aligned} \tag{14.11}$$

(14.11) is the Massey triple product of Floer homology in our case.

Theorem 14.12. *Let $s_{i,j} \in HF((L_i(v_i),\alpha_i),(L_j(v_j),\alpha_j))$. Then we have*

$$\langle \Phi_{1,2}(s_{1,2}), \Phi_{2,3}(s_{2,3}), \Phi_{3,4}(s_{3,4})\rangle = \pm\Phi_{1,4}(\mathfrak{m}_3(s_{1,2},s_{2,3},s_{3,4})).$$

Proof. Let v be a section of $\mathcal{E}(L_1(v_1),\alpha_1)$. We remark that $\mathfrak{m}_2(s_{1,2},s_{2,3}) = \mathfrak{m}_2(s_{2,3},s_{3,4}) = 0$ by degree reason. Then, by Theorem γ, we have

$$\begin{aligned}&\Phi_{1,4}(\mathfrak{m}_3(s_{1,2},s_{2,3},s_{3,4}))(v)\\ &\qquad = \mathfrak{m}_2(v,\mathfrak{m}_3(s_{1,2},s_{2,3},s_{3,4}))\\ &\qquad = \pm\mathfrak{m}_3(\mathfrak{m}_2(v,s_{1,2}),s_{2,3},s_{3,4}) \pm \mathfrak{m}_2(\mathfrak{m}_3(v,s_{12},s_{23}),s_{34})\\ &\qquad\quad \pm \overline{\partial}\mathfrak{m}_4(v,s_{1,2},s_{2,3},s_{3,4}) \pm \mathfrak{m}_4(\overline{\partial}v,s_{12,},s_{2,3},s_{3,4}).\end{aligned} \tag{14.13}$$

Let us consider current valued section $v\mapsto \mathfrak{m}_3(v,s_{1,2},s_{2,3})$, which we put $\Phi_{1,2,3}(s_{1,2},s_{2,3})$.

Lemma 14.14.

$$\overline{\partial}(\Phi_{1,2,3}(s_{1,2},s_{2,3})) = \pm\Phi_{1,2}(s_{1,2})\circ\Phi_{2,3}(s_{2,3}).$$

Proof. By Theorem γ, we have :

$$\begin{aligned}\overline{\partial}(\Phi_{1,2,3}(s_{1,2},s_{2,3})(v)) &= \overline{\partial}(\mathfrak{m}_3(v,s_{1,2},s_{2,3}))\\ &= \pm\mathfrak{m}_2(\mathfrak{m}_2(v,s_{1,2}),s_{2,3}) \pm \mathfrak{m}_3(\overline{\partial}v,s_{1,2},s_{2,3}).\end{aligned}$$

The lemma follows. □

By (14.13) and Lemma 14.14, we have

$$\begin{aligned}&\Phi_{1,4}(\mathfrak{m}_3(s_{1,2},s_{2,3},s_{3,4}))(v)\\ &\quad = \pm\,\Phi_{2,3,4}(s_{2,3},s_{3,4})(\Phi_{1,2}(s_{1,2})(v)) \pm \Phi_{34}(s_{34})(\Phi_{123}(s_{12},s_{23})(v))\\ &\qquad\quad \pm\,\overline{\partial}\mathfrak{m}_4(s_{12},s_{23},s_{34},v) \pm \mathfrak{m}_4(s_{1,2},s_{2,3},s_{3,4},\overline{\partial}v)\\ &\quad \equiv \pm\,\langle\Phi_{1,2}(s_{1,2}),\Phi_{2,3}(s_{2,3}),\Phi_{3,4}(s_{3,4})\rangle(v) \mod \operatorname{Im}\overline{\partial}.\end{aligned}$$

Theorem 14.12 follows. □

We can continue and can check the coincidence of higher Massey products in a similar way.

We next check that the map $\Phi_{i,j}$ is independent of the choice of the system of chains $\mathfrak{C}$. We assume that $\mathfrak{m}'_k$ is homotopy equivalent to $\mathfrak{m}_k$ in the sense of Definition 11.8 and $\mathfrak{h}_k$ be the homotpy between them. We use $\mathfrak{m}'_k$ in place of $\mathfrak{m}_k$ and obtain

$$\Phi'_{1,2} : HF^k((L_1(w_1),\alpha_1),(L_2(w_2),\alpha_2)) \to \operatorname{Ext}^k(\mathcal{E}(L_1(w_1),\alpha_1),\mathcal{E}(L_2(w_2),\alpha_2)).$$

Lemma 14.15. $\Phi_{1,2} = \Phi'_{1,2}$.

Proof. Let $s\in HF^k((L_1(w_1),\alpha_1),(L_2(w_2),\alpha_2))$ and v be a section of $\mathcal{E}(L_1(w_1),\alpha_1)$. Definition 11.8 implies

$$\overline{\partial}(\mathfrak{h}_2(v,s)) \pm \mathfrak{h}_2(\overline{\partial}v,s) = \mathfrak{m}_2(v,s) - \mathfrak{m}'_2(v,s) = \Phi_{1,2}(s)(v) - \Phi'_{1,2}(s)(v).$$

Thus $\Phi_{1,2}(s)$ represents the same cohomology class as $\Phi'_{1,2}(s)$. □

Now we are going to prove the main result of this chapter :

Theorem 14.16. *$\Phi_{i,j}$ is an isomorphism.*

Proof. We already proved that the rank of the left hand side of (14.5) coincides with the right hand side of (14.5) in §13. So it suffices to show that Φ_{ij} is injective. We use Axiom III here.

Remark 14.17. We did not use Axiom III in the construction of $\Phi_{i,j}$. (We mainly used Maurer-Cartan equation, which follows from Axiom II.) In fact Axiom I,II are satisfied if we put $\mathfrak{C} \equiv 0$. Then $\Phi_{i,j} = 0$. Hence the results of this section (except Theorem 14.16) is obviously satisfied. We need Axiom III to make sure that our operators are nontrivial.

We remark that there exists a nondegenerate pairing

$$\begin{aligned}\langle \cdot, \cdot \rangle : \mathrm{Ext}^k(&\mathcal{E}(L_1(w_1), \alpha_1), \mathcal{E}(L_2(w_2), \alpha_2)) \\ &\otimes \mathrm{Ext}^{n-k}(\mathcal{E}(L_2(w_2), \alpha_2), \mathcal{E}(L_1(w_1), \alpha_1)) \to \mathbb{C}.\end{aligned}$$

that is the Serre duality. (Note the canonical bundle of the torus is trivial.)

On the other hand, if $k = \eta(\tilde{L}_1, \tilde{L}_2) = n - \eta(\tilde{L}_2, \tilde{L}_1)$ then

$$\begin{aligned}HF^k((L_1(w_1), \alpha_1), (L_2(w_2), \alpha_2)) &\cong HF^{n-k}((L_2(w_2), \alpha_2), (L_1(w_1), \alpha_1)) \\ &\cong \bigoplus_{p \in L_1 \cap L_2} \mathbb{C}[p].\end{aligned}$$

We use it to define a perfect pairing

$$\langle \cdot, \cdot \rangle : \sum a_p[p] \otimes \sum b_p[p] \mapsto \sum a_p b_p$$

between $HF^k((L(w_1)_1, \alpha_1), (L_2(w_2), \alpha_2))$ and $HF^{n-k}((L_2(w_2), \alpha_2), (L_1(w_1), \alpha_1))$. Now Theorem 14.16 follows from the following :

Theorem 14.18.

$$\langle \Phi_{1,2}(s_{1,2}), \Phi_{2,1}(s_{2,1}) \rangle = C \langle s_{1,2}, s_{2,1} \rangle.$$

Here C is a nonzero constant.

Proof. We will calculate $\Phi_{1,2}(s_{1,2})$ and $\Phi_{2,1}(s_{2,1})$ more explicitely. We may regard $L(\tilde{L}_{\mathrm{pt}}, \tilde{L}_1, \tilde{L}_2) \cong V/\tilde{L}_{\mathrm{pt}}$. By definition, we may choose $\mathfrak{C}(\tilde{L}_{\mathrm{pt}}, \tilde{L}_1, \tilde{L}_2) \subset L(\tilde{L}_{\mathrm{pt}}, \tilde{L}_1, \tilde{L}_2)$ as a codimention k linear subspace where Q is positive definite. (We remark that Lemma 14.15 implies that we may take arbitrary $\mathfrak{C}$ satisfying Axioms I,II,III to prove Theorem 14.16.) Furthermore we may assume $\mathfrak{C}(\tilde{L}_{\mathrm{pt}}, \tilde{L}_1, \tilde{L}_2) \cap \Gamma \cong \mathbb{Z}^n$ by perturbing it a bit if necessary. We put $\mathfrak{C}(\tilde{L}_{\mathrm{pt}}, \tilde{L}_1, \tilde{L}_2) \cap \Gamma = \Gamma_1 \subset \Gamma/(\Gamma \cap \tilde{L}_{\mathrm{pt}})$. Let us consider the case $s_{1,2} = [p]$, where $p \in L_1 \cap L_2$. Let $\tilde{p} \in V$ be a lift of it. Then the support of the current $\Phi_{1,2}([p])$ is in

$$\begin{aligned}T &= \left\{ [v] \;\middle|\; \tilde{p} \in \hat{L}_{\mathrm{pt}}(v) + \mathfrak{C}(\tilde{L}_{\mathrm{pt}}, \tilde{L}_1, \tilde{L}_2) \right\} \\ &= \left\{ [v] \;\middle|\; v - [\tilde{p}] \in \mathfrak{C}(\tilde{L}_{\mathrm{pt}}, \tilde{L}_1, \tilde{L}_2) \right\} \qquad (14.19) \\ &\cong T^{n-k} \subset (V/\tilde{L}_{\mathrm{pt}})/(\Gamma/(\Gamma \cap \tilde{L}_{\mathrm{pt}})).\end{aligned}$$

We consider $[v] \in T$. Let $q_1 \in L_{\rm pt}(v) \cap L_1(w_1)$, $q_2 \in L_{\rm pt}(v) \cap L_2(w_2)$. We are going to calculate the $[q_2]$ coefficient of $\Phi_{1,2}([p])([q_1])$. It is zero unless

$$q_1, q_2 \in \tilde p + \mathfrak{C}(\tilde L_{\rm pt}, \tilde L_1, \tilde L_2) + \tilde L_{\rm pt} \mod \Gamma. \tag{14.20}$$

In case (14.20) holds, we choose lifts $\tilde q_1$,$\tilde q_2$ of q_1,q_2 to V such that

$$\tilde q_1, \tilde q_2 \in \tilde p + \mathfrak{C}(\tilde L_{\rm pt}, \tilde L_1, \tilde L_2) + \tilde L_{\rm pt}. \tag{14.21}$$

We may assume furthermore, $\tilde q_1 \in \hat L_1(\tilde p)$, $\tilde q_2 \in \hat L_2(\tilde p)$.

