Lagrangian Floer theory I

Kenji Fukaya

based on joing work with
Yong-Geun Oh, Kaoru Ono, H. Ohta

(X, ω) is a symplectic manifold

$\Updownarrow$

X is a 2n dimensional manifold

ω is a 2 form

$$d\omega = 0$$

ω^n never vanish

$H : X \to \boldsymbol{R}$ a smooth function

X_H the Hamiltonian vector field generated by H is defined by

$$\omega\left(X_H, V\right) = dH(V)$$

$\varphi : X \to X$ is called to be a Hamiltonian diffeomorphism

$$\Longleftrightarrow \quad H : X \times [0,1] \to \boldsymbol{R}, \quad H_t(x) = H(x,t)$$

$$\varphi_t : X \to X$$

$$\varphi_1 = \varphi, \quad \varphi_0(x) = x$$

$$\frac{d\varphi_t}{dt} = X_{H_t} \circ \varphi_t$$

$t \mapsto \varphi_t(x) = \ell(t)$ is a solution of Hamilton equation

$$\frac{d\ell}{dt} = X_{H_t} \circ \ell$$

$L \subset X$ is called a Lagrangian submanifold

if $\dim L = \frac{1}{2} \dim X$ $\omega\big|_L = 0$

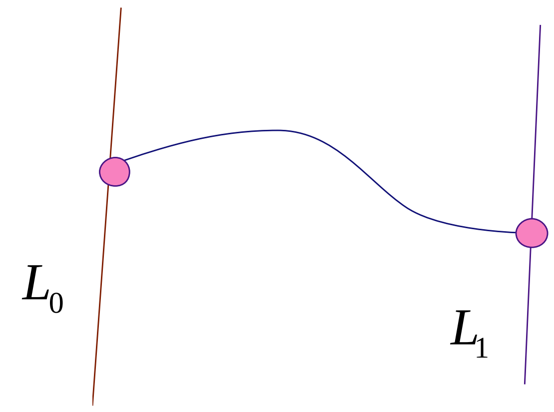
Problem

$$H : X \times [0,1] \to \boldsymbol{R} \qquad L_0, L_1 \subset X \qquad \text{given.}$$

Does there exists $\ell : [0,1] \to X$

$$\frac{d\ell}{dt} = X_{H_t} \circ \ell$$

$$\ell(0) \in L_0, \quad \ell(t) \in L_1$$



Example

$$X = T^*M, \quad q_0, q_1 \in M$$

$$L_0 = T^*_{q_0}M, \quad L_1 = T^*_{q_1}M$$

$$H_t(p,q) = \|p\|^2 + V(q)$$

Restatement of the Problem

$\varphi : X \to X$ — Hamiltonian diffeomorphism

$L_0, L_1 \subset X$ — Lagrangian submanifold

given.

$$\varphi(L_0) \cap L_1 \neq \emptyset \quad ?$$

In the first talk I assume:

- $$\int_{\Sigma} u^{*}\omega = 0$$

Here ω is a symplectic form.

$$u : (\Sigma, \partial\Sigma) \to (X, L)$$

is an arbitrary map from
Riemann surface with boundary Σ.

is satisfied, for example

$$X = T^* M, \quad L = M = \text{zero section}$$

Theorem (Floer$_{\text{+epsilon}}$)(1980’)

I For any pair of Lagrangian submanifold $L_1, L_2 \subset X$ satisfying

we have a Floer homology $HF(L_1, L_2)$.

II If L_1 is transversal to L_2

then $\#\left(L_1 \cap L_2\right) \geq \operatorname{rank} HF(L_1, L_2)$

III Floer homology is invariant of Hamiltonian diffeomorphisms:

$$HF(L_1, L_2) \cong HF(\varphi_1(L_1), \varphi_2(L_2))$$

IV $L_1 = L_2 = L \longrightarrow HF(L, L) \cong H(L)$

Lagrangian intersection

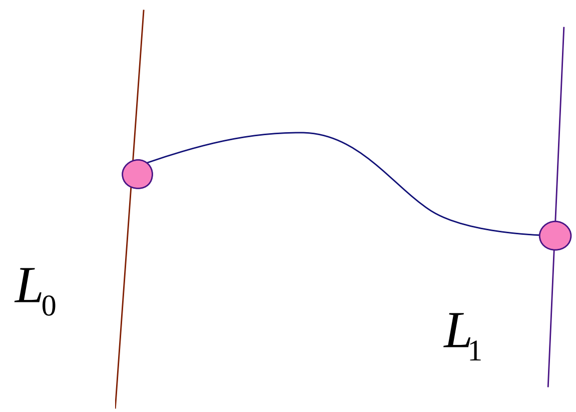
I, II, III, IV $\longrightarrow$ $\#\big(L \cap \varphi(L)\big) \geq \mathrm{rank}\ H(L)$

$$\#\big(L \cap \varphi(L)\big) \underset{\mathrm{II}}{\geq} \mathrm{rank}\ HF(L, \varphi(L)) \underset{\mathrm{IV}}{\geq} \mathrm{rank}\ H(L)$$

If L_1 is transversal to L_2 then $\#\big(L_1 \cap L_2\big) \geq \mathrm{rank}\ HF(L_1, L_2)$

Idea of the proof (variationa analysis)

$$\Omega(L_0, L_1) = \left\{ \ell : [0,1] \to X \,\middle|\, \ell(0) \in L_0,\ \ell(1) \in L_1 \right\}$$

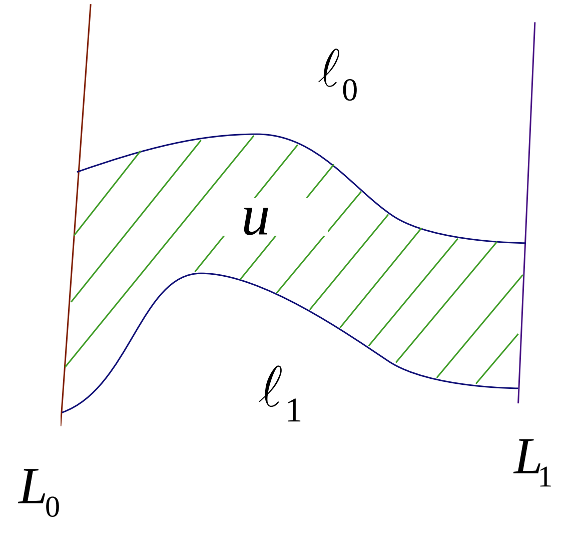


Fix a base point: $\ell_0 \in \Omega(L_0, L_1)$

Define $\boldsymbol{A} : \Omega(L_0, L_1) \to \boldsymbol{R}$

$\ell_s \in \Omega(L_0, L_1), \quad s \in [0,1]$ a path joining ℓ_0 to ℓ_1

$$\boldsymbol{A}(\ell_1) := \int_{[0,1]^2} u^* \omega \qquad u(s,t) = \ell_s(t)$$



Independent of path

In the first talk I assume:

- $$\int_{\Sigma} u^{*}\omega = 0$$

Here ω is a symplectic form.

$$u:(\Sigma,\partial\Sigma)\to(X,L)$$

is an arbitrary map from
Riemann surface with boundary Σ.

