Lagrangian Floer homology

Kenji FUKAYA

Based on joint work with
Oh-Ohta-Ono

Floer homology

(A. Floer, ~25 year ago)

Homology group of degree $\frac{\infty}{2}$ in an ∞ dimensional space

Realized by using Morse theory

Many examples

Gauge theory
(Law dimensional topology)

$$M \to HF(M)$$

3 manifold → group

- Yang-Mills Gauge theory (Floer)
- Seiberg-Witten Floer homology
  (Marcolli-Wang,Kronheimer-Mrowka, etc.)
- Heegard Floer homology
  (Ozsvath-Szabo.)

Symplectic Geometry

- Periodic Hamiltonian system

  $\mathrm{rank}\ HF(M) \geq \#(\text{closed orbit of periodic Hamiltonian system})$

- Lagrangian submanifold

  $$\mathrm{rank}\ HF(L_1; L_2) \geq \#(L_1 \cap L_2)$$

- Contact manifold (Eliashberg-Hofer-Givental)

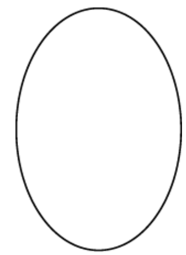
  $\mathrm{rank}\ HC(N) \geq \#(\text{closed orbit of Reeb vector field})$

(M, ω) Symplectic manifold

ω: 2 form, $d\omega = 0$, ω^n = volume form

Periodic Hamiltonian system

- $H : S^1 \times M \to \mathbb{R}$



$\ell : S^1 \to M$

Floer's chain complex $CF(X;H)$

- generator $\ell : S^1 \to M$ Periodic solution of Hamiltonian vector field

$$\frac{D\ell}{dt} = \sum_{i=1}^{n} \left(\frac{\partial H_t}{\partial q^i} \frac{\partial}{\partial p^i} - \frac{\partial H_t}{\partial p^i} \frac{\partial}{\partial q^i} \right)$$

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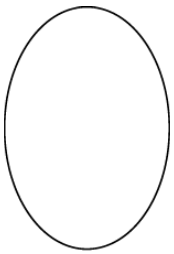

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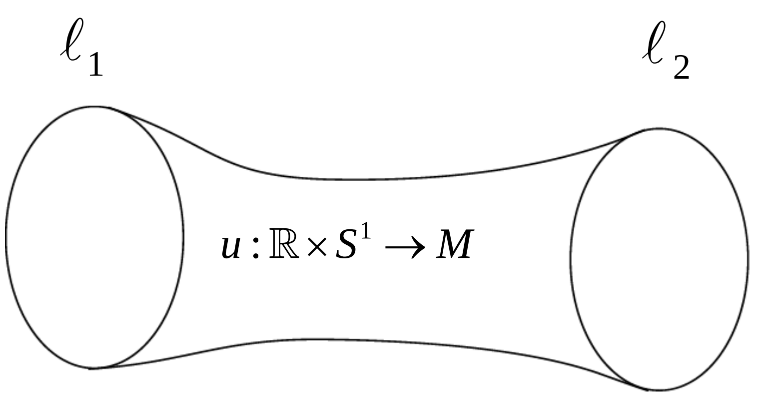
- Boundary operator

$$\langle \partial \ell_1, \ell_2 \rangle = \text{count } u : \mathbb{R} \times S^1 \to M$$

$$\frac{du}{d\tau} = J\left(\frac{du}{dt} - X_H\right)$$

$$\lim_{\tau\to-\infty} u(\tau,t) = \ell_1(t) \qquad \lim_{\tau\to+\infty} u(\tau,t) = \ell_2(t)$$



