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Loop space and holomorphic disc -summary-

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Abstract

We explain an application of L_∞ structure on the homology of free loop space and of the moduli space of pseudo-holomorphic disc, to symplectic topology of Lagrangian submanifold. Our result was anounced in [5] before. We present some more detail on the following two points :
(1). The way to use homotopy theory of L_∞ algebra :
(2). The construction of L_∞ structure on the homology of free loop space.

(2) uses "correspondence parametrized by an operad" which was introduced in [6] and chain level intersection theory in [9] §30.

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1 Introduction

This is an updated version of the earier anoucement paper by the author [5] which discuss an application of string topology (homology of loop space) together with Floer theory (molduli space of pseudo-holomorphic discs) to the study of Lagrangian submanifold. Especially we explain in more detail the way how to use homological algebra of L_∞ structure. The disussion of this point in [5] was sketchy. (See [5] Remark 8.3 however.) We also explain the way how to use 'correspondence parametrized by an operad' to work out the detail of the Chas-Sullivan's L_∞ structure on the homology of free loop space. We use singular homology here. We used de Rham cohomology in [6], since transversality is easier to handle when we use de Rham cohomology. When we study homology of loop space, de Rham theory is hard to apply. So we use

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singular homology and use the technique of [9] §30 to work out the chain level transversality.

In this article all manifolds are assumed to be connected.

The main result anounced in [5] is :

Theorem 1.1 *If L is an irreducible and oriented 3 manifold which is embeded in $\mathbb{C}^3$ as a Lagrangian submanifold, then L is diffeomorphic to the direct product $S^1 \times \Sigma$ of a circle S^1 and a surface Σ.*

There is a generalization to the case of aspherical manifold of higher dimension.

Theorem 1.2 *Let L be a compact n dimensional aspherical manifold which is embeded in $\mathbb{C}^n$ as a Lagrangian submanifold. We assume L is oriented and spin.*

Then a finite covering space $\tilde{L}$ of L is homotopy equivalent to the direct product $S^1 \times L'$ for some space L'.

Moreover, we may choose $\gamma \in \pi_1(S^1) \subset \pi_1(S^1 \times L') \subset \pi_1(L)$ such that the centralizer of γ in $\pi_1(L)$ is of finite index. Furthermore the Maslov index of γ is 2 and the symplectic area of the disc bounding γ is positive.

The key point to prove these theorems is to use the relation of pseudo-holomorphic disc to the string topology [2]. We remark that relation between pseudo-holomorphic curve and loop space was used in the pioneering work by Viterbo [12] before.

Let $\mathcal{L}(L)$ be a free loop space of a compact orientated n dimensional manifold L without boundary.

Theorem 1.3 *$H(\mathcal{L}(L);\mathbb{Q})[n-1]$ has a structure of L_∞ algebra.*

L_∞ algebra is a homotopy version of Lie algebra. (See §2.) $[n-1]$ above stands for degree shift. Namely

$$H(\mathcal{L}(M);\mathbb{Q})[n-1]_d = H_{d+n-1}(\mathcal{L}(M);\mathbb{Q}).$$

We use moduli space of (pseudo)holomorphic disc equation (sometimes with perturbation by Hamiltonian function), to prove the following two results. Following [9] we define univeral Novikov rint $\Lambda_{0,nov}$ as the totality of the series

$$\sum a_i T^{\lambda_i} e^{n_i} \tag{11}$$

such that $a_i \in \mathbb{Q}$, $\lambda_i \in \mathbb{R}_{\geq 0}$, $n_i \in 2\mathbb{Z}$ and that $\lim_i \lambda_i = \infty$. Here T and e are formal parameters. It is graded by defining $\deg e = 2$, $\deg T = 0$. The ideal which consists of elements (11) such that $\lambda_i > 0$ is denoted by $\Lambda_{0,nov}^+$. We define $\Lambda_{nov} = \Lambda_{0,nov}[T^{-1}]$.

Theorem 1.4 *Let $L \subset M$ be a compact Lagrangian submanifold of a compact (or convex) symplectic manifold M. We assume L is relatively spin. Then for each $E_0 > 0$, there exists*

$$\alpha \in H_{n-2}(\mathcal{L}(M); \Lambda^+_{0,nov})$$

such that

$$\sum \mathfrak{l}(e^\alpha) \equiv 0 \mod T^{E_0}. \tag{12}$$

Theorem 1.5 *Let L be as in Theorem* 1.4*. We assume that there exists a Hamiltonian diffeomorphism $\varphi : M \to M$ such that $\varphi(L) \cap L = \emptyset$. Then there exists*

$$\mathfrak{B} \in H_{n-1}(\mathcal{L}(M); \Lambda_{nov})$$

such that

$$\sum \mathfrak{l}(\mathfrak{B}, e^\alpha) \equiv [L] \mod \Lambda^+_{0,nov}. \tag{13}$$

There are several other applications of Theorems 1.4, 1.5, which will be explored elsewhere. (See for example [1], [3].)