Critical point of $\boldsymbol{A}$

$$\Longleftrightarrow \quad \ell(t) \equiv p, \quad p \in L_0 \cap L_1$$ constant path

Morse theory of $\boldsymbol{A}$ $\Longrightarrow$ Floer homology group $HF(L_0, L_1)$

$$HF(L_0, L_1) = \frac{\mathrm{Ker}\,\partial}{\mathrm{Im}\,\partial}$$

$$\partial : CF(L_0, L_1) \to CF(L_0, L_1)$$

$CF(L_0, L_1)$ free abelian group on critical point set of ***A***

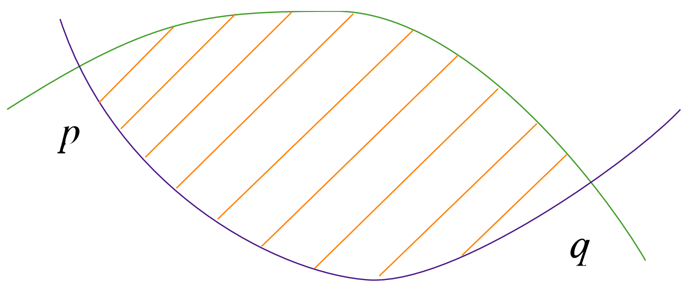
$$\mathrm{II}$$

$$L_0 \cap L_1 \qquad \ell_p(t) \equiv p$$

$$\partial \ell_p = \sum_q \langle \partial \ell_p, \ell_q \rangle \ell_q$$

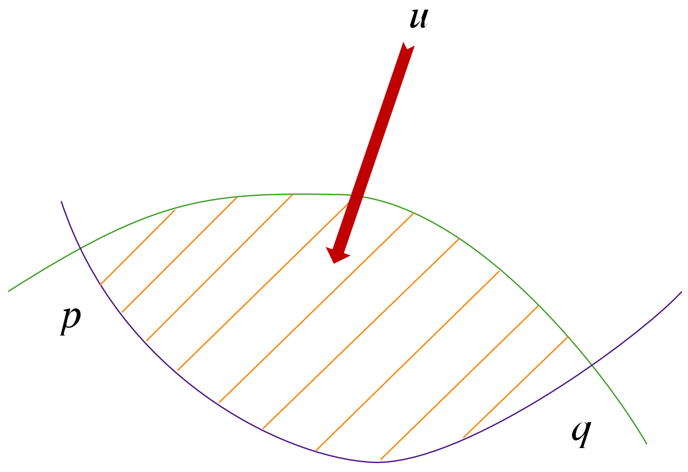
$\langle \partial \ell_p, \ell_q \rangle =$ # gradient lines of

A joining ℓ_p and ℓ_q



gradient lines of $\boldsymbol{A}$ joining ℓ_p and ℓ_q

⟺ holomorphic map $u : \boldsymbol{R} \times [0,1] \to X$



Theorem (Floer$_{\text{+epsilon}}$)(1980')

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we have a Floer homology $HF(L_1, L_2)$.

II If L_1 is transversal to L_2

then $\#\left(L_1 \cap L_2\right) \geq \operatorname{rank} HF(L_1, L_2)$

III Floer homology is invariant of Hamiltonian diffeomorphisms:

$$HF(L_1, L_2) \cong HF(\varphi_1(L_1), \varphi_2(L_2))$$

IV $L_1 = L_2 = L \longrightarrow HF(L, L) \cong H(L)$

$CF(L_0, L_1)$ free abelian group on critical point set of $\mathcal{A}$

$\parallel$

$L_0 \cap L_1$ $\ell_p(t) \equiv p$

$$\text{rank } HF(L_0, L_1) \geq \text{rank } CF(L_0, L_1) \geq \# L_0 \cap L_1$$

II

If L_1 is transversal to L_2

then $\#\left(L_1 \cap L_2\right) \geq \text{rank } HF(L_1, L_2)$

Theorem (Floer$_{\text{+epsilon}}$)(1980')

I For any pair of Lagrangian submanifold $L_1, L_2 \subset X$ satisfying

we have a Floer homology $HF(L_1, L_2)$.

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IV $L_1 = L_2 = L \longrightarrow HF(L, L) \cong H(L)$

Reduction to Morse theory (or to classical cohomology)

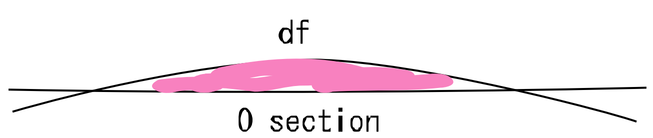
$L_1, L_2 \subset X$ assume L_2 is close to L_1

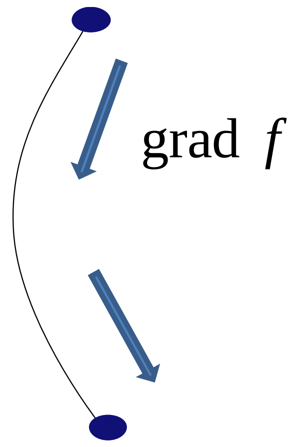
L_2 can be regarded as $\subseteq T^* L_1$ and as a graph of some closed one form u

(Weinstein)

Floer (1980's)

If $u = df$ then $HF(L_1, L_2)$ is Morse homology of f





In the next talk I will explain what we need to remove to study the case when is not necessarily satisfied.

In the rest of this talk I will explain another calculation when is satsfied.

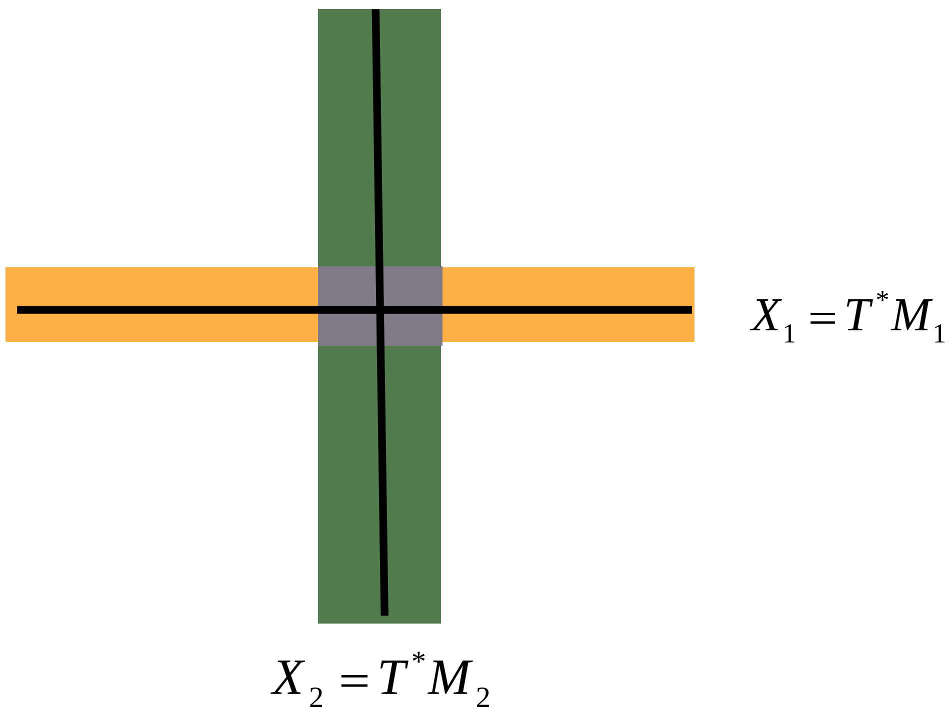
$$\boxed{\quad \int_{\Sigma} u^{*}\omega = 0 \quad}$$