We define $\tilde q_1(\gamma)$, $\tilde q_2(\gamma)$ by $\{\tilde q_1(\gamma)\} = \hat L_1(\tilde p) \cap \hat L_{\rm pt}(v+\gamma)$, $\{\tilde q_2(\gamma)\} \in \hat L_2(\tilde p) \cap \hat L_{\rm pt}(v+\gamma)$. We put

$$\Gamma'_1 = \{\gamma \in \Gamma'_1 | \pi(\tilde q_1(\gamma)) = q_1,\ \pi(\tilde q_2(\gamma)) = q_2\},$$

and

$$\begin{aligned}\Theta_0^{(k)}(v,\sigma)_{p,q_1,q_2} = \sum_{\gamma\in\Gamma'_1} &\exp(-2\pi Q(\tilde q_1(\gamma), \tilde p, \tilde q_2(\gamma)) \\ &+ 2\pi\sqrt{-1} H(\alpha_1, \sigma, \alpha_2; \tilde q_1(\gamma), \tilde p, \tilde q_2(\gamma))).\end{aligned} \tag{14.22}$$

Figure 16

Let $\mathfrak{C}_T$ be the k current such that

$$\int_{T^n} \mathfrak{C}_T \wedge u = \int_T u$$

for $n-k$ form u on $T^n = (V/\tilde L_{\rm pt})/(\Gamma/(\Gamma \cap \tilde L_{\rm pt}))$. We pull it back to $(T^{2n}, \Omega)^\vee$. Let $\mathfrak{C}_T^{(0,k)}$ be the $(0,k)$ component of the pull back. By definition, the $[q_2]$ coefficient of $\Phi_{12}([p])([q_1])$ around (v,σ) is $\Theta_0^{(k)}(v,\sigma)_{p,q_1,q_2}\mathfrak{C}_T^{(0,k)}$. We remark that

$$\Theta_0^{(k)}(v,\sigma)_{p,q_1,q_2} = \sum_{\gamma\in\Gamma'_1} C_\gamma \exp\left(2\pi\sqrt{-1}\sigma(q_2(\gamma) - q_1(\gamma))\right). \tag{14.23}$$

where C_γ is independent of σ.

We next consider $\Phi_{2,1}(s_{2,1})$. Note $L(\tilde L_{\rm pt}, \tilde L_1, \tilde L_2) \cong L(\tilde L_{\rm pt}, \tilde L_2, \tilde L_1) \cong V/\tilde L_{\rm pt}$. Then we may identify $Q|_{L(\tilde L_{\rm pt}, \tilde L_1, \tilde L_2)} = -Q|_{L(\tilde L_{\rm pt}, \tilde L_2, \tilde L_1)}$ by this isomorphism. Hence $\mathfrak{C}(\tilde L_{\rm pt}, \tilde L_1, \tilde L_2)$ is transversal to $\mathfrak{C}(\tilde L_{\rm pt}, \tilde L_2, \tilde L_1)$.

Let $p' \in L_1(w_1) \cap L_2(w_2)$. We consider $s_{2,1} = [p']$. Let $\tilde p' \in V$ be its lift. We put

$$\begin{aligned}T' &= \left\{[v] \;\middle|\; v - \tilde p' \in \mathfrak{C}(\tilde L_{\rm pt}, \tilde L_2, \tilde L_1)\right\} \\ &\cong T^k \subset (V/\tilde L_{pt})/(\Gamma/\Gamma \cap \tilde L_{pt}).\end{aligned} \tag{14.24}$$

Let $[v] \in T'$, and $q_1 \in L_{\rm pt}(v) \cap L_1(w_1)$, $q_2 \in L_{\rm pt}(v) \cap L_2(w_2)$. We define

$$\begin{aligned}\Theta_0^{(n-k)}(v,\sigma)_{p',q_2,q_1} = \sum_{\mu\in\Gamma_2} \exp(2\pi Q(q_1'(\mu), \tilde p', q_2'(\mu)) - \\ 2\pi\sqrt{-1} H(\alpha_1,\sigma,\alpha_2; q_1'(\mu), \tilde p', q_2'(\mu)))\,.\end{aligned} \tag{14.25}$$

Here $\{\tilde q_1'(\mu)\} = \hat L_1(\tilde p') \cap \hat L_{\rm pt}(v+\mu)$, $\{\tilde q_2'(\mu)\} = \hat L_2(\tilde p') \cap \hat L_{\rm pt}(v+\mu)$ and

$$\Gamma_2' = \{\mu \in \mathfrak{C}(\tilde L_{\rm pt}, \tilde L_2, \tilde L_1) \cap \Gamma \subset \Gamma/(\Gamma\cap\tilde L_{\rm pt}) | \pi(\tilde q_1'(\mu)) = q_1,\ \pi(\tilde q_2'(\mu)) = q_2\}.$$

By definition, the $[q_1]$ component of $\Phi_{21}([p'])([q_2])$ at (v,σ) is $\Theta_0^{(n-k)}(v,\sigma)_{p',q_2,q_1}\mathfrak{e}_{T'}^{(0,n-k)}$. (14.25) implies

$$\Theta_0^{(n-k)}(v,\sigma)_{p',q_2,q_1} = \sum_{\mu\in\Gamma_2'} C_\gamma' \exp\left(-2\pi\sqrt{-1}\sigma(\tilde q_2'(\mu) - \tilde q_1'(\mu))\right). \tag{14.26}$$

Let ω^n be the nontrivial holomorphic n form on $(T^{2n},\Omega)^\vee$. We have :

$$\begin{aligned}&\int_{(T^{2n},\Omega)^\vee} \tilde\Phi_0([p])\wedge\tilde\Phi_0([p']) \wedge \omega^n \\ &\qquad = C \sum_{v\in T\cap T'} \int \Theta_0^{(k)}(v,\sigma)_{p,q_1,q_2} \wedge \Theta_0^{(n-k)}(v,\sigma)_{p',q_2,q_1}\, d\sigma.\end{aligned} \tag{14.27}$$

By (14.23) and (14.26), we find that (14.27) is equal to

$$C \sum_{\substack{\gamma\in\Gamma_1',\\ \mu\in\Gamma_2'}} C_\gamma C_\mu \int \exp\left(2\pi\sqrt{-1}\sigma\left(\tilde q_2(\gamma) - \tilde q_1(\gamma) - \tilde q_2'(\mu) + \tilde q_1'(\mu)\right)\right) d\sigma. \tag{14.28}$$

The integral in (14.28) is zero unless

$$\tilde q_1(\gamma) - \tilde q_2(\gamma) - \tilde q_1'(\mu) + \tilde q_2'(\mu) = 0. \tag{14.29}$$

Lemma 14.30. *If (14.29) holds then we have :*

$$\tilde q_1(\gamma) - \tilde q_2(\gamma) = \tilde q_1'(\mu) - \tilde q_2'(\mu) = 0. \tag{14.31}$$

Proof. We define $\phi : V/\tilde L_{\rm pt} \to \tilde L_{\rm pt}$ as follows. Let $x \in V/\tilde L_{\rm pt}$. We put

$$\{\tilde q_1(x)\} = \tilde L_{\rm pt}(v+x) \cap \tilde L_1(\tilde p),$$
$$\{\tilde q_2(x)\} = \tilde L_{\rm pt}(v+x) \cap \tilde L_2(\tilde p).$$

Then we define

$$\phi(x) = \tilde q_2(x) - \tilde q_1(x).$$

Since $\tilde L_1 \cap \tilde L_{\rm pt} = \tilde L_2 \cap \tilde L_{\rm pt} = \tilde L_1 \cap \tilde L_2 = 0$, it follows that ϕ is an isomorphism. The lemma then follows from $\tilde q_1(\gamma) - \tilde q_2(\gamma) = \phi(\gamma)$, $\tilde q_1'(\mu) - \tilde q_2'(\mu) = \phi(\mu)$, $\gamma \in \mathfrak{C}(\tilde L_{\rm pt}, \tilde L_1, \tilde L_2)$, $\mu \in \mathfrak{C}(\tilde L_{\rm pt}, \tilde L_2, \tilde L_1)$, $\mathfrak{C}(\tilde L_{\rm pt}, \tilde L_1, \tilde L_2) \cap \mathfrak{C}(\tilde L_{\rm pt}, \tilde L_2, \tilde L_1) = \{0\}$. □

Since $\hat L_1(\tilde p) \cap \hat L_2(\tilde p) = \{\tilde p\}$, $\hat L_1(\tilde p') \cap \hat L_2(\tilde p') = \{\tilde p'\}$, it follows from (14.31) that

$$\tilde p = \tilde q_1(\gamma) = \tilde q_2(\gamma), \qquad \tilde p' = \tilde q_1'(\mu) = \tilde q_2'(\mu). \tag{14.32}$$

Therefore $[\tilde p] = [q_1] = [\tilde p']$. In this case, we may choose $\tilde p = \tilde p'$. Then (14.28) is $CC_0C_0{\rm Vol}(T^n)$ and is a constant. The proof of Therem 14.18 is complete. □

As we remarked before, Theorem 14.18 implies Theorem 14.16. □

§15. The general case.

In this section, we generalize Proposition 13.5 and prove Theorem δ. We first consider the case when $L_1(w_1)$ is transversal to $L_2(w_2)$. (We do not assume that they are transversal to L_{pt}.) We consider

$$\begin{aligned} T(L_i(w_i),\alpha_i) &= \Big\{ [v,\sigma] \mid w_i - v \in \tilde L_i + \tilde L_{\text{pt}},\ \alpha_i|_{\tilde L \cap \tilde L_{\text{pt}}} - \sigma|_{\tilde L\cap \tilde L_{\text{pt}}} = 0 \Big\} \\ &\subseteq (T^{2n},\Omega)^\vee. \end{aligned}$$

as in §4.

Lemma 15.1. *If $L_1(w_1)$ is transversal to $L_2(w_2)$ then $T(L_1(w_1),\alpha_1)$ is tranversal to $T(L_2(w_2),\alpha_2)$.*

Proof. Let $[v,\sigma] \in T(L_1(w_1),\alpha_1) \cap T(L_2(w_2),\alpha_2)$. We may identify

$$T_{[v,\sigma]}(T^{2n},\Omega)^\vee = V/\tilde L_{\text{pt}} \oplus L^*_{\text{pt}}.$$

Then

$$T_{[v,\sigma]}T(L_i(w_i),\alpha_i) = (\tilde L_i + \tilde L_{\text{pt}})/\tilde L_{\text{pt}} \oplus (\tilde L_i \cap \tilde L_{\text{pt}})^\perp.$$

The lemmma follows. □

By definition, the sheaf $\mathcal E(L_i(w_i),\alpha_i)$ is a direct image sheaf of a holomorphic vector bundle on $T(L_i(w_i),\alpha_i)$. Here we recall the following well-known lemma.

Let M be a complex manifold and N_i be complex submanifolds of codimention d_i. We assume that N_1 is transversal to N_2. Let $\mathcal E_i$ be a holomorphic vector bundle on N_i. Let $I_i : N_i \to M$ be the inclusion and $\pi_i : NN_i \to N_i$ be the normal bundle.

