Chapter 8. Floer theory of Lagrangian submanifolds over $\mathbb{Z}$

§34. Statement of the results in Chapter 8.

So far we have been studying Floer cohomology with rational coefficients. If we put some additional assumptions on our symplectic manifold, Floer cohomology can be defined over $\mathbb{Z}$ coefficients or $\mathbb{Z}_2$ coefficients. In this chapter, we discuss this point and its applications. We remark that in Chapters 3-5, we developed the homological algebra also with the coefficient ring of $\mathbb{Z}$ or $\mathbb{Z}_2$. So the necessary algebraic part of the story has already been generalized. We first recall the following :

Definition 34.1. Let J be an almost complex structure on M. We call J *spherically positive* if every J-holomorphic sphere $v : S^2 \to M$ with $c_1(M)[v] \le 0$ is constant. We denote by $\mathcal{J}^{c_1>0}_{(M,\omega)}$ or $\mathcal{J}^{c_1>0}_{\omega}$ the set of spherically positive almost complex structures which are compatible with ω.

We call a symplectic manifold (M,ω) *spherically positive* if there exists a compatible spherically positive almost complex structure J.

Remark 34.2.

(1) We recall that a symplectic manifold (M,ω) is (positively spherically) *monotone* if there exists $c > 0$ such that $[\omega] \cap \alpha = c c_1(M) \cap \alpha$ for any $\alpha \in \pi_2(M)$. It is easy to see that any monotone symplectic manifolds are spherically positive. For example $\mathbb{C}P^n$, $\mathbb{C}^n$, T^{2n} are spherically positive with respect to the standard complex structure. Fano manifold M is spherically positive. Hence for any Kähler form ω of it, (M,ω) is spherically positive. Any 4-dimensional symplectic manifold is also spherically positive with respect to Fredholm regular almost complex structures.

Furthermore any product of spherically positive symplectic manifold is spherically positve.

(2) The set $\mathcal{J}^{c_1>0}_{\omega}$ may be neither path connected nor dense in $\mathcal{J}_{\omega}$, except when $\dim M = 4$ or M is monotone.

(3) If $\psi : (M,\omega) \to (M',\omega')$ is a symplectic diffeomorphism, then it induces a bijection $\psi_* : \mathcal{J}^{c_1>0}_{(M,\omega)} \to \mathcal{J}^{c_1>0}_{(M',\omega')}$, $\psi_*(J) = J^{\psi}$ in an obvious way.

We recall that our construction of A_∞ structure uses the moduli space of J-holomorphic maps. We have proved independence of the isomorphism class (upto homotopy) of the A_∞ structure using the fact that we have not put any additional restriction on the almost complex structure in our Fredholm theory before this chapter. However our construction of fundamental chains over $\mathbb{Z}$ used in this chapter relies on the spherical positivity. As a result, the isomorphism classes of various A_∞ structures will depend on the choice of J or more precisely on the connected component of $\mathcal{J}^{c_1>0}_{\omega}$. (See (8), (9), (10) of Theorem 34.3.) Because of this, we will put J-dependence of various A_∞ objects explicit in our notations in this chapter.

For $R = \mathbb{Q}, \mathbb{Z}$ or $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, we put

$$\Lambda^R_{0,nov} = \left\{ \sum a_i T^{\lambda_i} e^{n_i} \;\middle|\; \lambda_i \in \mathbb{R}_{\ge 0},\, n_i \in \mathbb{Z},\, a_i \in R,\, \lim_{i\to\infty} \lambda_i = \infty \right\}.$$

We define Λ^R_{nov}, $\Lambda^{+R}_{0,nov}$ in the same way. (In Chapters 1,3,4,5,6,7, it is written as $\Lambda_{0,nov}(R)$ in place of $\Lambda^R_{0,nov}$.)

The first main result of this chapter is :

Theorem 34.3. *Assume (M,ω) is spherically positive and let $J \in \mathcal{J}^{c_1>0}_\omega$ be given. Then for any relatively spin Lagrangian submanifold $L \subset (M,\omega)$ we have the following :*

(1) *The filtered A_∞ algebra $(C(L,J),\mathfrak{m}^J_*)$ in Theorem* 10.11 *is defined over $\Lambda^{\mathbb{Z}}_{0,nov}$. We write it as $(C(L,J;\Lambda^{\mathbb{Z}}_{0,nov}),\mathfrak{m}^J_*)$.*

(2) *If we take the $\mathbb{Z}$-reduction, see* (7.13), *the cohomology ring of $(\overline{C}(L,J;\mathbb{Z}),\overline{\mathfrak{m}}^J_*)$ is isomorphic to the cohomology ring $H(L;\mathbb{Z})$ of L.*

(3) *$(C(L,J;\Lambda^{\mathbb{Z}}_{0,nov}),\mathfrak{m}^J_*)$ has a homotopy unit.*

(4) *Let $(L^{(1)},L^{(0)})$ be a relatively spin pair of clean intersection and $\{J_t\}_{t\in[0,1]}$ be a path in $\mathcal{J}^{c_1>0}_\omega$. Then, the A_∞ bimodule $C(L^{(1)},L^{(0)};\{J_t\}_t;\Lambda^{\mathbb{Z}}_{0,nov})$ is defined as a homotopy-unital filtered A_∞ bimodule over $C(L^{(1)},J_0;\Lambda^{\mathbb{Z}}_{0,nov})$ - $C(L^{(0)},J_1;\Lambda^{\mathbb{Z}}_{0,nov})$.*

(5) *Theorems* 11.18 *and* 11.43 (*sequence of obstruction classes*) *over $\Lambda^{\mathbb{Z}}_{0,nov}$ hold.*

(6) *For any field R,* (24.6.1) *and* (24.6.2) *in Theorem* 24.5 (*the spectral sequence calculating Floer cohomology*) *remain to be the case after replacing the coefficient ring $\Lambda^{\mathbb{Q}}_{0,nov}$ by $\Lambda^R_{0,nov}$. Theorem* 24.10 *also holds.*

(7) *If L is rational and rationally unobstructed with $\mathbb{Z}$-coefficients, then the spectral sequence converges with the coefficient ring $\Lambda^{\mathbb{Z}}_{0,nov}$. More specifically, the construction in* §25*, especially* (25.10)*, remains to hold.*

We next explain the $\mathbb{Z}$ version of Theorems 14.1 and 22.1,22.4. We prepare some notations.

Situation 34.4. (1) We first consider the situation of Theorem 14.1. Let $(M.\omega)$, (M',ω') be symplectic manifolds, L, L' be Lagrangian submanifolds of M, M', respectively. We consider a symplectic diffeomorphism $\psi : (M,\omega) \to (M',\omega')$ with $L' = \psi(L)$. Let $J \in \mathcal{J}^{c_1>0}_{(M,\omega)}$, $J' \in \mathcal{J}^{c_1>0}_{(M',\omega')}$. We assume that there exists a path $\{J_\tau\}_{\tau\in[0,1]}$ joining J^ψ to J' in $\mathcal{J}^{c_1>0}_{(M',\omega')}$.

(2) We next consider the situation of Theorem 22.4. Let $(M.\omega)$, (M',ω') be symplectic manifolds. Let $L^{(0)}, L^{(1)}$ be Lagrangian submanifolds of M and $L^{(0)\prime}, L^{(1)\prime}$ Lagrangian submanifolds of M'. We consider a symplectic diffeomorphism $\psi : (M,\omega) \to (M',\omega')$ with $L^{(i)\prime} = \psi(L^{(i)})$ $(i = 0,1)$. Let $\{J_t\}_{t\in[0,1]}$ and $\{J'_t\}_{t\in[0,1]}$ be paths in $\mathcal{J}^{c_1>0}_{(M,\omega)}$ and $\mathcal{J}^{c_1>0}_{(M',\omega')}$ respectively. Let $\{J_{\tau,t=i}\}_{\tau\in[0,1]}$ be paths in $\mathcal{J}^{c_1>0}_{(M',\omega')}$ joining J^ψ_i to J'_i $(i = 0,1)$. We assume that there is a family $\{J_{\tau,t}\}_{(\tau,t)\in[0,1]^2}$ of elements of $\mathcal{J}^{c_1>0}_{(M',\omega')}$ such that $J_{\tau,i} = J_{\tau,t=i}$, $J_{0,t} = J^\psi_t$, $J_{1,t} = J'_t$.

(3) We finally consider the situation of Theorem 22.14. Let $L^{(0)}, L^{(1)}$ be Lagrangian submanifolds of (M,ω). Let $\{\psi_\rho^{(i)}\}_\rho$ be Hamiltonian isotopies such that $\psi_0^{(i)} = id$. We put $L^{(i)\prime} = \psi_1^{(i)}(L^{(i)})$. $(i = 0, 1.)$ Let $\{J_t\}_{t\in[0,1]}$ be a path in $\mathcal{J}_\omega^{c_1>0}$. We define

$$J_{\tau,t=i} = J_t^{\psi_\tau^{(i)}} \in \mathcal{J}_\omega^{c_1>0}.$$

We also put

$$J_{\tau,t} = J_t^{\psi_{t\tau}^{(1)}\circ\psi_{(1-t)\tau}^{(0)}} \in \mathcal{J}_\omega^{c_1>0}, \quad J'_t = J_{1,t}.$$

Theorem 34.3 (continued)**.** (8) *Theorem* 14.1 *is generalized to this case in the following sense : We consider Situation* 34.4 (1)*. Then we have a homotopy-unital homotopy equivalence*

$$(\psi, \{J_\tau\}_\tau)_* : (C(L, J; \Lambda_{0,nov}^{\mathbb{Z}}), \mathfrak{m}_*^J) \to (C(L', J'; \Lambda_{0,nov}^{\mathbb{Z}}), \mathfrak{m}_*^{J'}),$$

(*See* §15.3)*. Theorem* 14.2 *is also generalized to this case in the similar sense.*

The homotopy class of $(\psi, \{J_\tau\}_\tau)_*$ *is depends only on the homotopy class of* ψ, $\{J_\tau\}_\tau$.

(9) *Theorem* 22.1 (*invariance of the filtered* A_∞ *bimodule under the action of symplectic diffeomorphisms*) *is generalized in the following sense : We consier Situation* 34.2 (2)*. Then we have a homotopy equivalence of filtered* A_∞ *bimodules*

$$(\psi, \{J_{\tau,t}\}_{\tau,t})_* : C(L^{(1)\prime}, L^{(0)\prime}; \{J_t\}_t; \Lambda_{0,nov}^{\mathbb{Z}}) \to C(L^{(1)\prime}, L^{(0)\prime}; \{J'_t\}_t; \Lambda_{0,nov}^{\mathbb{Z}})$$

over

$$(\psi, \{J_{t=1,\tau}\}_\tau)_* - (\psi, \{J_{t=0,\tau}\}_\tau)_*,$$

where

$$(\psi, \{J_{t=i,\tau}\}_\tau)_* : (C(L^{(i)}, J_i; \Lambda_{0,nov}^{\mathbb{Z}}), \mathfrak{m}_*^{J_i}) \to (C(L^{(i)\prime}, J'_i; \Lambda_{0,nov}^{\mathbb{Z}}), \mathfrak{m}_*^{J'_i})$$

is a homotopy equivalence in (8)*. The homotopy class of* $(\psi, \{J_{\tau,t}\}_{\tau,t})_*$ *depends only on the homotopy class of* $\psi, \{J_{\tau,t}\}_{\tau,t}$*. Consequently Theorem* 14.3 *is generalized to this case in the similar sense.*

(10) *Theorem* 22.4 (*invariance of the filtered* A_∞ *bimodule under the Hamiltonian isotopy*) *is generalized in the following sense. We consider Situation* 34.2 (3)*. Then there exists a* $\mathfrak{C}$*-weakly filtered homotopy equivalence of weakly filtered* A_∞ *bimodules*

$$(\psi_\rho^0, \psi_\rho^1)_* : C(L^{(1)\prime}, L^{(0)\prime}; \{J_t\}_t; \Lambda_{nov}^{\mathbb{Z}}) \to C(L^{(1)\prime}, L^{(0)\prime}; \{J'_t\}_t; \Lambda_{nov}^{\mathbb{Z}})$$

over

$$(\psi_1^1, \{J_{t=1,\tau}\}_\tau)_* - (\psi_1^0, \{J_{t=0,\tau}\}_\tau)_*.$$

(Here $\mathfrak{C}$ *is as in Theorem* 22.14*.) Homotopy class of* $(\psi_\rho^0, \psi_\rho^1)_*$ *depends only on homotopy class of* $(\psi_\rho^0, \psi_\rho^1)$*. Consequently,* 14.4 (*invariance of Floer cohomology*) *is generalized to this case in the similar sense.*

Remark 34.5. (1) We remark that in the situation of Theorem 22.1 (that is Situation 34.4 (2)) existence of two parametre family $\{J_{\tau,t}\}_{(\tau,t)\in[0,1]^2}$ of almost complex structures is assumed. On the other hand in the situation of Theorem 22.14 (that is Situation 34.4 (3)) existence of such a family is automatic.

(2) If M is monotone or $\dim_{\mathbb{R}} M = 4$ then $\mathcal{J}_{\omega}^{c_1>0}$ is connected. Hence the filtered A_∞ algebra and filtered A_∞ bimodule in Theorem 24.3 is well-defined up to homotopy equivalence in those cases. (Note $\mathcal{J}_{\omega}^{c_1>0}$ may not be simply connected in case $\dim_{\mathbb{R}} M = 4$. Hence homotopy type of homotopy equivalence in Theorem 34.3 may depend on the choice of paths of almost complex structure.)

(3) As we mentioned in the introduction, it is likely that spherical positivity can be removed from Theorem 34.3. Since the argument to dispose spherical positivity is more complicated, we postpone further study elsewhere.