2 L_∞ algebra an its homotopy theory.

In this section we review homotopy theory of L_∞ algebra and explain the way how to use it to deduce Theorems 1.1, 1.2 from Theorems 1.3, 1.4, 1.5. We first review the notion of L_∞ algebra. See [9] Chapter 8 §37, [11] etc., for more detail. Let C be a graded $\mathbb{Q}$ vector space. We shift its degree to obtain $C[1]$. Namely $C[1]^d = C^{d+1}$. Let $B_kC[1]$ be the tensor product of k copies of $C[1]$. We define an action of symmetric group $\mathfrak{S}_k$ of order $k!$ on it by

$$\sigma(x_1 \otimes \cdots \otimes x_k) = (-1)^{\sum_{i<j;\sigma(i)>\sigma(j)}(\deg x_i - 1)(\deg x_j - 1)} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}.$$

$E_kC[1]$ denotes the subspace of $B_kC[1]$ consisting of elements which are invariant of $\mathfrak{S}_k$ action. We put

$$[x_1, \cdots, x_k] = \sum_{\sigma \in \mathfrak{S}_k} (-1)^{\sum_{i<j;\sigma(i)>\sigma(j)}(\deg x_i - 1)(\deg x_j - 1)} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}.$$

They generate $EC[1]$.

The coalgebra structure

$$(x_1 \otimes \cdots \otimes x_k) \mapsto \sum_{i=1}^{k-1} (x_1 \otimes \cdots \otimes x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_k)$$

on $BC[1] = \oplus_{k=1}^{\infty} B_k C[1]$ induces a coalgebra structure Δ on $EC[1] = \oplus_{k=1}^{\infty} E_k C[1]$, which is graded cocommutative and coassociative.

An L_∞ structure of C is, by definition, a coderivation $\hat{d} : EC[1] \to EC[1]$ of degree one such that $\hat{d} \circ \hat{d} = 0$. We denote $Hom(E_\ell C[1], E_1 C[1])$ component of $\hat{d}$ by $\mathfrak{l}_\ell$. The sequence of operators $\mathfrak{l}_1, \mathfrak{l}_2, \cdots$ determines $\hat{d}$ uniquely.

$\mathfrak{l}_1$ satisfies $\mathfrak{l}_1 \circ \mathfrak{l}_1 = 0$ and hence defines a cohomology group $H(C, \mathfrak{l}_1)$. Then $\mathfrak{l}_2$ defines a structure of graded Lie algebra on $H(C, \mathfrak{l}_1)$. (We need sign. See for example [9] §37.)

L_∞ homomorphism is, by definition, a coalgebra homomorphism $\hat{\mathfrak{f}} : EC[1] \to EC'[1]$ which satifies $\hat{d} \circ \hat{\mathfrak{f}} = \hat{\mathfrak{f}} \circ \hat{d}$. The homomorphism $\hat{\mathfrak{f}}$ is determined uniquely by its $Hom(E_\ell C[1], E_1 C'[1])$ components ($\ell = 1, 2, \cdots,$), which we denote by $\mathfrak{f}_\ell$. The compositions of two L_∞ homomorphisms is defined in an obvious way.

If $\mathfrak{f} = (\mathfrak{f}_i)_{i=1}^{\infty}$ is an L^∞ homomorphism $C \to C'$ then $\mathfrak{f}_1$ is a chain map $(C, \mathfrak{l}_1) \to (C', \mathfrak{l}_1)$. We say $\mathfrak{f}$ is linear if $\mathfrak{f}_k = 0$ for $k \neq 1$.

An element $\mathbf{e} \in C$ is said to be central if $\mathfrak{l}_k([\mathbf{e}, x_2, \cdots, x_k])$ vanish for any k and x_i. We say $(C, \mathfrak{l}_*, \mathbf{e})$ a central L_∞ algebra if $(C, \mathfrak{l}_*)$ is an L_∞ algebra and $\mathbf{e} \in C$ is central. An L_∞ homomorphism $\mathfrak{f} = (\mathfrak{f}_i)_{i=1}^{\infty}$ is said to be central if $\mathfrak{f}_1(\mathbf{e}) = \mathbf{e}$ and $\mathfrak{f}_k([\mathbf{e}, x_2 \cdots, x_k]) = 0$ for $k \geq 2$.

The following L_∞ analogue of [9] Chapter 4 is basic to develop homotopy theory.