Lemma 15.2. *We assume $\Lambda^{d_1} NN_1|_{N_1\cap N_2}$ is trivial. Then we have*

$$\text{Ext}^k(I_{1*}\mathcal E_1, I_{2*}\mathcal E_2) \simeq \text{Ext}^{k-d_1}(\mathcal E_1|_{N_1\cap N_2}, \mathcal E_2|_{N_1\cap N_2}).$$

Proof. We consider the Kozul complex

$$0 \to \Lambda^{d_1} NN_1 \to \cdots \to \Lambda^1 NN_1 \to \mathbb C \to 0. \tag{15.3}$$

We take a tubular neighborhood of N_1 and pull back (15.3) there. We also extend $\mathcal E_1$ to a neighborhood of N and denote it by $\tilde{\mathcal E}_1$. Then

$$0 \to \pi_1^*\Lambda^{d_1} NN_1 \otimes \tilde{\mathcal E}_1 \to \cdots \to \pi_1^*\Lambda^1 NN_1 \otimes \tilde{\mathcal E}_1 \to \pi_1^*\mathbb C \otimes \tilde{\mathcal E}_1 \to I_{1*}\mathcal E_1 \to 0 \tag{15.4}$$

is a locally free resolution. Therefore, using the assumption that $\Lambda^{d_1} NN_1|_{N_1\cap N_2}$ is trivial, we have

$$\mathfrak{Ext}^k(I_{1*}\mathcal E_1, I_{2*}\mathcal E_2) \cong \begin{cases} 0 & \text{if } k \ne d_1, \\ I_*\mathfrak{Hom}(\mathcal E_1|_{N_1\cap N_2}, \mathcal E_2|_{N_1\cap N_2}) & \text{if } k = d_1. \end{cases}$$

Here $I : N_1 \cap N_2 \to M$ is the inclusion. The lemma follows easily. □

We put

$$\begin{aligned} W &= (\tilde L_1 + \tilde L_{\text{pt}}) \cap (\tilde L_2 + \tilde L_{\text{pt}}), \\ W_0 &= \{ w \in W \mid \forall v \in W, \omega(w,v) = 0\}. \end{aligned}$$

We remark

$$\begin{aligned}W_0 &= (\tilde L_1 \cap \tilde L_{\rm pt}) + (\tilde L_2 \cap \tilde L_{\rm pt}) \\ &= \{w \in W \mid \forall v \in W, \Omega(w,v) = 0\}.\end{aligned}$$

Since $\tilde L_{\rm pt} \subset W$ is Lagrangian linear space, we have $W_0 \subseteq \tilde L_{\rm pt} \subseteq W$. We define

$$\bar W = W/W_0. \tag{15.5}$$

$\bar W$ carries a symplectic structure. (In fact, $\bar W$ is the linear symplectic reduction of V with respect to W_0.) The B field on V induces one on $\bar W$.

We put $\bar\Gamma = (\Gamma \cap W)/(\Gamma \cap W_0)$. $\bar\Gamma$ is a lattice of $\bar W$. In a way similar to §4 we define :

$$\begin{aligned}\tilde L^-_{\rm pt} &= \tilde L_{\rm pt}/W_0 \subset \bar W, \\ \tilde L^-_{\rm st} &= ((\tilde L_{\rm st} + W_0) \cap W)/W_0, \\ \tilde L^-_i &= ((\tilde L_i + W_0) \cap W)/W_0.\end{aligned}$$

They are Lagrangian submanifolds by [MS2] Lemma 2.7. We define a mirror torus $(\bar W/\bar\Gamma, \bar\Omega)^\vee$ using $\tilde L^-_{\rm pt}$. It is easy to see

$$\bar W/\tilde L^-_{\rm pt} \cong W/\tilde L_{\rm pt} \subseteq V/\tilde L_{\rm pt}, \tag{15.6.1}$$

$$(\tilde L^-_{\rm pt})^* \cong W_0^\perp \subseteq \tilde L^*_{\rm pt}. \tag{15.6.2}$$

Hence $(\bar W/\bar\Gamma, \bar\Omega)^\vee \subseteq (T^{2n}, \Omega)^\vee$. We can prove the following lemma in a way similar to §4.

Lemma 15.6. *Connected components of* $T = T(L_1(w_1), \alpha_1) \cap T(L_2(w_2), \alpha_2)$ *are orbits of* $(\bar W/\bar\Gamma, \bar\Omega)^\vee$.

We decompose T as a union of complex subtori $T = \cup_\ell T_\ell$. Let $(v_\ell, \sigma_\ell) \in T_\ell$. We define an isomorphism $I_{(v_\ell,\sigma_\ell)} : (\bar W/\bar\Gamma, \bar\Omega)^\vee \to T_\ell$ by $I_{(v_\ell,\sigma_\ell)}(g) = g(v_\ell, \sigma_\ell)$.

Let $(u, \sigma) \in T_\ell$, $p_i \in L_{\rm pt}(u) \cap L_i(w_i)$. We can find a lift $\tilde p_i \in V$ of p_i such that

$$\tilde p_i - \tilde v_\ell \in (\tilde L_1 + L_{\rm pt}) \cap (\tilde L_2 + L_{\rm pt}) = W. \tag{15.7}$$

Moreover the lift $\tilde p_i \in V$ satisfying (15.7) is unique modulo the action of $W \cap \Gamma$. Hence

$$f_{\ell,i}(p_i) := [\tilde p_i - \tilde v_\ell] \in \bar W/\bar\Gamma$$

depends only on p_i, ℓ. We put

$$L^-_i(\overline w_i(v_\ell)) = \{f_{\ell,i}(p) | \exists u \; p_i \in L_{\rm pt}(u) \cap L_i(w_i)\}. \tag{15.8}$$

(It is easy to see that the right hand side of (15.8) is parallel to L^-_i.)

Using the splitting $V = \tilde L_{\rm st} \oplus \tilde L_{\rm pt}$, we have a projection $\pi_{\tilde L_{\rm st}} : V \to \tilde L_{\rm pt}$. We put

$$\bar\alpha_{i,\sigma_0} = \alpha_i - \pi^*_{\tilde L_{\rm pt}}(\sigma_0) \in (\tilde L^-_i)^* \subseteq \tilde L^*_i. \tag{15.9}$$

We remark that $\tilde L^-_i$ is transversal to $\tilde L^-_{\rm pt}$. Now, by Definition 4.12, we have the following :

Lemma 15.10.

$$I_{(v_\ell,\sigma_\ell)*}\mathcal{E}(L_i^-(\overline{w}_i(v_\ell)),\bar\alpha_{i,\sigma_0}) \cong \mathcal{E}(L_i(w_i),\alpha_i)|_{T_\ell}.$$

We also have :

Lemma 15.11.

$$\sharp\pi_0(T)\cdot\sharp(L_1^-(\overline{w}_1(v_\ell))\cap L_2^-(\overline{w}_2(v_\ell))) = \sharp(L_1(w_1)\cap L_2(w_2)).$$

Proof. Let $x \in L_1(w_1)\cap L_2(w_2)$. Then there exists unique $y \in (V/\Gamma)/(L_{\rm pt}/\Gamma)$ such that $x \in L_{\rm pt}(y)$. Then $[y,\sigma] \in T$ for some σ. Let $[y,\sigma] \in T_\ell$. Then $f_{\ell,1}(x) = f_{\ell,2}(x) \in L_1^-(\overline{w}_1(v_\ell))\cap L_2^-(\overline{w}_2(v_\ell))$.

On the other hand, given ℓ and $z \in L_1^-(\overline{w}_1(v_\ell))\cap L_2^-(\overline{w}_2(v_\ell))$, it is easy to find $x \in L_1(w_1)\cap L_2(w_2)$ such that $z = f_{\ell,1}(x) = f_{\ell,2}(x)$. We remark that $\sharp(L_1^-(\overline{w}_1(v_\ell))\cap L_2^-(\overline{w}_2(v_\ell)))$ is independent of ℓ. The proof of Lemma 15.11 is complete. □

On the other hand, we have :

Lemma 15.12.

$$\eta(\tilde L_1^-,\tilde L_2^-) + \dim\tilde L_1\cap\tilde L_{\rm pt} = \eta(\tilde L_1,\tilde L_2).$$

Proof. We devide

$$V \cong (W_0\oplus W^\perp)\oplus W/W_0$$

and split $\tilde L_i = \tilde L_i^1\oplus\tilde L_i^2$ respectively, where $\tilde L_i^1 = \tilde L_i\cap W_0$, $\tilde L_i^2 = \tilde L_i^- = (\tilde L_i\cap W)/(\tilde L_i\cap W_0)$, $(i = 1,2,{\rm pt})$. Note that $W_0\oplus W^\perp$, W/W_0 are symplectic vector spaces and $\tilde L_i^1$, $\tilde L_i^2$ are their Lagrangian subspaces.

It is easy to see from definition that

$$\eta(\tilde L_1^1,\tilde L_2^1,\tilde L_{\rm pt}^1) + \eta(\tilde L_1^2,\tilde L_2^2,\tilde L_{\rm pt}^2) = \eta(\tilde L_1,\tilde L_2,\tilde L_{\rm pt}).$$

The defininition in §2 of the Maslov index η between two Lagrangian subspaces depends on the choice of $\tilde L_{\rm pt}$. We use $\tilde L_{\rm pt}^1$, $\tilde L_{\rm pt}^2$ to define $\eta(\tilde L_1^1,\tilde L_2^1)$, $\eta(\tilde L_1^2,\tilde L_2^2)$. Now, by (2.16), we have

$$\eta(\tilde L_1^-,\tilde L_2^-) = \eta(\tilde L_1^2,\tilde L_2^2) = \eta(\tilde L_1^2,\tilde L_2^2,\tilde L_{\rm pt}^2),$$
$$\eta(\tilde L_1,\tilde L_2) = \eta(\tilde L_1,\tilde L_2,\tilde L_{\rm pt}) - \dim\tilde L_2\cap\tilde L_{\rm pt}.$$

Lemma 15.12 follows from Sublemma 15.13 and $\dim W_0 = \dim\tilde L_2\cap\tilde L_{\rm pt} + \dim\tilde L_1\cap\tilde L_{\rm pt}$. □

Sublemma 15.13.

$$\eta(\tilde L_1^1,\tilde L_2^1,\tilde L_{\rm pt}^1) = \dim W_0.$$

Proof. We may identify $W_0\oplus W^\perp = T^*W_0 = T^*\mathbb{R}^{m_1+m_2}$, and $\tilde L_1^1$, $\tilde L_1^2$ are conormal bundles of $\mathbb{R}^{m_1}\oplus 0$, $0\oplus\mathbb{R}^{m_2}$, respectively. We use the notation of §2. We let $\tilde L_0$ be the fiber of $T^*\mathbb{R}^{m_1+m_2}$. We perturb $\tilde L_1^1$ and $\tilde L_1^2$ so that they are graphs of