Note that for the case $R = \mathbb{Z}$, we do not have the canonical models (see §23) of filtered A_∞ algebra and filtered A_∞ bimodule. Thus we can not state the $\mathbb{Z}$-coefficients versions of Theorems A and F in terms of *cohomology* as in Chapter 1. However as we already discussed in §27.1 (see Lemma 27.3) we can construct a structure of filtered A_∞ algebra on the subcomplex of $C(L;\Lambda_{0,nov})$ if the inculusion is chain homotopy equivalence with respect to the classical boundary operator $\mathfrak{m}_{1,\beta_0}$. More precisely we have the following. Let $\overline{C}_0(L)$ is a subcomplex of $\overline{C}(L;\mathbb{Z})$ which induces an isomorphism on cohomology. We assume $\overline{C}(L;\mathbb{Z})/\overline{C}_0(L)$ is free as $\mathbb{Z}$ module. We put $C_0(L;\Lambda_{0,nov}^{\mathbb{Z}}) = \overline{C}_0(L)\otimes_{\mathbb{Z}}\Lambda_{0,nov}^{\mathbb{Z}}$.

Theorem As. *Assume (M,ω) is spherically positive and $J\in\mathcal{J}_{\omega}^{c_1>0}$. Let $L\subset (M,\omega)$ be a relatively spin Lagrangian submanifold. Then we can construct the filtered A_∞ algebra $(C_0(L,J;\Lambda_{0,nov}^{\mathbb{Z}}),\mathfrak{m}_*^J)$ on $\Lambda_{0,nov}^{\mathbb{Z}}$.*

Let us assume Situation 34.4 (1). *We assume ψ induces an isomorphism $\psi_* = \Pi\circ\psi_*\circ i : \overline{C}_0(L)\to\overline{C}_0(L')$. (Here $i : \overline{C}_0(L)\to C(L)$ is the inclusion and $\Pi : C(L')\to C_0(L')$ is the projection to the direct summand.) Then we have an isomorphism*

$$(\psi,\{J_\tau\}_\tau)_* : C_0(L,J;\Lambda_{0,nov}^{\mathbb{Z}})\to C_0(L',J';\Lambda_{0,nov}^{\mathbb{Z}})$$

of filtered A_∞ algebras. Its homotopy class depends only on the isotopy class of symplectic diffeomorphism $\psi : (M,L)\to(M',L')$ and of $\{J_\tau\}_\tau$.

The Poincaré dual $PD([L])\in C_0^0(L;\Lambda_{0,nov}^{\mathbb{Z}})$ of the fundamental cycle $[L]$ is the homotopy unit of our filtered A_∞ algebra. The homomorphism $(\psi,\{J_t\}_t)_*$ is homotopy-unital.

We next consider the situation of Theorem F. Let $L^{(1)}$ is of clean intersection to $L^{(0)}$. We consider a subcomplex $\overline{C}_0(L^{(1)}\cap L^{(0)};\mathbb{Z})$ of $\overline{C}(L^{(1)}\cap L^{(0)};\mathbb{Z})$. (Note the right hand side is an appropriate countably generated complex of the singular chain complex of $L^{(1)}\cap L^{(0)}$.) We use it to define $\overline{C}_0(L^{(1)},L^{(0)};\Lambda_{0,nov}^{\mathbb{Z}})$ in the same way as above.

Theorem Fs. *Assume (M,ω) is spherically positive and $\{J_t\}_{t\in[0,1]}$ is a path in $\mathcal{J}_\omega^{c_1>0}$. Let $(L^{(1)},L^{(0)})$ be a relatively spin pair of Lagrangian submanifolds of (M,ω). Assume that $L^{(0)}$ and $L^{(1)}$ intersect cleanly. Then $C_0(L^{(1)},L^{(0)};\{J_t\}_t;\Lambda_{0,nov}^{\mathbb{Z}})$ is a a homotopy-unital filtered A_∞ bimodule over $C_0(L^{(1)};J_1;\Lambda_{0,nov}^{\mathbb{Z}})$ - $C_0(L^{(0)};J_0;\Lambda_{0,nov}^{\mathbb{Z}})$.*

Remark 34.6. There are several results in the previous chapters whose validity for the $\mathbb{Z}$ coefficient in the spherically positive case is still not clear to us. We make some comments on them here. Among them, (1), (2) and (4) also apply to Theorem 34.7 below :

(1) Both the statement and the proof of Theorem 33.1 in §33 are based on the de Rham theory and so are valid for the coefficient ring $\mathbb{R}$. However it is clear from the proof given below that the induced A_∞ algebra $(\overline{C}(L;J;\Lambda_{0,nov}^{\mathbb{Z}}),\overline{\mathfrak{m}}_*^J)$ obtained by reducing the coefficient ring to $\mathbb{Z}$ is homotopy equivalent to the A_∞ algebra constructed in Theorem 9.8. In this sense, Theorem 33.1 (comparison of our filtered A_∞ algebra to it classical part) is also generalized over $\mathbb{Z}$.
(2) In a discussion of §13, we used the fact that our Novikov ring contains $\mathbb{Q}$. Therefore constructions of the operators $\mathfrak{q}$, $\mathfrak{p}$ do not apply for the $\Lambda_{0,nov}^{\mathbb{Z}}$ coefficient. Because of this, the proof of Theorem 13.41 is not generalized over the $\Lambda_{0,nov}^{\mathbb{Z}}$ coefficients, and hence not Theorem 13.41 itself. Similar remarks apply to (24.6.3) of Theorem 24.5.
(3) When L is irrational, it is not clear to us whether the convergence result of the spectral sequence in Theorem 24.5 holds over $\Lambda_{0,nov}^{\mathbb{Z}}$. (See the paragraph right after Theorem 24.10.)
(4) If the cohomology group $H(L;\mathbb{Z})$ is a torsion free $\mathbb{Z}$-module, then we may choose $C_0(L;\mathbb{Z}) = H(L;\mathbb{Z})$. Therefore, there is a structure of unital filtered A_∞ algebra on $H(L;\Lambda_{0,nov}^{\mathbb{Z}})$, which is well-defined up to isomorphism. In general, for any CW or simplicial decomposition of L we can define a structure of filtered A_∞ algebra on the cochain complex over $\Lambda_{0,nov}^{\mathbb{Z}}$ which is associated to the cellular or simplicial decomposition.

In the spherically positive case, we can also work over $\mathbb{Z}_2$ which enables us to generalize various results of Chapter 6 for L that is neither orientable nor relatively spin. Because the Maslov index can be odd for unorientable L, we need to enlarge our Novikov ring to $\Lambda_{0,nov}^{\mathbb{Z}_2}[e^{1/2}]$ so that it includes the square root $e^{1/2}$. Recall that $\deg e = 2$ in our definition.

Theorem 34.7. *Assume (M,ω) is spherically positive and $J \in \mathcal{J}_\omega^{c_1>0}$. Let $L \subset (M,\omega)$ be a Lagrangian submanifold, not necessarily relatively spin. Then, we can define a unital filtered A_∞ algebra $(C(L;J;\Lambda_{0,nov}^{\mathbb{Z}_2}[e^{1/2}]),\mathfrak{m}_*)$ as in Theorem* 10.11. *If $L^{(1)},L^{(0)}$ are Lagrangian submanifolds and $\{J_t\}_{t\in[0,1]}$ is a path of almost complex structures in $\mathcal{J}_\omega^{c_1>0}$, then the A_∞ bimodule $C(L^{(1)},L^{(0)};\{J_t\}_{t\in[0,1]},\Lambda_{0,nov}^{\mathbb{Z}_2}[e^{1/2}])$ over $(C(L^{(1)};J_t;\Lambda_{0,nov}^{\mathbb{Z}_2}[e^{1/2}]),\mathfrak{m}_*)$- $(C(L^{(0)};J_t;\Lambda_{0,nov}^{\mathbb{Z}_2}[e^{1/2}]),\mathfrak{m}_*)$ is defined.*

Theorems 14.1*,* 14.2*,* 14.3*,* 14.4*,* 22.1*,* 22.11 *and Theorem* 24.5*,* (24.6.1)*,* (24.6.2)*, and Theorem* 24.10 *are generalized to this case in the sense similar to that of Theorem* 34.3*.*

Theorems Bs, Cs, Ds, Es, Gs in the introduction directly follow from Theorems 34.3 and 34.7.

For the case $R = \mathbb{Z}_2$, we have the canonical model, since $\mathbb{Z}_2$ is a field. So Theorems A and F can be generalized directly to the situation of Theorem 34.3 over $\Lambda_{0,nov}^{\mathbb{Z}_2}[e^{1/2}]$.

Using Theorem 34.7 we have the following result which is similar to Corollary 24.20. This is nothing but Theorem L (1) in Chapter 1.

Corollary 34.8. *Assume* (M,ω) *is spherically positive and* $H^2(L;\mathbb{Z}_2) = 0$*. Suppose that* $L \subset M$ *is displaceable, i.e., there is a Hamiltonian diffeomorphism* ϕ *such that*

$$L \cap \phi(L) = \emptyset.$$

Then the Maslov index homomorphism $\mu_L : \pi_2(M,L) \to \mathbb{Z}$ *is not trivial.*

Proof. We will prove this by contradiction. Suppose to the contrary that μ_L is trivial for L. Since μ_L is trivial, all the obstruction classes for the Floer cohomology over $\mathbb{Z}_2$ coefficients lie in $H^2(L;\mathbb{Z}_2) = 0$ because $n - (n - 2 + \mu_L) = 2$. Since we assume $H^2(L;\mathbb{Z}_2) = 0$, all obstructions automatically vanish and so we derive $\mathcal{M}(L,J;\mathbb{Z}_2) \neq \emptyset$. Now let $b \in \mathcal{M}(L,J;\mathbb{Z}_2)$ and consider the corresponding Floer cohomology $HF((L,b),(L,b);J) \cong HF((L,b),(\phi(L);\phi_*(b);\phi_*J)$ is well defined. The assumption $L \cap \phi(L) = \emptyset$ then implies $HF((L,b),(L,b);J) = 0$. On the other hand, Theorem 24.12 is generalized to the $\mathbb{Z}_2$ coefficient for the spherically positive case and implies that $HF((L,b),(L,b);J)$ cannot be zero which gives rise to a contradiction. □

Proof of Theorem Ks. Note that $\mathbb{C}^n$ is noncompact but bounded at infinity. Therefore all the theorems in this book apply to compact Lagrangian submanifolds of $\mathbb{C}^n$. Obviously $\mathbb{C}^n$ is spherically positive (with resepct to the standard complex structure). For given compact Lagrangian submanifold L, we can easily find a Hamiltonian diffeomorphism $\phi : \mathbb{C}^n \to \mathbb{C}^n$ of compact support such that $L \cap \phi(L) = \emptyset$. Now Corollary 34.8 finishes the proof. □

We now point out some main ideas in the proofs of Theorems 34.3 and 34.7, which deal with the case of the $\mathbb{Z}$ or $\mathbb{Z}_2$ coefficients. Note that the only reason for the usage of rational cycles, instead of integral ones, in the stable map compactification comes from the fact that the (finite) automorphism groups of stable maps we use could be non-trivial. A simple examination of the constructions in [FuOn99II] or in Chapter 7 of this book, however, shows that all the construction would work with the $\mathbb{Z}$ or $\mathbb{Z}_2$ coefficients, *without assuming any other condition*, as long as all the stable maps involved in the construction have trivial automorphism groups.

In this respect, the following observation is important, although its proof immediately comes from the structure of $PSL(2;\mathbb{R})$.

Lemma 34.9. *The automorphism group of a semi-stable bordered Riemann surface $(\Sigma, \vec{z}) \in \mathcal{M}_{k+1}^{\rm main}$ of genus 0 is torsion free if $k \geq 0$ and if it has no sphere components. In particular, if a stable map $((\Sigma, \vec{z}), w) \in \mathcal{M}_{k+1}^{\rm main}(J;\beta)$ with $k \geq 0$ has no sphere bubbles, then the automorphism group of $((\Sigma, \vec{z}), w)$ is trivial.*

Here we specify J, the almost complex structure, in the notation $\mathcal{M}_{k+1}^{\rm main}(J;\beta)$ above, since the results of Chapter 8 (for example Theorem 34.11 below) may not hold for arbitrary J but is correct only after choosing J appropriately. (For the results of Chapter 3 $\sim$ 7, we may take any J, since the perturbation we use is an abstract perturbation.) An immediate corollary of Lemma 34.9 is the following :

Corollary 34.10. *Let $k \geq 0$. If an element $((\Sigma, \vec{z}), w) \in \mathcal{M}_{k+1}^{\rm main}(J;\beta)$ does not have sphere components, then $((\Sigma, \vec{z}), w) \in \mathcal{M}_{k+1}^{\rm main}(J;\beta)$ has trivial automorphism group. In particular, if (M, ω, J) does not allow any pseudo-holomorphic sphere then its automorphism group is trivial.*

Let us now assume that (M,ω) is spherically positive and $J \in \mathcal{J}_{\omega}^{c_1>0}$. We consider the moduli space

$$\mathcal{M}_{k+1}^{\rm main}(\beta; P_1, \cdots, P_k) := \mathcal{M}_{k+1}^{\rm main}(J;\beta) \times_{(ev_1,\cdots,ev_k)} (P_1 \times \cdots \times P_k).$$

We have a natural decomposition such that

$$\begin{aligned}&\mathcal{M}_{k+1}^{\rm main}(\beta; P_1, \cdots, P_k)\\ &\qquad = \mathcal{M}_{k+1}^{\rm main}(\beta; P_1, \cdots, P_k)_{\rm free} \cup \mathcal{M}_{k+1}^{\rm main}(\beta; P_1, \cdots, P_k)_{\rm fix}.\end{aligned}$$

Here $\mathcal{M}_{k+1}^{\rm main}(\beta; P_1, \cdots, P_k)_{\rm free}$ (resp. $\mathcal{M}_{k+1}^{\rm main}(\beta; P_1, \cdots, P_k)_{\rm fix}$) is the set of elements with trivial (resp. nontrivial) automorphism groups. The following theorem is the main step of the proofs of Theorems 34.3 and 34.7.