Definition 2.1 *Let C be an L_∞ algebra. We say an L_∞ algebra $\mathfrak{C}$ together with linear L_∞ homomorphims* $\mathrm{Incl} : C \to \mathfrak{C}$, $\mathrm{Eval}_{s=s_0} : \mathfrak{C} \to C$ *($s_0 = 0, 1$) to be a model of $[0,1] \times C$ if the following holds.*

(1) Incl *and* $\mathrm{Eval}_{s=s_0}$ *induces isomorphisms on cohomology groups.*
(2) $\mathrm{Eval}_{s=s_0} \circ \mathrm{Incl}$ *are identy for $s_0 = 0$ or 1.*
(3)

$$\mathrm{Eval}_{s=0} \oplus \mathrm{Eval}_{s=1} : \mathfrak{C} \to C \oplus C$$

is surjective.

If C is central then a model of $[0,1] \times C$ is said to be central if it has a central element and Incl, $\mathrm{Eval}_{s=s_0}$ *are central.*

In a way similar to [9] Chapter 4, we can prove various properties of it so that homotopy theory can be established. Especially we have the following :

(HT.1) Model of $[0,1] \times C$ exists for any C. We denote by $\mathfrak{C}$ a model of $[0,1] \times C$. If C is central then we may take a central model.

(HT.2) We say two L_∞ homomorphisms $\mathfrak{f}, \mathfrak{g} : C_1 \to C_2$ are homotopic to each other if there exists $\mathfrak{h} : C_1 \to \mathfrak{C}_2$ such that $\mathfrak{f} = \mathrm{Eval}_{s=0} \circ \mathfrak{h}$, $\mathfrak{g} = \mathrm{Eval}_{s=1} \circ \mathfrak{h}$. We write $\mathfrak{f} \sim \mathfrak{g}$. Central version is defined in the same way.

(HT.3) The relation $\sim$ is independent of the choice of $\mathfrak{C}_2$ and is an equivalence relation.
(HT.4) The relation $\sim$ is compatible with composition.
(HT.5) We say $\mathfrak{f} : C_1 \to C_2$ is a homotopy equivalence if there exists $\mathfrak{g} : C_2 \to C_1$ such that $\mathfrak{f} \circ \mathfrak{g} \sim id$ and $\mathfrak{g} \circ \mathfrak{f} \sim id$.

The next theorem is due to [10]. See [9] §37 and §23.

Theorem 2.2 *If C is an L_∞ algebra then there exists a struture of L_∞ algebra on $H(C, \mathfrak{l}_1)$ which is homtopy equivalent to C. If C is central so is $H(C, \mathfrak{l}_1)$.*

We next discuss Maurer-Cartan equation. Let C be an L_∞ algebra and

$$\alpha = \sum_i \alpha_i T^{\lambda_i} e^{n_i} \in C \,\hat{\otimes}\, \Lambda_{0,nov}^+$$

be an element of degree 1. We say that it satisfies Maurer-Cartan equation if

$$\sum_k \mathfrak{l}_k([\alpha, \cdots, \alpha]) = 0. \tag{21}$$

Note the right hand side converges in an appropriate adic topology since $\alpha \in \Lambda_{0,nov}^+$.

Equation (21) is the equation (12).

We denote the set of solutions of (21) by $\tilde{\mathcal{M}}(C)$.

We can prove the following properties of it in the same way as the A_∞ case in [9] Chapter 4.

(MC.1) If $\mathfrak{f} : C_1 \to C_2$ is an L_∞ homomophism then it induces $\mathfrak{f}_* : \tilde{\mathcal{M}}(C_1) \to \tilde{\mathcal{M}}(C_2)$.
(MC.2) Let $\alpha, \alpha' \in \tilde{\mathcal{M}}(C)$. We say that they are gauge equivalent to each other if there exists $\tilde{\alpha} \in \tilde{\mathcal{M}}(\mathfrak{C})$ such that $(\mathrm{Eval}_{s=0})_*(\tilde{\alpha}) = \alpha$ and $(\mathrm{Eval}_{s=1})_*(\tilde{\alpha}) = \alpha'$.
(MC.3) $\sim$ is an equivalence relation and is independent of $\mathfrak{C}$. We put $\mathcal{M}(C) = \tilde{\mathcal{M}}(C)/\sim$.
(MC.4) An L_∞ homomophism $\mathfrak{f} : C_1 \to C_2$ induces a map $\mathfrak{f}_* : \mathcal{M}(C_1) \to \mathcal{M}(C_2)$.
(MC.5) If $\mathfrak{f} \sim \mathfrak{g}$ then $\mathfrak{f}_* = \mathfrak{g}_* : \mathcal{M}(C_1) \to \mathcal{M}(C_2)$.

