How higher str. comes

Geometry : Space → invariant

① Space ⇒ number early 80'

Donaldson

② Space ⇒ groups late 80'

Floer

③ Space → higher alg. str. early 90'

well def. up to homotopy

Physics

Config. spaces Kontsevich

Moduli sp. in alg. geo Nakajima

Moduli sp in diff. geo. F.

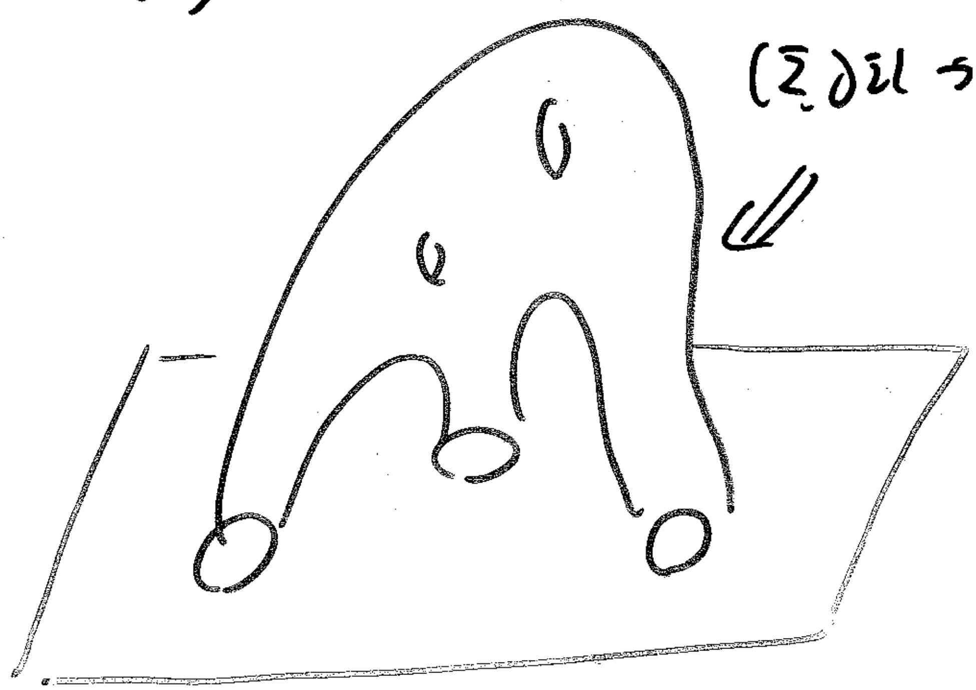
1′

(M, ω) symplectic $\quad J : TM \circlearrowleft$ almost cpx

$L \subset M$ Lag. sub.

$\beta \in H_2(M, L)$

$\mathcal{M}_{g=3, m=3}(\beta)$ moduli of holo $(\Sigma, \partial\Sigma) \to (M, L)$



Nondef.

$$\Psi = \sum_{g\,m,\,\beta} \# M_{g\,m}(\beta)\, s^{2g-2+m}\, q^{\beta \cap \omega}$$