Theorem 34.11. *Assume (M,ω) is spherically positive and $J \in \mathcal{J}_{\omega}^{c_1>0}$. Let $L \subset (M,\omega)$ be a Lagrangian submanifold and let $P_1, \cdots, P_k$ be given singular simplices and $\beta \in \Pi(M;L)$. Consider the moduli space $\mathcal{M}_{k+1}^{\rm main}(\beta; P_1, \cdots, P_k)$ and its decomposition given above. Then there exists a family of* **single valued** *piecewise smooth sections $\mathfrak{s}^{\epsilon}$ of the obstruction bundle in a Kuranishi neighborhood of $\mathcal{M}_{k+1}^{\rm main}(\beta; P_1, \cdots, P_k)$ and a decomposition*

$$\begin{aligned}&\mathcal{M}_{k+1}^{\rm main}(\beta; P_1, \cdots, P_k)^{\mathfrak{s}^{\epsilon}}\\ &\qquad = \mathcal{M}_{k+1}^{\rm main}(\beta; P_1, \cdots, P_k)_{\rm free}^{\mathfrak{s}^{\epsilon}} \cup \mathcal{M}_{k+1}^{\rm main}(\beta; P_1, \cdots, P_k)_{\rm fix}^{\mathfrak{s}^{\epsilon}}\end{aligned} \tag{34.12}$$

of the perturbed moduli space such that :

(34.13.1) $\mathcal{M}_{k+1}^{\rm main}(\beta;P_1,\cdots,P_k)^{\mathfrak{s}^\epsilon}_{\rm free}$ *is a PL manifold.*
(34.13.2) $\mathcal{M}_{k+1}^{\rm main}(\beta;P_1,\cdots,P_k)^{\mathfrak{s}^\epsilon}$ *has a triangulation compatible with the smooth structure on* $\mathcal{M}_{k+1}^{\rm main}(\beta;P_1,\cdots,P_k)^{\mathfrak{s}^\epsilon}_{\rm free}$.
(34.13.3) $\mathcal{M}_{k+1}^{\rm main}(\beta;P_1,\cdots,P_k)^{\mathfrak{s}^\epsilon}_{\rm fix}$ *is contained in a sub-complex of dimension*

$$\dim \mathcal{M}_{k+1}^{\rm main}(\beta;P_1,\cdots,P_k)^{\mathfrak{s}^\epsilon}_{\rm free} - 2.$$

(34.13.4) $\lim_{\epsilon\to 0}\mathfrak{s}^\epsilon = s$, *where* s *is the original Kuranishi map over* $\mathcal{M}_{k+1}^{\rm main}(\beta;P_1,\cdots,P_k)$ *which is constructed in Chapter 7.*

Once Theorem 34.11 (and the analogous statements for other moduli spaces used in previous chapters) is proven, the rest of the proofs of Theorems 34.4 and 34.7 are the straightforward analogs of the arguments used in the previous sections. We will not repeat them here. In order to construct a section whose zero set has a smooth triangulation, we work in the piecewise linear (or piecewise smooth) category. This is the reason why the section we obtain in Theorem 34.11 is piecewise smooth (and is not necessary smooth).

In §37, we describe an example which illustrates various constructions given in this book. There we study in detail the case of Lagrangian submanifold L of $\mathbb{C}^{n+1}$ that is homeomorphic to $S^1\times S^n$. We study the leading order contribution of the holomorphic discs to the matrix coefficients of the filtered A_∞ algebra associated to L. The result is simpler to describe in the language of filtered L_∞ algebras rather than that of filtered A_∞ algebras. The relevant filtered L_∞ algebra will be obtained by symmetrizing our filtered A_∞ algebra. Because of this, we describe the story of filtered L_∞ algebras and the symmetrization of filtered A_∞ algebras in §36 as much as we need in §37.

In §38-§43, we study the intersection theory of the class of Lagrangian submanifolds consisting the fixed point sets of anti-symplectic involutions. Such Lagrangian submanifolds naturally arise in the study of real algebraic geometry as the real point set of a complex algebraic variety. They are also previously studied by the second named author [Oh95I] for the real forms of compact Hermitian symmetric spaces in relation to the following conjecture.

Arnold-Givental conjecture 34.14. *Let* τ *be an anti-symplectic involution of* (M,ω), *i.e., be a diffeomorphism with* $\tau^*\omega=-\omega$. *Assume that* $L=\operatorname{Fix}\tau$ *is non-empty and* $\phi: M\to M$ *is a Hamiltonian diffeomorphism such that* L *intersects* $\phi(L)$ *transversely. Then we have*

$$\#(L\cap\phi(L)) \geq \sum \text{rank } H^*(L;\mathbb{Z}_2).$$

We will study this Arnold-Givental conjecture for the case of spherically positive symplectic manifolds. We first introduce the following analogue of spherical positivity.

Condition 34.15. Suppose $\tau : M \to M$ be an anti-symplectic involution with $\text{Fix}\ \tau \neq \emptyset$. We assume that there exists a spherically positive J such that $\tau^* J = -J$.

We note that when $L = \text{Fix}\ \tau$ for an anti-symplectic involution τ and a spherically positive J satisfies $\tau^* J = -J$, then L has the property that $\mu_L(w) > 0$ for any non-constant J-holomorphic disc w with its boundary lying in L.

Theorem 34.16. *We assume that* (M, ω) *safisfies Condition* 34.15. *We consider* $L = \text{Fix}\ \tau$. *Then* L *is unobstructed over* $\mathbb{Z}_2$ *with respect to* J. *Moreover we can choose a bounding cochain* $b \in \mathcal{M}(C(L, J; \Lambda_{0,nov}^{\mathbb{Z}_2}))$ *such that*

$$HF((L, b), (L, b); J; \Lambda_{0,nov}^{\mathbb{Z}_2}) \cong H^*(L; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} \Lambda_{0,nov}^{\mathbb{Z}_2}.$$

Theorem M in introduction is an immediate consequence of Theorem 34.16. Note that L is not necessarily assumed to be orientable or relatively spin. We remark that a symplectic manifold M is said to be (positively spherically) monotone if $[\omega] \cap \alpha = c c_1(M) \cap \alpha$ for any $\alpha \in \pi_2(M)$ where $c > 0$ is independent of α.

Theorem 34.17. *The Arnold-Givental conjecture holds for the following classes of* $((M, \omega), \tau)$.

(1) M *is (positively spherically) monotone.*
(2) *There is a complex structure* J *on* M *such that* ω *is its Kähler form and* (M, J) *is Fano.* τ *is an anti-holomorphic involution of* (M, J).
(3) $\dim_{\mathbb{R}} M = 4$.
(4) $((M, \omega), \tau)$ *is the product* $\prod_{i=1}^{k}((M_i, \omega), \tau_i)$, *where* $((M_i, \omega), \tau_i)$ *are one of the cases* (1), (2), (3) *above.*

Theorem 34.17 follows from Theorem 34.16 as we will explain at the end of §39.

Since the real forms of compact Hermitian symmetric spaces are (positively) monotone, we have the following slight generalization of the result from [Oh95I], in that it eliminates some restrictions posed on the real forms.

Corollary 34.18. *The Arnold-Givental conjecture holds for the real forms of any compact Hermitian symmetric spaces.*

The proof of Theorem 34.16 will be given in §38−§43. We like to note that spherical positivity of L is used only in §43. An idea of the proof of Theorem 34.16 is that for fixed point set $L = \text{Fix}\ \tau$ of anti-holomorphic involution τ and for J satisfying $\tau_* J = -J$, any J-holomorphic disc $\varphi : D^2 \to M$ attached to L comes in pair and their contributions cancel each other. (See §38 for the detailed description of this symmetry.) We remark that the $\mathbb{Z}_2$-coefficient is used in Theorem 34.16 : This is not only because we do not assume L is oriented but also because the above mentioned cancelation occurs only over the $\mathbb{Z}_2$-coefficient. Namely holomorphic discs in pair may or may not have opposite orientations. We discuss this orientation problem in §38, and §47 (Chapter 9) in detail.

As a byproduct of the constructions used in these sections, we can also prove the following results *over* $\mathbb{Q}$.

Definition 34.19. Let $\tau : M \to M$ be an anti-symplectic involution (that is $\tau^*\omega = -\omega$, $\tau^2 = id$). We assume that $L = \text{Fix}\ \tau$ is non empty, (hence is a Lagrangian submanifold). We say that L is *τ-relatively spin* if there exists a relative spin structure $[V, \sigma]$ such that the pull back $\tau^*[V, \sigma]$ is stably conjugate to $[V, \sigma]$. Such a relative spin structure is called a *τ-relatively spin structure.*

See Definition 44.7 and the top of §44.5 for the definition of the pull back $\tau^*[V, \sigma]$ of the relative spin structure by the anti-symplectic involution τ. (See also Definition 44.2 for relative spin structure and Definition 44.5 for the notion of stably conjugate.) For example, if L is spin, it is automatically τ-relatively spin. (Remark 44.18).

Theorem 34.20. *Let M be a symplectic manifold and τ an anti-symplectic involution. If $L = \text{Fix}\tau$ is non-empty, oriented, and τ-relatively spin, then the filtered A_∞ algebra $(C(L; \Lambda_{0,nov}^{\mathbb{Q}}), \mathfrak{m})$ in Theorem 10.11 can be chosen so that*

$$\mathfrak{m}_{k,\beta}(P_1, \cdots, P_k) = (-1)^{\epsilon} \mathfrak{m}_{k,\tau^*\beta}(P_k, \cdots, P_1) \tag{34.21}$$

where

$$\epsilon = \frac{\mu_L(\beta)}{2} + k + 1 + \sum_{1 \le i < j \le k} \deg' P_i \deg' P_j.$$

Here $\deg' = \deg - 1$ is the shifted degree. Theorem 34.20 will be proved in §38 and §47 and Theorem O in Chapter 1 is an immediate consequence of Theorem 34.20. (Note that we do *not* assume the spherical positivity of (M, ω) in Theorem 34.20 and in the rest of this section.)

Corollary 34.22. *Let τ and $L = \text{Fix}\ \tau$ be as in Theorem 34.20. Assume that the image of $c_1 : \pi_2(M) \to \mathbb{Z}$ is contained in $2\mathbb{Z}$ in addition. Then L is unobstructed over $\mathbb{Q}$ and so $HF(L, L)$ is defined. Moreover we may choose $b \in \mathcal{M}(L; \Lambda_{0,nov}^{\mathbb{Q}})$ so that the map*

$$\begin{aligned}(-1)^{k(\ell+1)}(\mathfrak{m}_2)_* : HF^k((L, b), (L, b); \Lambda_{0,nov}^{\mathbb{Q}}) \otimes HF^\ell((L, b), (L, b); \Lambda_{0,nov}^{\mathbb{Q}}) \\ \longrightarrow HF^{k+\ell}((L, b), (L, b); \Lambda_{0,nov}^{\mathbb{Q}})\end{aligned}$$

induces a graded commutative product.

We remark that we do not assert that Floer cohomology $HF((L, b), (L, b); \Lambda_{0,nov}^{\mathbb{Q}})$ is isomorphic to $H^*(L; \mathbb{Q}) \otimes \Lambda_{0,nov}^{\mathbb{Q}}$. (Namely we do not assert $\mathfrak{m}_1 = \overline{\mathfrak{m}}_1$.) Indeed, we will show in §44.6 Chapter 9 that for the case $L = \mathbb{R}P^{2n+1}$ in $\mathbb{C}P^{2n+1}$ the Floer cohomology group is not isomorphic to the classical cohomology group. (See Theorem 44.24.)

We remark that Theorem 34.20 and Corollary 34.22 can be applied to the real point set L of any Calabi-Yau manifold (defined over $\mathbb{R}$) if it is oriented and τ-relatively spin. In particular, such L is unobstructed and so the Floer cohomology $HF((L,b),(L,b);\Lambda^{\mathbb{Q}}_{0,nov})$ of L is defined for given $b \in \mathcal{M}(L;\Lambda^{\mathbb{Q}}_{0,nov})$.

Proof of Theorem 34.20 ⇒ Corollary 34.22. By assumption, the Maslov index of L modulo 4 is trivial. Therefore (34.21) implies $\mathfrak{m}_{0,\tau^*\beta}(1) = -\mathfrak{m}_{0,\beta}(1)$. If follows by the cancellation argument mentioned above that L is unobstructed. Then (34.21) implies

$$\mathfrak{m}_{2,\beta}(P_1,P_2) = (-1)^{1+\deg' P_1 \deg' P_2}\mathfrak{m}_{2,\tau_*\beta}(P_2,P_1). \tag{34.23}$$

We denote

$$P_1 \cup_Q P_2 := (-1)^{\deg P_1(\deg P_2+1)}\sum_{\beta}\mathfrak{m}_{2,\beta}(P_1,P_2).$$

Then a simple calculation shows that (34.23) gives rise to

$$P_1 \cup_Q P_2 = (-1)^{\deg P_1 \deg P_2} P_2 \cup_Q P_1.$$

Hence $\cup_Q$ is graded commutative. □

Remark 34.24. For the later purpose, we mention here that the same proof will show that in the situation of Corollary 34.22 we have $\mathfrak{l}_k = \bar{\mathfrak{l}}_k$ if k is even. Here $\mathfrak{l}_k$ is the symmetrization $(C(L,\Lambda^{\mathbb{Q}}_{0,nov}),\mathfrak{l}_k)$ of the filtered A_∞ algebra $(C(L,\Lambda^{\mathbb{Q}}_{0,nov}),\mathfrak{m}_k)$, and $\bar{\mathfrak{l}}_k$ is L_∞ structure obtained as the reduction of the coefficient of $(C(L,\Lambda^{\mathbb{Q}}_{0,nov}),\mathfrak{l}_k)$ to $\mathbb{Q}$. We refer to §36 for their precise definitions. Note that *over* $\mathbb{R}$ we may choose $\bar{\mathfrak{l}}_k = 0$ by Theorem V in Chapter 1. On the other hand, Theorem 36.19 shows that $\bar{\mathfrak{l}}_k = 0$ *over* $\mathbb{Q}$.

