# Lec案 vertion 1 ①

## §1 Survival toric geometry

$(X, \omega)$ symplectic manifold $\qquad 2n = \dim_{\mathbb{R}} X$

$T^n \circlearrowright X$ sym act

$\exists\ \pi : X \longrightarrow \mathbb{R}^n$ moment map

namely $\pi = (\pi_1 \cdots \pi_n)$

$(0 \cdots 0 \overset{k}{1} 0 \cdots 0) \longmapsto \dfrac{\partial}{\partial t_k}$ vector field

$\omega(V, \dfrac{\partial}{\partial t_k}) = (d\pi_k)(V)$ 　フゴーチェック

namely $\pi_k$ = moment map of $S^1 \hookrightarrow T^n$
k-th component

$P = \operatorname{Im} \pi$

∘ $P$ is convex ($\square$-1$\square$, Atiyah)
polygon

Example

$\mathbb{P}^n = \{[z_0 : z_1 : \cdots : z_n] \mid \sum |z_i|^2 = 1 \quad z_i \in \mathbb{C}\}$

$T^n = \{(\sigma_0, \cdots, \sigma_n) \mid |\sigma_i| = 1, \quad \sigma_i \in \mathbb{C}\} / S^1$

$S^1 = \{(\sigma, \cdots, \sigma)\}$

$T^n \circlearrowright \mathbb{P}^n \qquad \underbrace{(\sigma_0, \cdots, \sigma_n)[z_0, \cdots, z_n]}$

$= [\sigma_0 z_0 : \cdots : \sigma_n z_n]$

$\pi : \mathbb{P}^n \longrightarrow \mathbb{R}^n$

$[z_0 : \cdots : z_n] \longmapsto (|z_1|, \cdots, |z_n|)$

$T^n \cong \{(\sigma_1^{-1} \cdots \sigma_n^{-1}, \sigma_1, \cdots, \sigma_n) \mid \sigma_i \in \mathbb{C} \quad |\sigma_i| = 1\}$

$\parallel$

$(S^1)^n \quad \sigma_i$

$k$-th cpt $S^1 \subset (S^1)^n$

moment map of $S^1$ : $k$-th component $= \pi_k$.

$P = \{(t_1, \cdots, t_n) \mid 0 \le t_i \quad \sum t_i = 1\}$ simplex

Example 2

blow up

$\lambda : \mathbb{P}^2 \longrightarrow \Delta^2 \subset \mathbb{R}^2$

$\varepsilon$ nbd of

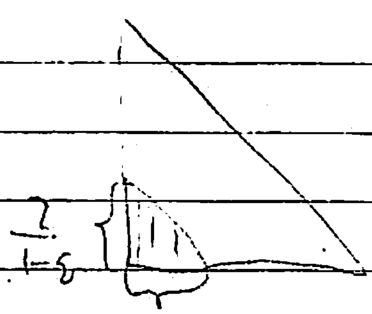
$$[1;0,0] = B_\varepsilon(0)$$
$$\cap$$
$$\mathbb{C}^2 \subset \mathbb{P}^2$$

$$\left\{ [1, z_1, z_2] \;\middle|\; \frac{|z_1|^2 + |z_2|^2}{1+|z_1|^2+|z_2|^2} < c \right\} \subset \mathbb{C}$$

$\big\downarrow \lambda$

$\mathbb{R}^2$

$$\frac{c}{1+c} = \varepsilon$$



$$(1-\varepsilon)c = \varepsilon$$

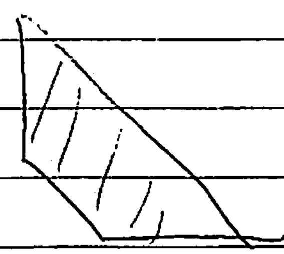
$\frac{\varepsilon}{1-\varepsilon}$

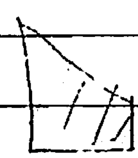
$\frac{\varepsilon}{1-\varepsilon}$

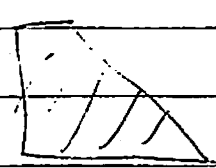
$(\mathbb{P}^2 \setminus B_\varepsilon(0))/\sim$