Nonconj

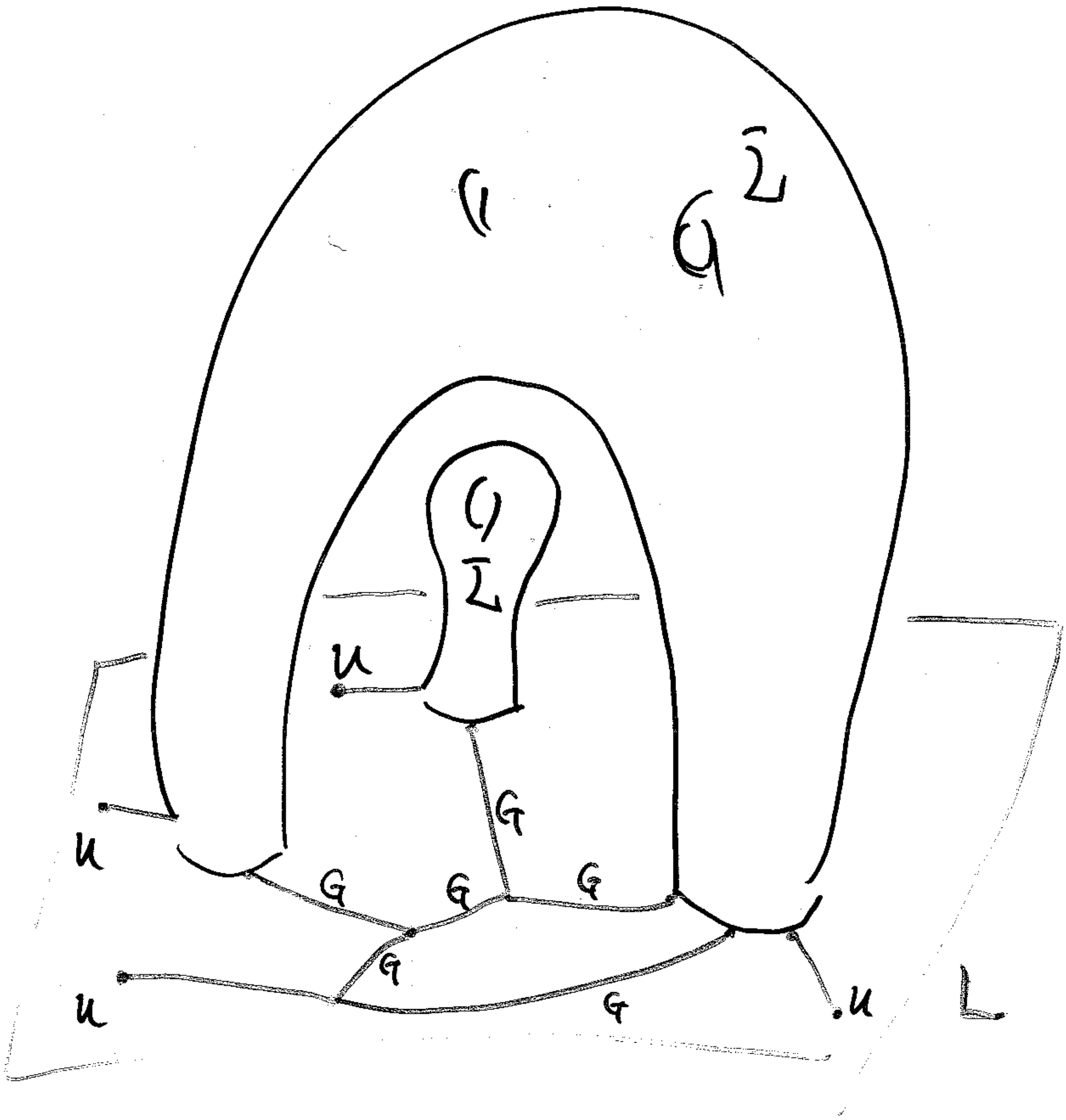
$\dim_{\mathbb{C}} M = 3$

$c^1 M = 0 \quad H_1(L:\mathbb{Q}) = 0$

$\Rightarrow \lim_{q \to 0} \Psi =$ Perturbative CS invariant

To count is too naive. ③

One needs



④

- G : propagator

$$G \in \Lambda^{n-1}(L \times L)$$

$$u \longmapsto \int_x G(y,x)\, u(x) = (d^* \Delta^{-1} u)(y)$$

$$= \left(\int_0^\infty d^* e^{-t\Delta} u \right)(y)$$

$u \in \Lambda L$

This actually works some how.

The purpose of this talk is to explain algebraic structure behind it.

$dG =$ id $-$ Harmonic projection

Notation

C graded vector space

$(C[1])^k = C^{k+1}$

$B_k C[1] = \overbrace{C[1] \otimes \cdots \otimes C[1]}^{k}$

$B_k^{cyc} C[1] = B_k C[1] / \sim$

$x_1 \otimes \cdots \otimes x_k = (-1)^* x_k \otimes x_1 \otimes \cdots \otimes x_{k-1}$

$* = (\deg x_k + 1) \sum_{i=1}^{k-1} (\deg x_i + 1)$

$E_k C[1] = B_k C[1] / \mathfrak{S}_k$ ← sym. group

$x_1 \otimes \cdots \otimes x_k \sim (-1)^* x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}$

