Mirror symmetry between toric A model and Landau-Ginzburg B model

at Keio Univ
2008 10/9

Joint with Oh-Ohta-Ono

§ Singularity (Saito) theory

$W(w_1 \cdots w_B, y_1 \cdots y_n)$ holo

$$\begin{array}{ccc} V \times U & \longrightarrow & \mathbb{C} \\ \cap \quad \cap & & \\ \mathbb{C}^B \quad \mathbb{C}^n & & \end{array}$$

$0 \in U$ is isolated critical point of $W(0, -) = W_0$

$W_w(y) = W(w, y)$

$\mathrm{Crit}\, W_w = \{ y \in U \mid \frac{\partial W_w}{\partial y_i} = 0,\ i = 1, \cdots, n \}$

$\# \mathrm{Crit}\, W_w < \infty$

germ of holo. functions

Definition

- $$\mathrm{Jac}\,W_w = \bigoplus_{y \in \mathrm{Crit}\,W_w} \frac{\mathcal{O}_y}{\left(\frac{\partial W_w}{\partial y_i} \,\middle|\, i = 1, \dots, n\right)}$$

- $$KS : T_w V \longrightarrow \mathrm{Jac}\,W_w$$
$$\frac{\partial}{\partial w_i} \longmapsto \left[\frac{\partial W_w}{\partial w_i}\right]$$

- W_w is miniversal $\iff$ KS is isomorphism

- If W_w is miniversal, $T_w V$ is a ring

$$KS\left(\frac{\partial}{\partial w_i} \circ \frac{\partial}{\partial w_j}\right) = KS\left(\frac{\partial}{\partial w_i}\right) \cdot KS\left(\frac{\partial}{\partial w_j}\right)$$

③

<u>Residue pairing</u>

Assume W_w is Morse.

$$\operatorname{Jac} W_w = \bigoplus_{y \in \operatorname{Crit} W} \mathbb{C}\, 1_y$$

$$\langle 1_y, 1_{y'} \rangle_{res} = \begin{cases} 0 & y \neq y' \\ \dfrac{1}{\det \operatorname{Hess} W_w} & y = y' \end{cases}$$

$\langle \ \rangle_{res}$ extends to all $w \in V$

metric on TV.

∇^{LV} Levi-Civita

④

Deep results by K. Saito, M. Saito

① ∇^{LC} is flat.

i.e. $\exists\, w_i$ coordinate flat coordinate

$$\nabla^{LC}_{\frac{\partial}{\partial w_i}} \frac{\partial}{\partial w_j} = 0$$

② $\exists\, \Phi : V \longrightarrow \mathbb{C}$

s.t.

$$\left\langle \frac{\partial}{\partial w_i} \circ \frac{\partial}{\partial w_j}, \frac{\partial}{\partial w_k} \right\rangle_{Res} = \frac{\partial^3 \Phi}{\partial w_i \partial w_j \partial w_k}$$

§ Gromov-Witten theory (closed string)

(X, ω) symplectic manifold, J almost complex str,

$\alpha \in H_2(X; \mathbb{Z})$

$$\overset{\circ}{\mathcal{M}}_\ell(X; \alpha) = \left\{ (u; z_1, \ldots, z_\ell) \;\middle|\; \begin{array}{l} u : S^2 \to X \text{ holomorphic} \\ z_i \in S^2 \quad z_i \neq z_j \\ [u] = \alpha \end{array} \right\} \Big/ \mathrm{Aut} = PSL(2, \mathbb{C})$$

Compactify (stable map) $\rightsquigarrow \mathcal{M}_\ell(X; \alpha)$

$$\begin{array}{cccc} ev : & \mathcal{M}_\ell(X; \alpha) & \longrightarrow & X^\ell \\ & \Psi & & \Psi \\ & (u; z_1, \ldots, z_\ell) & \longmapsto & (u(z_1), \cdots, u(z_\ell)) \end{array}$$

⑥

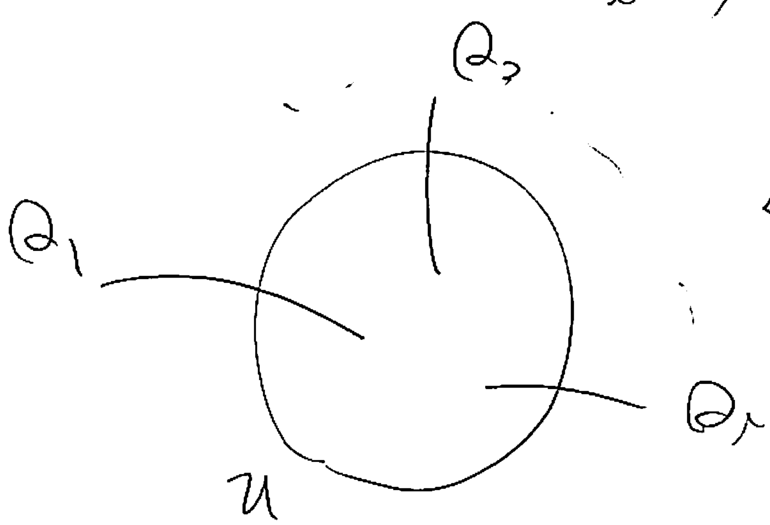
$$\Lambda_0 = \left\{ \sum a_i T^{\lambda_i} \;\middle|\; \begin{array}{l} a_i \in \mathbb{Q} \\ \lambda_i \geq 0, \ \lim \lambda_i \to \infty \end{array} \right\}$$