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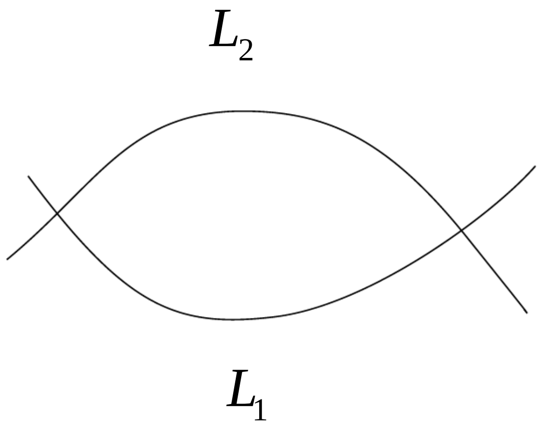

Lagrangian submanifold

- $L_1, L_2 \subset M$ $\dim L = \frac{1}{2}\dim M$ $\omega|_L = 0$

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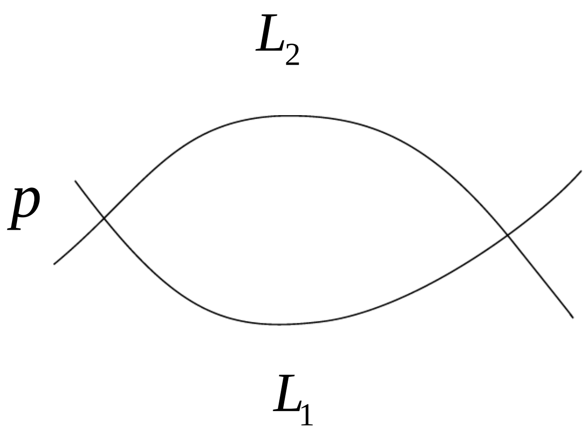

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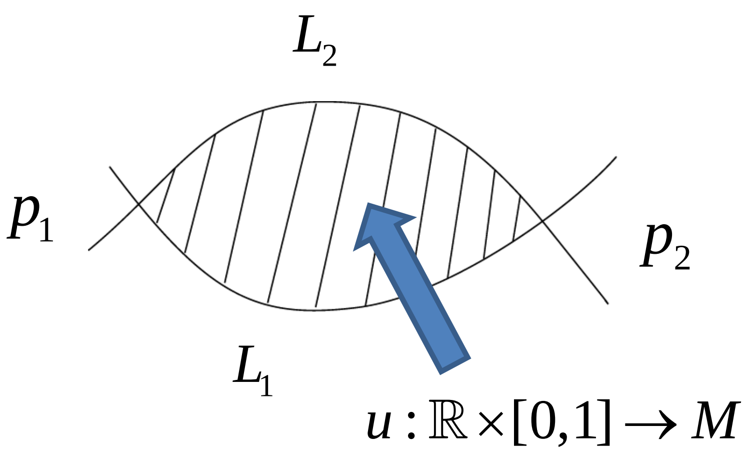

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- generator $\quad p \in L_1 \cap L_2$
- Boundary operator

$$\langle \partial p_1, p_2 \rangle = \text{count } u:\mathbb{R}\times[0,1]\to M$$

$$\frac{du}{d\tau} = J\frac{du}{dt}$$

$$\lim_{\tau\to-\infty} u(\tau,t) = p_1 \qquad \lim_{\tau\to+\infty} u(\tau,t) = p_2$$

$$u(\tau,0)\in L_1 \qquad u(\tau,1)\in L_2$$

$$(M,\omega)$$ Symplectic manifold

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Periodic Hamiltonian system

- $H : S^1 \times M \to \mathbb{R}$

$HF(M;H) = \dfrac{\text{Ker}\,\partial}{\text{Im}\,\partial}$ Floer homology

- Floer homology always exists .
- $HF(M;H) = H(M)$

  If M is compact.

Lagrangian submanifold

- $L_1, L_2 \subset M$ $\dim L = \frac{1}{2}\dim M$ $\omega|_L = 0$

$HF(L_1, L_2) = \dfrac{\text{Ker}\,\partial}{\text{Im}\,\partial}$ Floer homology

- Floer homology may or may not exist .
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  in some but not all cases.

$$(M, \omega)$$ Symplectic manifold

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  in some but not all cases.