We next consider equation (13). Let $(C, \mathfrak{l}, \mathbf{e})$ be a central L_∞ algebra and $\alpha \in \tilde{\mathcal{M}}(C)$. We consider an element $\mathfrak{B} \in C \,\hat{\otimes}\, \Lambda_{nov}$ such that

$$\sum_{k=0}^{\infty} \mathfrak{l}_{k+1}([\mathfrak{B}, \alpha, \cdots, \alpha]) \equiv \mathbf{e} \mod \Lambda_{0,nov}^+. \tag{22}$$

Lemma 2.3 *If $\mathfrak{f} : C \to C'$ be a central A_K homomorphism and if $\alpha \in \tilde{\mathcal{M}}(C)$, $\mathfrak{B} \in C \hat{\otimes} \Lambda_{nov}$ satisfies* (22) *then*

$$\mathfrak{f}_*(\mathfrak{B}) = \sum_{k=1}^{K} \mathfrak{l}_k([\mathfrak{B}, \alpha, \cdots, \alpha]) \in C' \hat{\otimes} \Lambda_{nov}$$

and $\mathfrak{f}_*(\alpha)$ satisfy (22) *also.*

We need also the following purely topological lemma. We decompose the free loop space as a disjoint union of components :

$$\mathcal{L}(L) = \bigcup_{[\gamma] \in \pi_1(L)/\sim} \mathcal{L}_{[\gamma]}(L).$$

Lemma 2.4 *Let L be an aspherical manifold. We assume that L is compact without boundary. Then, we have the following. ($n = \dim L$.)*

(1) *If $k \notin \{0, \cdots, n\}$, then*

$$H_k(\mathcal{L}_{[\gamma]}(L); \mathbb{Z}) = 0. \tag{23}$$

(2) *If $H_n(L_{[\gamma]}(L); \mathbb{Z}) \neq 0$ then the centralizer Z_γ of $\gamma \in [\gamma]$ is of finite index in $\pi_1(L)$.*

See [5] p270.

Assuming these results, the proof of Theorem 1.2 goes as follows.

We first remark that, in the case of L_∞ structure in string topology, our graded $\mathbb{Q}$ vector space C is a singular chain complex $S(\mathcal{L}(M))$ of the loop space. (Actually we use smooth singular chain complex of the space of piecewise smooth loops. We need to define such a notion carefully. We will discuss it in [7].) We apply cohomology notation. So the degree d chain will be regarded as a degree $-d$ cochain. We also shift the homology degree by $n-1$. (See Theorem 1.3.) Namely we regard

$$S_d(\mathcal{L}(M)) = (S(\mathcal{L}(M))[n-1])_{d+1-n}$$

and

$$(C[1])^d = C^{1+d} = (S(\mathcal{L}(M))[n-1])_{-1-d} = S_{n-2-d}(\mathcal{L}(M)). \tag{24}$$

We next use Theorems 1.3 and 2.2 to obtain an L_∞ structure on $H(\mathcal{L}(L); \mathbb{Q})$. It is central and $[L]$ (the fundamental class of the submanifolds $\cong L \subset \mathcal{L}(L)$ which is identified with the set of constant loops) is in the center. Using Theorem 1.4 and (MC.1) we obtain $\alpha \in \tilde{\mathcal{M}}(H(\mathcal{L}(L); \mathbb{Q}))$. We next use Theorem 1.5 and Lemma 2.3 to obtain $\mathfrak{B} \in H(\mathcal{L}(L); \mathbb{Q})$ such that

$$\sum_{k=0}^{\infty} \mathfrak{l}_{k+1}([\mathfrak{B}, \alpha, \cdots, \alpha]) \equiv [L] \mod \Lambda_{+,nov}. \tag{25}$$

We put :

$$\mathfrak{B} = \sum_i \mathfrak{B}_i T^{\lambda_i} e^{n_i}, \quad \alpha = \sum_i \alpha_i T^{\lambda_i} e^{m_i}.$$

$\alpha_i \neq 0$, $\mathfrak{B}_i \neq 0$.

By (24) we have

$$\deg[L] = -1.$$

(Here deg is the cohomology degree before shift, that is $d+1$ in (24).)

(25) implies that there exists i and $j(k)$ such that

$$(\deg \mathfrak{B}_i - 1) + \sum_k (\deg \alpha_{j(k)} - 1) + 1 = \deg[L] - 1 = -2. \tag{26}$$

Since L is orientable, its Maslov index is even. (Here the Maslov index is a homomorphim $\eta_L : \pi_2(\mathbb{C}^n, L) \to 2\mathbb{Z}$. See [9] Chapter 2 for example.) We can use it and the definition of $\mathfrak{B}$ to show that $\deg \mathfrak{B}_i$ is even and $\deg \alpha_{j(k)}$ is odd.