$$X_1 = T^* M_1, \quad X_2 = T^* M_2 \qquad\qquad p_1 \in M_1, \quad p_2 \in T^* M_2$$

$$X = \text{glue } X_1 \text{ and } X_2 \text{ at } p_1, p_2$$

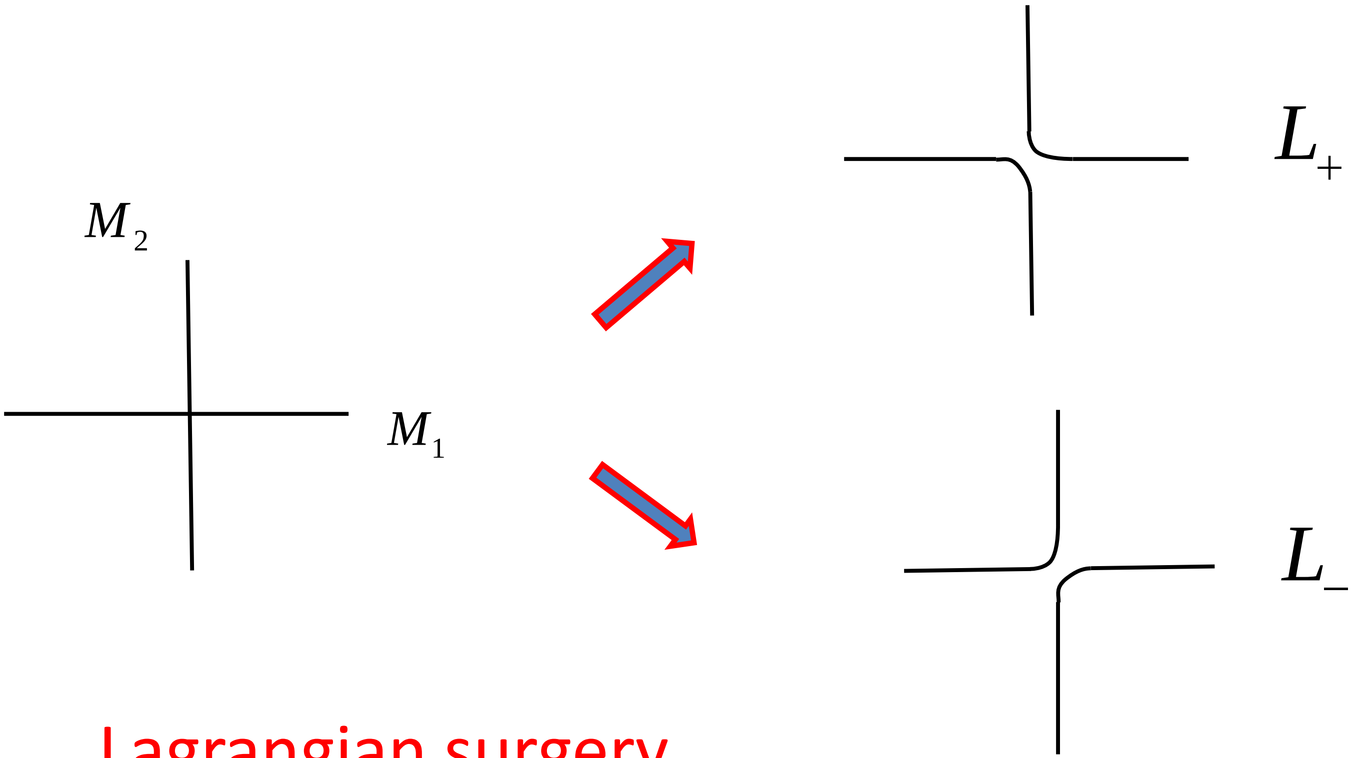
by exchanging fiber and base coordinate

$$X \supset M_1, M_2$$



$X \supset M_1 \cup M_2$ is singular at their intersection point.

$L_{\pm} = M_1 \#_{\pm} M_2$ is the connected sum, embedded in X



Lagrangian surgery

Theorem (F-Oh-Ohta-Ono)

There exists a long exact sequnce (for a Lagrangian submanifold L).

$$\to HF(L, M_1) \to HF(L, M_1 \#_+ M_2) \to HF(L, M_2) \to$$

$$\to HF(L, M_2) \to HF(L, M_1 \#_- M_2) \to HF(L, M_1) \to$$

Note: Proved before by P. Seidel
in case M_1 or M_2 is a sphere.

Corollarty

$$HF^d(M_1, M_1 \#_- M_2) = \begin{cases} H^d(M_1) & d \neq n = \dim M_1 \\ 0 & d = n \end{cases}$$

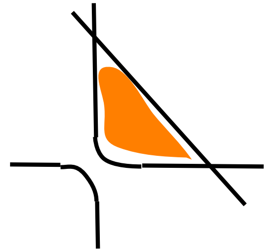
$$HF^d(M_1, M_1 \#_+ M_2) = \begin{cases} H^d(M_1) & d \neq 0 \\ 0 & d = 0 \end{cases}$$

In particular there is no Hamiltonian diffeomorphism $\varphi : X \to X$ such that $\varphi(M_1 \#_- M_2) = M_1 \#_+ M_2$

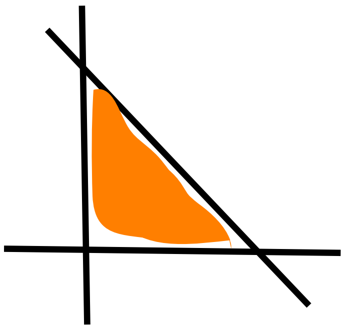
Note $HF(M_1, M_1) = H(M_1)$

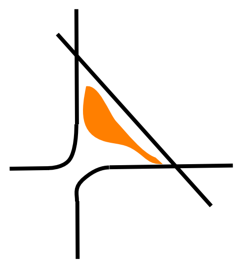
$HF(M_1, M_2) = \mathbb{Z}$

Idea of the proof



becomes
holomorphic 2 gon





Becomes S^{n-2} parametrized
family of holomorphic 2 gons

Holomorphic triangle

Lagrangian Floer theory II

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Ideal statement for Lagragian Floer homology $HF(L_1, L_2)$

I For any pair of Lagrangian submanifold $L_1, L_2 \subset X$

we have a Floer homology $HF(L_1, L_2)$.

II If L_1 is transversal to L_2

then $\#\left(L_1 \cap L_2\right) \geq \mathrm{rank}\ HF(L_1, L_2)$

III Floer homology is invariant of Hamiltonian diffeomorphisms:

$$HF(L_1, L_2) \cong HF(\varphi_1(L_1), \varphi_2(L_2))$$

IV $L_1 = L_2 = L \longrightarrow HF(L, L) \cong H(L)$

Floer (1980's)

$$L_1 = L \subset X \qquad \pi_2(X, L) = 0$$

$$L_2 = \varphi(L)$$

$$\Downarrow$$

I II III IV

Oh (1990's)

L is monotone + α

$$\Downarrow$$

I II III $\cancel{\text{IV}}$

Lagrangian intersection

$\mathrm{I},\mathrm{II},\mathrm{III},\mathrm{IV} \longrightarrow \#\big(L\cap\varphi(L)\big)\geq \mathrm{rank}\ H(L)$

$$\#\big(L\cap\varphi(L)\big) \underset{\mathrm{II}}{\geq} \mathrm{rank}\ HF(L,\varphi(L)) \underset{\mathrm{IV}}{\geq} \mathrm{rank}\ H(L)$$

This inequality can not be true in general !