$\Lambda = \Lambda_0[T^{-1}]$ field

$$\left(H^*(X;\Lambda_0)\right)^{\otimes \ell} \xrightarrow{GW_\ell} \Lambda_0$$

$$GW_\ell(Q_1 \cdots Q_\ell) \in \Lambda_0$$

$$= \sum_\alpha T^{\alpha \cap \omega} \int_{\mathcal{M}_\ell(X,\alpha)} ev^*(Q_1 \times \cdots \times Q_\ell) \quad \wedge$$



$\Leftarrow$ count with weight $T^{\alpha \cap \omega}$

$Q \in H(X;\Lambda_0)$

<u>Def</u> $\cup^Q : H(X;\Lambda_0) \otimes H(X;\Lambda_0) \to H(X,\Lambda_0)$

$$\langle a \cup^Q b, c \rangle_{PD} = \sum_{\ell} \frac{1}{\ell!} GW_{\ell+3}(a\, b\, c\, \overbrace{Q \cdots Q}^{\ell})$$

$\langle \quad \rangle_{PD}$ is Poincaré duality

◎ $(a \cup^Q b) \cup^Q c = a \cup^Q (b \cup^Q c)$

◎ $\langle a \cup^Q b, c \rangle_{PD} = \langle a, b \cup^Q c \rangle$

ℓ

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$\boxed{\text{Dubrovin}}$

$V = H(X; \Lambda_0)$ $H(X; \Lambda_0)$

$Q \in V$ $T_Q V$ is a ring $\cup^Q$

$T_Q V = H(X; \Lambda_0)$

○ $\langle\, ,\, \rangle_{PD}$ metric on V

flat

$\Uparrow$ trivial $\Leftarrow$ affin coordinate is flat coordinate

○ $\exists\, \Phi$

$$\left\langle \frac{\partial}{\partial w_i} \cup^Q \frac{\partial}{\partial w_j}, \frac{\partial}{\partial w_k} \right\rangle = \frac{\partial^3 \Phi}{\partial w_i \partial w_j \partial w_k}$$

$$\left(\Phi(Q) = \sum_{\ell} \frac{1}{\ell!} GW_\ell(Q \cdots Q) \right)$$

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Frobenius manifold structure
(= Saito flat structure)

$$\begin{cases} V : \text{ manifold} \\ T_w V \text{ has a ring structure } \circ \\ T_w V \text{ has a metric } \langle\ \rangle \\ \nabla^{LC} \text{ Levi-Civita is flat} \\ \\ \exists \Phi \end{cases}$$

$$\left\langle \frac{\partial}{\partial w_i} \circ \frac{\partial}{\partial w_j}, \frac{\partial}{\partial w_k} \right\rangle_{PD}$$

$$= \frac{\partial^3 \Phi}{\partial w_i \partial w_j \partial w_k}$$

Rough statement of Main theorem

X compact toric manifold

$\exists W \quad H(X) \times \Lambda_0^n \longrightarrow \Lambda_0$

s.t.

① W ⇝ Frobenius manifold structure on $H(X)$	$=$	② GW ⇝ Frobenius manifold structure on $H(X)$