Ideal statement for Lagragian Floer homology $HF(L_1, L_2)$

I For any pair of Lagrangian submanifold $L_1, L_2 \subset M$

we have a Floer homology $HF(L_1, L_2)$.

II If L_1 is transversal to L_2

then $\#\left(L_1 \cap L_2\right) \geq \text{rank } HF(L_1, L_2)$

III Floer homology is invariant of Hamiltonian diffeomorphisms:

$$HF(L_1, L_2) \cong HF(\varphi_1(L_1), \varphi_2(L_2))$$

IV $L_1 = L_2 = L \longrightarrow HF(L, L) \cong H(L)$

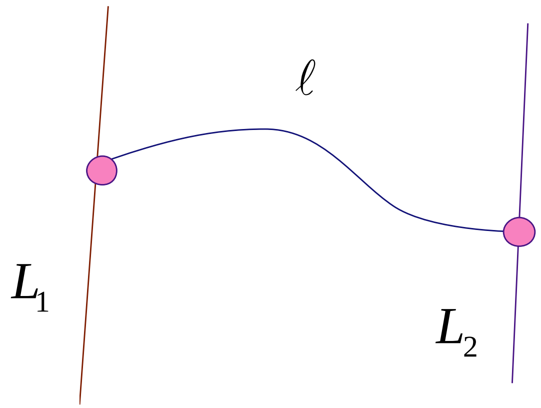
Hamiltonian diffeomorphisms: $\varphi_H : M \to M$

$$H : M \times \mathbb{R} \to \mathbb{R}, \quad H_t(x) = H(x,t)$$

$$\ell(0) = x, \quad \ell(1) = \varphi_H(x)$$

$$\ell : [0,1] \to M, \quad \ell(t) = (q(t), p(t))$$

$$\frac{Dp_i}{dt} = \frac{\partial H_t}{\partial q^i}, \quad \frac{Dq_i}{dt} = -\frac{\partial H_t}{\partial p^i}$$



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$$\varphi_H(L_1) \cap L_2 \quad \Longleftrightarrow$$

$$\ell(0) \in L_1, \quad \ell(1) \in L_2$$

Example $\quad L_1 = T_p^* M \subset T^* M \qquad L_2 = T_q^* M \subset T^* M$

Floer (1980's)

$$L_1 = L \subset M \qquad \pi_2(M, L) = 0$$

$$L_2 = \varphi(L)$$

$\Downarrow$

I II III IV

Oh (1990's)

L is monotone + α

$\Downarrow$

I II III ~~IV~~

Lagrangian intersection (1)

I, II, III, IV $\longrightarrow$ $\#\left(L \cap \varphi(L)\right) \geq \text{rank } H(L)$

$$\#\left(L \cap \varphi(L)\right) \quad \geq \quad \text{rank } HF(L, \varphi(L)) \quad \underset{\text{IV}}{\geq} \quad \text{rank } H(L)$$

II

If L_1 is transversal to L_2 then $\#\left(L_1 \cap L_2\right) \geq \text{rank } HF(L_1, L_2)$

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IV

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This inequality can not be true in general !

$L \subset \mathbb{C}^n$ $\longrightarrow$ $\exists \varphi \quad \varphi(L) \cap L = \varnothing$

$$0 = \#\left(L \cap \varphi(L)\right) \geq \operatorname{rank} H(L) > 0$$

Fukaya-Oh-Ohta-Ono (this century) $HF(L_1, L_2)$

I' For any (relatively) spin Lagrangian submanifold $L \subset M$

$\exists$ `Maurer-Cartan formal scheme' $\mathcal{M}(L)$ (can be empty.)

$\exists$ Floer homology $HF((L_1, b_1),(L_2, b_2))$

parametrized by $b_i \in \mathcal{M}(L)$.

