Loop space
and
holomorphic disc

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Many parts are joint work with
Oh-Ohta-Ono
\( L \): closed oriented manifold. 

We assume that it is spin.

\( \mathcal{L}(L) \): Loop space of \( L \)

\( S^1 \) acts on \( \mathcal{L}(L) \):

\[
(t \cdot \ell)(s) = \ell(t + s)
\]

**Theorem 1** (Chas-Sullivan, ......., F)

\( H_1(\mathcal{L}(L)) \) has a structure of \( L \) infinity algebra.

L infinity algebra = A homotopy version of Lie algebra
$M$ : closed symplectic manifold

$L$ : Lagrangian submanifold of $M$

$$\Lambda = \left\{ \sum a_i q^{\lambda_i} \mid \lambda_i \in \mathbb{R}, \quad \lambda_i \to +\infty \right\}$$

$$\Lambda_+ = \left\{ \sum a_i q^{\lambda_i} \mid \lambda_i \in \mathbb{R}_+, \quad \lambda_i \to +\infty \right\}$$

**Theorem 2**

*There exists $b$ in $H(\mathcal{L}(L); \Lambda_+)$ such that*

$$\sum_{k=1}^{\infty} \mathsf{I}_k(b \cdots b) = 0$$

An analogue of Maurer-Cartan equation

$$db + \frac{1}{2} \{b, b\} = 0$$
Theorem 3
If $F : M \to M$ be a Hamiltonian diffeomo. with

$$F(L) \cap L = \emptyset$$

then, there exists $B$ in $H(\mathcal{L}(L); \Lambda[q^{-1}])$ with

$$\sum_{k=0}^{\infty} l_{k+1}(B, b \ldots, b) \equiv [L] \mod \Lambda_+$$

$[L] = \text{the homology class of all constant loops}$
L infinity algebra

$C$ : graded vector space


$$E_k C[1] = \frac{C[1] \otimes \cdots \otimes C[1]}{S_k}$$

$S_k :$ symmetric group of order $k!$ acts by

$$\sigma(x_1 \otimes \cdots \otimes x_k) = \pm x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}$$
\[ EC[1] = \bigoplus E_k C[1] \] is a coalgebra (cocommutative and coassociative)

**Definition**

L infinity structure on \( C \) is a coderivation

\[ d : EC[1] \rightarrow EC[1] \]

such that

\[ d \circ d = 0. \]
It is equivalent to give series of operations

\[ 1_k : E_k C[1] \rightarrow C[1] \]

with

\[
\sum_{\sigma \in S_n} \sum_{k + \ell = n + 1} \pm \frac{n!}{k! \ell!} 1_{\ell} \left( 1_k (x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}) \otimes x_{\sigma(k+1)} \otimes \cdots \otimes x_{\sigma(n)} \right) = 0
\]

L infinity relation
Example of $L$ infinity relation

- $l_1 \circ l_1 = 0$  
  We have a homology group. $H(C;l_1)$

- If $l_k = 0$ for $k \neq 2$
  
  L infinity relation becomes
  
  $\pm l_2(l_2(x \otimes y) \otimes z) \pm l_2(l_2(y \otimes z) \otimes x) \pm l_2(l_2(z \otimes x) \otimes y) = 0$

  graded Jacobi
**L infinity structure on** $H(\mathcal{L}(L))$

Intersection theory of chains on loop space

$F : P \rightarrow \mathcal{L}(L)$ \hspace{1cm} $G : Q \rightarrow \mathcal{L}(L)$

$P = [0, 1]$ \hspace{1cm} $Q = [0, 1]$

$f\{P, Q\}$ \hspace{1cm} $\{P, Q\} = \text{one point}$
**Definition of** \( b \in H(\mathcal{L}(L)) \)

\( \beta \in \pi_2(M,L) \)

\[ P(\beta) = \left\{ u : (D^2, \partial D^2) \to (M,L) \mid u \text{ is holomorphic, } [u] = \beta \right\} \]

\[ f : P(\beta) \to \mathcal{L}(L) \]

\[ f(u) = u|_{\partial D^2} : S^1 \to L \]

\[ b = \sum_{\beta} (P(\beta), f) q^{\beta \cap \omega} \]

\( \omega \) : symplectic form,
\( b \) satisfies **Maurer-Cartan**

\[
\partial b + \mathbf{l}_2(b, b) = 0
\]

\( u \in u \in \partial P(\beta) \)

\( f(u) \in \{P(\beta_1), P(\beta_2)\} \)

\( u_i \in P(\beta) \quad \lim_{i \to \infty} u_i = u \in \partial P(\beta) \)
Maurer-Cartan equation

\[ x \mapsto d_b(x) = dx + \{b, x\} \quad \text{satisfies} \quad d_b \circ d_b = 0 \]