$d(x_1^2 + \cdots x_{m_1}^2) + 0$, $0 + d(x_{m_1+1}^2 + \cdots x_{m_2}^2)$ respectively. Note $\tilde L^1_{\rm pt} = W_0$ is the zero section. Hence $\eta_{L_0}(\tilde L^1_1, \tilde L^1_2) = m_2$, $\eta_{L_0}(\tilde L^1_2, \tilde L^1_{\rm pt}) = 0$, $\eta_{L_0}(\tilde L^1_{\rm pt}, \tilde L^1_1) = m_1$. Therefore,

$$\eta(\tilde L^1_1, \tilde L^1_2, \tilde L^1_{\rm pt}) = 2(m_1 + m_2) - (m_2 + 0 + m_1) = m_1 + m_2 = \dim W_0$$

as required. □

We apply Theorem 13.5 to $\mathcal{E}(L_i^-(\overline{w}_i(v_\ell)), \bar\alpha_{i,\sigma_0})$, $i = 1, 2$ and obtain

$$\begin{aligned}\operatorname{Ext}^{\eta(\tilde L_1^-, \tilde L_2^-)}(\mathcal{E}(L_1^-(\overline{w}_1(v_\ell)), \bar\alpha_{1,\sigma_0})), \mathcal{E}(L_2^-(\overline{w}_2(v_\ell)), \bar\alpha_{2,\sigma_0}))\\ = \mathbb{C}^{\sharp(L_1^-(\overline{w}_1(v_\ell)) \cap L_2^-(\overline{w}_2(v_\ell)))}.\end{aligned} \tag{15.14}$$

Lemmata 15.2, 15.11,15.12 and (15.14) imply Theorem δ, in the case when $L_1(w_1)$ is transversal to $L_2(w_2)$. □

Now we remove the assumption that $L_1(w_1)$ is transversal to $L_2(w_2)$. We define $\tilde L^-_{\rm pt}$, $\tilde L^-_{\rm st}$, $\tilde L^-_i$ by the same formula as before.

$T(L_1(w_1), \alpha_1)$ is no longer transversal to $T(L_1(w_2), \alpha_2)$. However they cleanly intersect to each other. We generalize Lemma 15.2 to handel this case.

Let M be a complex manifolds N_i be complex submanifolds of codimention d_i. We assume that N_1 cleanly intersect to N_2. We also assume that the dimension of $\dim N_1 \cap N_2$ is independent of the connected components. Let $\mathcal{E}_i$ be holomorphic vector bundles on N_i. We put $d_{12} = \dim_{\mathbb{C}} M + \dim_{\mathbb{C}} N_1 \cap N_2 - \dim_{\mathbb{C}} N_1 - \dim_{\mathbb{C}} N_2$. We consider the holomorphic vector bundle

$$\mathcal{N} = \frac{TM|_{N_1\cap N_2}}{TN_1|_{N_1\cap N_2} + TN_1|_{N_1\cap N_2}}$$

on $N_1 \cap N_2$. Note that the rank of $\mathcal{N}$ is d_{12}. We assume that $\mathcal{N}$ and $NN_1|_{N_1\cap N_2}$ are trivial bundles. Let $I_i : N_i \to M$ be inclusion. Under this assumption, we have :

Lemma 15.15. *We have*

$$\operatorname{Ext}^k(I_{1*}\mathcal{E}_1, I_{2*}\mathcal{E}_2) \simeq \bigoplus_\ell \left(\operatorname{Ext}^{k+d_{12}-d_1-\ell}(\mathcal{E}_1|_{N_1\cap N_2}, \mathcal{E}_2|_{N_1\cap N_2}) \otimes H^\ell(T^{d_{12}};\mathbb{C})\right).$$

We can prove Lemma 15.15 in a way similar to Lemma 15.12 by using Kozul resolution. We omit the proof.

We go back to our case of torus. We assume $T(L_1(w_1), \alpha_2) \cap T(L_2(w_2), \alpha_2) \ne \emptyset$. We put

$$\begin{aligned} d_{12} &= n + \dim_{\mathbb{C}}(T(L_1(w_1), \alpha_2) \cap T(L_2(w_2), \alpha_2) \\ &\qquad - \dim_{\mathbb{C}} T(L_1(w_1), \alpha_1) - \dim_{\mathbb{C}} T(L_2(w_2), \alpha_2), \\ h_{12} &= \dim_{\mathbb{C}} \tilde L_1^- \cap \tilde L_2^-, \\ d_i &= n - \dim_{\mathbb{C}} T(L_i(w_i), \alpha_i) = \dim \tilde L_i \cap \tilde L_{\rm pt}. \end{aligned}$$

Lemma 15.16. $d_{12} + h_{12} = \dim \tilde L_1 \cap \tilde L_2$.

Proof. We split V, $\tilde L_i$ as in the proof of Lemma 15.12. Then

$$\dim \tilde L_1^1 \cap \tilde L_2^1 = \dim \tilde L_1 \cap \tilde L_2 \cap \tilde L_{\rm pt} = \dim \frac{V}{\tilde L_1 + \tilde L_2 + \tilde L_{\rm pt}} = d_{12},$$
$$\dim \tilde L_1^2 \cap \tilde L_2^2 = \dim \tilde L_1^- \cap \tilde L_2^- = h_{12}.$$

The lemma follows. □

Lemma 15.17. $d_1 - d_{12} + \eta(\tilde L_1^-, \tilde L_2^-) = \eta(\tilde L_1, \tilde L_2)$.

Proof. We use the same notation as in the proof of Lemma 15.12. By (2.16), we have

$$\eta(\tilde L_1^-, \tilde L_2^-) = \eta(\tilde L_1^2, \tilde L_2^2, \tilde L_{\rm pt}^2) - \dim \tilde L_1^- \cap \tilde L_2^-,$$
$$\eta(\tilde L_1, \tilde L_2) = \eta(\tilde L_1, \tilde L_2, \tilde L_{\rm pt}) - \dim \tilde L_1 \cap \tilde L_2 - \dim \tilde L_2 \cap \tilde L_{\rm pt}.$$

Hence, using Lemma 15.16, we have :

$$\eta(\tilde L_1, \tilde L_2) - \eta(\tilde L_1^-, \tilde L_2^-) = \eta(\tilde L_1^1, \tilde L_2^1, \tilde L_{\rm pt}^1) - d_{12} - \dim \tilde L_2 \cap \tilde L_{\rm pt}.$$

We next calculate $\eta(\tilde L_1^1, \tilde L_2^1, \tilde L_{\rm pt}^1)$. We have $W_0 = \mathbb R^{d_{12}} \oplus \mathbb R^{m_1} \oplus \mathbb R^{m_2}$ and can identify $\tilde L_1^1$, $\tilde L_2^1$ with the conormal bundles of $\mathbb R^{d_{12}} \oplus \mathbb R^{m_1} \oplus 0$, $\mathbb R^{d_{12}} \oplus 0 \oplus \mathbb R^{m_2}$, respectively. We perturb them (while keeping the dimension of $\tilde L_i^1 \cap \tilde L_j^1$, $i, j = 1, 2, {\rm pt}$) so that $\tilde L_1^1$ is a graph of $d(x_{d_{12}+1}^2 + \cdots + x_{d_{12}+m_1}^2)$, and that $\tilde L_2^1$ is a graph of $d(x_{d_{12}+m_1+1}^2 + \cdots + x_{d_{12}+m_1+m_2}^2)$. Hence $\eta_{L_0}(\tilde L_1^1, \tilde L_2^1) = m_2$, $\eta_{L_0}(\tilde L_2^1, \tilde L_{\rm pt}^1) = 0$, $\eta_{L_0}(\tilde L_{\rm pt}^1, \tilde L_2^1) = m_1$. Therefoer $\eta(\tilde L_1^1, \tilde L_2^1, \tilde L_{\rm pt}^1) = 2 \dim W_0 - m_1 - m_2 = 2d_{12} + m_1 + m_2$.

On the other hand, $\dim \tilde L_2 \cap \tilde L_{\rm pt} = d_{12} + m_2$. Therefore

$$\eta(\tilde L_1, \tilde L_2) - \eta(\tilde L_1^-, \tilde L_2^-) = m_1.$$

We can see easily $m_1 + d_{12} = \dim \tilde L_1 \cap \tilde L_{\rm pt} = {\rm codim}_{\mathbb C}\, T(L_1(w_1), \alpha_1) = d_1$. Lemma 15.17 follows. □

We next generalize Lemma 15.11. Suppose $T = T(L_1(w_1), \alpha_1) \cap T(L_2(w_2), \alpha_2)$ is nonempty. We devide it to connected components as $T = \cup_\ell T_\ell$. We define $L_i^-(\overline w_i(v_\ell))$ in the same way as before.

Lemma 15.18. *We assume that $\alpha_1 - \alpha_2$ is zero on $\tilde L_1 \cap \tilde L_2$. Then the following condition* (15.19) *are equivalent to* (15.20)*.*

(15.19) $T = T(L_1(w_1), \alpha_1) \cap T(L_2(w_2), \alpha_2)$ *is nonempty and* $L_1^-(\overline w_1(v_\ell)) \cap L_2^-(\overline w_2(v_\ell))$ *is nonempty.*

(15.20) $L_1(w_1) \cap L_2(w_2)$ *is nonempty.*

Moreover, in case (15.19)*,* (15.20) *are satisfied, we have*

$$\sharp \pi_0(T) \cdot \sharp \pi_0(L_1^-(\overline w_1(v_\ell)) \cap L_2^-(\overline w_2(v_\ell))) = \sharp \pi_0(L_1(w_1) \cap L_2(w_2)).$$

The proof is a straight forward generalization of the proof of Lemma 15.11 and is omitted.

Now we are in the position to complete the proof of Theorem δ. We put

$$\begin{aligned} z_1 &= \sharp\pi_0(T(L_1(w_1),\alpha_2)\cap T(L_2(w_2),\alpha_2)),\\ z_2 &= \sharp\pi_0(L_1^-(\overline{w}_1(v_\ell))\cap L_2^-(\overline{w}_2(v_\ell))). \end{aligned}$$

Lemma 15.16 and 15.18 implies

$$HF^k((L_1(w_1),\alpha_1),(L_2(w_2),\alpha_2)) \cong (H^{k-\eta(\tilde L_1,\tilde L_2)}(T^{d_{12}+h_{12}};\mathbb{C}))^{\oplus z_1 z_2}. \tag{15.21}$$

In fact, if (15.20) is not satisfied then the left hand side is zero. On the other hand, in that case $z_1z_2 = 0$ since (15.19) is not satisfied.

If (15.20) is satisfied, then (15.21) follow from the definition of Floer homology, $z_1z_2 = \sharp\pi_0(L_1(w_1)\cap L_2(w_2))$ and $\dim L_1(w_1)\cap L_2(w_2) = d_{12}+h_{12}$.