$L\subset\mathbb{C}^n \longrightarrow \exists\varphi \quad \varphi(L)\cap L=\emptyset$

$$0=\#\big(L\cap\varphi(L)\big) \geq \mathrm{rank}\ H(L)>0$$

Fukaya-Oh-Ohta-Ono (this century) $HF(L_1, L_2)$

I' For any (relatively) spin Lagrangian submanifold $L \subset X$

$\exists$ `Maurer-Cartan formal scheme' $\mathcal{M}(L)$ (can be empty.)

$\exists$ Floer homology $HF((L_1, b_1), (L_2, b_2))$

parametrized by $b_i \in \mathcal{M}(L)$.

II If L_1 is transversal to L_2 then $\#\left(L_1 \cap L_2\right) \geq \operatorname{rank} HF(L_1, L_2)$

III Floer homology is invariant of Hamiltonian diffeomorphisms:

$$HF((L_1, b_1), (L_2, b_2)) \cong HF((\varphi_1(L_1), \varphi_{1*}(b_1)), (\varphi_2(L_2), \varphi_{2*}(b_2)))$$

IV' $L_1 = L_2 = L$ $\longrightarrow$ There exists a spectral sequence

$$E_2 = H(L) \quad \Rightarrow \quad E_\infty = HF(L, L)$$

Ideal statement

I

For any pair of Lagrangian submanifold $L_1, L_2 \subset X$

we have a Floer homology $HF(L_1, L_2)$.

Real statement

I'

For any (relatively) spin Lagrangian submanifold $L \subset X$

$\exists$ `Maurer-Cartan formal scheme' $\mathcal{M}(L)$ (can be empty.)

$\exists$ Floer homology $HF((L_1, b_1), (L_2, b_2))$

parametrized by $b_i \in \mathcal{M}(L)$.

Ideal statement

IV $L_1 = L_2 = L \longrightarrow HF(L,L) \cong H(L)$

Real statement

IV' $L_1 = L_2 = L \longrightarrow$ There exists a spectral sequence

$$E_2 = H(L) \quad \Rightarrow \quad E_\infty = HF(L,L)$$

Story can be re-written by
hotmotopy theory
of A inifinity algebra

A inifinity algebra and Maurer-Cartan scheme

(Stasheff)

(Filtered) A inifinity algebra

$$m_k : \underbrace{C \otimes \cdots \otimes C}_{k} \to C$$

$$\sum_{k_1+k_2=k+1} \pm m_{k_1}(x_1, \cdots, m_{k_2}(x_i, \cdots, x_{i+k_2-1}), \cdots x_k) = 0$$

Maurer-Cartan scheme

$$\mathcal{M}(C) = \left\{ b \in C^1 \,\middle|\, \sum_k m_k(b, \cdots, b) = 0 \right\} \Big/ \text{Gauge equivalence}$$

Theorem (Fukaya-Oh-Ohta-Ono)

$L \subset M$

: relatively spin Lagrangian submanifold. ⇒ $H(L)$ is a filtered A infinity algebra.

Theorem (Fukaya-Oh-Ohta-Ono)

(relatively spin)
pair of Lagrangian submanifold
$L_1, L_2 \subset M$
⇒ $\exists \quad CF(L_1; L_2)$

filtered A infinity $H(L_1) - H(L_2)$ bimodule.

These can be generalized to 3 or more Lagrangian submanifold.

For example it gives product $HF(L_1, L_2) \otimes HF(L_2, L_3) \to HF(L_1, L_3)$

In the rest of this talk I will show example where we can calculate Floer homology and show that it is nozero.

Toric manifolds

(X,ω) is a toric manifolds

$\Updownarrow$

T^n acts on (X,ω)

There exists a moment map $\pi = (H_1, \ldots, H_n) : X \to \boldsymbol{R}^n$

T^n action is generated by the Hamiltonian vector fields X_{H_i}

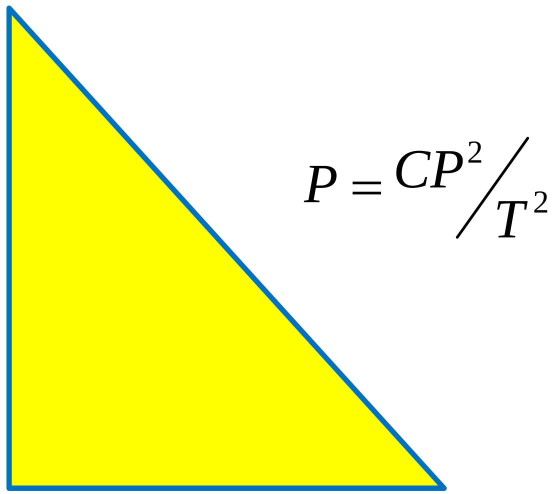
The image $\pi(X) \subset \boldsymbol{R}^n$ is a convex polytope, P.

$u \in \operatorname{Int} P$ $\Longrightarrow$ $\pi^{-1}(u)$ is a Lagrangian torus

is a orbit of the T^n action.

Put $L(u) := \pi^{-1}(u)$

Example



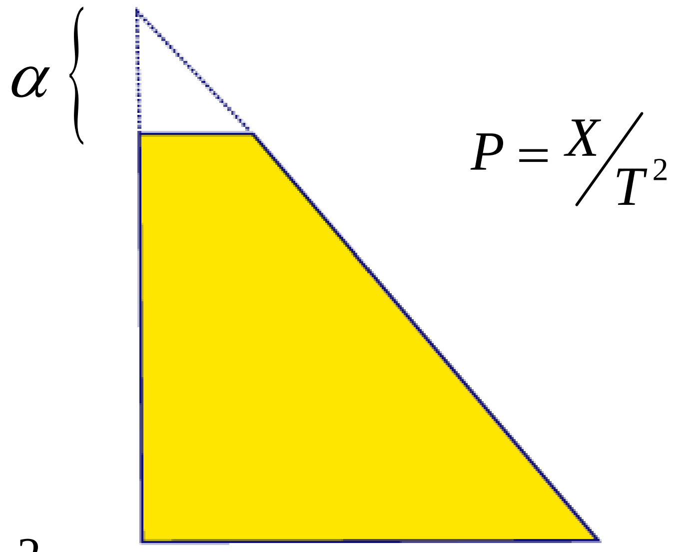
$$X = CP^2$$

$U(1)\times U(1) = T^2$ action on $X = CP^2$

$$\left(e^{2\pi\sqrt{-1}\theta_1}, e^{2\pi\sqrt{-1}\theta_2}\right)\bullet\left[x_0 : x_1 : x_2\right] = \left[x_0 : e^{2\pi\sqrt{-1}\theta_1}x_1 : e^{2\pi\sqrt{-1}\theta_2}x_2\right]$$

$P = {CP^2}/{T^2} =$ triangle $=\left\{(u_1,u_2)\in \boldsymbol{R}^2 \,\middle|\, 0\le u_1,u_2,\quad u_1+u_2\le 1\right\}$

Example



$X =$ one point blow up CP^2

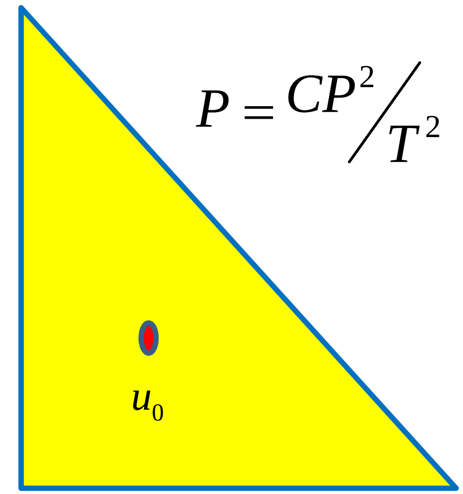
T^2 action on CP^2 induce action on X

if we blow up at one of the three fixed points.