Batyrev, Givental, Hori-Vafa

What is W in Theorem

⇑

Open closed Gromov Witten Potential

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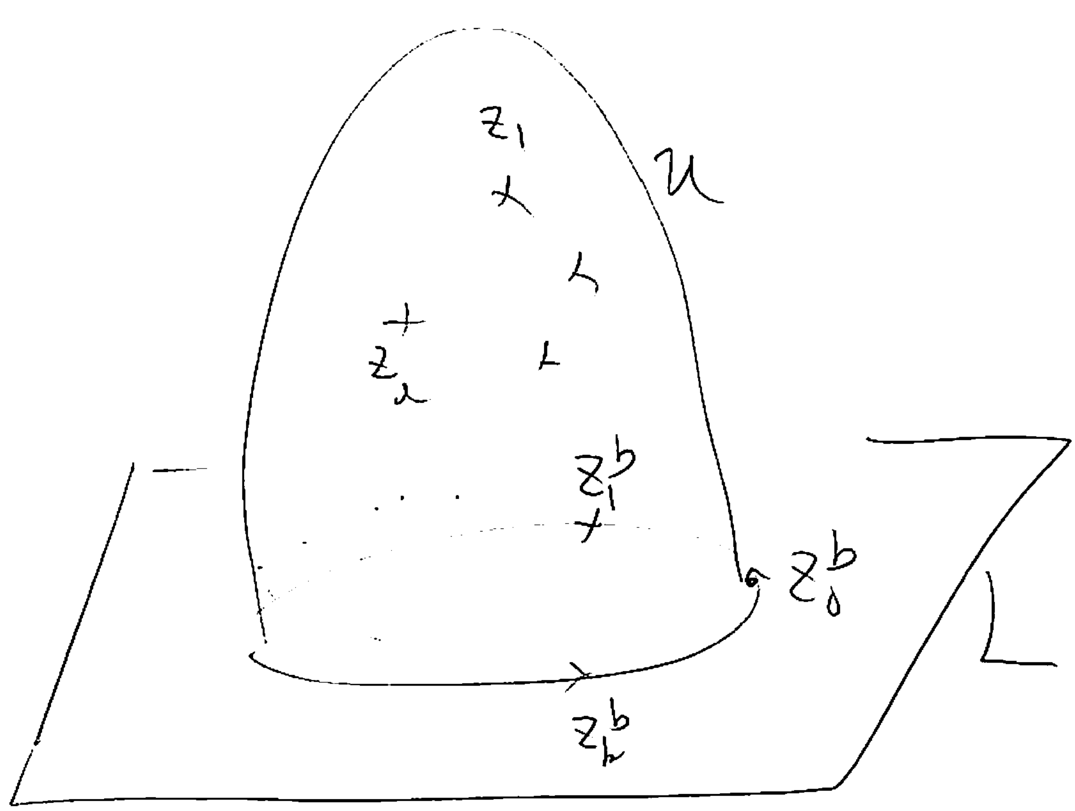
Potential function with bulk

$L \subset X$ Lagrangian submanifold



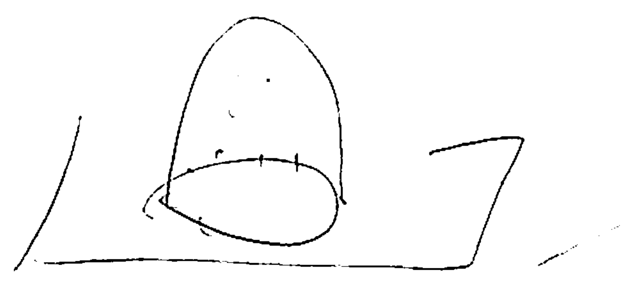
$\beta \in H_2(X, L; \mathbb{Z})$

$\mathring{\mathcal{M}}_{k+1,\ell}(\beta)$

$$= \left\{ (u ; z_1 \cdots z_\ell ; z_0^b \cdots z_k^b) \mid \star \right\} / PSL(2, \mathbb{R})$$

$$\star \begin{cases} u : (D^2, \partial) \longrightarrow (X, L) \text{ holomorphic} \\ u_*([D^2]) = \beta \\ z_1, \cdots, z_\ell \in \operatorname{Int} D^2 \qquad z_i \neq z_j \\ z_1^b \cdots z_\ell^b \in \partial D^2 \qquad z_i^b \neq z_j^b \end{cases}$$

respects cyclic order of ∂D^2.



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$\mathcal{M}_{k+1,\ell}(\beta)$ compactification

$$ev : \mathcal{M}_{k+1}(\beta) \longrightarrow X^{\ell} \times L^{k+1}$$

$$(u, z_1 \cdots z_\ell, z_0^b, \cdots, z_k^b) \longmapsto (u(z_1), \cdots, u(z_\ell)$$
$$u(z_1^b), \cdots, u(z_k^b), u(z_0^b))$$

$Q \in H(X)$
$b \in H^1(L)$

$$W(Q,b) = \sum_{k,\ell,\beta} \frac{T^{\beta \cap \omega}}{\ell!} \int_{\mathcal{M}_{k+1,\ell}(\beta)} ev^*(Q \cdots Q\, b \cdots b\, [L]) \in \Lambda_0$$

$[L]$ = volume form

$$H(X ; \Lambda_0) \times H^1(L ; \Lambda_0) \longrightarrow \Lambda_0$$

In general, this does not work so simple. (whole book ~ 1000 pages)

$\begin{cases} X \text{ toric} \\ L = L(u) \end{cases}$

$\mu : X \longrightarrow P \subseteq \mathbb{R}^n$ moment map

$L(u) = \mu^{-1}(u)$

$u \in \mathrm{Int}\, P$

$W_u(\mathfrak{b}, b)$ is well defined up to change of variables of

$b \in H^1(L(u) ; \Lambda_0) = \Lambda_0^n$

Thm (FOOO)

$\mathfrak{b} \in H(X; \Lambda_0)$ $\quad$ $W(\mathfrak{b}, -) : H^1(L) \to \Lambda_0$ of Int P

$$(H(X;\Lambda), \cup^{\mathfrak{b}}) \xrightarrow{KS} \mathrm{Jac}\, W(\mathfrak{b}, -)$$

$$\frac{\partial}{\partial w_i} \longmapsto \left[\frac{\partial W_{\mathfrak{b}}}{\partial w_i}\right]$$

$\left(\mathfrak{b} = \sum w_i \mathfrak{b}_i \quad \mathfrak{b}_i \text{ basis of } H(X;\mathbb{Z})\right)$

① KS is a ring isomorphism.