$* = \sum_{i<j,\ \sigma(i) > \sigma(j)} (\deg x_i + 1)(\deg x_j + 1)$

⑥

L C^∞ mfd $\partial L = \emptyset$ cpt
oriented

ΛL : de Rham complex

Def. $\mathrm{Hom}_{C^\infty}(B^{cyc}\Lambda L[2], \mathbb{R})$

$$= \bigoplus_k \Big\{ \varphi : B_k^{cyc}\Lambda L[2] \longrightarrow \mathbb{R}$$
$$\Big| \ \exists \omega \in \Lambda(L^k)$$
$$\varphi(u_1 \cdots u_k) = \int_{L^k} u_1(x_1) \cdots u_k(x_k)\, \omega(x_1 \cdots x_k) \Big\}$$

Fact
$\mathrm{Hom}_{C^\infty}(B^{cyc}\Lambda L[2], \mathbb{R})$ is $\underbrace{B\hat{\,}^{L} \text{ alg.}}_{\text{defined later}}$

Fix Riemannian metric on L ⑰

$\Downarrow$

$H_{DR}(L;\mathbb{R})$ has a structure of cyclic A_∞ algebra. ← defined later

<u>Fact</u> C cyclic A_∞ algebra. $\dim C < \infty$

$\Rightarrow$ $\mathrm{Hom}(B^{cyc}C[1], \mathbb{R})$ is $\underset{L}{BV}$ algebra

Thm $\exists$ $\overset{L}{B\cancel{\Omega}}_{\infty}$ homotopy equiv.

$$\mathrm{Hom}_{C^\infty}(B^{cyc}\Lambda L[2], \mathbb{R})$$
$$\to \mathrm{Hom}(B^{cyc}H(L)[2], \mathbb{R})$$

Cor $\mathrm{Hom}(B^{cyc}H(L)[2], \mathbb{R})$

is independant of Rie. metric up to homotopy equiv.

Remark ⑨

$\dim L = 3$, $H^1(L; D) = 0$ framing

Axelrod-Singer

$\Rightarrow$ Perturbative CS inv.

independent of Rie. metric

Prob.

Unify Corr. + Ax-Sing.

to obtain some structure

for general $H(L) \neq 0$.

Definitions

$\boxed{A_\infty \text{ algebra}}$ (Stasheff)

$$m_k : B_k C[1] \longrightarrow C[2] \qquad k \geq 1$$

$$\Updownarrow$$

$$\hat{d}_k : BC[2] \circlearrowleft \quad \text{coderivation}$$

$$\hat{d} = \sum \hat{d}_k$$

$$\hat{d}^2 = 0 \iff (C, \{m_k\}) \text{ is } A_\infty\text{-alg.}$$

$\boxed{\text{Cyclic } A_\infty \text{ alg}}$ (Kontsevich)

$(C, \{m_k\})$ A_∞ alg

$\langle \ \rangle : C \otimes C \longrightarrow \mathbb{R}$ inner prod.

s.t. $\langle x_0, m_k(x_1 \cdots x_k)\rangle$

$$= (-1)^* \langle x_k, m_k(x_0, \cdots, x_{k-1})\rangle$$

$$\# = (\deg x_k + 1) \sum_{i=0}^{k-1} (\deg x_i + 1)$$ (11)

Ex $C = \Lambda L$ de Rham

$$m_1(x) = (-1)^{\deg x} dx$$

$$m_2(x, y) = (-1)^{\deg x (\deg y + 1)} x \wedge y$$

$$\langle x, y \rangle = \int y \wedge x$$

$$m_k = 0 \quad k \geq 3$$

Fact 1

$(C, m, \langle\ \rangle)$ cyc A_∞-alg

$\Rightarrow$ $H(C) = \dfrac{\mathrm{Ker}\, m_1}{\mathrm{Im}\, m_1}$

has a str. of cyc. A_∞ alg.

Fact 2

$(C, m, \langle\ \rangle)$ cyc A_∞ alg.

$\dim C < \infty$, $\langle\ ,\ \rangle$ is nondegenerate

$\Rightarrow$ $\mathrm{Hom}(B^{cyc}C[2], \mathbb{R})$

is BL alg.