$P = X/T^2 =$ rectangle

$$= \{(u_1, u_2) \in \boldsymbol{R}^2 \mid 0 \le u_1, u_2, \quad u_1 + u_2 \le 1, \quad u_2 \le 1 - \alpha\}$$

Theorem (Cho-Oh + alpha)



$$X = CP^2$$

$u_0 = (1/3, 1/3) \in P$

$$\exists b \quad HF((L(u_0), b), (L(u_0), b)) = H(T^2)$$

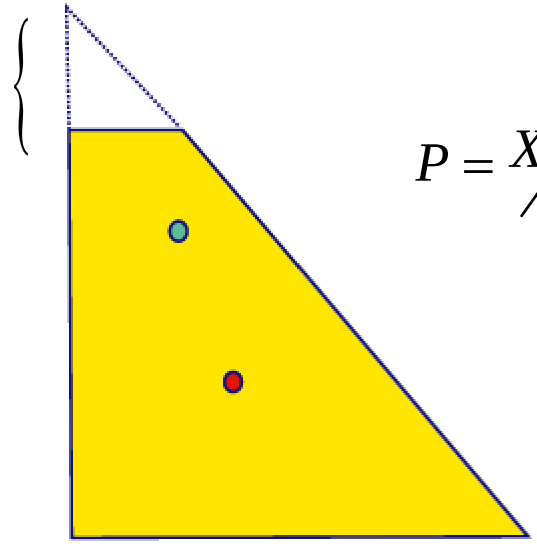
$u \neq u_0 (1/3, 1/3)$

$$\forall b \qquad HF((L(u_0), b), (L(u_0), b)) = 0$$

McDuff $\quad u \neq u_0 (1/3, 1/3) \Longrightarrow \exists \varphi : X \to X$

$$\varphi(L(u)) \cap L(u) = \emptyset$$

Example



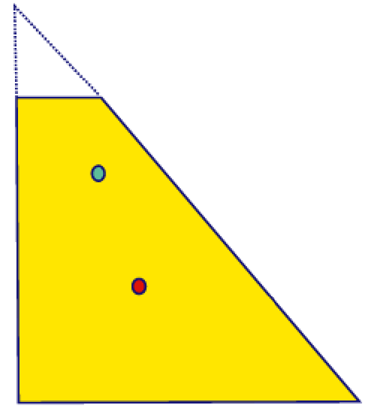
$$X = \text{ one point blow up } CP^2$$

$$P = \left\{ (u_1, u_2) \in \boldsymbol{R}^2 \,\middle|\, 0 \le u_1, u_2, \quad u_1 + u_2 \le 1, \quad u_2 \le 1 - \alpha \right\}$$

$$\alpha < 1/3$$

- $u_1 = (\alpha, 1 - 2\alpha) \in P$
- $u_0 = (1/3, 1/3) \in P$

Theorem (FOOO)



$u_1 = (\alpha, 1-2\alpha) \in P$

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$$\exists b \quad HF((L(u_0), b), (L(u_0), b)) = H(T^2)$$

$$\exists b \quad HF((L(u_1), b), (L(u_1), b)) = H(T^2)$$

$$\underset{\forall b}{u \neq u_0, u_1} \quad HF((L(u_0), b), (L(u_0), b)) = 0$$

McDuff $\quad u \neq u_0, u_1 \quad \Longrightarrow \quad \exists \varphi : X \to X$

$$\varphi(L(u)) \cap L(u) = \emptyset$$

We can find all the Lagrangian fiber $L(u)$ with nontrivial Floer homology

in any compact toric manifold, by solving explicitely calculable polynomial equations finitely many times.

What is he way to calculate Floer homology?

$$b \in H^1(L(u);\Lambda_0)$$

Λ_0 is the set of all formal sums (Universal Novikov ring)

$$c_1 T^{\lambda_1} + c_2 T^{\lambda_2} + c_3 T^{\lambda_3} + \cdots$$

$$c_i \in \boldsymbol{C} \qquad \lambda_i \in \boldsymbol{R}_{\geq 0}, \quad \lambda_i \to \infty$$

$$\sum_{k=1}^{\infty} m_k(b,\cdots,b) = \mathrm{PO}(b) \bullet PD[L(u)]$$

$\mathrm{PO} : H^1(L(u);\Lambda_0) \to \Lambda_0$ a function = potential function

= Landau-Ginzubrug potential

Theorem (FOOO)

$$HF((L(u),b),(L(u),b)) \neq 0$$

$$\Updownarrow$$

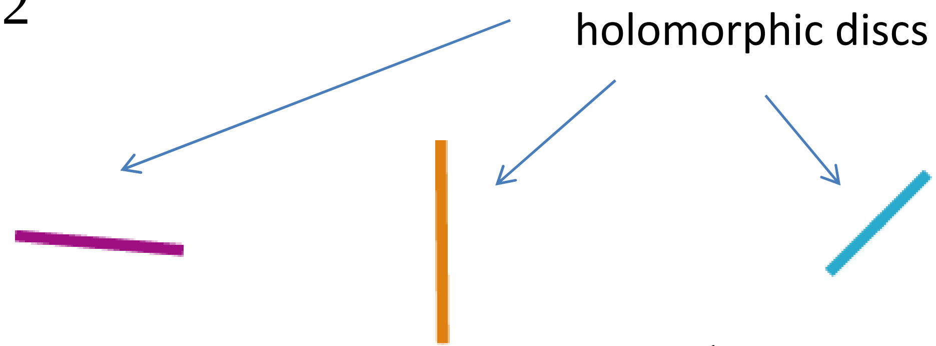
$b \in H^1(L(u);\Lambda_0)$ is a critical point of $\mathrm{PO} : H^1(L(u);\Lambda_0) \to \Lambda_0$

$$HF((L(u),b),(L(u),b)) = \frac{\ker \delta^b}{\mathrm{im}\,\delta^b}$$

$$\begin{aligned} \delta^b(c) &= \sum m_{k+l+1}(b^k c b^l) PD[L] \\ &= \frac{d}{dt}\mathrm{PO}(b+tc)\Big|_{t=0} PD[L] \end{aligned}$$