On the other hand, we have

$$\begin{aligned} &HF^k((L_1^-(\overline{w}_1(v_\ell)),\bar\alpha_{1,\sigma_0}),(L_2^-(\overline{w}_2(v_\ell)),\bar\alpha_{2,\sigma_0}))\\ &\qquad\cong (H^{k-\eta(\tilde L_1^-,\tilde L_2^-)}(T^{h_{12}};\mathbb{C}))^{\oplus z_2}, \end{aligned} \tag{15.22}$$

where α_{i,σ_0} is defined in the same way as (15.9).

We use Theorem 13.5 and obtain

$$\begin{aligned} &HF^k((L_1^-(\overline{w}_1(v_\ell)),\bar\alpha_{1,\sigma_0}),(L_2^-(\overline{w}_2(v_\ell)),\bar\alpha_{2,\sigma_0}))\\ &\qquad\cong \mathrm{Ext}^k(\mathcal{E}(L_1^-(\overline{w}_1(v_\ell)),\bar\alpha_{1,\sigma_0}),\mathcal{E}(L_2^-(\overline{w}_2(v_\ell)),\bar\alpha_{2,\sigma_0})). \end{aligned} \tag{15.23}$$

Hence, by Lemmata 15.15, 15.17 and (15.21), (15.22), (15.23), we have

$$HF^k((L_1(w_1),\alpha_1),(L_2(w_2),\alpha_2)) \cong \mathrm{Ext}^k(\mathcal{E}(L_1(w_1),\alpha_1),\mathcal{E}(L_2(w_2),\alpha_2)),$$

The proof of Theorem δ is now complete. □

Chapter 4. Lagrangian resolution.

§16. Lagrangian resolution.

The purpose of this section is to show that multi theta function $\mathfrak{m}_k$ describes various important properties of the sheaves on complex tori. In fact, in this section, we do not use so much the fact that our complex manifold is a torus. Many of the arguments of this section may be generalized to more general complex manifolds if we can construct $\mathfrak{m}_k$ satisfying Theorem γ on it. We study the derived category $\mathbb{D}(T^{2n},\Omega)^\vee$ of coherent sheaves of complex torus. For $\mathcal{F} \in Ob(\mathbb{D}((T^{2n},\Omega)^\vee))$, $u \in \mathbb{Z}$ let $\mathfrak{F}[u] \in Ob(\mathbb{D}((T^{2n},\Omega)^\vee))$ be the object obtained by shifting degree. Namely $H^k((T^{2n},\Omega)^\vee,\mathcal{F}[u]) \cong H^{k+u}((T^{2n},\Omega)^\vee,\mathcal{F})$. Roughly speaking, we construct objects such as

$$\bigoplus_a \mathcal{E}\left(L_{0,a}(w_{0,a}),\alpha_{0,a}\right)[-u(0,a)] \to \cdots \to \bigoplus_a \mathcal{E}\left(L_{I,a}(w_{I,a}),\alpha_{I,a}\right)[-u(I,a)].$$

(Note $\mathcal{E}[-k]$ is $\mathcal{E}$ which sits in degree k.) We consider

$$x_{i,j;a,b} \in HF^{i-j+u(j,b)-u(i,a)+1}\left((L_{i,a}(w_{i,a}),\alpha_{i,a}),(L_{j,b}(w_{j,b}),\alpha_{j,b})\right), \tag{16.1}$$

where $L_{j,b}(w_{j,b})$ are affine Lagrangian submanifolds transversal to each other and to $L_{\rm pt}$. For each $0 \le i < j \le k$, a,b we consider an equation

$$\sum_k \sum_{\substack{i=\ell(1)<\cdots<\ell(k+1)=j\\ a=c(1),\cdots,c(k+1)=b}} \begin{aligned}(-1)^\mu\mathfrak{m}_k(&x_{\ell(1),\ell(2);c(1),c(2)},\cdots,\\ &x_{\ell(k),\ell(k+1);c(k),c(k+1)}) = 0.\end{aligned} \tag{16.2}$$

The sign $(-1)^\mu$ is so that it will be equivalent to :

$$\sum_k \sum_{\substack{i=\ell(1)<\cdots<\ell(k+1)=j\\ a=c(1),\cdots,c(k+1)=b}} \tilde{\mathfrak{m}}_k\left[x_{\ell(1),\ell(2);c(1),c(2)}\Big|\cdots\Big|x_{\ell(k),\ell(k+1);c(k),c(k+1)}\right] = 0. \tag{16.3}$$

(Note that all terms are of degree $i-j+u(j,b)-u(i,a)-1$ *after shifted.*) Here $\tilde{\mathfrak{m}}_k$ is different from $\mathfrak{m}_k$ only by sign and is defined by

$$\mathfrak{m}_k = s^{-1}\circ\tilde{\mathfrak{m}}_k\circ(s\otimes\cdots\otimes s).$$

Definition 16.4. We say a system $\mathbb{L} = \left(\left((L_{i,a}(w_{i,a}),\alpha_{i,a})\right),(u(i,a)),(x_{i,j;a,b})\right)$ to be a *Lagrangian resolution*, if (16.2) is satisfied.

The following is a restatement of Theorem ϕ.

Theorem 16.5. *For any Lagrangian resolution* $\mathbb{L}$, *we can associate an object* $\mathcal{E}(\mathbb{L}) \in Ob\left(\mathbb{D}\left((T^{2n},\Omega)^\vee\right)\right)$.

Remark 16.6. We remark that a pair $(L(w), \alpha)$ may be regarded as a Lagrangian resolution. In that case, $\mathcal{E}(L(w), \alpha)$ in Theorem 16.5 coincides with one we constructed before.

Proof. We consider a direct sum of holomorphic vector bundles :

$$C(\mathbb{L}) = \Lambda^{(0,d)} \otimes \left(\bigoplus_{d,i,a} \mathcal{E}(L_{i,a}(w_{i,a}), \alpha_{i,a}) \right) \tag{16.7}$$

where the degree of an element of $\Lambda^{(0,d)} \otimes \mathcal{E}(L_{i,a}(w_{i,a}), \alpha_{i,a})$ is $d+i-u(i,a)$. We will define a boundary operator on (16.7) and will regard it as a complex of $\mathcal{O}_{(T^{2n},\Omega)^\vee}$ module sheaves. Let $i = \ell(1) < \cdots < \ell(k) = j$, $a = c(1), c(2), \cdots, c(k) = b$. We put $\vec{\ell} = (\ell(1), \cdots, \ell(k))$ $\vec{c} = (c(1), \cdots, c(k))$. Let

$$\begin{aligned} d(\vec{\ell}, \vec{c}) = &\sum_{s=1}^{k-1} (\ell(s) - \ell(s+1) + u(\ell(s+1), c(s+1)) - u(\ell(s), c(s)) + 1) \\ &\quad + 2 - k \\ = &\ \ell(1) - \ell(k) + c(k) - c(1) + 1. \end{aligned} \tag{16.8}$$

Hereafter we use $\tilde{\mathfrak{m}}_k$ in place of $\mathfrak{m}_k$ to simplify the sign convention.

Definition 16.9. We define a distribution valued homomorphism

$$\tilde{\mathfrak{m}}_{(\vec{\ell},\vec{c})} : \mathcal{E}(L_{i,a}(w_{i,a}), \alpha_{i,a}) \to \Lambda^{(0,d(\vec{\ell},\vec{c}))} \otimes \mathcal{E}(L_{j,b}(w_{j,b}), \alpha_{j,b}),$$

by

$$\tilde{\mathfrak{m}}_{(\vec{\ell},\vec{c})}(z) = \tilde{\mathfrak{m}}_k^{(0,d(\vec{\ell},\vec{c}))} \left[z | x_{\ell(1),\ell(2);c(1),c(2)} | \cdots | x_{\ell(k-1),\ell(k);c(k-1),c(k)} \right], \tag{16.10}$$

where $z \in \mathcal{E}(L_{i,a}(w_{i,a}), \alpha_{i,a})_{(v,\sigma)} \cong HF^n((L_{\mathrm{pt}}(v), \sigma), (L_{i,a}(w_{i,a}), \alpha_{i,a}))$.

(16.8) implies that the right hand side of (16.10) is in $\mathcal{E}(L_{j,b}(w_{j,b}), \alpha_{j,b})_{(v,\sigma)} \cong \Lambda^{(0,d(\vec{\ell},z))} \otimes HF^n(L_{\mathrm{pt}}(v), \sigma), (L_{j,b}(w_{j,b}), \alpha_{j,b}))$.

Definition 16.11. Let $S = (s_{i,a})$ be a smooth section of $C(\mathbb{L})$. We define a distribution valued section $\hat{\bar{\partial}} S$ of $C(\mathbb{L})$ by

$$\left(\hat{\bar{\partial}} S \right)_{j,b} = \bar{\partial} s_{j,b} + \sum_{\vec{\ell},\vec{c}} (-1)^{\deg S + d(\vec{\ell},\vec{c})} \tilde{\mathfrak{m}}_{(\vec{\ell},\vec{c})}(s_{i,a}), \tag{16.12}$$

where $\deg S$ is the degree of S as a distribution. Here the sum is taken over all $(\vec{\ell}, \vec{c})$ such that $i = \ell(1) < \cdots < \ell(k) = j$, $a = c(1), c(2), \cdots, c(k) = b$.

Lemma 16.13. *$\hat{\bar{\partial}} \circ \hat{\bar{\partial}}$ is well-defined and $\hat{\bar{\partial}} \circ \hat{\bar{\partial}} = 0$.*

Proof. Note that $\hat{\bar{\partial}} S$ is in general not well-defined for a distribution section S. However we can prove that $\hat{\bar{\partial}} \circ \hat{\bar{\partial}}(S)$ is well-defined for smooth S in a similar way to the proof of Lemma 14.2. We introduce some notations. Let $\ell_1(k_1) = \ell_2(1)$ and $c_1(k_1) = c_2(1)$. We put

$$(\vec{\ell}_1 \cup \vec{\ell}_2)(i) = \begin{cases} \ell_1(i) & \text{if } i \le k_1, \\ \ell_2(i - k_1 + 1) & \text{if } i > k_1. \end{cases}$$

$$(\vec{c}_1 \cup \vec{c}_2)(i) = \begin{cases} c_1(i) & \text{if } i \le k_1, \\ c_2(i - k_1 + 1) & \text{if } i > k_1. \end{cases}$$