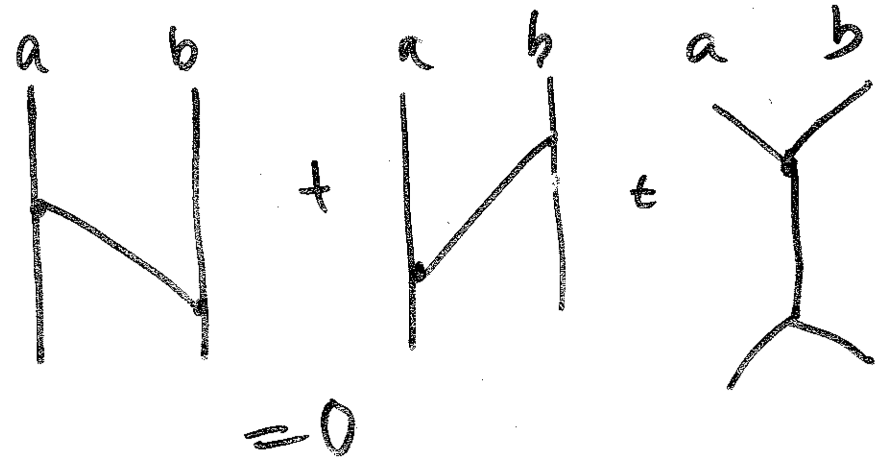
B~~V~~L alg. (c.f. Cielibak-Latshov) 13

D graded vector space

$\{\ \} : D \otimes D \longrightarrow D$

$\square : D \longrightarrow D \otimes D$

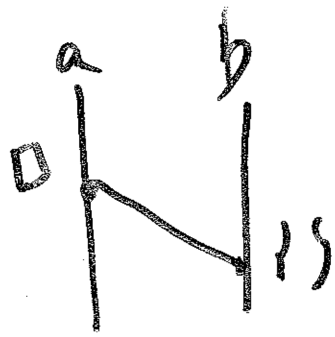
$d : D \longrightarrow D$

1. $d^2 = 0$
2. $\{\ ,\ \}$ derivation, Jacobi
3. $\square$ co-derivation, co~~commutative~~ Jacobi
   co associative
4. 

$= 0$

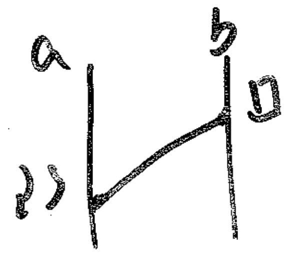
$$\Box a = \sum_c a^1_c \otimes a^2_c$$

$$(\Box : D \to D \otimes D)$$



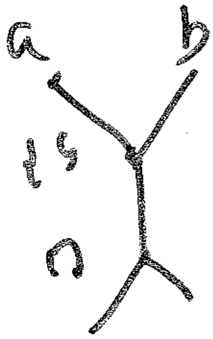
$$= \sum_c \pm\, a^1_c \otimes \{a^2_c, b\}$$

$$+$$



$$= \sum_c \pm\, \{a, b^1_c\} \otimes b^2_c$$

$$+$$



$$= \Box(\{a, b\})$$

$$\| \\ 0$$

Prop $\mathrm{Hom}_{C^\infty}(B^{cyc}\Lambda L(2), \mathbb{R})$ is BV alg. (This is rel. to String top.)

∵) element of $\mathrm{Hom}_{C^\infty}(B^{cyc}\Lambda L(2), \mathbb{R})$ is identified with

$$w \in \Lambda(L^k)$$

$$w_1, w_2 \in \Lambda(L^{k_1}), \Lambda(L^{k_2})$$

$$\{w_1, w_2\} \in \Lambda(L^{k_1+k_2-2})$$

$\{w_1, w_2\}(x_1 \cdots\cdots x_m)$ (GH bracket)

$$= \sum_i \int_{y \in L} \pm\, w_1(x_i \cdots x_j\, y)\, w_2(y\, x_{j+1} \cdots x_{i-1})$$

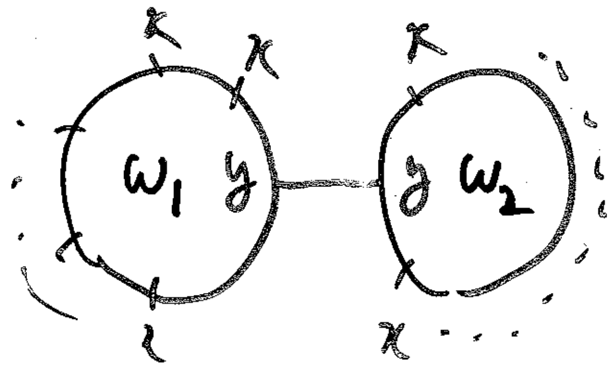
(16)

$$\square : \Lambda(L^k) \to \bigoplus_{\substack{k_1+k_2 \\ = k-2}} \Lambda(L^{k_1}) \hat{\otimes} \Lambda(L^{k_2})$$