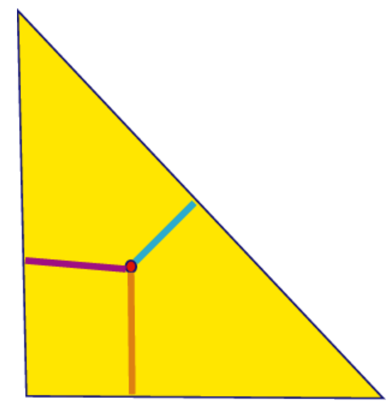
$\delta^b(c) \equiv 0$ $\Longleftrightarrow$ $b \in H^1(L(u);\Lambda_0)$ is a critical point of

$$X = CP^2$$



$$\mathrm{PO} = \mathrm{T}^{u_1} y_1 + T^{u_2} y_2 + T^{1-u_1-u_2} (y_1 y_2)^{-1}$$

$$b = (x_1, x_2) \in H^1(L(u); \Lambda_0) = \Lambda_0 \oplus \Lambda_0$$



$$y_1 = e^{x_1}, \quad y_2 = e^{x_2}$$

at $\quad u_0 = (1/3, 1/3) \in P$

$$y_1 = y_2 = \exp(2k\pi\sqrt{-1}/3), \quad k = 0, 1, 2$$

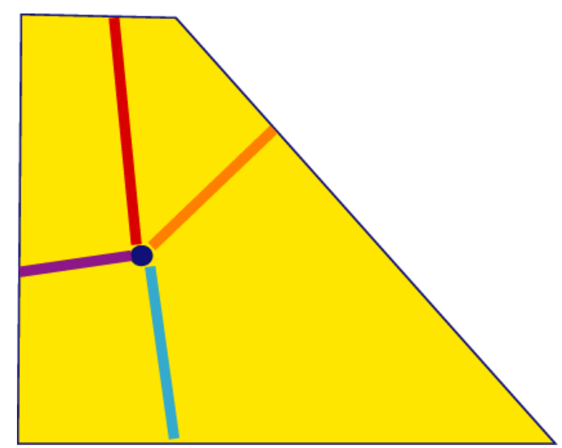
are critical point

$X =$ one point blow up CP^2

$$\mathrm{PO} = \mathrm{T}^{u_1} y_1 + T^{u_2} y_2 + T^{1-u_1-u_2}(y_1 y_2)^{-1} + T^{1-\alpha-u_2} y_2^{-1}$$

at $u_1 = (\alpha, 1-2\alpha) \in P$

$$\mathrm{PO} = \mathrm{T}^{\alpha}(y_1 + (y_1 y_2)^{-1} + y_2^{-1}) + (\text{higher order})$$



$$y_1 = -1 + (\text{higher order})$$

$$y_2 = 1 + (\text{higher order})$$

is a critical point

Find NONZERO cricical point of leading order

$$y_1 = e^{x_1}, \quad y_2 = e^{x_2}$$

Lagrangian Floer theory III

Kenji Fukaya

based on joing work with
Yong-Geun Oh, Kaoru Ono, H. Ohta

Mirror Symmetry (Physisists: Candelas etc. approx. 1990)

(X, ω)

symplectic manifold

count the number of (pseudo)holomorphic map

$S^2 \to X$ Gromov-Witten invariant

= by taking generating function

$(\hat{X}, J)$

complex manifold

product structure on sheaf cohomology

Yukawa-coupling

Calabi-Yau case. $c^1 = 0$

Why this is so nice ?

(X, ω)

count the number of (pseudo)holomorphic map

Global, Nonlinear

$S^2 \to X$

Gromov-Witten invariant

Difficult

$(\hat{X}, J)$

Local, Linear

product structure on sheaf cohomology

Easier

Yukawa-coupling

Homological mirror Symmetry (Kontsevitch. 1994)

(X, ω)

symplectic manifold

$L \subset X$

Lagrangian submanifold

Floer homology

$HF(L_1, L_2)$

(isomorphic)

$(\hat{X}, J)$

complex manifold

$E(L) \to \hat{X}$

Holomorphic vector bundle

(coherent sheaf)

(object of the derived category
of coherent sheaves)

$\mathrm{Ext}(E(L_1), E(L_2))$

Extention (sheaf cohomology)

Homological mirror Symmetry (Kontsevitch. 1994)

(X,ω)

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$\mathrm{Ext}(E(L_1), E(L_2))$

Extention (sheaf cohomology)

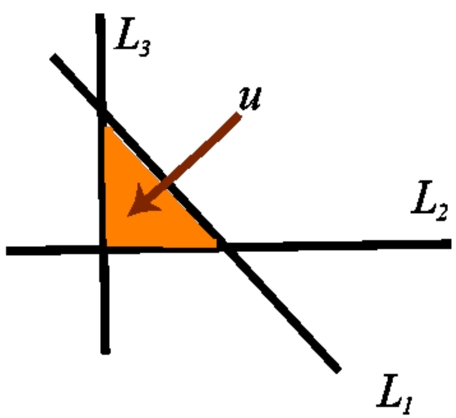
Isomorphism is functorial

Isomorphism is functorial

$$
\begin{tikzcd}
HF(L_1,L_2)\otimes HF(L_2,L_3) \arrow[d] \arrow[r, phantom, "\cong"] & \mathrm{Ext}(E(L_1),E(L_2))\otimes \mathrm{Ext}(E(L_2),E(L_3)) \arrow[d] \\
HF(L_1,L_2) \arrow[r, phantom, "\cong"] & \mathrm{Ext}(E(L_1),E(L_3))
\end{tikzcd}
$$

Product in Floer homology

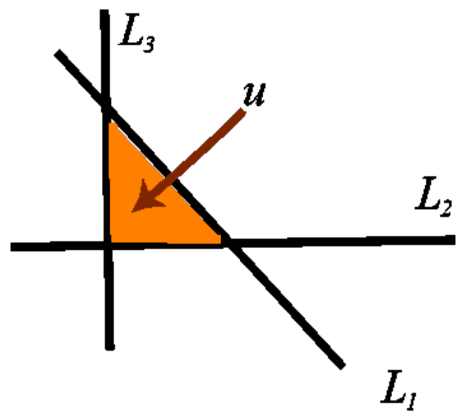
= count the triangle



Yoneda pairing

$$HF(L_1, L_2) \otimes HF(L_2, L_3) \cong \mathrm{Ext}(E(L_1), E(L_2)) \otimes \mathrm{Ext}(E(L_2), E(L_3))$$

$$HF(L_1, L_2) \cong \mathrm{Ext}(E(L_1), E(L_3))$$



Product in Floer homology

= count the triangle

Yoneda pairing

Easier to Calculate

Difficult to Calculate

Homological Mirror symmetry is proved in the case

- Elliptic curve (Kontsevich(94), Polishchuk-Zaslow(97))
- Tori (of higher dimension) partially (Fukaya(98),Kontsevich-Soibelman(00))
  T^4 (Abouzaid-Smoth(08))
- Quartic (Seidel (03))
- Toric (complex) - Landau-Ginzburg (symplectic) (Auroux-Bondal-Kazarkov, Ueda, Abouzaid, …)
- Toric (symplectic) - Landau-Ginzburg (complex) (Cho-Oh, FOOO, …….)
- Cotangent bundle (Nadler-Zaslow, Fukaya-Seidel-Smith)
- Genus 2 surface (Seidel)
- K3 (Fukaya (on progress))

There are various approaches toward the proof of homological Mirror symmetry

I will explain an approach based on

Family of Floer homologies

SYZ dual torus fibration

(Strominger - Yau - Zaslow)

Family of Floer homologies

SYZ dual torus fibration

↓

If family version of Floer homologies can be constructed
in an ideal way,
then we can prove homological mirror symmetry conjecture

Strominger-Yau-Zaslow Conjecture

$$(X,\omega) \Longleftrightarrow (\hat{X}, J)$$

symplectic manifold
(2n dimensional)

complex manifold
(2n dimensional)

$$\begin{tikzcd}
T^n \arrow[d] & & (T^n)^* \arrow[d] \\
X \arrow[dr] & & \hat{X} \arrow[dl] \\
& B^n &
\end{tikzcd}$$

Dual torus fibration

Lagrangian fibration = completely integrable system

(X,ω) symplectic manifold (2n dimensional)

$\pi = (H_1, \cdots, H_n) : X \to \boldsymbol{R}^n$ $\qquad \left[X_{H_i}, X_{H_j} \right] = 0$

X_{H_i} Hamiltonian vector field generated by H_i

Theorem (Liouville-Arnold) If $\pi^{-1}(u) = L(u)$ is compact, then

$$L(u) = T^n \quad \text{Lagrangian torus.}$$

globalize

$\pi : X \to B^n$ Fiber bundle, fiber = Lagrangian T^n

$$(X,\omega) \longleftrightarrow (\hat{X}, J)$$

$$\begin{tikzcd}
T^n \arrow[d] & & (T^n)^* \arrow[d] \\
X \arrow[dr] & & \hat{X} \arrow[dl] \\
& B^n &
\end{tikzcd}$$

Dual torus fibration

taking dual torus of each fiber

$$T^n = \boldsymbol{R}^n / \Gamma$$

Lattice

$$T^n = \left(\boldsymbol{R}^n\right)^* / \Gamma^*$$

Dual lattice

Why ?