$\sim$ $S^1 = \{(\sigma^{-2}, \sigma, \sigma) \mid \sigma \in S^1\}$ action

blow up of $\mathbb{P}^2$

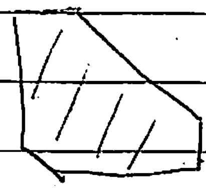
Rel $\mathbb{P}^2 \# \overline{\mathbb{P}^2}\left(\frac{\varepsilon}{1-\varepsilon}\right) \xrightarrow{\mu}$ 

Blow up at $[0:1:0]$ $\longrightarrow$ 

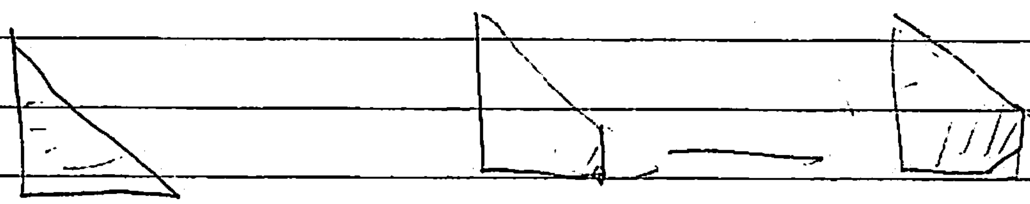
$[0:0:1]$ $\longrightarrow$ 

2 pts blow up

3 pt blow up



Iterated blow up



§2 Problem

$(X, \omega) \xrightarrow{\pi} P \subset \mathbb{R}^n$ toric mfd

$\cup$

$T^n$

$u \in \mathrm{Int}\, P$

$\pi^{-1}(u) = T^n$ orbit $\cong T^n$

(Action is free on $\pi^{-1}(\mathrm{Int}\, P)$)

Put $L(u) = \pi^{-1}(u)$

$L(u)$ is Lagrangian mfd.

(i.e. $\omega|_{L(u)} = 0$)

Problem

Find all $u \in \operatorname{Int} P$ st.

$L(u)$ is non-displaceable.

Def $L \subset X$ is nondisplaceable

$\Longleftrightarrow \forall \varphi : X \to X$ Haml. diffeo.

$\varphi(L) \cap L \neq \phi$

Note

$\varphi : X \to X$ Haml diff.

$\Longleftrightarrow \quad \exists f : [0,1] \times X \to \mathbb{R}$

$\exists \varphi_t : X \to X \quad t \in [0,1]$

st $\left.\dfrac{d\varphi_t}{ds}\right|_t = X_{f_t}$

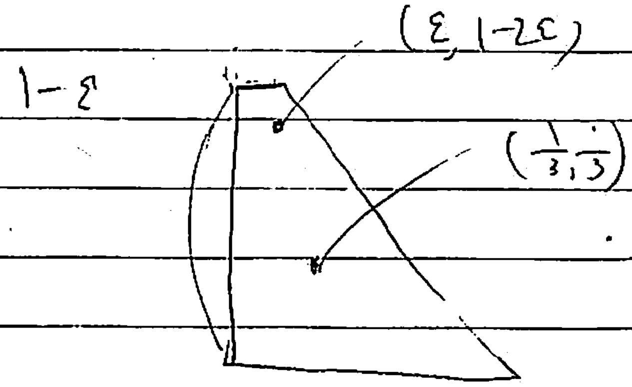
$\omega(V, X_{f_t}) = df_t(V)$,

Sample of answer

① $X = \mathbb{P}^n$

$L(u)$ is nondisplaceable $\iff$ $u = \left(\frac{1}{n+1}, \cdots, \frac{1}{n+1}\right)$

② $X$ one pt blow up $\mathbb{P}^2$




$\varepsilon < \frac{1}{3}$

$L(u)$ is nondisplaceable

$\iff$ $u = \left(\frac{1}{3}, \frac{1}{3}\right)$

or

$u = (\varepsilon, 1-2\varepsilon)$

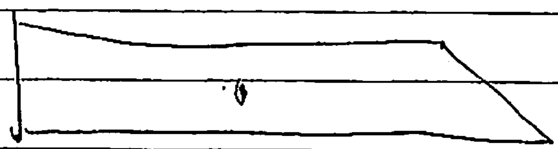
$\varepsilon = \frac{1}{3}$

$(X, \omega)$ is monotone

$\left( \Leftrightarrow \ [\omega] = c(c_1(X)) \right)$

$\Leftrightarrow \quad u = \left(\frac{1}{3}, \frac{1}{3}\right)$

$\varepsilon > \frac{1}{3}$



$\Leftrightarrow \quad u = \left( \frac{1+\varepsilon}{4}, \frac{1-\varepsilon}{2} \right)$

2 pt blow up etc will be discussed

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## Method of proof

Lagrangian Floer theory (Floer, Oh, FOOO)

General

$L \subset X$ relatively spin Lag sub

$\rightsquigarrow$ $\mathcal{M}_{weak}(L)$ set.