Open string \quad \leftrightarrow \quad Gauge theory

Loop space

Witten (1992), Cattaneo Frohlich, Jurg, and F.
Applications:

Theorem 4
Let \( L \) be a Lagrangian submanifold of \( \mathbb{C}^n \) such that
\[
\pi_k(L) = 1 \quad k \neq 1.
\]
Then there exists \( \Gamma \subseteq \pi_1(L) \) a finite index subgroup such that
\[
\Gamma \cong \mathbb{Z} \times G
\]
Corollary 5

Let $L$ be a Lagrangian submanifold of $C^3$ assume $L$ is irreducible,
then

$$L \cong \Sigma \times S^1$$

(diffeomorphic).

Corollary 6 (Audin’s conjecture)

Let $T^n$ be a Lagrangian submanifold of $C^n$.
Then the Maslov index homomorphism

$$\pi_1(L) \to 2\mathbb{Z}$$

is surjective.

Independently claimed by Cielibak and Eliashberg.
Idea of the proof

\[ \pi_k(L) = 1 \quad k \neq 1 \]

Contradiction by comparing degree

\[ H(\mathcal{L}(L)) \text{ is small} \]

Need nontrivial element in the left hand side.

The right hand side is non zero.

\[ \sum_{k=0}^{\infty} 1_{k+1}(B, b \ldots, b) \equiv [L] \mod \Lambda_+ \quad \text{(Theorem 3)} \]
Theorem 3

If $F : M \to M$ be a Hamiltonian diffeomorphism, with

$$F(L) \cap L = \emptyset$$

then, there exists $B$ in $H(\mathcal{L}(L); \Lambda[q^{-1}])$ with

$$\sum_{k=0}^{\infty} \lambda_{k+1}(B, b \cdots b) \equiv [L] \mod \Lambda_+$$

$[L] = $ the homology class of all constant loops
Theorem 7
Let \( L \) be a Lagrangian submanifold of \( M \). Assume
- \( \dim L \) is even.
- \( L \) admits a metric of negative sectional curvature.

Let \( F : M \rightarrow M \) Hamiltonian diffeo. with
- \( F(L) \cap L \).

Then
\[
\#(F(L) \cap L) \geq \sum_{k} \text{rank } H_{k}(L)
\]

There is an earlier related work by Viterbo
Theorem 8 (Based on the work by Cho-Oh,Cho)

Let $M$ be a toric Fano manifold with moment map $\text{pr} : M \to B$. Put $T(v) = \text{pr}^{-1}(v)$.

Then there exists $v \in B$ such that for any Hamiltonian diffeomorphism $F : M \to M$,

$$F(T(v)) \cap T(v) \neq \emptyset$$

There is also an earlier related work by Entov-Polterovich
Application to the structure of super potential

\[ b = \sum_{\beta} (P(\beta), f)q^{\beta \cap \omega} = \sum_{\beta} b(\beta)q^{\beta \cap \omega} \]

\[ b(\beta) \in H(\mathcal{L}(L)) \]

\[ b(\beta) \text{ can be viewed as a (super) function on cohomology group of } L \]

Super potential of \( L \)
Iterated integral of Chen

\[ b(\beta) = (P(\beta), f), \quad f : P(\beta) \to L(L) \]

Put: \[ f(x) = \ell_x : S^1 \to L \quad (x \in P(\beta)) \]

\[ ev_k : P(\beta) \times \left( S^1 \right)^k \to L^k \]

\[ ev_k(x; t_1, \cdots, t_k) = (\ell_x(t_1), \cdots, \ell_x(t_k)) \]

\[ \nu_i : \text{Harmonic forms on } L \]
\[ \Psi_{\beta,k}(v_1, \ldots, v_k) = \int_{P(\beta) \times (S^1)^k} ev_k^*(v_1 \wedge \cdots \wedge v_k) \]

\[ \Psi = \sum_{\beta,k} \frac{1}{k!} q^{\omega \cap \beta} \Psi_{\beta,k} \]

super potential: \( H(L; \mathcal{Q}) \to \Lambda \)
Theorem 9

There exist Laurent polynomials \( \psi_i(y_1, \cdots, y_a, z_1, \cdots, z_c) \) such that

\[
\Psi = \sum_i q^{\lambda_i} \psi_i
\]

\( b \in H(\mathcal{L}(L)) \) is well defined up to gauge equivalence.

\[\downarrow\]

\( \Psi \) is well defined up to change of variables.
Example (Hori-Vafa, Cho-Oh)