Theorem 34.20 and Corollary 34.22 can be applied also to the diagonal of square of a symplectic manifold. Namely we consider the following situation. Let (N,ω_N) be a symplectic manifold. We consider the product

$$(M,\omega_M) = (N \times N, \omega_N \otimes 1 - 1 \otimes \omega_N).$$

The involution $\tau : M \to M$, $\tau(x,y) = (y,x)$ is anti-symplectic and its fixed point set L is the diagonal

$$\{(x,x) \mid x \in N\} \cong N.$$

We note that the natural map $i_* : H_*(\Delta,\mathbb{Q}) \to H_*(N\times N;\mathbb{Q})$ is injective and so the spectral sequence collapses at E_2-term by Theorem 24.5, which in turn induces the natural isomorphism $H(N;\mathbb{Q})\otimes\Lambda_{0,nov} \cong HF(L,L)$. We also remark that the image of Maslov index $\mu_L : \pi_2(M,L) \to \mathbb{Z}$ is automatically in $4\mathbb{Z}$ for this case. Therefore we can apply Corollary 34.22 and derive a graded commutative product

$$\cup_Q : HF((L,b),(L,b);\Lambda^{\mathbb{R}}_{0,nov})\otimes HF((L,b),(L,b);\Lambda^{\mathbb{R}}_{0,nov}) \to HF((L,b),(L,b);\Lambda^{\mathbb{R}}_{0,nov})$$

given above. In fact, we can prove that the following stronger statement.

Proposition 34.25. *The product $\cup_Q$ coincides with the quantum cup product on (N, ω_N) under the natural isomorphism $HF((L,b),(L,b);\Lambda_{0,nov}^{\mathbb{Q}}) \cong H(N;\mathbb{Q}) \otimes \Lambda_{0,nov}^{\mathbb{Q}}$.*

We will prove Proposition 34.25 also in §38.

We remark that for the case of diagonals, $\mathfrak{m}_k$ ($k \geq 3$) define a quantum (higher) Massey product. It was discussed formally in [Fuk97III]. We made it rigorous here.

Remark 34.26. The authors thank Cheol-Hyun Cho for some helpful discussion concerning Proposition 34.25.

There are related works by Welschinger on real pseudo-holomorphic discs in symplectic 4-manifolds. See [Wel05], for example.

As we mentioned already, the proof of Theorems 34.16 and 34.17 are based on the $\mathbb{Z}_2$-symmetry on the moduli space of pseudo-holomorphic discs. To work out the details of this idea we first need to choose an almost complex structure J on M for which τ is anti-holomorphic. We need to choose J generic enough so that the assumptions imply absence of the moduli space of pseudo-holomorphic discs of "wrong dimension". This is necessary so that we can use this almost complex structure J to apply Theorems 34.3 and 34.7. We discuss this point in §39.

The other important point (which the authors overlooked in the year-2000 preprint version [FOOO00] of this book) is that the canonical involution on the moduli space of pseudo-holomorphic discs induced by τ may have a fixed point. The pseudo-holomorphic disc corresponding to a fixed point of this involution is called *lantern*, and is studied in detail in §40. There we construct another involution on the moduli space of lanterns which we use for a similar cancellation process. (We remark that we do *not* need to study lanterns to prove Theorems 34.3, 34.7, 34.20 and Corollary 34.22.) We construct a sequence of involutions on the interior of our moduli space in §41 and show in §42 that they can be regarded as involutions on the spaces with Kuranishi structure. (See §A1.3.) In §43, we study the boundary of the moduli space and complete the proof of Theorem 34.16. Numerical positivity of $L = \text{Fix}\ \tau$ is used only in this section. See §43.1 for the reason we need this unpleasant condition.

Remark 34.27. The authors thank U. Frauenfelder for pointing out an error in the year-2000 preprint version mentioned above. The idea applied here to rectify this error is due to the 4th named author and has been used by Frauenfelder also in his paper [Fra04] which discusses a particular case of the Arnold-Givental conjecture.

§35. Single-valued perturbation.

The purpose of this section is to prove Theorem 34.11 and apply it to prove Theorem 34.4 etc. In §35.1-3, we work with a global quotient by a finite group and an orbi-bundle and explain how to construct a *single-valued* section that has the properties required in Theorem 34.11. We use the stratification of the orbifold by the isotropy group. On each of the stratum we can rather easily construct a section whose zero set has the required dimension since each stratum is actually a manifold. The main point of our construction is how we paste those strata-wise sections to produce a global single-valued piecewise smooth section whose zero set carries a smooth triangulation.

This problem is thus naturally related to the theory of stratified set and to the singularity theory of smooth maps. The proof in §35.1-3 is based on several machinery established in the theory of stratified space.

In Appendix §A3 of the year 2000 preprint version [FOOO00], we provide another argument to find a single valued section whose zero set carries a triangulation, by using the *normally polynomial perturbations*, while we will use the *normally conical perturbations* introduced in §35.4 below. The approach of [FOOO00] is based on the triangulability of real analytic sets and of Whitney stratified spaces. In that sense it is closer to the method described in [FuOn01]. An advantage of this approach is that we only need to use the *statement* of the results established in the theory of Whitney stratified spaces : namely, Whitney stratified space has a C^0-triangulation. Although this result is rather difficult but is well-established. (In the approach of §35.1-3 we need to go back to the *proof* of this result and uses some of the ideas used in the proof of this result.)

Unfortunately, there is one drawback of the approach in §A3 of [FOOO00]. Namely the triangulation of Whitney stratified space is of C^0 but not smooth. This is a well-known fact, which is related to the various basic points (and difficulty) in the singularity theory of differentiable mapping. (For example, this fact is related to the reason why the set of stable map germs is not dense in C^∞ topology.) However, the triangulation of real algebraic set is sufficiently 'close to smooth' so that we may apply the construction of previous chapters. Because of this drawback we present the alternative argument in §35.1-3, by which we can actually construct smooth singular chains and the argument of the previous chapters directly apply. (As we mentioned in §1.6, we put the manuscript [FOOO08II] describing the normally polynomial perturbations in the first named author's home page, since the results and the proofs therein are correct and have own interest.)

We next explain in §35.4, generalization of the results of §35.1-3 to the case of Kuranishi structure.

Using the results of §35.1-4, the proof of Theorem 34.4 etc. are completed in §35.5. For this purpose we need to study an equivariant index of the linearization of the Cauchy-Riemann operator. Sperical positivity is used only in this part.

35.1. Single-valued piecewise smooth section of orbi-bundle : Statement of the result.

Let M be a smooth manifold and G a finite group acting effectively on it. The quotient $X = M/G$ defines an orbifold. (Such an orbifold is said to be a *global quotient.*) Let $E \to M$ be a Γ equivariant vector bundle on it. It is, by definition, an orbi-bundle E/G on X. A *single valued section* of this orbi-bundle is, by definition, a G-equivariant section of $E \to M$.

Let Γ be an abstract finite group. We denote

$$X^{\cong}(\Gamma) = \{p \in M \mid I_p \cong \Gamma\}/G \tag{35.1}$$

where I_p is the isotropy group of p, i.e.,

$$I_p = \{\gamma \in G \mid \gamma p = p\}. \tag{35.2}$$

We remark that $X^{\cong}(\Gamma)$ is a smooth *manifold*. In §A1.6, we define a standard stack structure on it. (See Example-Definition A1.83 and Definition A1.88.)

The set $\{X^{\cong}(\Gamma) \mid \Gamma\}$ defines a stratification of our space of X. (We refer [Math73] for the basic facts on Whitney stratifications.) Our stratification is a Whitney stratification. It is actually better than the usual Whitney stratification. Namely for our stratification, the normal cone exists and is locally trivial in the C^∞ sense. (We will define this notion later in Definition 35.12.) This is a consequence of Lemma A1.100. In general the normal cone of a Whitney stratified space is locally trivial only in C^0-sense.

Example 35.3. Let $C_a = \{(tx, ty, t) \mid -1 \le x \le 1, 0 \le y \le 1 + a - a|x|, t \ge 0\}$ and we put

$$X = \{(x, y, z, w) \mid (x, y, z) \in C_w, 0 < w < 1\}.$$

Using the fact that C_a is not affine isomorphic to C_b for $a \ne b$, we can prove that a neighborhood of w axis in X is not diffeomorphic to the product $\mathbb{R} \times Z$ for any $Z \subset \mathbb{R}^3$.

The fact that each of our strata has a normal cone which is locally trivial in C^∞ sense can be used to show that X has a smooth triangulation. (See Definition 35.12 for the definition of local triviality.) (We remark that the fact orbifold has a smooth triangulation is well-known, of course.) We can use our smooth stratification of X to define the notion of piecewise smoothness of sections of E/G : a smooth triangulation of X induces a smooth G-equivariant triangulation of M. A G-equivariant section of E that is piecewise smooth with respect to such a triangulation is identified with a piecewise smooth section of $E/G \to X$. (We remark that in this section we never use multi-valued sections and use only single valued sections.)

We now define the notion of locally trivial stratification (in C^∞ sense) suitable for our purpose.

Definition 35.4. An *affine polygon* $\mathbb{P}$ is a closed subset of $\mathbb{R}^n$ defined by a finite number of inequalities of the type

$$a_1x_1 + \cdots + a_nx_n \leq c. \tag{35.5}$$

A compact and connected subset P of a smooth manifold N is said to be a *locally polygonal set* if, at each $p \in P$, there exist a chart $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ of a neighborhood of p in N and an affine polygon $\mathbb{P} \subset \mathbb{R}^n$ such that

$$\varphi(P \cap U) = \varphi(U) \cap \mathbb{P}. \tag{35.6}$$

We say $(U, \varphi, \mathbb{P})$ as above a *chart* of P.

We can define the notion of *face* (of arbitrary codimension) of an affine polygon in an obvious way. Let P be a locally polygonal set. A closed subset Q of P is said to be a *face* of P if :

(35.7.1) For each $p \in Q$ there exists a chart $(U, \varphi, \mathbb{P})$ of P such that

$$\varphi(Q \cap U) = \left(\bigcup_{a\in A} \mathbb{P}_a\right) \cap \varphi(U)$$

where $\{\mathbb{P}_a\}_{a\in A}$ is a subset of the set of faces of $\mathbb{P}$.
(35.7.2) No proper closed subset $R \subset Q$ with $\dim R = \dim Q$ has property (35.7.1).

We remark that in many cases $\varphi(Q \cap U)$ is an intersection of $\varphi(U)$ with a single face of $\mathbb{P}$. However it may happen that two different faces of $\varphi(U) \cap \mathbb{P}$ are connected somewhere away from U. This is the reason why we allow $\varphi(Q \cap U)$ to be a union of several components in (35.7.1).

Let $F : X \to Y$ be a continuous map between two closed subsets of Euclidean spaces. We say F is *smooth* if it extends to a smooth map between open subsets of Euclidean spaces.

Using this definition and a local chart, we can define the notion of smoothness of a map $F : P \to Q$ between locally polygonal subsets. The notion of diffeomorphism between locally polygonal subsets is defined also in the same way. We remark that our locally polygonal subset is a *closed* subset and that a smooth map is assumed to be extended to its neighborhood. This is a much stronger notion than the smoothness that is usually used in the study of stratified sets where each stratum is a *locally closed* set and a smooth map is assumed to be smooth only on each of the stratum (and not necessarily on its closure).

An (abstract) *locally polygonal space* is a topological space equipped with a homeomorphism to a locally polygonal subset of some manifold, which we call a chart. A smooth map or diffeomorphism between them are defined by using the same notion for locally polygonal subsets. For each point p of a locally polygonal set P, the tangent space T_pP is defined even when p lies in its boundary or corners. (We again recall that a smooth map on a closed subset is assumed to be extended to its neighborhood.)

Let Y be a locally compact Hausdorff space with a stratification

$$Y = \bigcup_{a\in A} Y_a : \tag{35.8}$$

Namely

(35.9.1) each Y_a is a locally closed set.
(35.9.2) $\overline{Y}_a \setminus Y_a$ is a union of strata, Y_b.

Definition 35.10. The stratification (35.8) is said to be a *locally polygonal stratification*, if the following holds :

(35.11.1) For each a, we are given a locally polygonal space P_a and a homeomorphism φ_a from P_a to a closure of Y_a.
(35.11.2) If Y_b is a stratum contained in $\overline{Y}_a \setminus Y_a$, then there exists a face P_{ab} of P_a such that $\varphi_a(P_{ab}) = \overline{Y}_b$ and the composition

$$\varphi_b^{-1} \circ \varphi_a|_{P_{ab}} : P_{ab} \to P_b$$

is a diffeomorphism.

The *dimension* of Y is defined to be the maximum dimension of P_a's. For the spaces Y, Y' with locally polygonal stratification $\{Y_a\}_{a\in A}$ and $\{Y'_b\}_{b\in B}$, we say a homeomorphism $F : Y \to Y'$ between them is a *diffeomorphism* if there exists a bijection $a \mapsto b = b(a)$ from A to B such that F induces a diffeomorphism from P_a to $P'_{b(a)}$ for each a. A continuous map $F : Y \to M$ from a space Y with locally polygonal stratification to a smooth manifold M is said to be smooth if its restriction to each P_a is smooth. We say F is a *submersion* if its restriction to each stratum Y_a is a submersion.

We next define the notion of local triviality. We can define the product of locally polygonal stratifications in an obvious way. We can also restrict a locally polygonal stratification to an open set. Let $\mathbb{D} \subset \mathbb{R}^n$ be a *compact* affine convex polygon. We define a cone $C\mathbb{D}$ of $\mathbb{D}$ as the space

$$C\mathbb{D} = \{(tz, t) \in \mathbb{R}^{n+1} \mid z \in \mathbb{D}, t \in [0, 1)\}.$$

We can easily see the following : For each locally polygonal set P and any $p \in P$ contained in the interior of k dimensional face, we have a chart $(U, \varphi, C\mathbb{P} \times \mathbb{R}^k)$ such that $\varphi(P \cap U) = C\mathbb{P} \times \mathbb{R}^k$. In fact the claim is obvious for the case of affine polygons in Euclidean space and then the general case is easily reduced to this case.