② $\langle \alpha, \beta \rangle_{PD} = \langle KS(\alpha), KS(\beta) \rangle_{\text{Residue}}$

①

Q-D $\Rightarrow$ Batyrev, Givental
X Fano $\quad$ Cho-Oh

Cor

1. $W(Q, \ldots)$ (open-closed GW invariant) is a miniversal family

2. $\langle\ ,\ \rangle_{PD}$ on $TH(X;\Lambda_0)$ is flat metric

3.

$$\sum_{j,k} \frac{\partial^3 \Phi}{\partial w_i \partial w_j \partial w_k} \frac{\partial W}{\partial w_l} g^{kl}$$

$$= \frac{\partial W}{\partial w_i} \frac{\partial W}{\partial w_j} \quad \text{mod} \left(\frac{\partial W}{\partial y_i} \right)$$

$b = \sum x_i e_i$ $\quad e_i$ basis of $H^1(T^n;\mathbb{Z})$
$y_i = \exp(x_i)$

Application to Lag Floer theory

X toric $\quad L(u) \subset X \quad L(u) = \mu^{-1}(u)$

$\mathfrak{b} \in H^{ev}(X;\Lambda_0)$

$b \in H^1(L(u);\Lambda_0)$

$HF(L(u),(\mathfrak{b},b))$ Floer homology

$\star$ $HF(L(u),(\mathfrak{b},b)) \neq 0$

$\iff$ b is a critical point

$W_u(\mathfrak{b}, -)$

Thm If $W(\mathfrak{b},-)$ is Morse

$$\#\{(u,b) \mid HF(L(u),(\mathfrak{b},b)) \neq 0\} = \sum \operatorname{rank} H(X)$$

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How we obtain u, b
for given critical point of W

$W_u(\theta, b)$ open closed Gromov Witten inv. of $L(u)$,

$W_{u'} \hookrightarrow W_u$ $\quad u \in \mathrm{Int} P \subset \mathbb{R}^n$

$u' = (u_1', \cdots, u_n')$
$u = (u_1, \cdots, u_n)$

$H^1(L(u))$ basis e_i

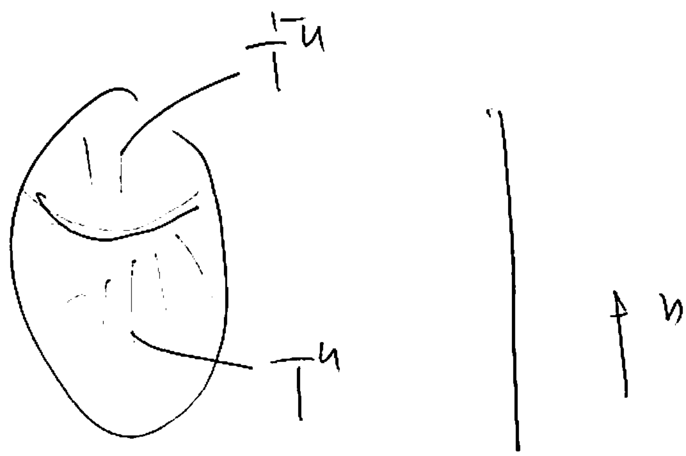
$b = \sum x_i e_i \quad x_i \in \Lambda_0 \quad y_i = e^{x_i}$

☆

$$\boxed{W_u(w, T^{u_1'-u_1}y_1 \cdots T^{u_n'-u_n}y_n) = W_{u'}(w, y_1 \cdots y_n)}$$

ex.

$\mathbb{P}^1$



$$W_u = T^u y + T^{1-u} y^{-1}$$

$$y = T^{u'-u} y'$$

$$W_u(y) = W_{u'}(y')$$

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$$\frac{\partial W_u}{\partial y} = T^u - T^{1-u} y^{-2} = 0$$

$$y = T^{\frac{1}{2}-u}$$

$$h = \sum x_i e_i \qquad y_i = e^{x_i}$$

$\log y$ exists in Λ_0 only $u = \frac{1}{2}$

$(\log T \notin \Lambda_0)$

$$\frac{\partial W_u}{\partial y_i} = 0 \quad \text{sol} \quad y_1, \dots, y_m$$

$\Rightarrow \exists!\, u$ st $\log y_1, \dots, \log y_m$ exists

$$HF(L(u), \sum \log y_i e_i) \neq 0.$$

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Application to quasi homomorphism

Entov-Polterovich improved by Ostrover (Mcduff)

$+\varepsilon$

$[\omega] \in H(X; \mathbb{Q})$

$(H(X, \Lambda), \nu^{Q})$

assum $\cong \Lambda \times$ somethg

$\Downarrow$

$C \geq |\psi(g_1 g_2) - \psi(g_1) - \psi(g_2)|$

$\Downarrow$

EP associate

$\psi : \widetilde{Ham} \longrightarrow \mathbb{R}$ quasi homo

$\parallel$

Universal cover of group of Hamiltonian differ

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Let X compact toric mfd

$$(H(X), \cup^{\theta}) \cong \mathrm{Jac}\, W(\theta)$$

$$\cong \prod_{y \in \mathrm{crt}\, W(\theta, -)} \frac{\mathcal{O}_y}{\left(\frac{\partial W(\theta, -)}{\partial y_i}\right)}$$