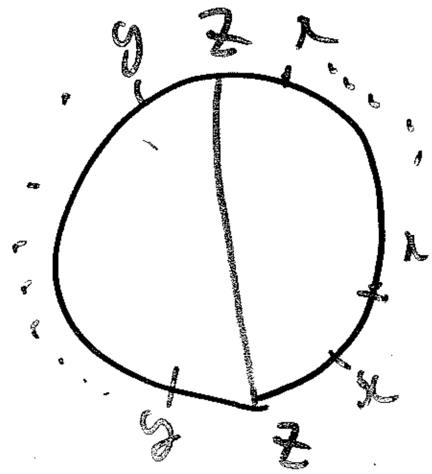
$$\Lambda(L^{k_1}) \hat{\otimes} \Lambda(L^{k_2}) \cong \Lambda(L^{k_1+k_2})$$

$$(\square w)(x_1 \cdots x_{k_1} y_1 \cdots y_{k_2})$$

$$= \sum_{i,j} \int_{z \in L} \pm\, w(x_i \cdots x_{i-1} z\, y_j \cdots y_{j-1} z)$$



$\{w_1, w_2\}$



$\square w$

<u>Fact 2</u> $(C, m, \langle\ \rangle)$ cyclic A_∞ alg., $\langle\ \rangle$ non degenerate, $\dim C < \infty$

$\Rightarrow$ $Hom(B^{cyc}C[1], \mathbb{R})$ is BV alg.

$\because$) Replace $\int$ by Σ //

<u>Thm</u>

$\exists\ \Phi :$ $Hom_{C\infty}(B^{cyc}\Lambda L[1], \mathbb{R})$
$\to Hom(B^{cyc}HL[1], \mathbb{R})$

$\underbrace{BV_\infty \text{ homotopy equivalence}}$

let us define it.

BV$_\infty$ algebra (cf. Cielibak-Latshev)

D graded vect. space

s formal parameter (string coupling constant)

$$P_{k,l} : E_k D[2] \longrightarrow E_l D[2][[s]] \quad (k, l \geq 1)$$

$$\Downarrow$$

$$\hat{P}_{k,l} : ED[2][[s]]$$

$$\hat{P}_{k,l}(x_1 \otimes \cdots \otimes x_n)$$

$$= \sum_{i_1 < \cdots < i_k} P_{k,l}(x_{i_1} \cdots x_{i_k}) x_{j_1} \cdots x_{j_{n-k}}$$

$$\left(\begin{array}{l} \{i_1 \cdots i_k, j_1 \cdots j_{n-k}\} = \{1, \cdots, n\} \\ j_1 < \cdots < j_{n-k} \end{array} \right)$$

① $\hat{d} = \sum \hat{P}_{k,\ell}$

$$\boxed{\hat{d}^2 = 0}$$

② $P_{k,\ell} \equiv 0 \mod S^{k+\ell-2}$

Remark $\hat{P}_{k\ell}$ is <u>not</u> (co)derivation

<u>Example</u>

$(D, d, \{S, \square\})$ $B\overset{L}{\nabla}$ alg

$P_{11} = d \qquad P_{21} = S\{\ \}$

$P_{12} = S\square \qquad P_{k\ell} = 0$ other

$B\overset{L}{\nabla}_\infty$ alg.

D, D' BV_∞^L alg.

BV_∞^L homomorphism

$$\varphi_{k,l} : E_k D[1] \longrightarrow E_l D'[1][[S]]$$

$$\hat{\varphi} : ED[1] \to ED'[1][[S]]$$

$$\hat{\varphi}(x_1 \cdots x_n)$$

$$= \sum \pm \varphi_{k_1, l_1}(x_{i(1,1)} \cdots x_{i(1,k_1)})$$

$$\cdots\cdots$$

$$\varphi_{k_m l_m}(x_{i(m,1)} \cdots x_{i(m,k_m)})$$

$\{1, \cdots, n\} = I_1 \cup \cdots \cup I_m$ disjoint

$$I_j = \{ i(j,1), \cdots, i(j, k_j) \}$$

$$\sum k_j = n$$

ℓ's are arbitrary

$\{\varphi_{k,\ell}\}$ is $B\Omega_\infty^L$ homo

$\Longleftrightarrow$ ① $\hat{\varphi} \circ \hat{d} = \hat{d} \circ \hat{\varphi}$