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$$HF((L_1, b_1),(L_2, b_2)) \cong HF((\varphi_1(L_1), \varphi_{1*}(b_1)),(\varphi_2(L_2), \varphi_{2*}(b_2)))$$

IV' $L_1 = L_2 = L$ ⟶ There exists a spectral sequence

$$E_2 = H(L) \quad \Rightarrow \quad E_\infty = HF(L, L)$$

Ideal statement

I For any pair of Lagrangian submanifold $L_1, L_2 \subset M$

we have a Floer homology $HF(L_1, L_2)$.

Real statement

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Story can be re-written by hotmotopy theory of A inifinity algebra

A inifinity algebra and Maurer-Cartan scheme

(Stasheff)

(Filtered) A inifinity algebra

$$m_k : \underbrace{C \otimes \cdots \otimes C}_{k} \to C$$

$$\sum_{k_1+k_2=k+1} \pm m_{k_1}(x_1,\cdots,m_{k_2}(x_i,\cdots,x_{i+k_2-1}),\cdots x_k) = 0$$

Maurer-Cartan scheme

$$\mathcal{M}(C) = \left\{ b \in C^1 \,\middle|\, \sum_k m_k(b,\cdots,b) = 0 \right\} \Big/ \text{Gauge equivalence}$$

Theorem (Fukaya-Oh-Ohta-Ono)

$L \subset M$

: relatively spin Lagrangian submanifold. ⇒ $H(L)$ is a filtered A infinity algebra.

Theorem (Fukaya-Oh-Ohta-Ono)

(relatively spin)
pair of Lagrangian submanifold
$L_1, L_2 \subset M$

⇒ $\exists \quad CF(L_1; L_2)$

filtered A infinity $H(L_1) - H(L_2)$ bimodule.

These can be generalized to 3 or more Lagrangian submanifold.

For example it gives product $HF(L_1, L_2) \otimes HF(L_2, L_3) \to HF(L_1, L_3)$

(potential) applications of Lagrangian Floer theory

- Lagrangian intersection.
- Restriction (or potentially classification) of Lagrangian submanifold.
- Homological Mirror symmetry

Lagrangian intersection (1)

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This inequality can not be true in general !

$L \subset \mathbb{C}^n$ $\longrightarrow$ $\exists \varphi \quad \varphi(L) \cap L = \emptyset$

$$0 = \#\left(L \cap \varphi(L)\right) \quad \geq \quad \operatorname{rank} H(L) > 0$$

Lagrangian intersection (2)

Arnold-Givental conjecture

$$\#\left(L \cap \varphi(L)\right) \geq \operatorname{rank} H(L;\mathbb{Z}_2)$$

if $L = \left\{x \in M \,\middle|\, \tau(x) = x\right\}$ where $\tau : M \to M$ is an anti-holomorphic involution.

Theorem (Fukaya-Oh-Ohta-Ono) AG conjecture is true if M is spherically positive.

Ok: in case M is a product of manifolds which satisfies one of the following:

- spherially monotone: $c^1 = \lambda[\omega]$ $\lambda > 0$ on $\pi_2(M)$
- $\dim_{\mathbb{C}} M = 2$
- M is Fano.

Lagrangian intersection (3) (FOOO)

(1) If $H^{\mathrm{ev}}(M;\mathbb{Q}) \to H^{\mathrm{ev}}(L;\mathbb{Q})$ is surjective then $\#\big(L \cap \varphi(L)\big) \ge \operatorname{rank} H(L;\mathbb{Q})$

(2) If $\mathcal{M}(L) \neq \varnothing$ then

$$\#\big(L \cap \varphi(L)\big) \ge \operatorname{rank} H(L;\mathbb{Q}) - 2 \operatorname{rank} \operatorname{Ker}\big(H_*(L;\mathbb{Q}) \to H_*(M;\mathbb{Q})\big)$$

Proof use

IV' $L_1 = L_2 = L$ ⟶ There exists a spectral sequence

$$E_2 = H(L) \quad \Rightarrow \quad E_\infty = HF(L,L)$$

and open closed Gromov-Witten theory.