In fact using the notation of §4, we have

$$\deg \mathfrak{B}_i = n - 2 - \dim \mathcal{N}(\beta_i) + 1 = \eta_L(\beta_i) - 2$$

if $\mathfrak{B}_i = [\mathcal{N}(\beta_i)]$ and

$$\deg \alpha_i = n - 2 - \dim \mathcal{M}(\beta_i) + 1 = \eta_L(\beta_i) + 1$$

if $\alpha_i = [\mathcal{M}(\beta_i)]$. (Here $\beta_i \in \pi_2(\mathbb{C}^n, L) \cong \pi_1(L)$.)

We remark that

$$\mathfrak{B}_i \in S(\mathcal{L}_{[\partial\beta_i]}(L); \mathbb{Q}), \qquad \alpha_i \in S(\mathcal{L}_{[\partial\beta_i]}(L); \mathbb{Q}),$$

in the above situation. (See §4.)

By Lemma 2.4 and (24), the degree of nonzero element of $C \cong H(\mathcal{L}(L); \mathbb{Q})$ is in $\{-1, \cdots, n-1\}$. Therefore, there exists $\alpha_{j(k)} \neq 0$ such that

$$\deg \alpha_{j(k)} = -1. \tag{27}$$

It follows that $\eta_L(\beta_{j(k)}) = 2$. We put $[\gamma] = \partial\beta_{j(k)} \in \pi_1(L)/\sim$. In particular $[\gamma] \neq [1]$.

We have

$$0 \neq \alpha_{j(k)} \in H_n(\mathcal{L}_{[\gamma]}(L); \mathbb{Q}).$$

Proposition 2.4 (2) now implies that the centrilizer Z_γ is of finite index in $\pi_1(L)$.

Since $\eta_L(\gamma) = 2$, we have an exact sequence

$$1 \to K \to Z_\gamma \to \mathbb{Z} \to 1$$

Here $Z_\gamma \to \mathbb{Z}$ is $\frac{1}{2}\eta_L$. Therefore γ and K generate Z_γ. Moreover γ commutes all the element of K. Hence $Z_\gamma \cong K \times \mathbb{Z}$. This implies Theorem 1.2.

Let us consider the case when $L \subset \mathbb{C}^3$ is irreducible and oriented. Gromov proved $H_1(L) \neq 0$. It follows that L is sufficiently large in the sense of Waldhausen. Therefore, by Thurston's classical result L has a geometrization. We can use it to deduce Theorem 1.1 from Theorem 1.2 easily.

3 Correcponsdence parametrized by operad and String topology

The proof of Theorem 1.3 is based on the construction of Chas-Sullivan [2] which defines a structure of graded Lie algebra of loop space homology. There are various other constructions, such as [4], which works at least in the cohomology level. As is clear from the discussion of the last subsection, we need not only graded Lie algebra structure but also L_∞ structure of $H(\mathcal{L}(L);\mathbb{Q})$. So we work on the chain level. The main difficulty to work out the story of string topology in the chain level is the transversality issue. We will explain a brief outlie of it. (The detail will appear in [7].) The idea is to use correspondence parametrized by operad, which was introduced in [6], together with chain level intersection theory in singular homology, which was developed in [9] Chapter 7 §30. Note in [6] §12, we used de-Rham cohomology. It is not easy to apply in our circumstances since de Rham cohomology of loop space is not defined in the usual sense.

We do not explain the basic idea by Chas-Sullivan to construct Lie bracket (loop bracket is one we use here), and refer [2]. We restrict ourselves to explain the method to realize Chas-Sullivan's idea in the chain level.

We first review the notion of operad, which is a variant of one introduced by P. May. (See [11].)

Definition 3.1 *Pseudo-operad is a system $(\mathfrak{P}_n, \circ_i)$ where : $\mathfrak{P}_n$ is a topological space on which the symmetric group $\mathfrak{S}_n$ of order $n!$ acts freely, and*

$$\circ_i : \mathfrak{P}_m \times \mathfrak{P}_n \to \mathfrak{P}_{n+m-1}$$

is a continuous map for $i = 1, \cdots, m$. They are supposed to satisfy the following axioms.