We calculate

$$\begin{aligned}
&((\hat{\bar{\partial}} \circ \hat{\bar{\partial}})S)_{j,b} \\
&= \bar{\partial}\bar{\partial} s_{j,b} + \sum_{\vec{\ell},\vec{c}} (-1)^{\deg S + d(\vec{\ell},\vec{c})} \bar{\partial} \circ \tilde{\mathfrak{m}}_{(\vec{\ell},\vec{c})}(s_{i,a}) \\
&\quad + \sum_{\vec{\ell},\vec{c}} (-1)^{\deg S + 1 + d(\vec{\ell},\vec{c})} \tilde{\mathfrak{m}}_{(\vec{\ell},\vec{c})} \circ \bar{\partial}(s_{i,a}) \\
&\quad + \sum_{\substack{\vec{\ell} = \vec{\ell}_1 \cup \vec{\ell}_2 \\ \vec{c} = \vec{c}_1 \cup \vec{c}_2}} (-1)^{d(\vec{\ell}_1,\vec{c}_1)} \left(\tilde{\mathfrak{m}}_{(\vec{\ell}_1,\vec{c}_1)} \circ \tilde{\mathfrak{m}}_{(\vec{\ell}_2,\vec{c}_2)} \right) (s_{i,a}) \\
&= (-1)^{d(\vec{\ell},\vec{c})} \left(\sum_{\vec{\ell},\vec{c}} \left(\bar{\partial} \tilde{\mathfrak{m}}_{(\vec{\ell},\vec{c})} \right) (s_{i,a}) \right. \\
&\quad \left. + \sum_{\substack{\vec{\ell} = \vec{\ell}_1 \cup \vec{\ell}_2 \\ \vec{c} = \vec{c}_1 \cup \vec{c}_2}} (-1)^{d(\vec{\ell}_2,\vec{c}_2)} \left(\tilde{\mathfrak{m}}_{(\vec{\ell}_2,\vec{c}_2)} \circ \tilde{\mathfrak{m}}_{(\vec{\ell}_1,\vec{c}_1)} \right) (s_{i,a}) \right).
\end{aligned} \tag{16.14}$$

(Note that our convention in (7.20) is that we take wedge with $\mathfrak{b}$ from the right.) On the other hand, (16.2) implies

$$\begin{aligned}
&\sum_{\vec{\ell},\vec{c}} (-1)^{(d_2)} (\tilde{\mathfrak{m}}^{(0,d_2)} \circ \tilde{\mathfrak{m}}^{(0,d_1)}) [s_{i,a} | x_{\ell(1),\ell(2);c(1),c(2)} | \\
&\qquad\qquad \cdots | x_{\ell(k-1),\ell(k);c(k-1),c(k)}] \\
&\qquad = \sum_{\substack{\vec{\ell} = \vec{\ell}_1 \cup \vec{\ell}_2 \\ \vec{c} = \vec{c}_1 \cup \vec{c}_2}} (-1)^{d(\vec{\ell}_2,\vec{c}_2)} \left(\tilde{\mathfrak{m}}_{(\vec{\ell}_2,\vec{c}_2)} \circ \tilde{\mathfrak{m}}_{(\vec{\ell}_1,\vec{c}_1)} \right) (s_{i,a}).
\end{aligned} \tag{16.15}$$

Theorem γ (+ the sign convention in (7.20)) and (16.15) imply that (16.14) vanishes. Lemma 16.13 is proved. □

We remark that $\hat{\bar{\partial}}$ is an $\mathcal{O}_{(T^{2n},\Omega)^\vee}$ module homomorphism. ($\bar{\partial}$ is an $\mathcal{O}_{(T^{2n},\Omega)^\vee}$ module homomorphism and $\tilde{\mathfrak{m}}_{(\vec{\ell},\vec{c})}$ does not contain derivative and so is an $\mathcal{O}_{(T^{2n},\Omega)^\vee}$ module homomorphism.) Thus we are almost done. Namely we obtain a "complex of $\mathcal{O}_{(T^{2n},\Omega)^\vee}$ module sheaf", which will give an element of $Ob\left(\mathbb{D}\left((T^{2n},\Omega)^\vee\right)\right)$.

However there is one trouble. Namely $\tilde{\mathfrak{m}}_{(\vec{\ell},\vec{c})}$ is distribution valued. Hence, we need to be careful to choose the regularity we assume to associate $\mathcal{O}_{(T^{2n},\Omega)^\vee}$ module sheaf to a holomorphic bundle $\mathcal{E}(L_i,\alpha_i)$. Namely if we consider the sheaf of all smooth sections S then $\tilde{\mathfrak{m}}_{(\vec{\ell},\vec{c})}(S)$ is not smooth and hence $\hat{\bar{\partial}}$ does not give a sheaf homomorphism. On the other hand, if we consider the sheaf of all distribution valued sections S, then $\tilde{\mathfrak{m}}_{(\vec{\ell},\vec{c})}(S)$ is not well-defined in general.

To overcome this trouble, we are going to replace $\tilde{\mathfrak{m}}_k^{(0,d)}$, (which are singular), by smooth one. In §§7,8,9, we constructed a family $\mathfrak{b}_k^{(d)}$ of integral currents on $L(1,\cdots,k)$ (we use $\mathfrak{b}_k^{(d)}$ in place of $\mathfrak{c}_k^{(d)}$ since we are using $\tilde{\mathfrak{m}}_k^{(0,d)}$ in place of $\mathfrak{m}_k^{(0,d)}$.) $\mathfrak{b}_k^{(d)}$ is obtained by solving the equation :

$$d\mathfrak{b}_k^{(d)} + \sum (-1)^{d_2}\mathfrak{b}_{k_2}^{(d_2)}\circ\mathfrak{b}_{k_1}^{(d_1)} = 0, \tag{16.16}$$

inductively. We are going to use smooth forms in place of integral currents. First we take a smooth d form $\mathfrak{b}_{smooth}(1,2,3)$ for each $\deg(1,2,3) = d$. More precisely, we choose $\mathfrak{b}'_{smooth}(1,2,3)$ first so that the following is satisfied.

(16.17.1) The support $supp\,(\mathfrak{b}'_{smooth}(1,2,3))$ is contained in

$$\Big\{ [v_1,v_2,v_3] \in L(1,2,3) \mid Q[v_1,v_2,v_3] \ge \delta \left\|[v_1,v_2,v_3]\right\|^2 \Big\}.$$

(16.17.2) Let $r>0$. Then $\mathfrak{b}'_{smooth}(1,2,3)$ is invariant of $[v_1,v_2,v_3] \mapsto [rv_1,rv_2,rv_3]$.
(16.17.3) $\mathfrak{b}'_{smooth}(1,2,3)$ is smooth outside origin.
(16.17.4) $d\mathfrak{b}'_{smooth}(1,2,3) = 0$. And $\mathfrak{c}'_{smooth}(1,2,3)$ represent the same element as $\mathfrak{c}_{smooth}(1,2,3)$ in $H^d_{DR}(S(Q,1,2,3),\mathbb{R})$

To remove the singularity at the origin we replace it by $\mathfrak{b}_{smooth}(1,2,3)$ such that :

(16.18.1) The support $supp\,(\mathfrak{b}_{smooth}(1,2,3))$ is contained in

$$\Big\{ [v_1,v_2,v_3] \in L(1,2,3) \mid Q[v_1,v_2,v_3] \ge \delta \left\|[v_1,v_2,v_3]\right\|^2 - C \Big\}.$$

for some constant C.
(16.18.2) Let $r>0$. Then $\mathfrak{b}_{smooth}(1,2,3)$ is invariant of $[v_1,v_2,v_3] \mapsto [rv_1,rv_2,rv_3]$, outside a compact set $\mathcal{K}(1,2,3)$.
(16.18.3) $\mathfrak{b}_{smooth}(1,2,3)$ is smooth.
(16.18.4) $\mathfrak{b}_{smooth}(1,2,3) - \mathfrak{b}'_{smooth}(1,2,3) = d\,(\Delta\mathfrak{b}(1,2,3))$, where $\Delta\mathfrak{b}(1,2,3)$ is supported on $\mathcal{K}(1,2,3)$.

We next construct $\mathfrak{b}_{smooth}(1,\cdots,k+1)$ inductively. We can solve (16.16) inductively in the same way as the proof of Theorem α in §7. We then obtain $\mathfrak{b}_{smooth}(1,\cdots,k+1)$ which is smooth and satisfies (16.16). Also we may choose it so that the following is satisfied :

(16.19.1) The support $supp\,(\mathfrak{b}_{smooth}(1,\cdots,k+1))$ is contained in

$$\Big\{ [v_1,\cdots,v_{k+1}] \in L(1,\cdots,k+1) \mid Q[1,\cdots,k+1] \ge \delta \left\|[1,\cdots,k+1]\right\|^2 - C \Big\}.$$

(16.19.2) Let $r>0$. Then $\mathfrak{b}_{smooth}(1,\cdots,k+1)$ is invariant of $[v_1,\cdots,v_{k+1}] \mapsto [rv_1,\cdots,rv_{k+1}]$, outside a set $\mathcal{K}(1,\cdots,k+1)$. $\mathcal{K}(1,\cdots,k+1)$ defined inductively, on k, by

$$\begin{aligned}\mathcal{K}(1,\cdots,k+1) =& \mathcal{K}_0(1,\cdots,k+1) \\ & \cup \bigcup_{1\le \ell<m\le k+1} \mathcal{K}(1,\cdots,\ell,m,\cdots,k+1)\times L(\ell,\cdots,m) \\ & \cup \bigcup_{1\le \ell<m\le k+1} L(1,\cdots,\ell,m,\cdots,k+1)\times \mathcal{K}(\ell,\cdots,m),\end{aligned}$$

where $\mathcal{K}_0(1,\cdots,k+1)$ is compact.

We now use $\mathfrak{b}_{smooth}(1,\cdots,k+1)$ in place of $c(1,\cdots,k+1)$ in Definitions 10.3 and 10.7 to obtain $\tilde{\mathfrak{m}}^{(0,d)}_{k,smooth}$. We remark that Axiom I (I.3) was used to show that $\tilde{\mathfrak{m}}^{(0,d)}_{k}$ converges. However we can use (16.19.1) and the exponential decay estimate to prove that $\tilde{\mathfrak{m}}^{(0,d)}_{k,smooth}$ is a (homomorphism bundle valued) smooth d form.

We want to use $\tilde{\mathfrak{m}}^{(0,d)}_{k,smooth}$ in place of $\tilde{\mathfrak{m}}^{(0,d)}_{k}$ to construct $(C(\mathbb{L}),\hat{\bar{\partial}}_{smooth})$. However $x_{i,j;a,b}$ satisfies (16.2) for $\tilde{\mathfrak{m}}^{(0,d)}_{k}$ but not for $\tilde{\mathfrak{m}}^{(0,d)}_{k,smooth}$. So we need to replace $x_{i,j;a,b}$ by $x'_{i,j;a,b}$ satisfying (16.2) for $\tilde{\mathfrak{m}}^{(0,d)}_{k,smooth}$.