Restriction of Lagrangian submanifold

Theorem (Gromov) If $L \subset \mathbb{C}^n$ is a Lagrangian sumanifold then $H^1(L;\mathbb{Q}) \neq 0$

More over: $E : H_2(\mathbb{C}^n; L) = H_1(L) \to \mathbb{R}$ $E(\beta) = \int_\beta \omega$ is non zero.

Restriction of Lagrangian submanifold

Theorem (Gromov) If $L \subset \mathbb{C}^n$ is a Lagrangian sumanifold then $H^1(L;\mathbb{Q}) \neq 0$

Use

$$L \subset \mathbb{C}^n \longrightarrow \exists \varphi \quad \varphi(L) \cap L = \emptyset$$

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More over: $E : H_2(\mathbb{C}^n; L) = H_2(L) \to \mathbb{R}$ $\quad E(\beta) = \int_\beta \omega$ is non zero.

- $L \subset M$ is said exact if $E(\beta) = \int_\beta \omega \equiv 0$
- $\mu : H_2(M;L) \to \mathbb{Z}$ $\quad E(\beta)$ ``$=$'' $\int_\beta c^1(M)$ Maslov class

Conjecture If $L \subset \mathbb{C}^n$ is a Lagrangian sumanifold then $\mu : H_2(\mathbb{C}^n; L) \to \mathbb{Z}$ is nonzero.

Theorem (Fukaya-Oh-Ohta-Ono, Biran-Cielibak) Conjecture is OK if $H^2(L;\mathbb{Q}) = 0$

Theorem (Fukaya)

If $L \subset \mathbb{C}^n$ is a spin Lagrangian sumanifold $\pi_k(L) = 0$ for $k = 2, 3, \ldots$

then $\tilde{L} \sim S^1 \times L'$ and $\mu(S^1) = 2$

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homotopy equivalent

finite covering of *L*

How one can calculate Lagrangian Floer homology ?

- Reduction to Morse theory (or to classical cohomology)
- Cutting and pasting.
- Mirror symmetry.

Reduction to Morse theory (or to classical cohomology)

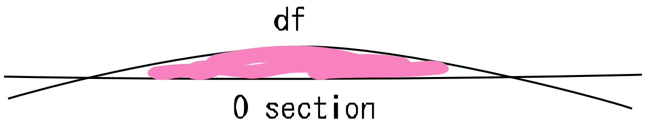
$L_1, L_2 \subset M$ assume L_2 is close to L_1

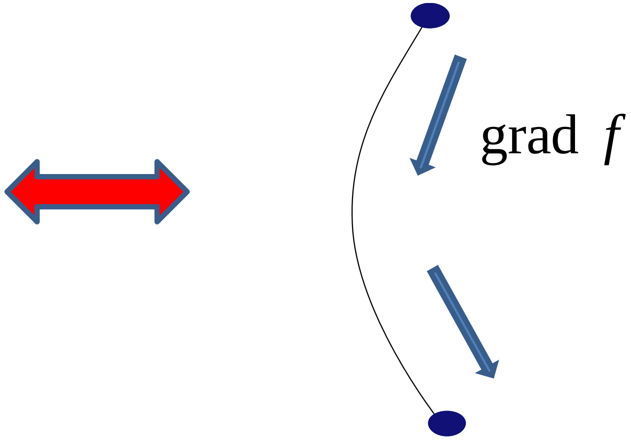
L_2 can be regarded as $\subseteq T^* L_1$ and as a graph of some closed one form u

(Weinstein)

Floer (1980's)