(1) (Associativity) *For* $(\mathfrak{x},\mathfrak{y},\mathfrak{z}) \in \mathfrak{P}_m \times \mathfrak{P}_n \times \mathfrak{P}_p$ *we have*

$$(\mathfrak{x} \circ_j \mathfrak{y}) \circ_i \mathfrak{z} = \begin{cases} (\mathfrak{x} \circ_i \mathfrak{z}) \circ_{j+p-1} \mathfrak{y} & 1 \le i \le j-1, \\ \mathfrak{x} \circ_j (\mathfrak{y} \circ_{i-j+1} \mathfrak{z}) & j \le i \le j+n-1, \\ (\mathfrak{x} \circ_{i-n+1} \mathfrak{z}) \circ_j \mathfrak{y} & j+n \le i \le n+m-1. \end{cases} \tag{31}$$

(2) (Equivariance) *Let* $(\mathfrak{x},\mathfrak{y}) \in \mathfrak{P}_m \times \mathfrak{P}_n$ *and* $\rho \in \mathfrak{S}_m$, $\sigma \in \mathfrak{S}_n$. *Then we have*

$$\rho(\mathfrak{x}) \circ_i \sigma(\mathfrak{y}) = (\rho \circ_i \sigma)(\mathfrak{x} \circ_{\rho(i)} \mathfrak{y}) \tag{32}$$

where $\rho \circ_i \sigma \in \mathcal{S}_{n+m-1}$ *is defined by*

$$(\rho \circ_i \sigma)(k) = \begin{cases} \rho(k) & k < i,\ \rho(k) < \rho(i), \\ \rho(k) + m & k < i,\ \rho(k) \ge \rho(i), \\ \rho(k-m) & k \ge i+m,\ \rho(k-m) < \rho(i), \\ \rho(k-m)+m & k \ge i+m,\ \rho(k-m) \ge \rho(i), \\ \sigma(k-i+1) + \rho(i) - 1 & i \le k \le i+m-1. \end{cases}$$

There is an operad which cotroll the L_∞ structure in string topology. Such an operad was introdued by Voronov [13] and is called Cacuti opeard. We can find $(\overline{\mathfrak{P}}_n, \circ_i)$ which has the above properties by a minor modification of Cacuti opeard. We also can define the following diagram :

$$\begin{array}{ccccc} & & \mathfrak{P}_n & & \\ & & \uparrow{\scriptstyle \pi_0} & & \\ \mathcal{L}(L)^n & \xleftarrow[\pi_2]{} & \overline{\mathfrak{P}}_n(L) & \xrightarrow[\pi_1]{} & \mathcal{L}(L) \end{array} \tag{33}$$

(See [13] Theorem 2.3.)

However Cacuti opeard itself is not enough for our purpose, since we need an operad which has a fundamental chain.

An n dimensional (locally) finite simplicial complex is said to have a fundamental cycle, if each of its n-simplex is identified with a standard simplex and is given an orientation such that the sum $\sum_i \pm\Delta_i^n$ of all n simplices (with this orientation) is a cycle as a singular chain. Here $\pm$ is determined by the compatibility of the given orientation and the orientation induced by the identification with the standard simplex. A pair $(P, \partial P)$ of an n dimensional simplicial complex P and its $n-1$ dimensional subcomplex ∂P is said to have a relative fundamental cycle, if ∂P has a fundamental cycle in the above sense and if each of n-simplex P is given an orientation such that the boundary of the sum $\sum_i \pm\Delta_i^n$ of all n simplices of P is the fundamental class of ∂P as a singular chain.

Definition 3.2 *A piecewise differentiable L_∞ operad is a system $\{\mathfrak{P}_n, \circ_i\}$ ($n \ge 2$) such that $(\mathfrak{P}_n, \partial\mathfrak{P}_n)$ is a simplicial complex with relative fundamental cycle and*

$$\circ_i : \mathfrak{P}_n \times \mathfrak{P}_m \to \partial\mathfrak{P}_{n+m-1}$$

is a simplicial map, which is an isomorphism to its image. We assume that there exists a free $\mathfrak{S}_n$ action on $\mathfrak{P}_n$. We assume the following axioms.

(1) (Associativity) *Definition* 3.1 (1) *is satisfied.*
(2) (Equivalence) *Definition* 3.1 (2) *is satisfied.*
(3) (Maurer-Cartan) *We have*

$$[\partial\mathfrak{P}_n] = \sum_{m=1}^{n-2} \sum_{\sigma\in\mathfrak{S}_n/(\mathfrak{S}_m\times\mathfrak{S}_{n-m})} \sigma \cdot [\mathfrak{P}_{m+1} \circ_1 \mathfrak{P}_{n-m}]. \tag{34}$$

The terms of the right hand sides intersect each other only at their boundaries.