For this purpose, we use the fact that the A_∞ structure determined by $\tilde{\mathfrak{m}}^{(0,d)}_{k,smooth}$ is homotopy equivalent to one determined by $\tilde{\mathfrak{m}}^{(0,d)}_{k}$, (Theorem β). Namely in the same way as the proof of Theorem 11.12, we find $\tilde{\mathfrak{h}}^{(0,d)}_{k}$ such that

$$\bar{\partial}\tilde{\mathfrak{h}}^{(0,d)}_{k} + \sum_{\substack{d_1+d_2=d+1\\ k_1+k_2=k+1}} \pm\tilde{\mathfrak{h}}^{(0,d_2)}_{k_2}\circ\tilde{\mathfrak{m}}^{(0,d_1)}_{k} - \sum_{\substack{d_1+d_2=d+1\\ k_1+k_2=k+1}} \pm\tilde{\mathfrak{m}}^{(0,d_1)}_{k,smooth}\circ\tilde{\mathfrak{h}}^{(0,d_2)}_{k_1} = 0. \tag{16.20}$$

(Sign is as in Definition 11.8.) Now we put

$$x'_{i,j;a,b} = \sum_{\vec{\ell},\vec{c}} \tilde{\mathfrak{h}}_k\left[x_{\ell(1),\ell(1);c(2),c(2)}|\cdots|x_{\ell(k),\ell(k+1);c(k),c(k+1)})\right]. \tag{16.21}$$

Then (16.2) and $d=0$ case of (16.20) imply

$$\sum_{\vec{\ell},\vec{c}} \tilde{\mathfrak{m}}_{k,smooth}\left[x'_{\ell(1),\ell(2);c(1),c(2)}|\cdots|x'_{\ell(k),\ell(k+1);c(k),c(k+1)}\right] = 0 \tag{16.22}$$

Hence we can use $x'_{i,j;a,b}$ and $\tilde{\mathfrak{m}}_{k,smooth}$ to construct $(C(\mathbb{L}),\hat{\bar{\partial}}_{smooth})$. (We remark that $\tilde{\mathfrak{m}}_{k,smooth}$ satisfies the Maurer-Cartan equation.) We then take the sheaf of smooth sections and regard $(C(\mathbb{L}),\hat{\bar{\partial}}_{smooth})$ as a chain complex of $\mathcal{O}_{(T^{2n},\Omega)^\vee}$ module sheaves. Note that the difference of $(C(\mathbb{L}),\hat{\bar{\partial}}_{smooth})$ from the direct sum of Dolbeault complex is degree zero term with smooth coefficient. Hence by usual Fredholm theory (elliptic estimate) we find that the cohomology sheaf of $(C(\mathbb{L}),\hat{\bar{\partial}}_{smooth})$ is coherent. We now define :

Definition 16.23. $\mathcal{E}(\mathbb{L}) = (C(\mathbb{L}),\hat{\bar{\partial}}_{smooth})$.

The proof of Theorem 16.5 and of Theorem ϕ is complete. □

We turn to the calculation of $\text{Ext}(\mathcal{E}(\mathbb{L}^{(1)}),\mathcal{E}(\mathbb{L}^{(2)}))$. Here $\mathbb{L}^{(k)}$ is as in Theorem γ. We first consider the case when $\mathbb{L}^{(1)}$ consists of a single pair (L_0,α_0). We put $\mathbb{L}^{(2)} = \mathbb{L} = (((L_{i,a}(w_{i,a}),\alpha_{i,a})),(u(i,a)),(x_{i,j;a,b}))$.

We define a chain complex $C(\mathbb{L}^{(1)},\mathbb{L}^{(2)})$ as in introduction. Namely we put :

$$C^{\ell}\left((L_0,\alpha_0),\mathbb{L}\right) = \bigoplus HF^{\ell-i+u(i,a)}\left((L_0,\alpha_0),(L_{i,a}(w_{i,a}),\alpha_{i,a})\right) \tag{16.24}$$

For $S=(s_{i,a})\in C^{\ell}\left((L_0,\alpha_0),\mathbb{L}\right)$, and put

$$\begin{aligned}(\delta S)_{j,b} = \overline{\partial} s_{j,b} + \sum_{\vec\ell,\vec c}(-1)^{\deg S + d(\vec\ell,\vec c)}\tilde{\mathfrak{m}}_k\left[s_{i,a},x_{\ell(1),\ell(2);c(1),c(2)}\right| \\ \cdots\left|x_{\ell(k-1),\ell(k);c(k-1),c(k)}\right].\end{aligned} \tag{16.25}$$

Lemma 16.26. $\delta\delta=0$.

The proof is a straight forward calculation using A_∞ formulae and (16.2). We omit it.

Theorem γ in this case is as follows :

Proposition 16.27. *The cohomology of* $\left(C^{\ell}\left((L_0,\alpha_0),\mathbb{L}\right),\delta\right)$ *is isomorphic to* $\text{Ext}(\mathcal{E}(L_0,\alpha_0),\mathcal{E}(\mathbb{L}))$.

Proof. To prove Proposition 16.27, We construct $\left(C^{\ell}\left((L_0,\alpha_0),\mathbb{L}\right),\delta_{smooth}\right)$ by replacing $\tilde{\mathfrak{m}}_k$, $x_{\ell(1),\ell(2);c(1),c(2)}$, with $\tilde{\mathfrak{m}}_{k,smooth}$, $x'_{\ell(1),\ell(2);c(1),c(2)}$.

Lemma 16.28. $\left(C^{\ell}\left((L_0,\alpha_0),\mathbb{L}\right),\delta_{smooth}\right)$ *is chain homotopy equivalent to* $\left(C^{\ell}\left((L_0,\alpha_0),\mathbb{L}\right),\delta\right)$.

Proof. Let $\mathfrak{h}_k$ satisfy (16.20). We define a homomorphism $H:\left(C^{\ell}\left((L_0,\alpha_0),\mathbb{L}\right),\delta\right)\to\left(C^{\ell}\left((L_0,\alpha_0),\mathbb{L}\right),\delta_{smooth}\right)$ by

$$H(S)_{i,a} = \sum_{\vec\ell,\vec c}\tilde{\mathfrak{h}}_k\left[s_{i,a}|x_{\ell(1),\ell(2);c(1),c(2)}|\ \cdots|x_{\ell(k-1),\ell(k);c(k-1),c(k)}\right],$$

where $S=(s_{i,a})$. By using (16.2), (16.20), (16.21) and (16.22), we find that H is a chain map. Since $\tilde{\mathfrak{h}}_1^{(0)}$ is the identity, it follows that H is an isomorphism. Lemma 16.28 follows. □

We next define a chain map

$$\Psi:(C^*((L_0,\alpha_0),\mathbb{L}),\delta_{smooth})\to(\Gamma(Hom(\mathcal{E}(L_0,\alpha_0),C(\mathbb{L}))),\hat{\overline{\partial}}_{smooth}). \tag{16.29}$$

Here $\Gamma(Hom(\mathcal{E}(L_0,\alpha_0),C(\mathbb{L})))$ is the vector space of smooth sections of the bundle $Hom(\mathcal{E}(L_0,\alpha_0),C(\mathbb{L}))$. We put

$$\begin{aligned}(\Psi S)_{j,b}(v,\sigma) = \sum_{\vec\ell,\vec c}&(-1)^{d(\vec\ell,\vec c)+\deg S} \\ &\tilde{\mathfrak{m}}^{(0,d(\vec\ell,\vec c))}_{k+1,smooth}\left[x_0(v,\sigma)|s_{i,a}|x'_{\ell(1),\ell(1);c(2),c(2)},\right. \\ &\left.\cdots|x'_{\ell(k-1),\ell(k);c(k-1),c(k)}\right].\end{aligned} \tag{16.30}$$

where $\deg S$ is the degree as differential form. □

Lemma 16.31. Ψ *is a chain map.*

Proof. The proof is an easy calculation using A_∞ formula, (16.20),(16.20),(16.22). □

We now prove that Ψ induces an isomorphism in cohomologies. The proof is by induction on I (the number of i's). In case when $I = 1$, Ψ coincides with the composition of the map (14.5) and the isomorphism H. Hence Ψ induces an isomorphism by Theorem 14.16. Let us assume that Ψ induces an isomorphism for $I-1$. We consider the case $I = I_1 + I_2$. Let $\mathbb{L}_{i\le I_1}$ be the part of $\mathbb{L}$ for $i \le I_1$ and $\mathbb{L}_{i\ge I_1+1}$ be the part of $\mathbb{L}$ for $i \ge I_1+1$. Then we have an exact sequence of chain complexes

$$\begin{aligned} 0 \to (C((L_0,\alpha_0),\mathbb{L}_{i\ge I_1+1}),\hat{\hat{\partial}}) &\to (C((L_0,\alpha_0),\mathbb{L}),\hat{\hat{\partial}}) \\ &\to (C((L_0,\alpha_0),\mathbb{L}_{i\le I_1}),\hat{\hat{\partial}}) \to 0. \end{aligned}$$

On the other hand we have an exact sequence

$$\begin{aligned} 0 \to (C^*((L_0,\alpha_0),\mathbb{L}_{i\ge I_1+1}),\delta_{smooth}) &\to (C^*((L_0,\alpha_0),\mathbb{L}),\delta_{smooth}) \\ &\to (C^*((L_0,\alpha_0),\mathbb{L}_{i\le I_1}),\delta_{smooth}) \to 0, \end{aligned}$$

of chain complexes. We compare two long exact sequences obtained from short exact sequences. Then, the induction hypothesis and five lemma imply that Φ induces an isomorphism for I. The proof of Proposition 16.27 is complete. □

Now we prove Theorem γ in the general case. We put

$$\begin{aligned} \mathbb{L}^{(1)} &= \left(\left((L'_{i,a}(w'_{i,a}),\alpha'_{i,a})\right),(u'(i,a)),(y_{i,j;a,b})\right), \\ \mathbb{L}^{(2)} &= \left(\left((L_{i,a}(w_{i,a}),\alpha_{i,a})\right),(u(i,a)),(x_{i,j;a,b})\right). \end{aligned}$$

We define $\Psi : (C(\mathbb{L}^{(1)},\mathbb{L}^{(2)}),\hat{\hat{\partial}}) \to (C^*(\mathbb{L}^{(1)},\mathbb{L}^{(2)}),\delta_{smooth})$ in a similar way to (16.30). We split

$$\mathbb{L}^{(1)} = \mathbb{L}^{(1)}_{i\le I_1} \oplus \mathbb{L}^{(1)}_{i\ge I_1+1}.$$

We have two exact sequences of chain complexes :

$$\begin{aligned} 0 \to (C(\mathbb{L}^{(1)}_{i\ge I_1+1},\mathbb{L}^{(2)}),\hat{\hat{\partial}}) &\to (C(\mathbb{L}^{(1)},\mathbb{L}^{(2)}),\hat{\hat{\partial}}) \\ &\to (C(\mathbb{L}^{(1)}_{i\le I_1},\mathbb{L}^{(2)}),\hat{\hat{\partial}}) \to 0. \end{aligned}$$

$$\begin{aligned} 0 \to (C^*(\mathbb{L}^{(1)}_{i\ge I_1+1},\mathbb{L}^{(2)}),\delta_{smooth}) &\to (C^*(\mathbb{L}^{(1)},\mathbb{L}^{(2)}),\delta_{smooth}) \\ &\to (C^*(\mathbb{L}^{(1)}_{i\le I_1},\mathbb{L}^{(2)}),\delta_{smooth}) \to 0. \end{aligned}$$

Hence we can use induction on the number of components of $\mathbb{L}^{(1)}$ and obtain Theorem γ in general. □

We finally give some examples of Theorem 16.5 and briefly explain how it may be related to Lagrangian surjery.