② $\varphi_{k,\ell} \equiv 0 \mod S^{k+\ell-2}$

$\exists$ notion of homotopy $\varphi \sim \varphi'$
between two $B\mathfrak{D}^t_\infty$ hom's

$\varphi \sim \varphi'$, $\varphi' \sim \varphi'' \Rightarrow \varphi \sim \varphi''$
etc

$\exists$ notion of homotopy equiv.
between two $B\mathfrak{D}^t_\infty$ alg.

Th'm $\varphi : D \to D'$ $B\mathfrak{D}^t_\infty$ hom

$$\varphi_{1,1} : \frac{\operatorname{Ker} P_{1,1}}{\operatorname{Im} P_{1,1}} \cong \frac{\operatorname{Ker} P'_{1,1}}{\operatorname{Im} P'_{1,1}}$$

$\Rightarrow \varphi$ has homotopy inverse

Th^m

23

$\exists \Phi : Hom(B^{cyc}\Lambda L[1], \mathbb{R})$
$\to Hom(B^{cyc}H(L)[1], \mathbb{R})$

BV_b^L hom. equiv.

$\Phi_{k,l} : \Lambda(L^{n_1+1}) \otimes \cdots \otimes \Lambda(L^{n_k+1})$
$\to E_l\, Hom(B^{cyc}H(L)[1], \mathbb{R})$

$w_1, \cdots, w_k \in \Lambda(L^{n_1+1}), \cdots, \Lambda(L^{n_k+1})$

$\vec{u}_i \in B^{cyc}_{m_i}(H(L)[1])$
$\|$
$u_{i,1} \otimes \cdots \otimes u_{i,m_i}$

u_{ij}
harmonic form
on L.

$\Phi_{k,\ell}(w_1 \cdots w_k)(\vec{u}_1 \cdots \vec{u}_\ell) \in \mathbb{R}[[S]]$

$\parallel$

$\sum_{r} C_r(w_1 \cdots w_k \vec{u}_1 \cdots \vec{u}_\ell) S^r$

Let us define it

C_r is a sum over (Σ, Γ)

- Σ surface with ℓ boundary
- $\Gamma \subset \Sigma$ graph

$r = k - \chi(\Sigma)$.

Condition for (Σ, Γ) 25

$$C^0(\Gamma) = C^0_{exe}(\Gamma) \cup C^0_w(\Gamma) \cup C^0_\wedge(\Gamma)$$

set of vertex

1. $C^0_{exe}(\Gamma) = \Gamma \cap \partial\Sigma$

$\partial\Sigma = \partial_1\Sigma \cup \cdots \cup \partial_\ell\Sigma$

$\partial_j\Sigma \cap \Gamma$ is m_j points

$\vec{u}_j = u_{j,1} \otimes \cdots \otimes u_{j,m_j}$

2. $\# C^0_w(\Gamma) = k$ $\quad k = \#$ of w's

$$C^0_w(\Gamma) \hookrightarrow \{w_1, \cdots, w_k\}$$

$$\begin{tikzcd} & v \arrow[r, leftrightarrow] & w_i \\ C_w^0(n) \arrow[ur] & & \end{tikzcd}$$

$\exists$ v has n_i+1 edges

($w_i \in \Lambda C L^{n_i+1}$)

3 $C_n^0(n) \ni v$

$\exists$ v has 3 edges

4 If D is a connected component of $\Sigma \setminus \Gamma$

$D \cong D^2$ disk

$D \cap \partial\Sigma \cong [0,1)$ arc.