In [7], we will prove :

Proposition 3.3 *There exists a piecewise differentiable L_∞ operad $\{\mathfrak{P}_n, \circ_i\}$ and simplicial maps $\pi_{(n)} : \mathfrak{P}_n \to \overline{\mathfrak{P}}_n$ such that $\pi_{(n)}$ is $\mathfrak{S}_n$ equivariant and commutes with $\circ_i$.*

Now correspondence (on loop space) parametrized by piecewise differentiable L_∞ operad $\{\mathfrak{P}_n, \circ_i\}$ is a system of diagrams :

$$\begin{tikzcd} & \mathfrak{P}_n & \\ \mathcal{L}(L)^n & \mathfrak{P}_n(L) \arrow[l, "\pi_2"'] \arrow[r, "\pi_1"'] \arrow[u, "\pi_0"'] & \mathcal{L}(L) \end{tikzcd} \tag{35}$$

More precisely we need to use Moore loop space (that is the space of the pair (ℓ, l) where ℓ is a loop of L and l is a positive number, which we regard as the length of our path. Moore loop space is used to go around the trouble of nonassociativity of the product of loops which is caused by the parametrization. The space $\mathbb{R}_+$ which parametrizes l is noncompact. So our $\mathfrak{P}_n$ is noncompact. Its fundamental chain is a locally finite chain.

The axioms which (35) are supposed to satisfy is similar to [6] Definition 6.1, (which is the A_∞ case). We omit it.

We can prove that such a correspondence induces an L_∞ structure on $S(\mathcal{L}(L);\mathbb{Q})$ of appropriate singular chain complex on $\mathcal{L}(L)$. Namely the operation $\mathfrak{l}_n$ is defined roughly by

$$\mathfrak{l}_n(P_1, \cdots, P_n) = (\pi_1)_* \left(\mathfrak{P}_n(L)_{\pi_2} \times_{\mathcal{L}(L)^n} (P_1 \times \cdots \times P_k)\right).$$

Actually we need to organize carefully the way how to perturb fiber product in the right hand side to achieve transversality. This is similar to the argument discussed in detail in [9] §30.

The diagram (35) is constructed by using the definition of Cacuti operad and the map $\pi_{(n)} : \mathfrak{P}_n \to \overline{\mathfrak{P}}_n$.

This is a brief sketch of the construction of L_∞ structure. (Actually we first construct L_K strucuture. Then we use the homological algebra trick developped in [9] §30 to construct L_∞ structure. For the purpose of our application, that is to prove Theorem 1.2, we can use L_K structure (for sufficiently large K) in place of L_∞ structure.)

The proof that $[L]$ is central uses the notion, homotopy center, and an argument similar to [9] §31.

4 Pseudo-holomorphic disc and Maurer-Cartan equation

In this section we give an outlie of the proof of Theorems 1.4 and 1.5. To prove Theorem 1.4 we use moduli space of pseudo-holomorphic discs. We fix a compatible almost complex structure J on M. Let $\beta \in \pi_2(M, L)$.

Definition 4.1 *We consider a map $u : D^2 \to M$ with the following properties.*

(1) *u is J-holomorphic.*
(2) *$u(\partial D^2) \subset L$.*
(3) *The homotopy type of u is β.*

Let $\mathrm{Int}\tilde{\mathcal{M}}(\beta)$ *be the space of all such maps u.*

We consider the group G of all biholomorphic maps $v : D^2 \to D^2$ with $v(1) = 1$. The group G acts on $\tilde{\mathcal{M}}(\beta)$ by $v \cdot u = u \circ v^{-1}$. Let $\mathrm{Int}\mathcal{M}(\beta)$ *be the quotient space.*

We can compactify $\mathrm{Int}\mathcal{M}(\beta)$ *by including stable maps, see* [9] §2, *to obtain $\mathcal{M}(\beta)$.*

In [9] §29, it is proved that $\mathcal{M}(\beta)$ is a space with Kuranishi structure in the sense of [8] and hence has a virtual fundamental chain. Using the contractibility of G, we can define

$$ev : \mathcal{M}(\beta) \to \mathcal{L}(L) \quad : [u] \mapsto u|_{\partial D^2}$$

and may regard the virtual fundamental chain of $\mathcal{M}(\beta)$ as a singular chain of the loop space $\mathcal{L}(L)$. In other words, there is a way to choose representative u for each $[u]$ so that ev is well defined (strongly continuous and smooth) map. We denote by $[\mathcal{M}(\beta)] \in S(\mathcal{L}(L))$ the virtual fundamental chain of $\mathcal{M}(\beta)$ and put

$$\alpha = \sum_{\beta} T^{\beta\cap\omega} e^{\mu_L(\beta)/2} [\mathcal{M}(\beta)]. \tag{41}$$

Here ω is the symplectic form and μ_L is the Maslov class. (See [9] Chapter 2 for example.)

If we discuss naively we can "show"

$$\partial\alpha + \frac{1}{2}\mathfrak{l}_2(\alpha,\alpha) = 0. \tag{42}$$

More precisely, we have

$$\partial\mathcal{M}(\beta) = \bigcup_{\beta_1+\beta_2=\beta} \mathcal{M}(\beta_1) \times_L (S^1 \times \mathcal{M}(\beta_2))$$

as an equality of spaces with Kuranishi structure. (See [5].) Here we use

$$[u] \mapsto u(1), \quad (t,[u]) \mapsto u(t)$$

to define the fiber product in the right hand side.