Example 16.32. Suppose $\eta(\tilde L_1,\tilde L_2)=0$. We choose

$$x=\sum_{p\in L_1(w_1)\cap L_2(w_2)} c_p[p]\in HF^0((L_1(w_1),\alpha_1),(L_2(w_2),\alpha_2))$$

$(L_1(w_1),\alpha_1),(L_2(w_2),\alpha_2),x$ determine a Lagrangian resolution $\mathbb{L}$. Then $\mathcal{E}(\mathbb{L})\in Ob\left(\mathbb{D}\left((T^{2n},\Omega)^\vee\right)\right)$ is determined by the complex

$$E(L_1(w_1),\alpha_1)\overset{\Phi(x)}{\longrightarrow}E(L_2(w_2),\alpha_2).$$

Example 16.33. (Compare [Pl].) Suppose $\eta(\tilde L_1,\tilde L_2)=1$. We choose an element $x\in HF^1((L_1(v_1),\alpha_1),(L_2(v_2),\alpha_2))$. The triple $(L_1(w_1),\alpha_1),(L_2(w_2),\alpha_2),x$ determine a Lagrangian resolution $\mathbb{L}$. ($u(2)=1$.) We have an exact sequence

$$0\to\mathcal{E}(L_2(w_2),\alpha_2)\to\mathcal{E}(\mathbb{L})[-1]\to\mathcal{E}(L_1(w_1),\alpha_1)\to 0. \tag{16.34}$$

which corresponds to $x\in \mathrm{Ext}^1(\mathcal{E}(L_1(w_1),\alpha_1),\mathcal{E}(L_2(w_2),\alpha_2))$. To show (16.34) we consider the operator $\hat{\bar\partial}$ we used in the proof of Theorem 16.5. We find that

$$\hat{\bar\partial}(s_1,s_2)=\left(\bar\partial s_1,\bar\partial s_2+m_2^{(1)}(s_1,x)\right),$$

where

$$s_i\in\Lambda^{(0,d)}\otimes\mathcal{E}(L_i(w_i),\alpha_i).$$

Therefore we have an exact sequence

$$0\to\bigoplus_d\left(\mathcal{E}(L_2(w_2),\alpha_2)\otimes\Lambda^{(0,d)}\right)\to C^*(\mathbb{L})\to\bigoplus_d\left(\mathcal{E}(L_1(w_2),\alpha_1)\otimes\Lambda^{(0,d)}\right)\to 0$$

where $C^*(\mathbb{L})$ determines the object $\mathcal{E}(\mathbb{L})[-1]$. (16.34) follows.

Example 16.35. Let $\eta^*(\tilde L_1,\tilde L_2)=\eta^*(\tilde L_3,\tilde L_4)=1$ and $\eta^*(\tilde L_i,\tilde L_j)=0$ for other $i<j$. We put $u(1)=0$, $u(2)=u(3)=1$, $u(4)=u(5)=2$, and consider

$$x_{ij}\in HF^{d_{ij}}\left((L_i(w_i),\alpha_i),(L_j(w_j),\alpha_j)\right),$$

where $d_{12}=d_{34}=1$, $d_{23}=d_{24}=d_{13}=d_{14}=d_{35}=d_{45}=0$. Our equation (16.2) is

$$\begin{cases}\mathfrak{m}_3(x_{23},x_{34},x_{45})\pm\mathfrak{m}_2(x_{24},x_{45})\pm\mathfrak{m}_2(x_{23},x_{35})=0\\ \mathfrak{m}_4(x_{12},x_{23},x_{34},x_{45})\pm\mathfrak{m}_3(x_{13},x_{34},x_{45})\pm\mathfrak{m}_3(x_{12},x_{24},x_{45})\\ \qquad\qquad\pm\mathfrak{m}_2(x_{13},x_{35})\pm\mathfrak{m}_2(x_{14},x_{45})=0.\end{cases} \tag{16.36}$$

They are third and fourth order equations of $L_1\bullet L_2+L_2\bullet L_3+L_3\bullet L_4+L_4\bullet L_5$ $+L_1\bullet L_3+L_1\bullet L_4+L_2\bullet L_4+L_3\bullet L_5$ variables. (The number of equations is $L_2\bullet L_5+L_1\bullet L_5$.) We have a diagrams of exact sequences :

Diagram 2, 3

Here $\mathcal{E}_i = \mathcal{E}(L_i(w_i), \alpha_i)$. The extension of the first line in Diagram 2 is given by x_{12}. The composition $\mathcal{E}_2 \to \mathcal{F} \to \mathcal{E}_3$ is x_{23}. There exists a lift $\mathcal{F} \to \mathcal{E}_3$ of x_{23} since $\mathrm{Ext}^1(\mathcal{E}_1, \mathcal{E}_3) = 0$. The lift is not unique. The ambiguity is controlled by $x_{13} \in Hom(\mathcal{E}_1, \mathcal{E}_3)$. The extension of the first horizontal line in Diagram 3 is given by $x_{34} \in \mathrm{Ext}^1(\mathcal{E}_3, \mathcal{E}_4)$. Note that we can find $\mathcal{H}$ as in the second horizontal line of Diagram 3, since $\mathrm{Ext}^1(\mathcal{F}, \mathcal{E}_4) = 0$. (The extension $\mathcal{H}$ such that Diagram 3 commutes is not unique. The ambiguity is controlled by x_{14} and x_{24}.) The equations (16.36) give a condition for the map $x_{45} \in Hom(\mathcal{E}_4, \mathcal{E}_5)$ to extend to $\mathcal{H} \to \mathcal{E}_4$. It seems possible but complicated to identify this obstruction as an element of $Hom(\mathcal{E}_1, \mathcal{E}_5) \oplus Hom(\mathcal{E}_2, \mathcal{E}_5)$. We do not need to do so since we can construct Diagrams 2,3 directly from

Figure 17

We next describe how Theorems 16.5 is related to the study of Lagrangian submanifolds in tori. We will explain what Examples 16.32 is expected to correspond in the mirror. We regard $L_1(w_1) \cup L_2(w_2)$ as a singular Lagrangian submanifold in (T^{2n}, Ω). We suppose $B = 0$ for simplicity. We can perform Lagrangian surgery at each $p \in L_1(w_1) \cap L_2(w_2)$ and obtain a smooth Lagrangian submanifold $L \subseteq (T^{2n}, \omega)$. (Figure 16.) (In fact there are two different ways to perform the Lagrangian surgery. The discussion below is applied to only one of them. See [FKOOO] Chapter 7.)

Figure 18

In case $n = 2$, $L_1(w_1) \bullet L_2(w_2) = 1$, we obtain a genus 2 Lagrangian surface in T^4. (Flat connections α_1, α_2 induce a flat connection α on L.)

"Proposition 16.37". *The bundle $\mathcal{E}(L, \alpha)$ mirror to (L, α) is equal to $\mathcal{E}(\mathbb{L})$ with $c_p \neq 0$.*

We put the proposition in the quote since the rigorous definition of the mirror bundle is not given in this paper. We explain an argument which justify the proposition. Let $(v, \sigma) \in (T^{2n}, \Omega)^\vee$. The fiber $\mathcal{E}(\mathbb{L})_{(v,\sigma)}$ is a cohomology of the complex

$$HF((L_{\mathrm{pt}}(v), \sigma), (L_1(w_1), \alpha_1)) \overset{m_2(\bullet, x)}{\longrightarrow} HF((L_{\mathrm{pt}}(v), \sigma), (L_2(w_2), \alpha_2)) \tag{16.38}$$

We remark that the isomorphism class of the complex (16.38) is independent of c_p as far as it is nonzero.

On the other hand, the fiber $\mathcal{E}(L, \alpha)_{(v,\sigma)}$ is expected to be isomorphic to the Floer homology $HF^0((L_{\mathrm{pt}}(v), \sigma), (L, \alpha))$. Floers chain complex to calculate it is (as graded abelian group) :

$$\begin{aligned} &CF((L_{\mathrm{pt}}(v), \sigma), (L, \alpha)) \\ &\quad = \bigoplus_{p \in L_{\mathrm{pt}}(\sigma) \cap L} Hom\left(\mathcal{L}(\sigma)_p, \mathcal{L}(\alpha)_p\right) \\ &\quad \cong Hom\left(\mathcal{L}(\sigma)_x, \mathcal{L}(\alpha_1)_x\right) \oplus Hom\left(\mathcal{L}(\sigma)_y, \mathcal{L}(\alpha_2)_y\right) \\ &\quad \cong HF\left((L_{\mathrm{pt},\sigma}), (L_1, \alpha_1)\right) \oplus HF\left((L_{\mathrm{pt},\sigma}), (L_2, \alpha_2)\right). \end{aligned} \tag{16.39}$$

Floer's boundary operator of $CF((L_{\rm pt}(v),\sigma),(L,\alpha))$ is obtained by counting the number of holomorphic 2 gons bounding $L_{\rm pt}(v)$ and L. As we can guess from Figure 18, and is proved in [FKOOO] Chapter 7, such 2 gon will become a holomorphic triangle used in the definition of $\mathfrak{m}_2$. Thus we find that the boundary operator in (16.38) and the map in (16.39) coincides. It "implies"

$$\mathcal{E}(\mathbb{L})_{(v,\sigma)} \cong \mathcal{E}(L,\alpha)_{(v,\sigma)}. \tag{16.40}$$

We finally remark that there is one important point of view which is not studied in this paper. That is, in this paper, we fix Ω and regard $\mathfrak{m}_k$ as a function on Abelian variety. In the theory of theta function, it is more important to regard it as a function of Ω (the moduli parameter of Abelian variety). This point of view is important also for Mirror symmetry. Note that we can generalize the equation

$$\bar\partial \tilde{\mathfrak{m}}_k^{(0,d)} + \sum (-1)^{d_2} \tilde{\mathfrak{m}}_{k_2}^{(0,d_2)} \circ \tilde{\mathfrak{m}}_{k_1}^{(0,d_1)} = 0 \tag{16.41}$$

so that $\bar\partial$ include derivative with respect to Ω. (Ω moves on the space $\{\Omega | \tilde L_i|_{\Omega} = 0, i = 1,\cdots,k+1\}$.) In the case when $d=0$ and the case when the image of the wall is compact in T^{2n}, (16.41) can be regarded as an equation to control wall crossing of our multi theta function $\tilde{\mathfrak{m}}_k$. (Here we regard it as a function of Ω.) The wall crossing studied in [Bo], [GZ] seems to be more directly related to it. We leave systematic study of multi theta function as a function of Ω, as a target of the future research.

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