We say a locally polygonal stratification is a *polygonal stratification* if each of P_a in (35.11.1) is diffeomorphic to a compact convex affine polygon. If Z has a polygonal stratification, its cone has a locally polygonal stratification.

We now define local triviality of a locally polygonal stratification inductively over the dimension.

Definition 35.12. In dimension 0 there is no condition. Assume that we have defined local triviality up to dimension $n-1$. Let (35.8) be a locally polygonal stratification of Y of dimension n. We say that it is *locally trivial* (*in* C^∞ *sense*) if the following holds : We use the notation of (35.11.1).

(35.13.1) Let $x \in Y_a$. Then there exists a space N with locally polygonal stratifications $\{N_a\}_{a\in A}$ and an open neighborhood U of $\varphi_a^{-1}(x)$ in P_a and a diffeomorphism from $U \times N$ to a neighborhood of Y in x.
(35.13.2) Moreover N is diffeomorphic to a cone CZ of a space Z of dimension $\le n-1$ with locally trivial polygonal stratification such that {the cone point} $\times U$ is mapped to an open subset of Y_a.

We note that stratification of a locally polygonal space by its faces is locally trivial in the above sense.

Example 35.14. Consider the family of 4 lines

$$L_1 : x = 0, \quad L_2 : y = 0, \quad L_3 : x = y, \quad L_4(a) : y = ax$$

in $\mathbb{R}^2$, where $a \in (0,1)$. This is a classical example of Whitney. We can easily see that, for $a_1 \ne a_2$, there exists no diffeomorphism of $\mathbb{R}^2$ which simultaneously sends L_i to L_i $(i = 1,2,3)$ and $L_4(a_1)$ to $L_4(a_2)$.

Let us take a nonconstant smooth map $f : \mathbb{R} \to (0,1)$. We put

$$P_i = L_i \times \mathbb{R}, \qquad i = 1,2,3,$$

and

$$P_4 = \{(x,y,z) \mid (x,y) \in L_4(f(z))\}.$$

Each of them is divided into two by z axis, which we denote $P_i^\pm$ respectively. These 8 strata together with 8 open sets P_i $i = 5,\cdots,12$ obtained by decomposing $\mathbb{R}^3$ by $P_i^\pm$ $(i = 1,2,3,4)$ and the z-axis, P_{13}, defines a stratification. They induce a Whitney stratification.

It is also a locally trivial locally polygonal stratification. To see this, we remark that the diffeomorphism in our sense of stratified set is a continuous map which is a diffeomorphism on each stratum. In other words, it is assumed that a diffeomorphism on each stratum is extended to a diffeomorphism to its neighborhood. However those extensions are not required to satisfy any consistency conditions between different strata.

For example, the stratum

$$P_5 = \{(x,y,z) \mid 0 \le y \le f(z)x\}$$

is diffeomorphic to

$$\{(x,y,z) \mid 0 \le y \le x\}$$

and hence is a locally polygonal space.

However the similar but slightly different example, $X \subset \mathbb{R}^4$ provided in Example 35.3 is not diffeomorphic to any locally polygonal set.

We have thus defined the notion of locally trivial locally polygonal stratification. We go back to the case of a global quotient $X = M/G$ of our interest.

Let us decompose $X^{\cong}(\Gamma)$ into the connected components

$$X^{\cong}(\Gamma) = \bigcup_i X^{\cong}(\Gamma; i). \tag{35.15}$$

The following lemma is the main reason why we have introduced various notions.

Lemma 35.16. *For each global quotient $X = M/G$, the stratification*

$$X = \bigcup_{\Gamma, i} X^{\cong}(\Gamma; i)$$

defines a locally trivial locally polygonal stratification of the underlying topological space $|X|$.

Proof. This is an immediate consequence of Lemma A1.100. □

Remark 35.17. For this lemma, we do not need to assume X to be a global quotient. We just state Lemma 35.16 for the case of global quotient because we prove Lemma A1.100 only in that case.

Let $[p] \in X^{\cong}(\Gamma; i)$ where $p \in M$ and $X = M/G$. We have a Γ action on the fiber E_p of our vector bundle E. We put

$$E_p^{\Gamma} = \{v \in E_p \mid \forall \gamma \in \Gamma \ \gamma v = v\}. \tag{35.18}$$

Its dimension depends only on Γ, i but independent of p. We define

$$d(\Gamma; i) = \dim X^{\cong}(\Gamma; i) - \dim E_p^{\Gamma}. \tag{35.19}$$

In subsections §35.2-3 we will prove the following :

Proposition 35.20. *For each C^0-section s of the orbi-bundle $E/G \to X$, there exists a sequence of single valued piecewise smooth sections s_ϵ converging to s in the C^0-sense such that the following holds :*

(35.21.1) *$s_\epsilon^{-1}(0)$ has a smooth triangulation. Namely for each simplex the embedding $\Delta \to X$ locally lifts to a map to M which is a smooth embedding.*

(35.21.2) *$s_\epsilon^{-1}(0) \cap X^{\cong}(\Gamma)$ is a PL manifold, such that each simplex is smoothly embedded into $X^{\cong}(\Gamma)$.*

(35.21.3) *If Δ is a simplex of* (35.21.1) *whose interior intersects with $X^{\cong}(\Gamma)$, then the intersection of Δ with $X^{\cong}(\Gamma)$ is Δ minus some faces and is smoothly embedded in $X^{\cong}(\Gamma)$.*

(35.21.4)

$$\dim s_\epsilon^{-1}(0) \cap X^{\cong}(\Gamma; i) = \dim X^{\cong}(\Gamma; i) - \dim E_p^{\Gamma}.$$

Remark 35.22. We remark that if s is a single-valued section and if $[p] \in X^{\cong}(\Gamma)$ then $s(p) \in E_p^{\Gamma}$. So the dimension given in (35.21.4) is optimal.

We will use Lemma 35.16 in the proof of Proposition 30.20.

35.2. System of tubular neighborhoods.

The proof of Proposition 35.20 is closely related to the proof of existence of a triangulation of the space with Whitney stratification. (See [Gor78]). Especially, we use the notion of system of tubular neighborhoods introduced by Mather [Math73]. Mather introduced this notion to prove the famous first isotopy lemma. The first isotopy lemma implies that Whitney stratification has a C^0 locally trivial normal cone (See [Math73] 2.7). Note that existence of smooth triangulation of an orbifold is well-known which is not what we intend to prove. In order to show that the zero set of our section has a smooth triangulation, we use various constructions appearing in the proof of existence of a C^0-triangulation on the Whitney stratified space.

As we mentioned in the last subsection, we have a C^∞ locally trivial tubular neighborhood in our situation. This property makes the system of tubular neighborhoods (or the system of normal cones) in this case carry properties better than that of [Math73].

Following §II in [Math73], we define a tubular neighborhood of the stratum$X^{\cong}(\Gamma)$ in X by a quadruple $(\pi_\Gamma, N_{X^{\cong}(\Gamma)}X, \sigma, \phi)$ satisfying

(35.23.1) $\pi_\Gamma : N_{X^{\cong}(\Gamma)}X \to X^{\cong}(\Gamma)$ is a vector bundle in the sense of stack. (See Definition A1.89.)
(35.23.2) $\sigma : X^{\cong}(\Gamma) \to \mathbb{R}_+$ is a smooth positive function.
(35.23.3) $\phi : B_\sigma(\Gamma)/\Gamma \to U^{\cong}(\Gamma)$ is a diffeomorphism onto a neighborhood $U^{\cong}(\Gamma)$ of $X^{\cong}(\Gamma)$ in X. Here

$$B_\sigma(\Gamma) = \{v \in N_{X^{\cong}(\Gamma)}X \mid \|v\| < \sigma(\pi_\Gamma(v))\}.$$

We define

$$\pi_\Gamma : U^{\cong}(\Gamma) \to X^{\cong}(\Gamma)$$

as the composition $\pi_\Gamma \circ \phi^{-1}$. We also define

$$\rho'_\Gamma : U^{\cong}(\Gamma) \to \mathbb{R}$$

by

$$\rho'_\Gamma(\phi(v)) = \|v\|^2.$$

We remark that both maps are smooth and π_Γ is a submersion. Moreover the pair

$$(\pi_\Gamma, \rho'_\Gamma) : U^{\cong}(\Gamma) \setminus X^{\cong}(\Gamma) \to X^{\cong}(\Gamma) \times \mathbb{R}_{>0}$$

defines a submersion.

We need to adjust them so that they become compatible for different Γ's.

Remark 35.24. In the situation of [Math73], the maps π, ρ are smooth only in the interior of the stratum.

Definition 35.25. *A system of tubular neighborhoods* of our stratification $\{X^{\cong}(\Gamma) \mid \Gamma\}$ is a family $(\pi_\Gamma, \rho'_\Gamma)$ such that

$$\pi_{\Gamma'} \circ \pi_\Gamma = \pi_{\Gamma'} \tag{35.26.1}$$

$$\rho'_{\Gamma'} \circ \pi_\Gamma = \rho'_{\Gamma'} \tag{35.26.2}$$

holds for $\Gamma' \supset \Gamma$. Here we assume the equalities (35.26.1), (35.26.2) whenever both sides are defined.

Proposition 35.27. *There exists a system of tubular neighborhoods.*

The proof is actually the same as that of Corollary 6.5 of [Math73]. Mather proved the existence of a system of tubular neighborhoods for the space with Whitney stratification. In his case the situation is less tame than our case since the normal cone exists only in C^0 sense. In our case, the proof is easier since the normal cone we produce by Lemma A1.100 is already smooth. For the sake of completeness, we give the proof of Proposition 35.27 later in this subsection (Proposition 35.33).

We next define the notion of a family of lines following Goresky [Gor78]. For $\epsilon > 0$ we put :

$$S^\Gamma(\epsilon) = \{p \in U^{\cong}(\Gamma) \mid \rho_\Gamma = \epsilon^2\} \tag{35.28}$$

$$U^{\cong}(\Gamma; \epsilon) = \{p \in U^{\cong}(\Gamma) \mid \rho_\Gamma < \epsilon^2\}. \tag{35.29}$$

We need to modify ρ'_Γ to ρ_Γ in the following way. (See the lines 2 -5 from the bottom of [Gor78] p 193. We warn that our notation ρ_Γ corresponds to Goresky's ρ'_Γ and ρ'_Γ to Goresky's ρ_Γ.) We took $\rho_\Gamma(p)$ before as $\|v\|^2$ for $p = \phi(v)$ where $\| \cdot \|$ is an appropriate norm. We take a function $f_\Gamma : X^{\cong}(\Gamma) \to \mathbb{R}_+$ that goes to zero on the boundary. Then we define $\rho_\Gamma(x) = (f_\Gamma \circ \pi_\Gamma(x))\rho'_\Gamma(x)$. In this way we may not need to use σ given in (35.23.2).

Figure 35.1

Definition 35.30. A family of smooth maps

$$r_\Gamma(\epsilon) : U^{\cong}(\Gamma) \setminus X^{\cong}(\Gamma) \to S^{\Gamma}(\epsilon)$$

is said to be a *family of lines* if the following holds for $\Gamma' \supset \Gamma$:

(35.31.1) $r_{\Gamma'}(\epsilon') \circ r_\Gamma(\epsilon) = r_\Gamma(\epsilon) \circ r_{\Gamma'}(\epsilon') \in S^{\Gamma}(\epsilon) \cap S^{\Gamma'}(\epsilon')$ for all $\epsilon', \epsilon > 0$.
(35.31.2) $\rho_{\Gamma'} \circ r_\Gamma(\epsilon) = \rho_{\Gamma'}$.
(35.31.3) $\rho_\Gamma \circ r_{\Gamma'}(\epsilon) = \rho_\Gamma$.
(35.31.4) $\pi_{\Gamma'} \circ r_\Gamma(\epsilon) = \pi_{\Gamma'}$.
(35.31.5) If $0 < \epsilon < \epsilon' < \delta$ then $r_\Gamma(\epsilon') \circ r_\Gamma(\epsilon) = r_\Gamma(\epsilon')$.
(35.31.6) $\pi_\Gamma \circ r_\Gamma(\epsilon) = \pi_\Gamma$.
(35.31.7) We define

$$h_\Gamma : U^{\cong}(\Gamma; \epsilon) \setminus X^{\cong}(\Gamma) \to S^{\Gamma}(\epsilon) \times (0, \epsilon)$$

by

$$h_\Gamma(p) = (r_\Gamma(\epsilon)(p), \sqrt{\rho_\Gamma(p)})$$

and extend it to $U^{\cong}(\Gamma; \epsilon)$ by setting $h_\Gamma(p) = (p, 0)$ on $X^{\cong}(\Gamma)$. Then h_Γ induces a diffeomorphism from $U^{\cong}(\Gamma; \epsilon)$ to the mapping cone of

$$\pi_\Gamma|_{S^\Gamma(\epsilon)} : S^{\Gamma}(\epsilon) \to X^{\cong}(\Gamma).$$

We remark that the diffeomorphism in (35.31.7) is the one in the sense of Definition 35.10.

We remark that the above definition except (35.31.7) exactly coincides with that of Goresky [Gor78]. The condition (35.31.7) is stronger than the corresponding one from [Gor78]. This is because in our situation the normal cone is smooth and diffeomorphic to a neighborhood $U^{\cong}(\Gamma)$ of $X^{\cong}(\Gamma)$.