We can work out the transversality issue carefully and perturb the moduli spaces $\mathcal{M}(\beta)$ (by choosing appropriate multisection) in a way compatible to the definition of our L_∞ structure, so that (12) holds, instead of (42). This is an outline of the proof of Theorem 1.4.

We can also prove that the element $\tilde{\mathcal{M}}(H(\mathcal{L}(L);\mathbb{Q}))$ is independent of various choices up to gauge equivalence.

To define $\mathfrak{B}$ (and prove Theorem 1.5), we perturb pseudo-holomorphic curve equation by using Hamiltonian function as follows.

We take a smooth function $H : [0,1] \times M \to \mathbb{R}$ and put $H_t(x) = H(t,x)$. It generates time dependent Hamiltonian vector field X_{H_t} by

$$X_{H_t} = J\,\mathrm{grad}\,H_t. \tag{43}$$

Let φ be the time one map of X_{H_t}. By the assumption of Theorem 1.5 we may choose φ (and H) so that $\varphi(L) \cap L = \emptyset$.

For each positive R, we take a smooth function $\chi_R : \mathbb{R} \to [0,1]$ such that

(1) $\chi_R(t) = 1$, if $|t| > R$,
(2) $\chi_R(t) = 0$, if $|t| < R - 1$,
(3) The C^k norm of χ_R is bounded uniformly on R.

We then consider a map $u = u(\tau, t) : \mathbb{R} \times [0,1] \to M$ with the following properties.

$$\frac{\partial u}{\partial \tau}(\tau,t) = J\left(\frac{\partial u}{\partial t}(\tau,t) - \chi_R(\tau) X_H(\varphi(\tau,t))\right), \tag{44a}$$

$$u(\tau,0), u(\tau,1) \in L, \tag{44b}$$

$$\int_{\mathbb{R}\times[0,1]} u^*\omega < \infty. \tag{44c}$$

We denote by $\mathcal{N}(R)$ the set of all such u.

We can show that $u : \mathbb{R} \times [0,1] \to M$ can be compactified to a map $(D^2, \partial D^2) \to (M, L)$. Hence it determines a class $[u] \in \pi_2(M, L)$. Let $\mathcal{N}(R, \beta)$ be the stable map compactification of the set of all $u \in \mathcal{N}(R)$ whose homotopy class is β. We put

$$\mathcal{N}(\beta) = \bigcup_{R\in[0,\infty)} (\{R\} \times \mathcal{N}(R,\beta)).$$

Using $\varphi(L) \cap L = \emptyset$, we can prove that $\mathcal{N}(R, \beta) = \emptyset$ for $R > R(\beta)$. Hence $\mathcal{N}(\beta)$ is compact. (See [5] §3.)

We define a map $ev : \mathcal{N}(\beta) \to \mathcal{L}(L)$ by $ev(u) = u|_{\partial D^2}$. The space $\mathcal{N}(\beta)$ has a Kuranishi structure and hence the virtual fundamental chain $[\mathcal{N}(\beta)] \in S(\mathcal{L}(L))$ is defined. We put

$$\mathfrak{B} = \sum_\beta T^{\beta\cap\omega} e^{\mu_L(\beta)/2} [\mathcal{N}(\beta)]. \tag{45}$$

We can prove the following equality (of spaces with Kuranishi structure).

$$\begin{aligned}\partial\mathcal{N}(\beta_0) = &\bigcup_\beta \mathcal{N}(\beta) \times_L (S^1 \times \mathcal{M}(-\beta)) \\ &\cup \bigcup_\beta (S^1 \times \mathcal{N}(\beta)) \times_L \mathcal{M}(-\beta)) \cup L,\end{aligned} \tag{46}$$

if $\beta_0 = 0 \in \pi_2(M, L)$ and

$$\begin{aligned}\partial\mathcal{N}(\beta) = &\bigcup_{\beta_1+\beta_2=\beta} \mathcal{N}(\beta_1) \times_L (S^1 \times \mathcal{M}(\beta_2)) \\ &\cup \bigcup_{\beta_1+\beta_2=\beta} (S^1 \times \mathcal{N}(\beta_1)) \times_L \mathcal{M}(\beta_2),\end{aligned} \tag{47}$$

if $\beta \cap \omega \le 0$ and $\beta \ne \beta_0$. (See [5] p 269.)

Naively speaking (namely modulo transversality), Formulas (46), (47) imply

$$\partial\mathfrak{B} + \mathfrak{l}_2([\mathfrak{B}, \alpha]) \equiv [L] \mod \Lambda^+_{0,nov}.$$

We can then prove Theorem 1.5 again by carefully choosing the perturbation.

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