Given complex manifold $(\hat{X}, J)$

Use the statement of homological mirror symmetry conjecture

To look for its Mirror (X, ω) symplectic manifold

Skyscraper sheaf

$(\hat{X}, J)$ complex manifold $p \in \hat{X}$

$\boldsymbol{F}_p$ a coherent sheaf on $\hat{X}$

$$\boldsymbol{F}_p(U) = \begin{cases} 0 & \text{if} \quad p \notin U \\ \boldsymbol{C} & \text{if} \quad p \in U \end{cases}$$

Moduli of skyscraper sheaves $=$ $\hat{X}$

(X, ω) Mirror symplectic manifold of $(\hat{X}, J)$

Looking for the Lagrangian submanifold $L_p \subset X$

such that L_p becomes the skyscraper sheaf $\boldsymbol{F}_p$

$$E(L_p) = \boldsymbol{F}_p$$

Homological mirror Symmetry (Kontsevitch. 1994)

(X, ω)

symplectic manifold

$L \subset X$

Lagrangian submanifold

Floer homology

$HF(L_1, L_2)$

(isomorphic)

$(\hat{X}, J)$

complex manifold

$E(L) \to \hat{X}$

Holomorphic vector bundle

(coherent sheaf)

(object of the derived category
of coherent sheaves)

$\mathrm{Ext}(E(L_1), E(L_2))$

Extention (sheaf cohomology)

What we know about L_p

$$E(L_p) = \boldsymbol{F}_p$$

$$HF(L_1, L_2)$$

(isomorphic)

$$\mathrm{Ext}(E(L_1), E(L_2))$$

$$HF(L_p, L_p) \cong \mathrm{Ext}(\boldsymbol{F}_p, \boldsymbol{F}_p)$$

$$H(T^n; \boldsymbol{C}) \cong \mathrm{Ext}(\boldsymbol{F}_p, \boldsymbol{F}_p) \cong HF(L_p, L_p) \Leftarrow H(L_p; \boldsymbol{C})$$

well known

Conjecture : L_p is a Lagrangian T^n

L_p is a Lagrangian T^n

Moduli of skyscraper sheaves $= \hat{X}$

$\hat{X} =$ Moduli of Lagrangian T^n

actually we need a bit more precise

Fukaya-Oh-Ohta-Ono (this century) $HF(L_1, L_2)$

I' For any (relatively) spin Lagrangian submanifold $L \subset M$

$\exists$ `Maurer-Cartan formal scheme' $\mathcal{M}(L)$ (can be empty.)

$\exists$ Floer homology $HF((L_1, b_1), (L_2, b_2))$

parametrized by $b_i \in \mathcal{M}(L)$.

II If L_1 is transversal to L_2 then $\#\left(L_1 \cap L_2\right) \geq \operatorname{rank} HF(L_1, L_2)$

III Floer homology is invariant of Hamiltonian diffeomorphisms:

$$HF((L_1, b_1), (L_2, b_2)) \cong HF((\varphi_1(L_1), \varphi_{1*}(b_1)), (\varphi_2(L_2), \varphi_{2*}(b_2)))$$

IV' $L_1 = L_2 = L$ $\longrightarrow$ There exists a spectral sequence

$$E_2 = H(L) \quad \Rightarrow \quad E_\infty = HF(L, L)$$

Ideal statement

I

For any pair of Lagrangian submanifold $L_1, L_2 \subset M$

we have a Floer homology $HF(L_1, L_2)$.

Real statement

I'

For any (relatively) spin Lagrangian submanifold $L \subset M$

$\exists$ `Maurer-Cartan formal scheme' $\mathcal{M}(L)$ (can be empty.)

$\exists$ Floer homology $HF((L_1, b_1), (L_2, b_2))$

parametrized by $b_i \in \mathcal{M}(L)$.

Statement to be modified

Lagrangian submanifold $L \subset X$

$\exists$ a holomorphic vecture bundle. $E(L) \to \hat{X}$

Precise statement

(relatively) spin Lagrangian submanifold $L \subset M$

and $b \in \mathcal{M}(L)$

$\exists$ a holomorphic vecture bundle. $E(L,b) \to \hat{X}$

Statement to be modified

$$\hat{X} = \text{ Moduli of Lagrangian } T^n$$

Precise statement

$\exists$ family of Lagrangian tori $\{L(u) | u \in B\}$

$$\hat{X} = \bigcup_{u \in B} \mathcal{M}(L(u))$$

$$
\begin{tikzcd}
T^n \arrow[d] & & (T^n)^* \arrow[d] \\
X \arrow[dr] & & \hat{X} \arrow[dl] \\
& B^n &
\end{tikzcd}
$$

Dual torus fibration

Stronminger Yau Zaslow Picture

$$\hat{X} = \bigcup_{u \in B} \mathcal{M}(L(u)) \qquad \mathcal{M}(L(u)) = (T^n)^*$$

$$X = \bigcup_{u \in B} L(u)$$

Family of Floer homology picture.

``Theorem'' (family of Floer cohomology)

Family of Lagrangian submanifolds $\{L(u) \mid u \in B\}$

Another Lagrangian submanifold L

$\Downarrow$

On $\hat{X} = \bigcup_{u \in B} \mathscr{M}(L(u))$

$\exists$ a holomorphic vecture bundle $E(L) \to \hat{X}$

such that $E(L)_{(L(u),b)} = HF((L(u),b),L)$

This idea of the proof of homological Mirror symmetry works in the case of $X = T^{2n}$

What is the difficulty to go beyond the case of tori ?

$$\begin{tikzcd} T^n \arrow[d] & & (T^n)^* \arrow[d] \\ X \arrow[dr] & & \hat{X} \arrow[dl] \\ & B^n & \end{tikzcd}$$

Dual torus fibration

Actually $T^n \longrightarrow X \longrightarrow B^n$ has a singular fiber.

Need to generalize this theorem to include singular fiber.