If $u = df$ then $HF(L_1, L_2)$ is Morse homology of f





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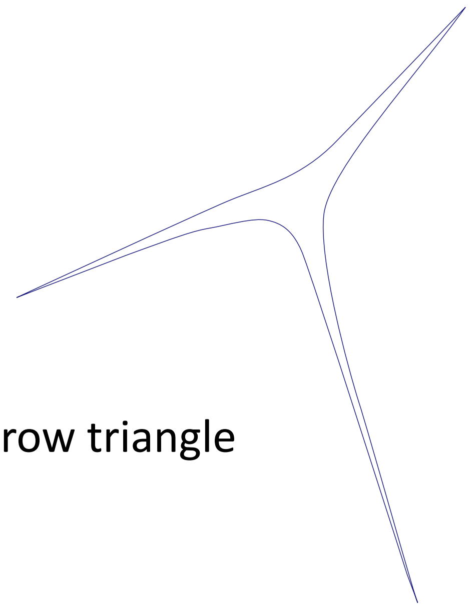
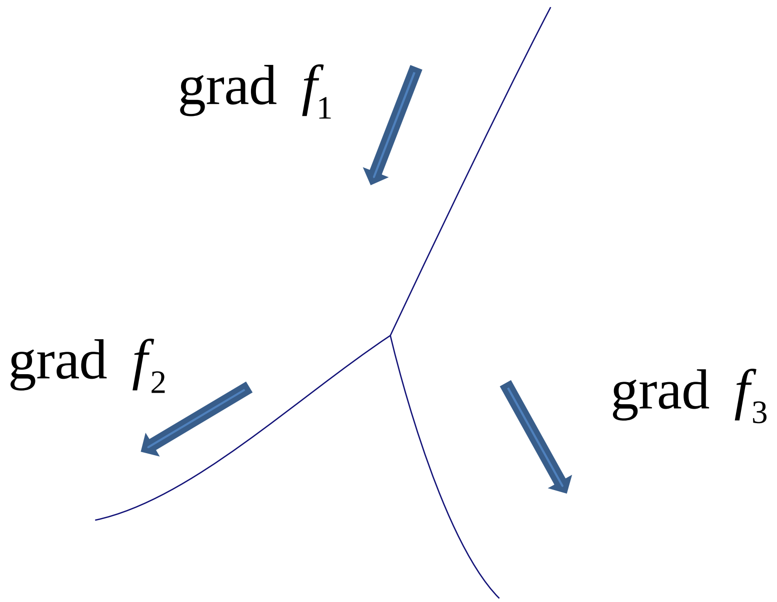
Fukaya-Oh (1990's)

generalizes it and includes product structures $m_k : \underbrace{C \otimes \cdots \otimes C}_{k} \to C$



very narrow triangle



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Kontsevich-Soibelman (2000) $T^* L \Rightarrow T^n$ - bundle over B

2000's includes singular fiber, can use also cylinder $\mathbb{R} \times S^1$

relation to tropical geometry

....

Cutting and pasting.

$L_1, L_2, L_3 \subset M$ L_1, L_2 intersects transversaly at one point

$L_1 \#_\varepsilon L_2$ Lagrangian surgery (diffeomorphic to the connected sum).