$\Lambda \times$ Something $= (H(X), \cup^{\theta})$

$\hookrightarrow$ corresponds to some critical point of $W(\theta, -)$

non degenerate

$$\Downarrow$$

$(L(u), b)$ s.t. $HF(L(u), (\theta, b)) \neq 0$

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$$\begin{tikzcd}
 & \Lambda \times \square \cong (\mathcal{H}(X;\Lambda_0), \cup^Q) \arrow[dl] \arrow[dr, "EP"] & \\
y \in \mathrm{Crit}(W(\mathfrak{b}-)) \arrow[d] & & \psi : \widetilde{\mathcal{H}am} \to \mathbb{R} \\
(u,b) \text{ st.} & & \text{quasi-homomorphism}
\end{tikzcd}$$

$HF(L(u), (\mathfrak{b}, b)) \neq 0$

Relation $L(u) \hookrightarrow \psi$

$\mathrm{Supp}\, \psi_n = L(\beta, u)$

Thm (FOOO)

$f : X \longrightarrow \mathbb{R}$ $\int f \omega^n = 0$

$f \equiv 0$ on a nbd of $L(u)$.

X_f Hamiltonian vector field

$\varphi(t) = \exp(tX_f) \in \widetilde{Ham}$

$\Downarrow$

$\psi(\varphi(t)) = 0$

Roughly Support of ψ = $L(u)$

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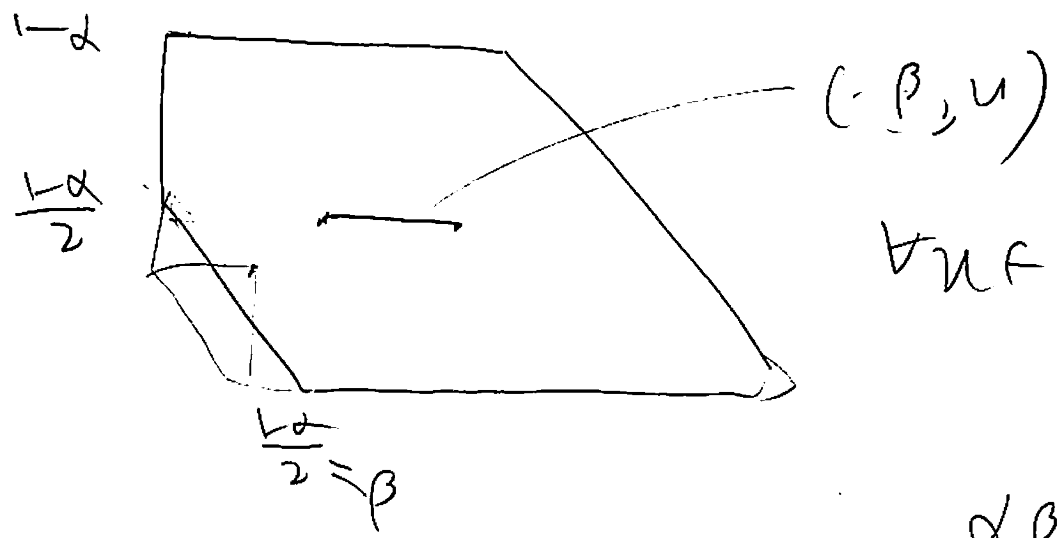
Example

X 2 point blow up



$\forall u \in (\beta, \frac{1-\beta}{2}) \cap \mathbb{Q}$

$\alpha, \beta \in \mathbb{Q}$

$\exists\, \varphi_u : \widetilde{\mathrm{Ham}} \longrightarrow \mathbb{R}$ quasi-homo.

" $\mathrm{supp}\, \varphi_u = L(\beta, u)$ "

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Kaledkar Panteu

$\forall N$, $u_1 \cdots u_N$

$\mathbb{Z}^N \subset \widetilde{\hat{H}am}$

$(\psi_{u_1} \cdots \psi_{u_N}) \; \hat{Ha} \longrightarrow \mathbb{R}^n$

is injective $\mathbb{Z}^N$.

ψ_u are all independent.