$\cup$

$b \rightsquigarrow HF((L,b))$.

$\exists\ HF((L,b)) \neq 0$

$\Rightarrow$ $L$ is nondisplaceable

(Non displaceable $\Leftrightarrow$ $\not\exists\ \psi : X \to X$ Hamilton.

$\psi(L) \cap L = \phi$)

Special in our case $L(u)$ : $T^n$ orbit

CX toric

① $H^1(T^n; \Lambda_0) \subset \mathcal{M}_{wk}(T^n)$

③ $\exists$ PO : $H^1(T^n; \Lambda_0) \to \Lambda_0$

st.

$HF((L, b)) \neq 0$ $\quad b \in H^1 \in \mathcal{M}$

$\Leftrightarrow$ $b$ is a critical point of PO.

$\Lambda_0 = \left\{ \sum a_i T^{\lambda_i} \;\middle|\; \begin{array}{l} \lambda_0 < \lambda_1 < \cdots \quad \lambda_i \in \mathbb{R}_{\geq 0} \\ \lim \lambda_i = \infty \\ a_i \in R \end{array} \right\}$

(or $\Lambda_0^R$)

$R \supset \mathbb{Q}$ field

$\Lambda_+ = \{ \sum \mid \lambda_i \gneq 0 \}$

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(A part of) def of PO and its calculation.

$L \subset X$ (general) $\beta \in \pi_2(X; L)$

Def $\overset{\circ}{\mathcal{M}}_{k+1}(L; \beta)$

$= \{ (u, z_0 \ldots z_k) \mid$ $u : (D^2, \partial D^2) \to (X, L)$ holo. $z_0, \ldots z_k \in \partial D^2$ respects cycl. order $\} / \sim$

$\sim$ $PSL(2,\mathbb{R}) = \mathrm{Aut}(D^2)$ act

$(u; z_0 \ldots z_k) \sim (u \circ \varphi; \varphi^{-1}(z_0) \ldots \varphi^{-1}(z_k))$

Can be compactified to $\mathcal{M}_{k+1}(L; \beta)$

It has a fundamental chain of dim

$k+1+n+\mu(\beta)-3$ $\mu(\beta)$ = Maslov index

$3 = \dim PSL(2,\mathbb{R})$

Special in our case $L = \Phi(V) = T^n \subset X$ toric

○ $T^n \circlearrowleft X$ $T^n$ act induce

$T^n$ act on $\mathcal{M}_{k+1}(L;\beta)$:

It is free! (since $T^n$ act on $L(u)$ is free.)

May choose pert $T^n$ invariant

$$\begin{array}{ccc} \mathcal{M}_{k+1}(L;\beta)^s & \xrightarrow{ev_0} & T^n \\ (u, z_0, \dots, z_k) & \longmapsto & u(z_0) \end{array}$$

$T^n$ equiv

perturbed moduli space

(zero of multi-section)

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Cor (n≥2) $L = L(u) \cong T^n$

① $\dim M_1(L;\beta) = n + \mu(\beta) - 2 \geq n$

if non empty

② $ev_1 : M_1(L,\beta) \longrightarrow L$

is submersion

∵) ③ ⇒ ② obvious

③ follows from $T^n$ equivariance

$M_1(L;\beta) = \phi$ if $\mu(\beta) < 2$

Cor $\partial M_1(L;\beta) = \phi$ if $\mu(\beta) = 2$

Thm

$M_1(L;\beta)$ has a fundamental cycle.

Def. $\underbrace{n_\beta}_{\mu(\beta)=2}$ = degree of ev

$$M_1(L;\beta) \longrightarrow L$$

PO is defined as follows

$b = \sum x_i e_i$ $\quad$ $e_1 \cdots e_n$ basis of $H_1(T^n)$

$x_i \in \Lambda_0$.