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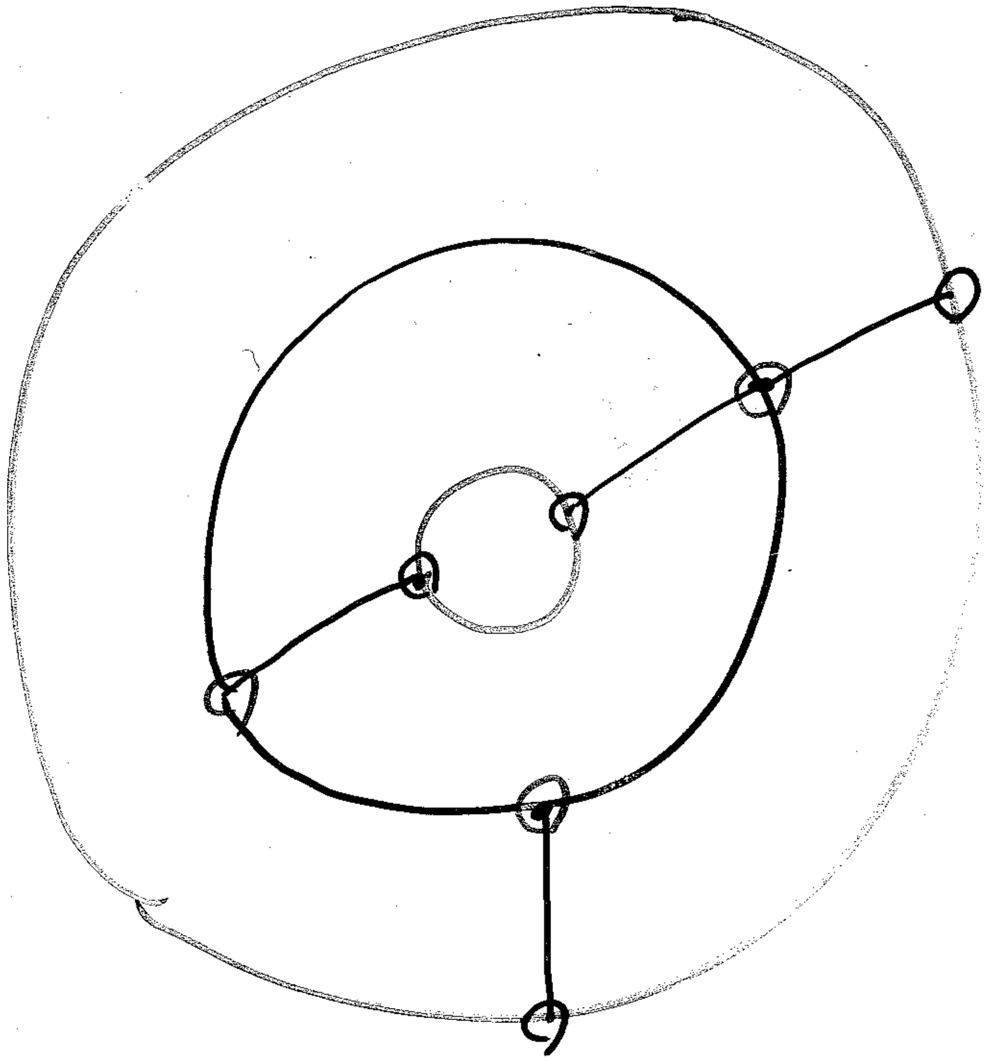
5 If $S^1 \hookrightarrow \Gamma$ cycle