(Biran – Entov – Polterovich

found such so many ψ

in case $D^{2n} \subset \mathbb{C}^n$), ①

Def. of $T_{ac}W$

$\Lambda\{y_1 \cdots y_n^{-1}\}$ ← completion of $\Lambda[y_1 \cdots y_n^{-1}]$

$$= \left\{ \sum a_k y_1^{i_{k,1}} \cdots y_m^{i_{k,m}} \;\middle|\; \begin{array}{l} a_k \in \Lambda \\ i_{k,i} \in \mathbb{Z} \end{array} \right\}$$

☆

$$a_k = \sum T^{\lambda_{k,i}} a_{k,i} \qquad \begin{array}{l} a_{k,i} \in \mathbb{Q} \\ \lambda_{k,i} \in \mathbb{R} \\ \lambda_{k,i} \to \infty \end{array}$$

$$v_T(a_k) = \inf\{\lambda_{k,i} \mid a_{k,i} \neq 0\}$$

$$☆ \iff \lim_{k\to\infty} v_T(a_k) = \infty$$

②

$u = (u_1 \cdots u_n) \in \mathbb{R}^n \qquad P = \mu(X) \subset \mathbb{R}^n$

$y_i^u = T^{u_i} y_i$

$$\Lambda^P\{y_1 \cdots y_n^{-1}\} = \bigcap_{u \in P} \Lambda\{y_1^u \cdots (y_n^u)^{-1}\}$$

ex. $\displaystyle\sum_{n \geq 0} T^n y^{n^2} = f$

$u < 0 \qquad y_u = T^{-u} y$

$$f = \sum T^{n - un^2} (y^u)^{n^2} \notin \Lambda_0\{y^u, y^{u-1}\}$$

$\to -\infty$

f converges on $\{u \mid u \geq 0\}$

③

0. $W(Q, -) \in \Lambda^p\{y_1 \cdots y_r^{-1}\}$

$$\Lambda \otimes \mathrm{Jac}\,W = \frac{\Lambda^p\{y_1 \cdots y_r^{-1}\}}{\left(y_i \frac{\partial W}{\partial y_i}\right)}$$

①

Idea of the proof.

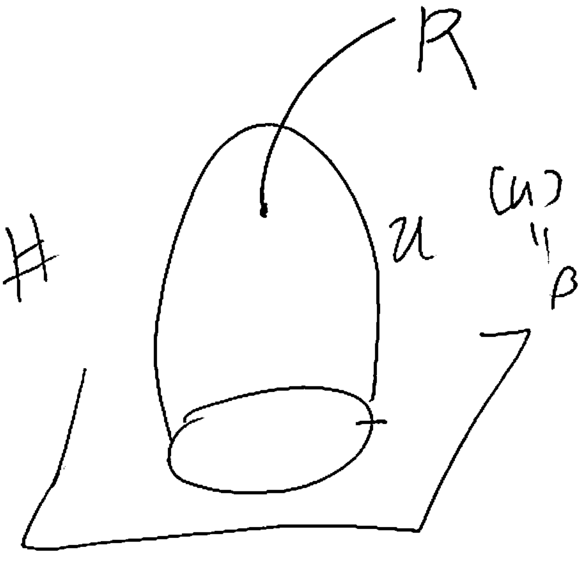
$Q = 0$

$$KS : H(X;\Lambda) \longrightarrow \operatorname{Jac} W$$

$$\frac{\partial}{\partial w_i} \longmapsto \left[\frac{\partial W}{\partial w_i}\right]$$

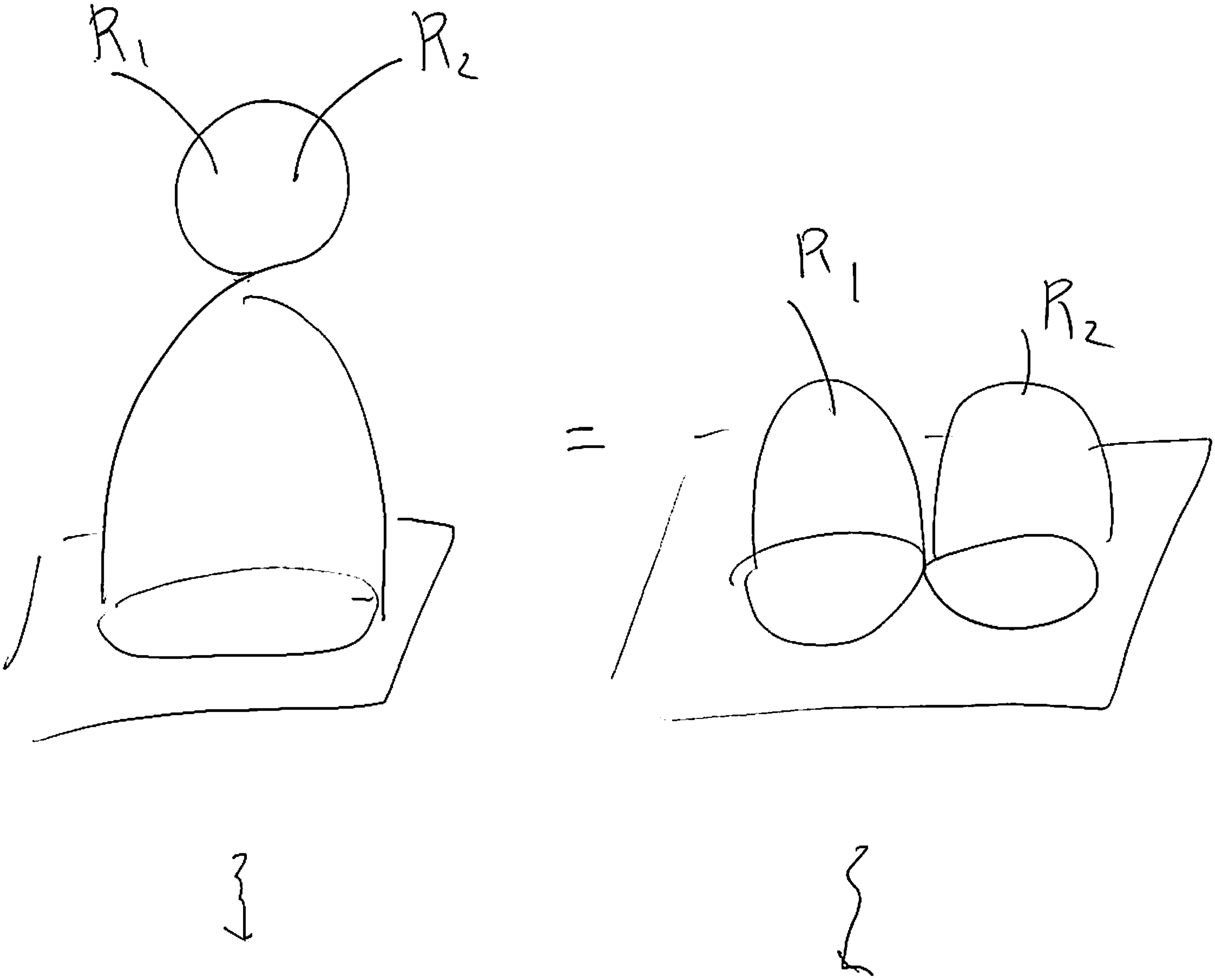
$$\begin{array}{c} \frac{\partial}{\partial w_i} \\ \cup \\ R \end{array}$$