$\text{moment map } = \Pr$

$M = S^2$

$B = [0,1]$

$\Pr^{-1}(\nu) = S^1(\nu)$

$\Psi_{S^1(\nu)}(x) = e^x q^\nu + e^{-x} q^{1-\nu}$ for $x \in H^1(S^1(\nu))$

area = $\nu$
\[ M = CP^2 \quad \xrightarrow{Pr = \text{moment map}} \quad \text{Triangle in } R^2 \]

\[ T^2(v) = Pr^{-1}(v) \]

\[ \Psi_{T^2(v)}(x_1, x_2) = e^{x_1}q^{v_1} + e^{x_2}q^{v_2} + e^{-x_1-x_2}q^{1-v_1-v_2} \]
3 holomorphic discs (of Maslov index 2)

\[ \Psi_{T^2(v)}(x_1, x_2) = e^{x_1} q^{v_1} + e^{x_2} q^{v_2} + e^{-x_1-x_2} q^{1-v_1-v_2} \]
Theorem 8

Let $M$ be a toric Fano manifold with moment map $pr : M \to B$. Put $T(v) = pr^{-1}(v)$. Then there exists $v \in B$ such that for any Hamiltonian diffeomorphism $F : M \to M$,

$$F(T(v)) \cap T(v) \neq \emptyset$$
Idea of the proof of Theorem 8

Floer homology $HF((T(v), x),(T(v), x))) \neq 0$ (Floer, Oh, Oh-Ohta-Ono-F)

Super potential $\Psi_{T(v)}(x)$ has a critical point at $x$.

Here $\Psi_{T(v)}(x)$ is regarded as $x \in H^1(T(v)) \otimes \Lambda$ a function of $= \mathcal{M}(L)$
Lagrangian Floer theory (FOOO)

\( (M, L) \quad \rightarrow \quad \text{Set } \mathcal{M}(L) \)

\( x_i \in \mathcal{M}(L_i) \quad \rightarrow \quad HF((L_1, x_1), (L_2, x_2)) \)

\( F : M \rightarrow M \quad \rightarrow \quad F^* : \mathcal{M}(L) \rightarrow \mathcal{M}(F(L)) \)

\( F^* : \mathcal{M}(L) \rightarrow \mathcal{M}(F(L)) \)

\( HF((L, x), (L, x)) \quad = \quad HF((L, x), (F(L), F^*(x))) \)

\( L_1 \cap L_2 \quad \rightarrow \quad \#(L_1 \cap L_2) \quad \geq \quad \text{rank } HF((L_1, x_1), (L_2, x_2)) \)
Example (Cho-Oh)

\[ M = CP^2 \]

\[ \Psi_{T^2(v)}(x_1, x_2) = e^{x_1} q^{v_1} + e^{x_2} q^{v_2} + e^{-x_1-x_2} q^{1-v_1-v_2} \]

\[ v_1 = v_2 = \frac{1}{3} \]

\[ \Psi_{T^2(v)} = \left( e^{x_1} + e^{x_2} + e^{-x_1-x_2} \right) q^{\frac{1}{3}} = \left( y_1 + y_2 + y_1^{-1} y_2^{-1} \right) q^{\frac{1}{3}} \]

\[ \frac{\partial \Psi_{T^2(v)}}{\partial y_1} = \frac{\partial \Psi_{T^2(v)}}{\partial y_2} = 0 \quad \Rightarrow \quad y_1 = y_2, \quad y_1^3 = 1 \]
Example

\[ \nu = (0,0) \]

\[ \Psi_{T(\nu)}(x_1, x_2) = (e^{x_1} + e^{x_2} + e^{-x_1} + e^{-x_2})q + e^{-x_1-x_2}q^{1+\alpha} \]

Critical point

\[ x_1 = x_2 = \log \left( 1 + \frac{q^\alpha}{2} - \frac{3q^{2\alpha}}{8} + \cdots \right) \]
Related work and Generalization

- $M$ Calabi-Yau and $L$ zero Maslov (eg. special)
  - The same structure theorem as Theorem 9 but the sum
  $\Psi = \sum q^{\lambda_i} \psi_i$ becomes infinite sum.

(Pseudo)-holomorphic map from bordered Riemann surface with higher genus.

Involutive bi-Lie-infinity structure in place of L infinity structure.
  (algebraic part: Cielibak-F- Latshov)

- Relation to Pertubative Chern-Simons.
  Some relation is visible but not yet clear.
Example

\[\Psi_{T(v)}(x_1, x_2) = \left( e^{x_1} + e^{-x_2} + e^{-x_1-x_2} \right) q + e^{x_2} q^{1+\alpha}\]