$PO(b)$

$$= \sum_{\mu(\beta)=2} T^{\beta\cap\omega} \, n_\beta \, \exp\Big(\sum x_i \langle \underbrace{\partial\beta}_{H_1(T^n)}, \underbrace{e_i}_{H^1(T^n)} \rangle\Big)$$

Cho-Oh's thm

$P$ moment polytope

$\partial P = \partial_1 P \cup \cdots \cup \partial_m P$ $\quad D_i = \pi^{-1}(\partial_i P)$ $\cdot$ codim 2 in $X$

$\beta_1 \cdots \beta_m \in \pi_2(X, T^n)$

st $\quad \beta_i \cap D_j = \delta_{ij}$

$$\left(\begin{array}{ccccccc}
H_2(X) & \to & H_2(X, X \setminus D) & \to & H_1(X \setminus D) & & \\
\| & & \| & & \| & & \\
 & & H_2(X, T^n) & & H_1(T^n) & & \\
 & & \| & & \| & & \\
0 \to \mathbb{Z}^{m-n} & \longrightarrow & \mathbb{Z}^m & \longrightarrow & \mathbb{Z}^n & \to 0 & \\
 & & \beta_1 & \longmapsto & \partial \beta_1 & & \\
 & & \vdots & & & & \\
 & & \beta_m & \longmapsto & \partial \beta_m & &
\end{array}\right)$$

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Thm (Ch. Oh)

① $ev_0 : \mathcal{M}_1(\beta_i) \longrightarrow T^n$ deg 1 diffeo

$\cup$

$\mathring{\mathcal{M}}_1(\beta_i)$

② $\mu(\beta) = 2$

$\mathring{\mathcal{M}}_1(\beta) \neq \phi$ $\Rightarrow$ $\beta = \beta_1$ or ... or $\beta_m$

X Fano $\Leftrightarrow$ $u : S^2 \longrightarrow X$ holo

$u =$ const

or

$c^1 \cap u > 0$

Cor 1

$X$ Fano

$\lambda(\beta) \geq 2$

$M(\beta) \neq \emptyset \iff \beta = \beta_1 \cdots \beta_n$

Cor 2

$X$ Fano

$\Rightarrow PO(b) = \sum_{i=1}^{m} T^{\beta_i \cap \omega} \exp(x_j \langle \partial\beta_i, e_j \rangle)$

$b = \sum x_i e_i$

Write

$$\partial \beta_i = \sum v_{ij} e_j$$

$$y_j = e^{x_j}$$

Then

$$PO(b) = \sum_{i=1}^{m} T^{\beta_i \cap w} y_1^{v_{i1}} \cdots y_n^{v_{in}}$$

Put $\quad l_i(u) = \beta_i \cap w$ $\quad$ (regard $\beta_i \in H_2(X, L(u))$, $u \in P$

$$PO(b) = \sum_{i=1}^{m} T^{l_i(u)} y^{\vec{v}_i}$$

$$\left( \vec{v}_i \in \mathbb{Z}^n \quad y^{\vec{v}_i} = \prod y_j^{v_{ij}} \right)$$

**Lemma** $\operatorname{grad} l_i = \vec{v}_i$.

$$P = \{u \mid l_i(u) \geq 0 \quad i = 1, \dots, n\}$$

**Example**

$X = \mathbb{P}^n$

$P = \{(u_1, \dots, u_n) \mid 0 \leq u_i, \ \sum u_i \leq 1\}$

$\partial_i P = \{u \in P \mid u_i = 0\} \quad i \neq 0$

$\partial_0 P = \{ \quad \mid \sum u_i = 1\}$

$$\begin{cases} l_i(u) = u_i & v_i = (0 \cdots 0 \overset{i}{1} \cdots 0) \\ l_0(u) = 1 - \sum u_i & v_0 = (-1 \cdots\cdots -1) \end{cases}$$

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$$PO(u,b) = \sum_{i=1}^{n} T^{u_i} y_i + T^{1-u_1-\cdots-u_n} (y_1 \cdots y_n)^{-1}$$