$$KS(R) = \sum T^{\beta\cap\omega} \#$$



③

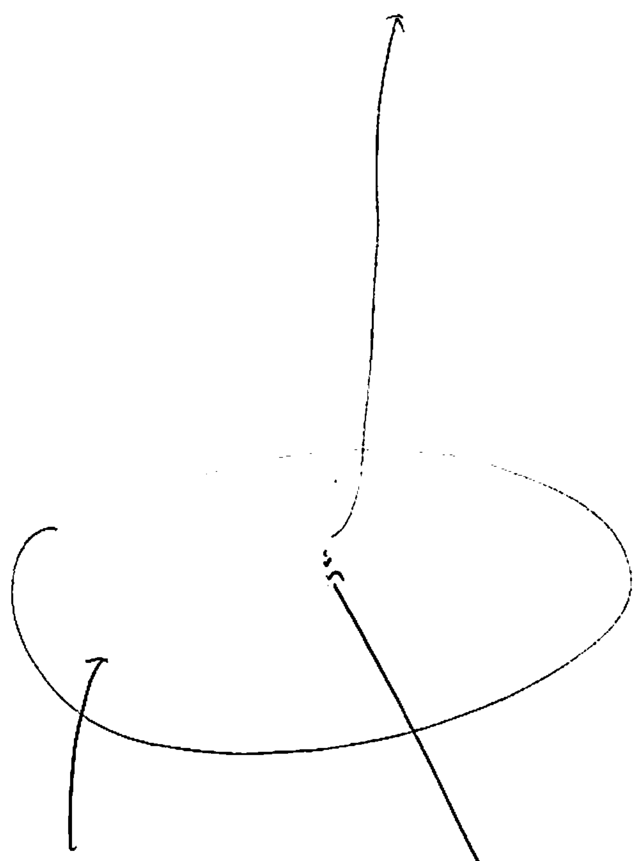
∘ KS is a ring homo.



$KS(R_1 \cup^{g} R_2) \qquad KS(R_1)\, KS(R_2)$

③

$(H(X:\Lambda), U^\theta)$ $\quad$ $(H(X:\mathbb{Q}), U)$ usual eg



generic point $\quad$ $T = 0$

$Sp(\Lambda_0)$

$\cup$

$Sp\{\sum a_i T^{\lambda_i} \mid \lambda_i \geq 0\}$

④

$$\begin{tikzcd}
H(X,\Lambda) \arrow[r, "KS"] & \overline{Jac}(W) \otimes \Lambda \\
H(X;\Lambda_0) \arrow[u] \arrow[r, "KS"'] \arrow[d, "T=0"] & Jac(W) \arrow[u] \arrow[d, "T=0"] \\
H(X;\mathbb{C}) \arrow[r, "\sim"] & {} \\
 & \dfrac{\mathbb{C}[z_1 \cdots z_m]}{(SR + \text{Linear rel})} \arrow[u, "\wr"]
\end{tikzcd}$$

classical
result in toric geometry

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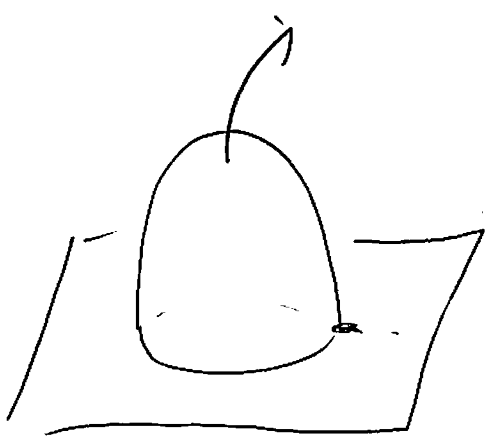
Residue Pairing = Poincaré duality

$$H^*(X;\Lambda) \xrightarrow{KS_*} \bigoplus_{p \in \mathrm{Crit}\, W} \Lambda$$