``Theorem''(family of Floer cohomology)

Family of Lagrangian submanifolds $\left\{ L(u) \middle| u \in B \right\}$

Another Lagrangian submanifold L

$\Downarrow$

On $\hat{X} = \bigcup_{u \in B} \mathscr{M}(L(u))$

$\exists$ a holomorphic vecture bundle $E(L) \to \hat{X}$

such that $E(L)_{(L(u),b)} = HF((L(u),b),L)$

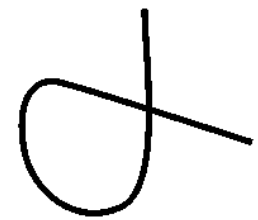
``Theorem'' (to be written up)

$$T^2 \longrightarrow X \longrightarrow B^2$$

n=2, Lagrangian fibration which has a singular fiber for a finitely many point $S \subset B^2$

Singular fiber is immersed S^2 with one self intersection. $B_0 = B^2 - S$

$L \subset X$ another Lagrangian submanifold



singular fiber

The holomorphic vecture bundle $E(L)_0 \to \hat{X}_0$

on $$\hat{X}_0 = \bigcup_{u \in B_0} \mathscr{M}(L(u))$$

$$E(L)_{(L(u),b)} = HF((L(u),b),L)$$

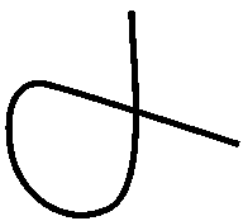
can be extended to the compactification.

Compactification of $\hat{X}_0 = \bigcup_{u \in B_0} \mathcal{M}(L(u))$

Works by Gross-Siebert, Kontsevitch-Soibelman.
(complex sides of the story.)

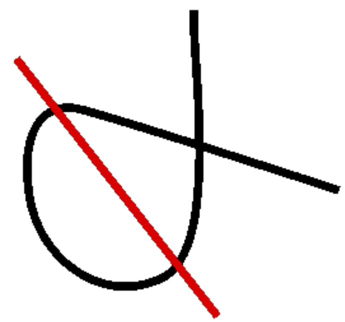
Sketch of the proof

Immersed Lagrangian Floer theory (Akaho-Joyce)



$= L'$

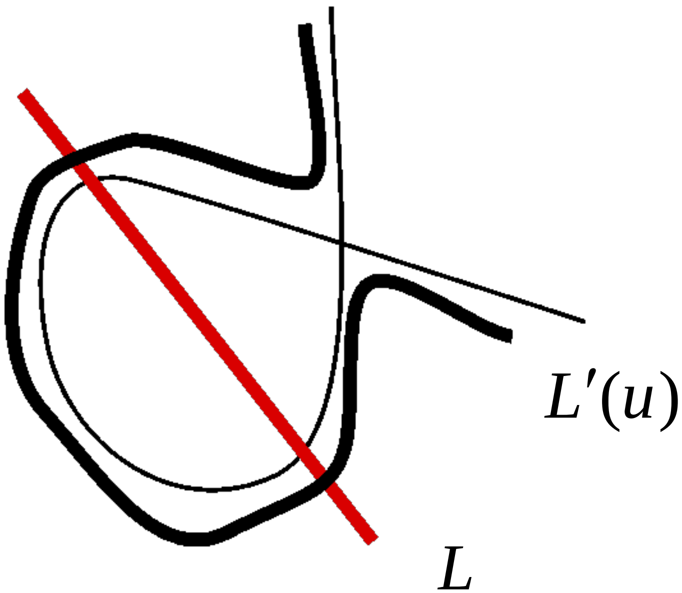
singular fiber



L'

L

$HF(L', L)$ parametrized by two variables. x_+, x_-

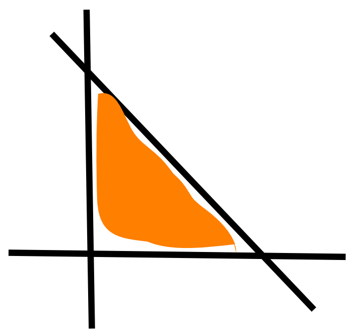


two parameter family of Lagrangian torus $L'(u)$

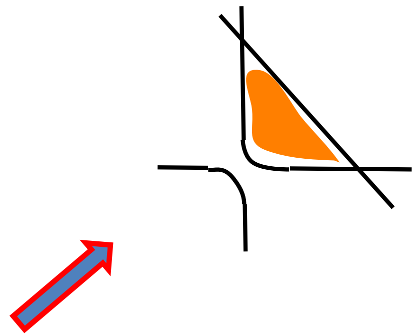
(obtained by Lagrangian surgery.) $u = (u_1, u_2)$

$$x_1 = u_1, \quad x_2 = u_1(1 - u_2)$$

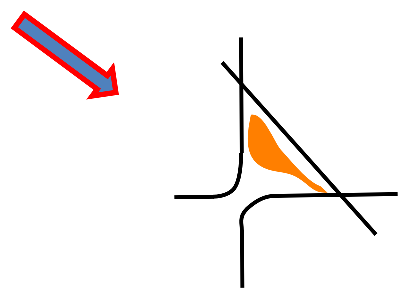
Coordinate change is determined by bifurcation of holomorphic discs



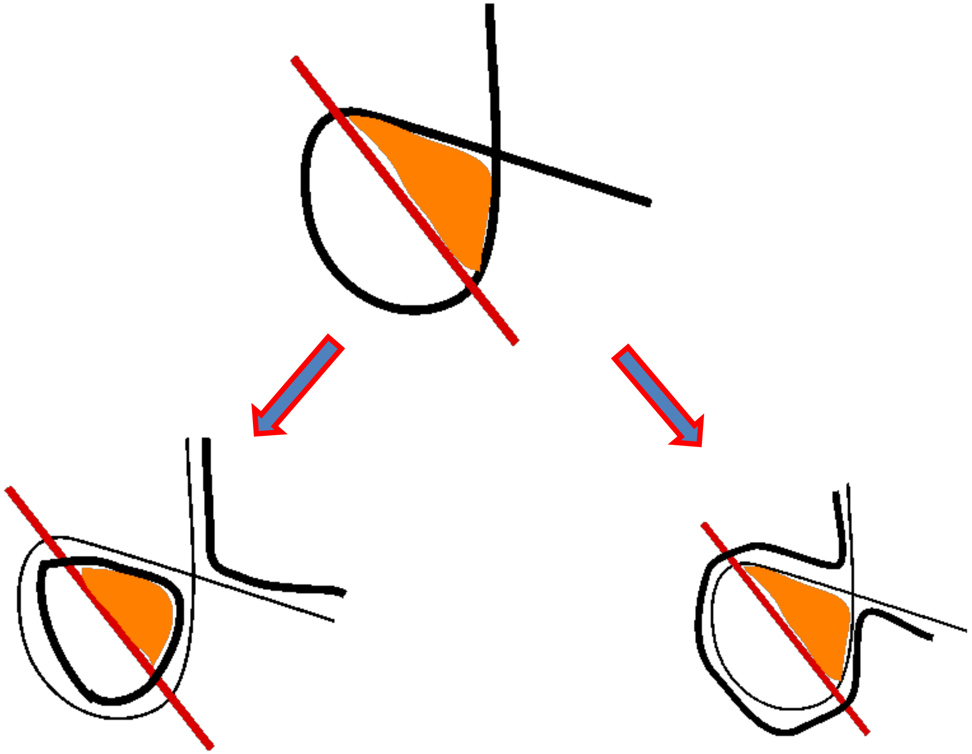
Holomorphic triangle



becomes
holomorphic 2 gon



Becomes S^{n-2} parametrized
family of holomorphic 2 gons



becomes
holomorphic 2 gon

Becomes $S^0 = 2$ point
parametrized family of holomorphic 2 gons

$$x_1 = u_1$$

$$x_2 = u_1 - u_1 u_2$$

Each of the terms corresponds to the holomorphic 2 gon