$y_i = e^{x_i}$ $b = \sum x_i e_i$ $y_i \in \Lambda_0 \setminus \Lambda_+$

$n=2$

$$PO = T^{u_1} y_1 + T^{u_2} y_2 - T^{1-u_1-u_2} (y_1 y_2)^{-1}$$

$HF((L(u), b)) \neq 0 \Rightarrow \frac{\partial PO}{\partial y_j} = 0$ at $b$

ExEx

If $u_1 < u_2, 1-u_1-u_2$ $\quad y \in \Lambda_0 \setminus \Lambda_{-}$

$$\frac{\partial p_0}{\partial y_1} = T^{u_1} \cdots \mod \Lambda_- T^{u_1}$$

∴ No crit. pt

If

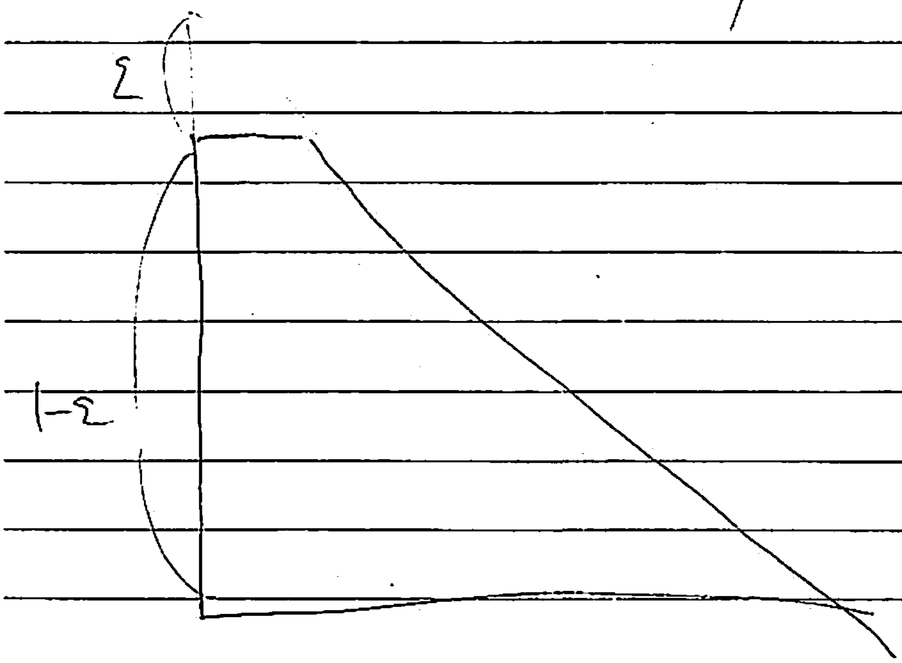
$u_1 = u_2 = 1-u_1-u_2 \quad (= \frac{1}{3})$

$$p_0 = T^{\frac{1}{3}}\left(y_1 + y_2 + (y_1 y_2)^{-1}\right)$$

$\frac{\partial p_0}{\partial y_i} = 0 \iff$ $\quad 1 - y_1^{-1} y_2^{-2} = 0$

$1 - y_1^{-2} y_2^{-1} = 0$

$\iff y_1 = y_2 \quad y_1^3 = 1$




$$W_0 = T^{u_1} y_1 + T^{u_2} y_2 + T^{1-u_2-2\varepsilon} y_2^{-1} + T^{1-u_1-u_2}(y_1 y_2)^{-1}$$

So, $u_1 = u_2 = \frac{1}{3}$ $\quad \varepsilon \ll \frac{1}{2}$

$$W_0 = T^{\frac{1}{3}}(y_1 + y_2 + y_1^{-1} y_2^{-1}) + T^{\frac{2}{3}-2\varepsilon} y_2^{-1}$$

$$\frac{\partial W_0}{\partial y_1} = 0 \Leftrightarrow 1 - y_1^{-2} y_2^{-1} = 0$$

$$1 - y_1^{-1} y_2^{-2} - T^{\frac{1}{3}-2\varepsilon} y_2^{-2} = 0$$

Sol!

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$\delta = \frac{2}{3} - \varepsilon$

$y_2 = y_1^{-2}$

$1 - y_1^3 - T^{\delta} y_1^4 = 0$

$y_1 = \exp\left(\frac{2\pi\sqrt{-1}\, i}{3}\right) + \text{higher order}$ $i = 0, 1, 2$

or

$y_1 = T^{-\delta} + \cdots$ higher order

does not satisfy $y_j \in \Lambda_0 \setminus \Lambda_+$

They are not

$x_2 = \log y_1$ $b = x_1 e_1 + x_2 e_2$

$HF(L(u), L) \neq 0$