Proposition 35.32. *There exists a family of lines.*

We can prove Proposition 35.32 in the same way as [Gor78] except that we need some extra argument to check (35.31.7). Instead of working this out, we give a slightly different self-contained proof of Propositions 35.27 and 35.32 below, which exploits the special case of orbifolds (or the spaces with locally trivial locally polygonal stratification). The proof below is simpler than those of Mather or of Goresky. This is because we have already proved that there exists a normal cone which is C^∞ locally trivial. For the cases studied by Mather or Goresky, proving existence of C^0 trivial normal cone is one of the main goals of their study. So our proof here rather goes in the opposite direction to their study.

We will prove the relative versions of Propositions 35.27 and 35.32 below which include the propositions themselves.

Hereafter we write (π, ρ, r) in place of $\{(\pi_\Gamma, \rho_\Gamma, r_\Gamma) \mid \Gamma \subset G\}$ for simplicity. We also write $\pi \circ r = \pi$ etc. in place of (31.3) etc. by an abuse of notations.

Proposition 35.33. *Let X be a global quotient and K its compact subset. Assume that there exists a system of tubular neighborhoods (π,ρ) and family of lines r in a neighborhood U of K. Then there exist π, ρ and r on X that coincide with the given ones in a neighborhood of K respectively.*

For the proof of Proposition 35.33, we generalize it to the following relative version. (Compare this with §5 [Math73] where a similar procedure of the proof is applied.) We recall that a map from a stratified space to a manifold is called a submersion if its restriction to each stratum is a submersion.

Proposition 35.34. *Let X be a global quotient and K its compact subset. Let U_1, U_4 be open subsets of X such that $U_4 \supset K$ and $U_1 \supset \overline{U}_4$. Assume that there exists a system of tubular neighborhoods (π,ρ) and family of lines r on U_1. We also assume that there exists a smooth proper submersion $pr_N : X \setminus U_4 \to N$ where N is a manifold. We assume $pr_N \circ \pi = pr_N$, and $pr_N \circ r = pr_N$ on $U_1 \setminus U_4$.*

Then there exist open sets U_2, U_3 with $U_j \supset \overline{U}_{j+1}$ for $j = 1,2,3$ and there exist π, ρ and r on X which coincide with the given ones on U_1. In addition, $pr_N \circ r = pr_N$ and $pr_N \circ \pi = pr_N$ hold on $U_2 \setminus U_3$.

Again we write $pr_N \circ \pi = pr_N$ etc. in place of $pr_N \circ \pi_\Gamma = pr_N$ etc. by an abuse of notations.

Proof. We may assume that $X^{\cong}(\Gamma,i) \subset U_4$ if and only if $X^{\cong}(\Gamma,i) \subset U_1$ for each Γ. We put

$$d = \dim X - \inf\{\dim X^{\cong}(\Gamma,i) \mid X^{\cong}(\Gamma,i) \text{ is not contained in } U_1\}. \tag{35.35}$$

The proof is given by an induction over d. If d is 0, there is nothing to prove.

We assume that Proposition 35.34 is proved when (35.35) is $d-1$ or smaller and prove the case of d. Let $X^{\cong}(\Gamma,i)$ be a stratum of dimension $\dim X - d$. Since this is the stratum of smallest dimension in $X \setminus U_4$, it follows that it is a smooth manifold outside U_4. We put $X_0^{\cong}(\Gamma,i) = X^{\cong}(\Gamma,i) \setminus U_{3.5}$. Here $U_{3.5}$ is a neighborhood of K, which is slightly bigger than U_4.

We take the tubular neighborhood (conical neighborhood) $U(X_0^{\cong}(\Gamma,i))$ which is diffeomorphic to $N_{X_0^{\cong}(\Gamma,i)}X/\Gamma$ by Lemma A1.100. We take and fix a diffeomorphism between them.

We consider the submersion

$$\partial N_{X_0^{\cong}(\Gamma,i)}X/\Gamma \longrightarrow X_0^{\cong}(\Gamma,i) \xrightarrow{pr_N} N. \tag{35.36}$$

We may take the projection $\pi_\Gamma : N_{X_0^{\cong}(\Gamma,i)}X/\Gamma \to X_0^{\cong}(\Gamma,i)$ so that $\mathrm{pr}_N \circ \pi_\Gamma = pr_N$, by choosing the Riemannian metric we use to prove Lemma A1.100 so that each of the fibers of pr_N are totally geodesic.

We will now modify the tubular neighborhood and the family of lines on $(U_1 \setminus U_{3.5}) \cap \partial N_{X_0^{\cong}(\Gamma,i)}X/\Gamma$ (which is given by assumption) so that we can apply the induction hypothesis to $\partial N_{X_0^{\cong}(\Gamma,i)}X/\Gamma$ and the fibration

$$\partial N_{X_0^{\cong}(\Gamma,i)}X/\Gamma \longrightarrow X_0^{\cong}(\Gamma,i).$$

In fact by assumption there exist a tubular neighborhood and a family of lines (π',ρ',r') on the set $\partial N_{X_0^{\cong}(\Gamma,i)}X/\Gamma \cap U_1$. In particular, there exists

$$\pi'_\Gamma : \partial N_{X_0^{\cong}(\Gamma,i)}X/\Gamma \to X_0^{\cong}(\Gamma,i),$$

such that if $\Gamma^\circ \supset \Gamma$, the map π'_Γ is consistent with π'_{Γ°, ρ'_{Γ° and r'_{Γ° in the sense of (35.26) and (35.31.4). (Note that (Γ°,Γ) here corresponds to (Γ,Γ') in (35.26) and (35.31.4).)

We note that π'_Γ may not coincide with π_Γ given by Lemma A1.100. But we can modify and glue them as follows : The difference between two projections (π'_Γ above and π_Γ) can be chosen to be arbitrarily small (in C^1 sense), by taking the tubular neighborhood small. Then we can use the minimal geodesic of a Riemannian metric on $X_0^{\cong}(\Gamma,i)$, to find an isotopy between them. Hence by a standard argument we can glue them.

Thus we can apply our induction hypothesis to

$$\partial N_{X_0^{\cong}(\Gamma,i)}X/\Gamma \to X_0^{\cong}(\Gamma,i)$$

and obtain the system (π,r,ρ) on $\partial N_{X_0^{\cong}(\Gamma,i)}X/\Gamma$. Since $N_{X_0^{\cong}(\Gamma,i)}X/\Gamma \cong U(X_0^{\cong}(\Gamma))$ is a cone of $\partial N_{X_0^{\cong}(\Gamma,i)}X/\Gamma$, the system (π,r,ρ) on $\partial N_{X_0^{\cong}(\Gamma,i)}X/\Gamma$ induces one on $N_{X_0^{\cong}(\Gamma,i)}X/\Gamma$ in an obvious way. It commutes with the projection $U(X_0^{\cong}(\Gamma)) \to N$, since $U(X_0^{\cong}(\Gamma)) \to N$ factors through π_Γ.

Recall that we have (π,r,ρ) on U_1 by assumption. On $U_1 \cap U(X_0^{\cong}(\Gamma))$, this system may not coincide with the one we constructed above. We now explain how we adjust this system to carry out the gluing process.

We first remark that the projection $\pi_\Gamma : \partial N_{X_0^{\cong}(\Gamma,i)}X/\Gamma \to X_0^{\cong}(\Gamma,i)$ coincides for the two systems on U_1 since they are already arranged so when we apply the induction hypothesis above. Since both systems on U_1 are the cone of the same system (π,ρ,r) on $\partial N_{X_0^{\cong}(\Gamma,i)}X/\Gamma$ by the same map, it follows that they are the same as an abstract structure.

However, the diffeomorphism from the cone of $\partial N_{X_0^{\cong}(\Gamma,i)}X/\Gamma \cap U$ to $U(X_0^{\cong}(\Gamma,i)) \cap U$ (which exists by (35.31.7)) may not coincide. (Note this map is defined by r, the family of lines of each of the structures.)

We can however show that they are isotopic by the same method as before. Namely we go to the branched covering M and join the two diffeomorphisms by the minimal geodesic of a G-equivariant Riemannian metric that is totally geodesic along the fiber of pr_N. Therefore we can glue them by the standard method.

Now we have extended the systems (π, ρ, r) to a neighborhood of $X^{\cong}(\Gamma, i)$. We repeat the same construction for each $X^{\cong}(\Gamma, i)$ with $\dim X^{\cong}(\Gamma, i) = \dim X - d$.

We thus reduce the problem to the case when d is strictly smaller. The proof of Proposition 35.34 is now finished by induction. □

35.3. Single valued piecewise smooth section of orbi-bundle : Proof.

Let us fix a sufficiently small $d > 0$ and put

$$\mathrm{Int} X^{\cong}(\Gamma) = X^{\cong}(\Gamma) \setminus \bigcup_{\Gamma' \supset \Gamma} \mathrm{Int}\, U^{\cong}(\Gamma'; d). \tag{35.37}$$

We remark that by the definition of system of tubular neighborhoods we have the following :

$$U^{\cong}(\Gamma_1) \cap U^{\cong}(\Gamma_2) \neq \emptyset \quad \Rightarrow \quad \Gamma_1 \subset \Gamma_2 \text{ or } \Gamma_2 \subset \Gamma_1. \tag{35.38}$$

It follows from (35.38) that $\mathrm{Int}X^{\cong}(\Gamma)$ is a smooth manifold with corners. The codimension k corner of $\mathrm{Int}X^{\cong}(\Gamma)$ is a union of

$$X(\Gamma; \Gamma_1, \Gamma_2, \cdots, \Gamma_k) = X^{\cong}(\Gamma) \cap \bigcap_i S^{\Gamma_i}(d) \tag{35.39}$$

where

$$\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_k \supset \Gamma. \tag{35.40}$$

Figure 35.2

Later in this section, we will first define our section s_ϵ on $\bigcup_\Gamma \mathrm{Int} X^{\cong}(\Gamma)$ and then extend it so that its zero set is a cone with respect to the family of lines. Thus our proof is an analog to the proof given in §3-5 [Gor78].

For the construction of our section s_ϵ, we define a decomposition

$$E_p = E_p^\Gamma \oplus E_p^\perp \tag{35.41}$$

on $X(\Gamma;\Gamma_1,\Gamma_2,\cdots,\Gamma_k)$ where

$$E_p^\Gamma = \bigoplus_{i=1}^{k} E_p(\Gamma_i) \oplus E_p(\Gamma)$$

and $E_p^\perp$ be its complement to E_p^Γ in E_p andfor $k = 0,1,2,\cdots$.

If Γ is maximal, we just set $E_p(\Gamma) = E_p^\Gamma$ on $X^{\cong}(\Gamma)$. Using local triviality of $E(\Gamma)$, we can extend our subbundle $E(\Gamma)$ to the neighborhood $U^{\cong}(\Gamma;d)$ for a sufficiently small d so that

$$E_p(\Gamma) \subset E_p^{\Gamma'}$$

for $p \in U^{\cong}(\Gamma;d) \cap X^{\cong}(\Gamma')$. Here we note $\Gamma \supset \Gamma'$: We take a Γ-invariant connection ∇ of E on $U^{\cong}(\Gamma;d)$ so that each of E^Γ is a totally geodesic subbundle and that the curvature of ∇ is zero on each fiber of $U^{\cong}(\Gamma;d) \to X^{\cong}(\Gamma)$. Then we can use the parallel transport with respect to ∇ along the path contained in the fiber of $U^{\cong}(\Gamma;d) \to X^{\cong}(\Gamma)$ to extend $E(\Gamma)$ to $U^{\cong}(\Gamma;d)$.

We next consider $p \in X(\Gamma;\Gamma_1)$. We may assume that $E_p(\Gamma_1)$ is defined. Then we define $E_p(\Gamma)$ as the orthonormal complement of $E_p(\Gamma_1)$ in E_p^Γ. We extend them to its neighborhood. We thus obtain

$$E_p \cong E_p(\Gamma_1) \oplus E_p(\Gamma) \oplus E_p^\perp.$$

We can continue by a downward induction on $\#\Gamma$, k and obtain the decomposition (35.41).

To perform our construction of s_ϵ we also need the following lemma.

Lemma 35.42. *Let $f : M \to N$ be a proper submersion between smooth manifolds and F be a vector bundle on M. We fix a smooth triangulation of N. Let s be a section of F. Then there exists a family s^ϵ of piecewise smooth sections of F such that*

(35.43.1) *s^ϵ converges to s in C^0 topology.*
(35.43.2) *s^ϵ is of general position to 0.*
(35.43.3) *$f : (s^\epsilon)^{-1}(0) \to N$ is piecewise linear with respect to some smooth triangulation of $(s^\epsilon)^{-1}(0)$ and a subdivision of given triangulation of N.*

Figure 35.3.

Proof. Since f is a submersion we may choose a triangulation of M and a subdivision of the given one on N with respect to which f is piecewise linear. (See [Mun66], [Whi40].) In other words, there exist simplicial complexes K_M, K_N and homeomorphisms $i_M : |K_M| \to M$, $i_N : |K_N| \to N$ with the following properties :

(35.44.1) The homeomorphisms i_M and i_N restrict to diffeomorphisms onto its image to each simplex.
(35.44.2) $i_N^{-1} \circ f \circ i_M$ is induced from a simplicial map. Namely it sends a simplex of K_M to a simplex of K_N and is affine on each simplex.

We next take a smooth triangulation of the total space F of our vector bundle so that the projection $F \to M$ is piecewise linear. By taking an appropriate subdivision of the simplicial decomposition, we may approximate our section s by a section $s^\epsilon : M \to F$ which is piecewise linear, C^0 close to s, and of general position to the zero section. (Existence of such s^ϵ is a standard result of piecewise linear topology. See, for example, [Hud69].) Then (35.44.1) and (35.44.2) are satisfied. Since s^ϵ is piecewise linear, which is affine on each simplex, it follows that the intersection of $(s^\epsilon)^{-1}(0)$ with each simplex is affine. Hence we can find a subdivision of K_M and K_N such that $(s^\epsilon)^{-1}(0)$ is a subcomplex and the restriction of f to $(s^\epsilon)^{-1}(0)$ is piecewise linear. □

We remark that (35.43.3) implies that the mapping cone of $f : (s^\epsilon)^{-1}(0) \to N$ has a smooth triangulation.