$\Rightarrow \exists\, v \in C^0_w(\Gamma)$

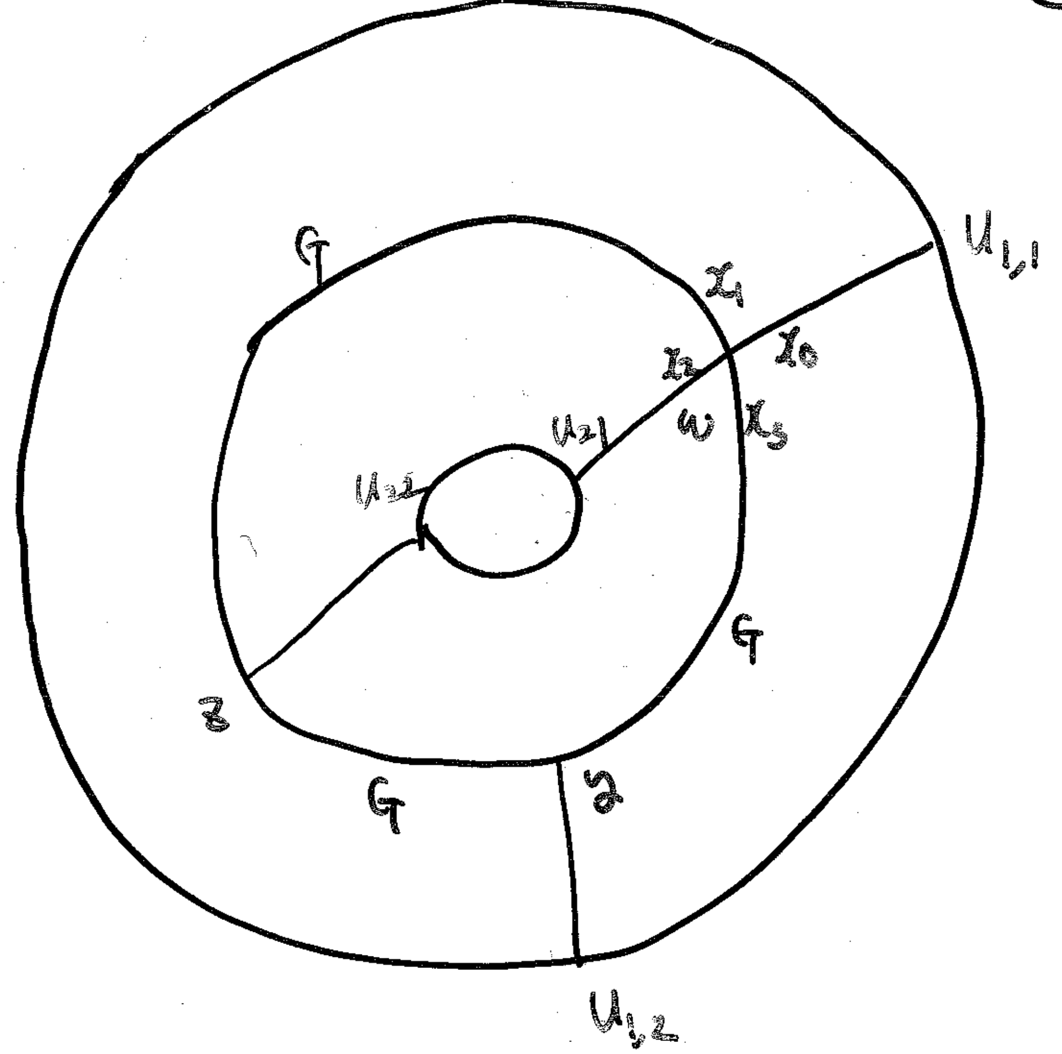
$v \in S^1$.

Example $\bar{\Sigma} =$ (circled picture) $\cong S^1 \times [0,1]$ 28



- $C^0_n(\Gamma)$
- $C^0_w(\Gamma)$
- $C^0_{ext}(\Gamma)$

$\partial \bar{\Sigma}$



$$\int_{L^6} w(x_0 x_1 x_2 x_3)\, u_{11}(x_0)\, u_{12}(y)\, u_{21}(x_2)$$

$$u_{22}(z)\, G(y\, x_3)\, G(x_1\, z)$$

$$G(z\, y)$$

$$\parallel$$

$$C(\Sigma, \Gamma;\, w_1 \cdots w_k, \vec{u}_1 \cdots \vec{u}_\ell)$$

$u \in \Lambda L$

$w \in \Lambda L^4$

$G \in \Lambda L^2$

<u>Def.</u>

$$\Phi_{k,l}(w_1 \cdots w_k)(\vec{u}_1 \cdots \vec{u}_l)$$

$$= \sum_{(\bar{z},r)} S^{-x(\bar{z})+k} \, c(\bar{z}, r ; w_1 \cdots, w_k, \vec{u}_1 \cdots \vec{u}_l)$$

—

<u>Claim</u> This is BV_∞ homo.

$\mathcal{M}_{g,k}(\beta) \leftarrow$ 

$$\Downarrow$$

Chain on $(\Omega L)^k$ $\quad$ ΩL : free loop space

$$\Downarrow \text{ Chen's Iterated integral}$$

$$E_k \mathrm{Hom}_{cb}(B^{cyc}\Lambda L[1], \mathbb{R})$$

$$\Downarrow \quad \Phi_{k,\ell}$$

$$E_\ell \mathrm{Hom}(B^{cyc}H(L)[1], \mathbb{R})$$