Fukaya-Oh-Ohta-Ono

Boundary operator for $HF(L_1 \#_\varepsilon L_2, L_3)$

$\Updownarrow$

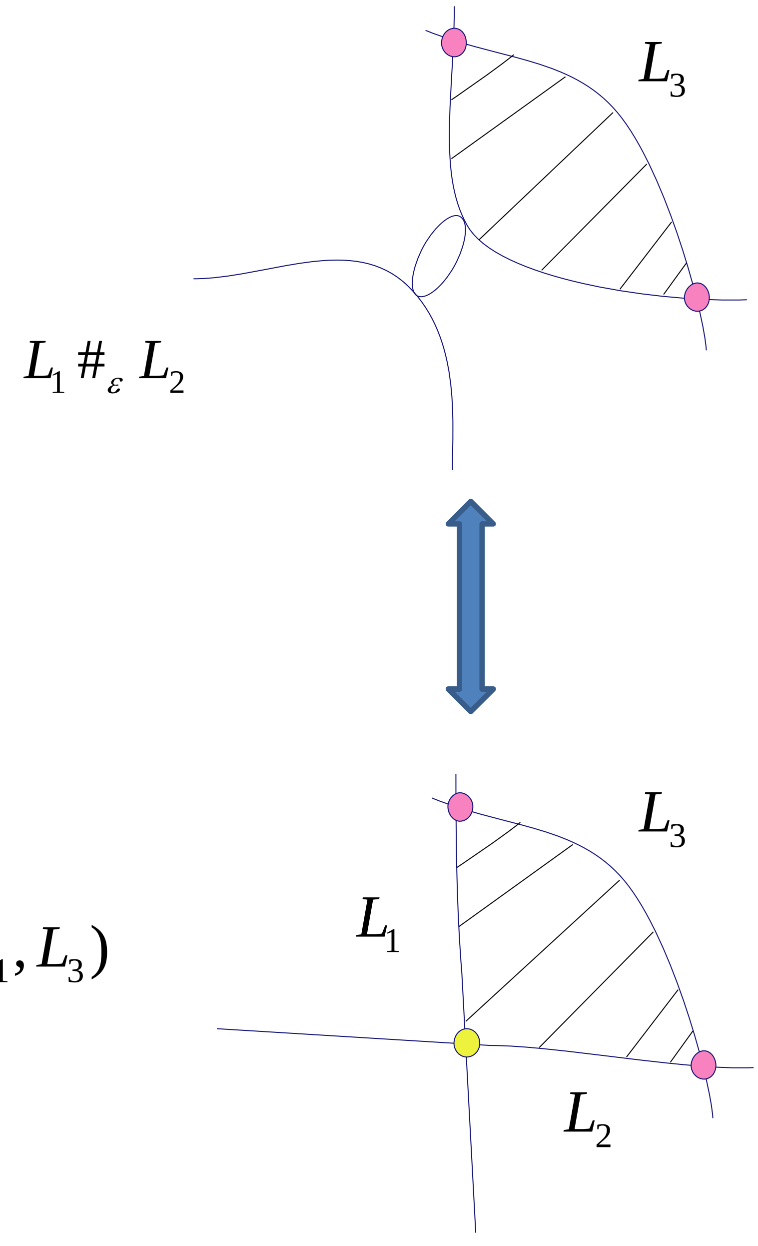
Product $HF(L_1, L_2) \otimes HF(L_2, L_3) \to HF(L_1, L_3)$

Boundary operator for

$$HF(L_1 \#_\varepsilon L_2, L_3)$$

Product

$$HF(L_1, L_2) \otimes HF(L_2, L_3) \to HF(L_1, L_3)$$



Homological Mirror symmetry (Kontsevich 1994)

(M, ω) symplectic manifold

(L, b)

$L \subset M$ Lagrangian sub-manifold

$b \in \mathcal{M}(L)$ Maurer-Cartan scheme

$HF((L_1, b_1), (L_2, b_2))$ Floer homology

(X, J) complex manifold

$E(L, b) \to X$

Holomorpic vector bundle

$\mathrm{Ext}(E(L_1, b_1), E(L_2, b_2))$ Sheaf cohomology

$HF((L_1, b_1), (L_2, b_2)) \simeq \mathrm{Ext}(E(L_1, b_1), E(L_2, b_2))$ reduces the

calculation of Floer cohomology to one of sheaf cohomology.

Homological Mirror symmetry is proved in the case

- Elliptic curve (Kontsevich(94), Polishchuk-Zaslow(97))
- Tori (of higher dimension) partially (Fukaya(98),Kontsevich-Soibelman(00))
  T^4 (Abouzaid-Smoth(08))
- Quartic surface (Seidel (03))
- Toric (complex) - Landau-Ginzburg (symplectic) (Auroux-Kazarkov-Orlov, Ueda, Abouzaid, …)
- Toric (symplectic) - Landau-Ginzburg (complex) (Cho-Oh, FOOO, …….)
- Cotangent bundle (Nadler-Zaslow, Fukaya-Seidel-Smith)
- Genus 2 surface (Seidel)
- K3 (Fukaya (on progress))