We now start the construction of our section s_ϵ. We will put

$$s_\epsilon = \bigoplus_i s_\epsilon^{\Gamma_i} \oplus s_\epsilon^{\Gamma} \oplus 0 \tag{35.45}$$

according to our decomposition (35.41). Note the $E_p^\perp$-component is necessarily zero because of the G-invariance. (In other words it is zero since s_ϵ is single-valued.)

We will construct s_ϵ^Γ by the downward induction over the order of Γ.

Let Γ be maximal. We consider the vector bundle $E^\Gamma \to X^{\cong}(\Gamma)$. We remark that this is a vector bundle on a manifold and is not an orbi-bundle. So we can take a smooth section s_ϵ^Γ which is transversal to 0.

We extend s_ϵ^Γ to a section of $E(\Gamma)$ on $U^{\cong}(\Gamma)$ so that it is constant along the fiber of π_Γ.

We next consider $X^{\cong}(\Gamma) \cap S^{\Gamma_1}(d)$ assuming

$$\partial \text{ Int } X^d(\Gamma) = \bigcup_{\Gamma_1 \supset \Gamma} X(\Gamma;\Gamma_1) \tag{35.47}$$

and the right hand side are disjoint with Γ_1 maximal. By assumption we have defined s^{Γ_1} already. We now apply Lemma 35.42 to

$$\pi_\Gamma : (s^{\Gamma_1})^{-1}(0) \cap X(\Gamma;\Gamma_1) \to (s^{\Gamma_1})^{-1}(0) \cap X(\Gamma_1)$$

and the bundle $E(\Gamma) \to (s^{\Gamma_1})^{-1}(0) \cap X(\Gamma;\Gamma_1)$. We then obtain s^Γ on $(s^{\Gamma_1})^{-1}(0) \cap X(\Gamma;\Gamma_1)$. We extend it to $X(\Gamma;\Gamma_1) \setminus (s^{\Gamma_1})^{-1}(0)$ in an arbitrary way. (It does not matter how we extend since it will not change the zero set.)

We have thus defined $s_\epsilon = s^{\Gamma_1} \oplus s^\Gamma$ on (35.47). Note on

$$\text{Int } X^d(\Gamma) \setminus \partial \text{ Int } X^d(\Gamma)$$

E is decomposed to $E^\Gamma = E(\Gamma)$ and $E^\perp$. (We remark that we decompose E^Γ to $E(\Gamma_1) \oplus E(\Gamma)$ only at their boundaries.) On the boundary we defined the section of $E^\Gamma \cong E(\Gamma) \oplus E(\Gamma_1)$ already which is of general position relative to zero. We can then extend it to $\text{Int } X^d(\Gamma)$ so that it is of general position to zero. (Note $E^\perp$ component is necessarily zero again.) We then extend this to its neighborhood so that it is constant in each of π_Γ fibers.

Now the main induction step goes as follows : Assuming $s^{\Gamma'}$ is defined for $\#\Gamma' > \#\Gamma$, we consider Γ. Take the decomposition

$$\partial \text{Int } X^d(\Gamma) = \bigcup X(\Gamma;\Gamma_1,\Gamma_2,\cdots,\Gamma_k) \tag{35.48}$$

of the boundary of $\partial \text{Int } X^d(\Gamma)$. We then define s^Γ on $X(\Gamma;\Gamma_1,\Gamma_2,\cdots,\Gamma_k)$ by a downward induction on k.

We consider a chain of isotropy groups $\Gamma_1,\cdots,\Gamma_k$ given as in (35.40) for which $X(\Gamma;\Gamma_1,\Gamma_2,\cdots,\Gamma_k)$ is nonempty. Let k be maximal among such choices. We now apply Lemma 35.42 to

$$\begin{aligned} \pi_{\Gamma_k} &: (s^{\Gamma_1} \oplus \cdots \oplus s^{\Gamma_k})^{-1}(0) \cap X(\Gamma;\Gamma_1,\Gamma_2,\cdots,\Gamma_k) \\ &\to (s^{\Gamma_1} \oplus \cdots \oplus s^{\Gamma_k})^{-1}(0) \cap X(\Gamma_k;\Gamma_1,\Gamma_2,\cdots,\Gamma_{k-1}), \end{aligned} \tag{35.49}$$

and $E(\Gamma) \to X(\Gamma;\Gamma_1,\Gamma_2,\cdots,\Gamma_k)$. Here we remark that the well-definedness of (35.49) is a consequence of compatibilities of π and r stated in Definitions 35.25 and 35.30.

We thus obtain s^{Γ} on $(s^{\Gamma_1}\oplus\cdots\oplus s^{\Gamma_k})^{-1}(0)\cap X(\Gamma;\Gamma_1,\Gamma_2,\cdots,\Gamma_k)$ which we extend to $X(\Gamma;\Gamma_1,\Gamma_2,\cdots,\Gamma_k)$ in an arbitrary way.

Now we can extend s^{Γ} to various $X(\Gamma;\Gamma_1,\Gamma_2,\cdots,\Gamma_\ell)$ by a downward induction on ℓ using an appropriate relative version of Lemma 35.42. Namely we assume s^{Γ} is defined on $X(\Gamma;\Gamma_1,\Gamma_2,\cdots,\Gamma_k)$ for $k>\ell$ then s^{Γ} is defined on $\partial X(\Gamma;\Gamma_1,\Gamma_2,\cdots,\Gamma_\ell)$. Then we extend it to $X(\Gamma;\Gamma_1,\Gamma_2,\cdots,\Gamma_\ell)$ by applying a relative version of Lemma 35.42 to

$$\begin{aligned}&(s^{\Gamma_1}\oplus\cdots\oplus s^{\Gamma_\ell})^{-1}(0)\cap X(\Gamma;\Gamma_1,\Gamma_2,\cdots,\Gamma_\ell)\\ &\to (s^{\Gamma_1}\oplus\cdots\oplus s^{\Gamma_\ell})^{-1}(0)\cap X(\Gamma_\ell;\Gamma_1,\Gamma_2,\cdots,\Gamma_{\ell-1})\end{aligned}$$

and a bundle $E(\Gamma)$.

Figure 35.4

We thus constructed s^{Γ} on (35.30). Again we extend s_{Γ} to $\operatorname{Int} X^d(\Gamma)$ so that it is of general position to 0.

Thus we have constructed s_ϵ on the union

$$\bigcup_{\Gamma}\operatorname{Int} X^d(\Gamma). \tag{35.50}$$

We will extend this to X as follows : We first remark

$$X\setminus\bigcup_{\Gamma}\operatorname{Int} X^d(\Gamma)=\bigcup_{\Gamma}\left((U^{\cong}(\Gamma;d)\cap\pi_{\Gamma}^{-1}(\operatorname{Int} X^d(\Gamma)))\setminus X^{\cong}(\Gamma)\right).$$

Let

$$p\in(U^{\cong}(\Gamma;d)\cap\pi_{\Gamma}^{-1}(\operatorname{Int} X^d(\Gamma)))\setminus X^{\cong}(\Gamma)$$

and consider

$$r_\Gamma(d)(p) \in S^\Gamma(d) \subseteq \bigcup_{\Gamma' \subset \Gamma,\ \Gamma \neq \Gamma'} \operatorname{Int} X^d(\Gamma')$$

and

$$\pi_\Gamma(p) \in \operatorname{Int} X^d(\Gamma).$$

Figure 35.5

By construction, $s^\Gamma(\pi_\Gamma(p))$ coincides with $s^\Gamma(r_\Gamma(d)(p))$ under the identification

$$E_{\pi_\Gamma(p)}(\Gamma) \cong E_{r_\Gamma(d)(p)}(\Gamma).$$

We now put

$$s_\epsilon(p) = s^\Gamma(\pi_\Gamma(p)) + \frac{\sqrt{\rho_\Gamma(p)}}{d} \sum_{\Gamma'' \neq \Gamma} s^{\Gamma''}(r_\Gamma(d)(p)). \tag{35.51}$$

This section coincides with previously defined one when $\sqrt{\rho_\Gamma(p)} = 0$ or d. Hence it defines a piecewise smooth section on X.

We remark that by definition

$$s_\epsilon^{-1}(0) \cap (U^{\cong}(\Gamma; d) \cap \pi_\Gamma^{-1}(\operatorname{Int} X^d(\Gamma)))$$

is the cone of the map

$$\pi_\Gamma : s_\epsilon^{-1}(0) \cap S^\Gamma(d) \to X^{\cong}(\Gamma).$$

Since we constructed our section applying Lemma 35.42 repeatedly, it follows that this cone is has a smooth triangulation. (We use (35.31.7) and the argument of §3,4 [Gor78] for this.) On (35.50) we have a transversality and hence (35.21.2) holds.

(35.21.3) and (35.21.4) are also obvious from construction. The proof of Proposition 35.20 is now complete. □

35.4. Single valued perturbation of a space with Kuranishi structure.

In this subsection we generalize Proposition 35.20 to the case of the obstruction bundle of Kuranishi structure.

Proposition 35.52. *Let X be given a Kuranishi structure that has a tangent bundle in the sense of Definition* A1.14*. Let $\{(V_p, E_p, \Gamma_p, \psi_p, s_p)\}_{p\in P}$ be a good coordinate system. Then there exists a family of piecewise smooth sections $\mathfrak{s}^\epsilon = \{\mathfrak{s}^\epsilon_p\}$ parameterized by ϵ so that $X^{\mathfrak{s}^\epsilon} = \bigcup_p (\mathfrak{s}^\epsilon_p)^{-1}(0)$ has the following properties for any Γ.*

(35.53.1) $X^{\mathfrak{s}^\epsilon}_{\cong}(\Gamma) := \bigcup_p (\mathfrak{s}^\epsilon_p)^{-1}(0) \cap V_p^{\cong}(\Gamma)$ *is a PL manifold.*
(35.53.2) *The dimension of $X^{\mathfrak{s}^\epsilon}_{\cong}(\Gamma)$ is $d(\Gamma; p; k)$, which depends only on the connected component of $X^{\mathfrak{s}^\epsilon}_{\cong}(\Gamma)$.*
(35.53.3) $\bigcup_p (\mathfrak{s}^\epsilon_p)^{-1}(0)/\Gamma_p$ *has a triangulation compatible with the smooth structures of $X^{\mathfrak{s}^\epsilon}_{\cong}(\Gamma)$.*
(35.53.4) $\lim_{\epsilon\to 0} \mathfrak{s}^\epsilon = s$*, where s is the Kuranishi map of the given Kuranishi structure, and the convergence is a C^0 convergence.*

For the proof we need the following relative version of Proposition 35.20. In the next proposition we say a piecewise smooth single-valued section of an orbi-bundle $E/G \to X = M/G$ to be *normally conical* if the following holds :

(1) There is a decomposition of $X = M/G$ to

$$\bigcup_\Gamma \mathrm{Int}X^{\cong}(\Gamma) \cup \bigcup_\Gamma (X^{\cong}(\Gamma) \setminus \mathrm{Int}X^{\cong}(\Gamma))$$

as in (35.37).
(2) On $\mathrm{Int}X^{\cong}(\Gamma)$ the $E^\perp$-component s is of general position to 0. (The E^Γ component is necessarily 0.)
(3) On $X^{\cong}(\Gamma) \setminus \mathrm{Int}X^{\cong}(\Gamma)$, the section s is given by (35.51).

Proposition 35.20'. *Let X be a global quotient, K a compact subset and U a neighborhood of K. Let s be a C^0-section of the orbi-bundle $E/G \to X$. We assume that s satisfies* (35.21) *on U and is normally conical in the above sense. Then there exists a sequence of single-valued piecewise smooth sections s_ϵ converging to s in C^0 sense satisfying* (35.21) *such that $s_\epsilon = s$ on K.*

The proof is the same as Proposition 35.20 and is omitted.

Proof of Proposition 35.52. The proof is by induction on $p \in P$ with respect to the order $<$. If p is minimal, we apply Proposition 35.20 to obtain $\mathfrak{s}_p^\epsilon$. Let us assume that we have $\mathfrak{s}_q^\epsilon$ for every $q < p$. We consider $\mathfrak{s}_q^\epsilon$ and the image $\phi_{pq}(V_{pq})$. We restrict $\mathfrak{s}_q^\epsilon$ on the image $\phi_{pq}(V_{pq})$ and use the embedding $\hat{\phi}_{pq}$ to obtain a section of $E_q|_{\phi_{pq}(V_{pq})} \to V_{pq}$. We can extend it to its neighborhood, so that the compatibility in the sense of Definition A1.21 is satisfied.

We remark that the required properties (35.53.1) - (35.53.4) above are satisfied on the tubular neighborhood $N_{\phi_{pq}(V_{pq})}$ if it is satisfied by $\mathfrak{s}_q^\epsilon$.

Now we can use Propositions 35.20' to obtain the section $\mathfrak{s}_p^\epsilon$. The proof of Proposition 35.52 is complete. □

35.5. Proof of Theorem 34.11.

Now we will use Proposition 35.52 to prove Theorem 34.11. Let J be a spherically positive compatible almost complex structure on (M,ω).