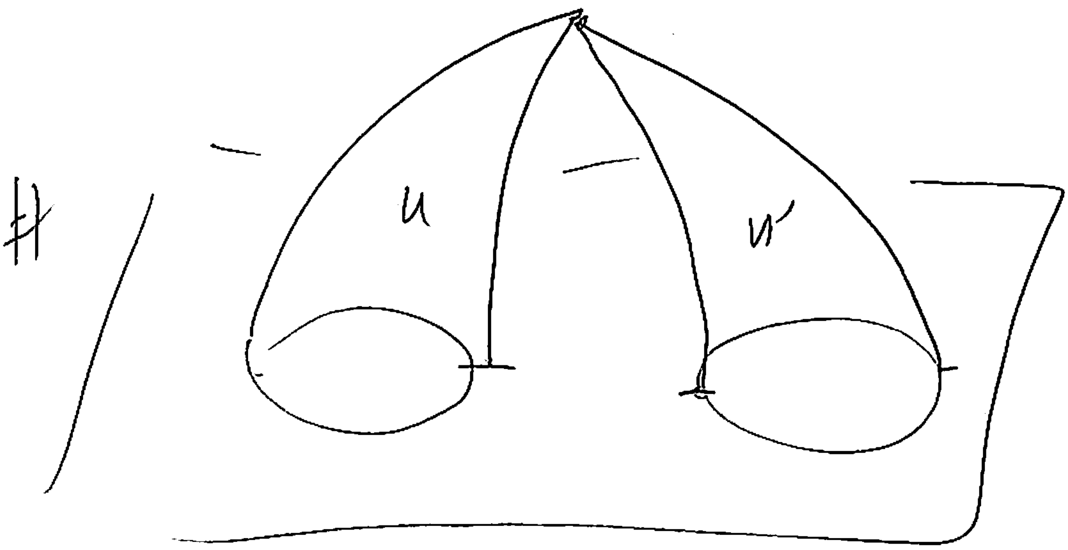
$\wr$ dual

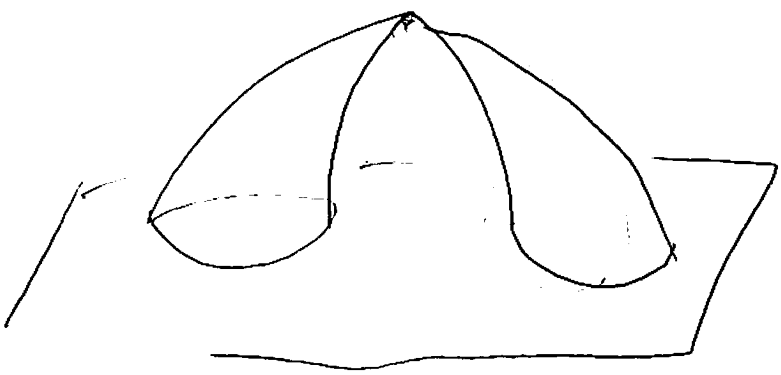
$$H_*(X;\Lambda) \xleftarrow{KS^*} \bigoplus_{p \in \mathrm{Crit}\, W} \Lambda$$

 $\longleftarrow\!\shortmid\ 1_p$

$$\langle KS^*(1_p), KS^q(1_p)\rangle$$

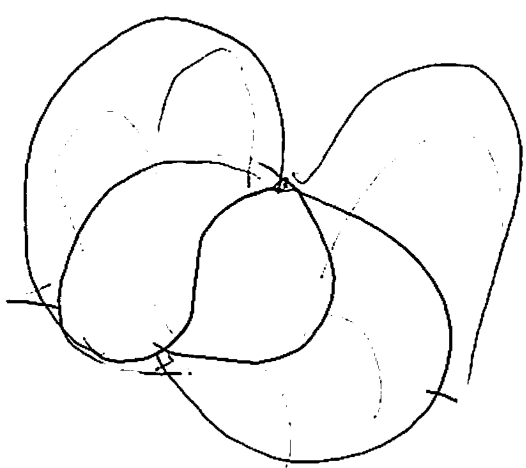
$$= \sum T^{u\cap w + u'\cap w}$$





$\langle KS(1), KS(1) \rangle_{PD}$

$$\|$$



P_i basis $H(L)$

$g_{ij} = \langle P_i, P_j \rangle_{PD}$

$$\sum g^{ij} \langle m_2([pt], P_i), m_2([pt], P_j) \rangle_{PD}$$

$\not{L}$