Let L be a Lagrangian submanifold of M and consider the moduli space

$$X = \mathcal{M}^{\text{main}}_{k+1,\ell}(\beta; P_1, \cdots, P_k)$$

as in §9. Here $\mathcal{M}^{\text{main}}_{k+1,\ell}(\beta)$ is the moduli space of genus zero stable maps $(\Sigma, \partial\Sigma) \to (M, L)$ with $k+1$ boundary marked points, ℓ interior marked points and of homology class β. And

$$\mathcal{M}^{\text{main}}_{k+1,\ell}(\beta; P_1, \cdots, P_k) = \mathcal{M}^{\text{main}}_{k+1,\ell}(\beta) \times_{L^k} (P_1 \times \cdots \times P_k).$$

We constructed a Kuranishi structure on X in Chapter 7, §29. We apply Proposition 35.52 to obtain $\mathfrak{s}^\epsilon$. Then the perturbed moduli space $\mathcal{M}_{k+1}(\beta; P_1, \cdots, P_k)^{\mathfrak{s}^\epsilon}$ has a smooth triangulation. Here we remark that

$$\mathcal{M}^{\text{main}}_{k+1}(\beta; P_1, \cdots, P_k) = \mathcal{M}^{\text{main}}_{k+1,0}(\beta; P_1, \cdots, P_k).$$

(In this subsection we will deal with the moduli space $\mathcal{M}^{\text{main}}_{k+1,0}(\beta; P_1, \cdots, P_k)$, since the operators $\mathfrak{q}$ defined via the moduli space $\mathcal{M}^{\text{main}}_{k+1,\ell}(\beta; P_1, \cdots, P_k)$ are not defined over $\mathbb{Z}$ because of the absence of a necessary algebraic counterpart of handling the cyclic symmetry. (See §13.))

We put

$$\mathcal{M}^{\text{main}}_{k+1}(\beta; P_1, \cdots, P_k)^{\mathfrak{s}^\epsilon}_{\text{free}} = X^{\mathfrak{s}^\epsilon}_{\cong}(1).$$

This space is smooth and of correct dimension.

We now study $X^{\mathfrak{s}^\epsilon}_{\cong}(\Gamma)$ for $\Gamma \neq \{1\}$. First note

$$\mathcal{M}^{\text{main}}_{k+1}(\beta; P_1, \cdots, P_k)^{\mathfrak{s}^\epsilon}_{\text{fix}} = \bigcup_{\Gamma \neq \{1\}} X^{\mathfrak{s}^\epsilon}_{\cong}(\Gamma).$$

Let $((\Sigma, \vec{z}), v, \vec{x})$ be an element of $\mathcal{M}^{\rm main}_{k+1}(\beta; P_1, \cdots, P_k)$: Namely Σ is a genus zero bordered Riemann surface, $v : (\Sigma, \partial\Sigma) \to (M, L)$ is pseudo-holomorphic, $\vec{z} = (z_0, \cdots, z_k)$ are boundary marked points of Σ, and

$$\vec{x} = (x_1, \cdots, x_k), \quad x_i \in P_i \quad \text{satisfying } v(z_i) = f_i(x_i).$$

Here $f_i : P_i \to L$ are smooth singular chains of L which we write just as P_i by an abuse of notation.

We recall that the genus of Σ is zero and the disc components cannot have non-trivial automorphism groups since it comes at least one special points, i.e., either marked or nodal points, on the boundary. Therefore for every non-trivial element $\varphi \in \mathrm{Aut}((\Sigma, \vec{z}), v)$ and any sphere component $S^2_i \cong S^2 \subset \Sigma$ preserved by φ, the automorphism φ acts as the multiplication by $e^{2\pi\sqrt{-1}\ell/k}$,

$$z \mapsto e^{2\pi\sqrt{-1}\ell/k} z$$

with the identification $S^2_i = \mathbb{C} \cup \{\infty\}$. And φ interchanges the other components : This is because any finite subgroup of $PSL(2;\mathbb{C}) = \mathrm{Aut}(S^2)$ which fix ∞ is conjugate to such a group.

As a consequence, the quotient space $(\overline{\Sigma}, \vec{\overline{z}}) = \Sigma/\mathrm{Aut}((\Sigma, \vec{z}), v)$ is again a (pointed) bordered Riemann surface of genus zero. The pseudo-holomorphic map v induces a map $\overline{v} : \Sigma/\mathrm{Aut}((\Sigma, \vec{z}), v) \to M$ on it. We call $((\overline{\Sigma}, \vec{\overline{z}}), \overline{v}, \vec{x})$ the *reduced model* of $((\Sigma, \vec{z}), v, \vec{x})$. We remark that the reduced model may be unstable. Namely there may appear a sphere component with two singular points where the map $\overline{v}$ is trivial. Even in the case the reduced model is stable, it may have a nontrivial automorphism.

Remark 35.54. We remark that the notions of "trivial automorphism" and "somewhere injective" are two different notions : Somewhere injectivity implies triviality of the automorphism group, but not the other way around. For example, there is a branched covering $S^2 \to S^2$ with no nontrivial automorphism. *For the abstract perturbation $\mathfrak{s}^\epsilon$, its transversality to zero is related to the existence of nontrivial automorphism but not to the somewhere injectivity.* (Somewhere injectivity is essential if one uses perturbations only of J to achieve transversality.)

We now compare the virtual dimension of $((\Sigma, \vec{z}), v, \vec{x}) \in \mathcal{M}^{\rm main}_{k+1}(\beta; P_1, \cdots, P_k)$ with that of its reduced model. We begin with the discussion of the deformation complex of a multiple sphere. Let $\alpha \in \pi_2(M)$ and $\widetilde{\mathcal{M}}^{\rm reg}(M; \alpha)$ be the set of pseudo-holomorphic maps $u : S^2 \to M$ with $[u] = \alpha$. For $u \in \widetilde{\mathcal{M}}^{\rm reg}(M; \alpha)$ we define $\mathfrak{R}_m(u) \in \widetilde{\mathcal{M}}^{\rm reg}(M; m\alpha)$ by $\mathfrak{R}_m(u)(z) = u(z^m)$.

For each $v \in \widetilde{\mathcal{M}}^{\rm reg}(M; m\alpha)$, we consider the linearization

$$D_v\overline{\partial} : \Gamma(S^2; v^*TM) \to \Gamma(S^2; \Lambda^{0,1}(v^*TM))$$

of the Cauchy-Riemann section $\overline{\partial}$. We denote by $C(v) = (C_0(v), C_1(v), D_v\overline{\partial})$ the elliptic complex, where we write

$$C_0(v) = \Gamma(S^2; v^*TM), \quad C_1(v) = \Gamma(S^2; \Lambda^{0,1}(v^*TM)).$$

We consider the assignment of the pull-back complex $C(\mathfrak{R}_m(u))$ to $u \in \widetilde{\mathcal{M}}^{\rm reg}(M;\alpha)$ on which the group $\mathbb{Z}_m$ acts. We regard this assignment of $\mathbb{Z}_m$-modules as a family $\mathbb{Z}_m$-equivariant index over $\widetilde{\mathcal{M}}^{\rm reg}(M;\alpha)$.

Lemma 35.55. *The index of $C(\mathfrak{R}_m(u))$ as a $\mathbb{Z}_m$-module is*

$$2c_1(M)(\alpha)\,\mathrm{Reg}_{\mathbb{Z}_m} \oplus 2n\mathbf{1}.$$

Here $\mathrm{Reg}_{\mathbb{Z}_m}$ is the regular representation of $\mathbb{Z}_m$ and $\mathbf{1}$ is its trivial representation.

Proof. Let γ be an element of $\mathbb{Z}_m$ with $\gamma \neq$ unit. We use the Lefschetz fixed point formula by Atiyah-Bott [AtBo67] to obtain

$$\sum_{*=0,1} (-1)^* \,\mathrm{Tr}\,(\gamma : H^*(C(\mathfrak{R}_m(u)) \to H^*(C(\mathfrak{R}_m(u))) = 2n.$$

(Note there are only two fixed points of γ and we take the trace over $\mathbb{R}$.) On the other hand, the numerical index of $D_{\mathfrak{R}_m(u)}\overline{\partial}$ is $2n + 2mc_1(M)(\alpha)$, which coincides with the super-trace of the unit element $e \in \mathbb{Z}_m$. The lemma follows immediately. □

We remark that we can also prove Lemma 35.55 by directly calculating the kernel and cokernel without using [AtBo67].

In particular, Lemma 35.55 implies that the $\mathbb{Z}_m$-invariant part of the index of $C(\mathfrak{R}_m(u))$ for the $\mathbb{Z}_m$-cover of a holomorphic sphere is equal to the index of $C(u)$ for its reduced model. We will use this fact in the proof of Proposition 35.63 coming later. Let $((\Sigma,\vec z), v, \vec x) \in \mathcal{M}^{\rm main}_{k+1}(\beta : P_1, \cdots, P_k)$ and $((\overline{\Sigma}, \overline{\vec z}), \overline{v}, \vec x)$ be its reduced model. We first recall the definition of the deformation complex of $((\Sigma,\vec z), v, \vec x)$ which is an elliptic complex acted upon by the group Γ of automorphisms of $((\Sigma,\vec z), v)$.

We decompose Σ into irreducible components $\Sigma = \bigcup_a \Sigma_a$ where Σ_a is a sphere or a disc and put $v_a = v|_{\Sigma_a}$. We consider the elliptic complex

$$C(v_a) = (C_0(v_a), C_1(v_a), D_{v_a}\overline{\partial})$$

where

$$D_{v_a}\overline{\partial} \;:\; C_0(v_a) = \Gamma(\Sigma_a, \partial\Sigma_a; v_a^*TM, v_a^*TL) \to C_1(v_a) = \Gamma(\Sigma_a; \Lambda^{0,1}(v_a^*TM)).$$

(The boundary condition v_a^*TL is empty if $\Sigma_a = S^2$.) For each singular point $z_i^{\rm sing}$ we take $z_{i,1}^{\rm sing} \in \Sigma_{a(i,1)}$, $z_{i,2}^{\rm sing} \in \Sigma_{a(i,2)}$ which are $z_i^{\rm sing}$ in Σ. We put

$$\begin{cases} \widetilde C_0((\Sigma,\vec z),v)^+ = \bigoplus_a C_0(v_a) \\ \widetilde C_0((\Sigma,\vec z),v) = \left\{ (W_a) \in \bigoplus_a C_0(v_a) \,\middle|\, W_{a(i,1)}(z_{i,1}^{\rm sing}) = W_{a(i,2)}(z_{i,2}^{\rm sing}) \right\}. \end{cases} \tag{35.56}$$

We put $\widetilde C_1((\Sigma,\vec z),v) = \bigoplus_a C_1(v_a)$. The operators $D_{v_a}\overline\partial$ induce

$$D_v\overline\partial : \widetilde C_0((\Sigma,\vec z),v) \to \widetilde C_1((\Sigma,\vec z),v).$$

Let $\mathrm{Aut}(\Sigma,\vec z)$ be the group of all automorphisms of $(\Sigma,\vec z)$. We have a canonical homomorphism of its Lie algebra $\mathrm{aut}(\Sigma,\vec z)$ into $\widetilde C_0((\Sigma,\vec z),v)$: Note that by the definition of $\mathrm{Aut}(\Sigma,\vec z)$ any element of $\mathrm{aut}(\Sigma,\vec z)$ has its value zero at the singular points. The stability condition implies that this homomorphism is injective and so we may regard $\mathrm{aut}(\Sigma,\vec z)$ as a subspace of $\widetilde C_0((\Sigma,\vec z),v)$. Moreover the image of $\mathrm{aut}(\Sigma,\vec z)$ lies in the kernel of $D_v\overline\partial$. Therefore we have the following complex

$$0 \to \mathrm{aut}(\Sigma,\vec z) \to \widetilde C_0((\Sigma,\vec z),v) \to \widetilde C_1((\Sigma,\vec z),v). \tag{35.57}$$

We put

$$\begin{aligned}\widetilde C_0((\Sigma,\vec z),v,\vec x) = \Big\{ \Big((W_a),(\mathrm{v}_i)\Big) \,\Big|\, & (W_a) \in \widetilde C_0((\Sigma,\vec z),v),\ \mathrm{v}_i \in T_{x_i}P_i, \\ & W_{a_i}(z_i) = (d_{x_i}f_i)(\mathrm{v}_i)\Big\}.\end{aligned}$$

Since $\mathrm{Aut}(\Sigma,\vec z)$ fixes the marked points $\vec z$, it induces an action on $\mathcal M_{k+1}^{\rm main}(\beta : P_1,\cdots,P_k)$ and so its Lie algebra $\mathrm{aut}(\Sigma,\vec z)$ injects to $\widetilde C_0((\Sigma,\vec z),v,\vec x)$. This leads us to define

$$\begin{aligned} C_0((\Sigma,\vec z),v,\vec x) &:= \widetilde C_0((\Sigma,\vec z),v,\vec x)/\,\mathrm{aut}(\Sigma,\vec z) \\ C_1((\Sigma,\vec z),v,\vec x) &:= \widetilde C_1((\Sigma,\vec z),v). \end{aligned}$$

Here $z_i \in \Sigma_{a_i}$. The operator $D_v\overline\partial$ also induces a homomorphism $C_0((\Sigma,\vec z),v,\vec x) \to C_1((\Sigma,\vec z),v,\vec x)$.

We denote by

$$C((\Sigma,\vec z),v,\vec x), \quad \widetilde C((\Sigma,\vec z),v,\vec x)$$

the complexes

$$\begin{aligned} D_v\overline\partial &: C_0((\Sigma,\vec z),v,\vec x) \to C_1((\Sigma,\vec z),v,\vec x), \\ D_v\overline\partial &: \widetilde C_0((\Sigma,\vec z),v,\vec x) \to \widetilde C_1((\Sigma,\vec z),v,\vec x), \end{aligned}$$

respectively. The group $\Gamma = \mathrm{Aut}(\Sigma,\vec z)$ acts on these complexes in an obvious way.

To describe the relation between the deformation complex of $((\Sigma,\vec z),v,\vec x)$ and that of its reduced model, we need one more notation.

Definition 35.58. We call a point $z \in \Sigma$ a *free fixed point* if the following holds :

(35.59.1) z is not a singular point.

(35.59.2) z is on a sphere component S^2_a of Σ such that there exists $\gamma \in \Gamma$ which preserves S^2_a and acts nontrivially on S^2_a. Moreover $\gamma(z) = z$.

For $\Gamma' \subset \Gamma$ we denote by $F(\Gamma')$ the set of all z satisfying (35.59) for $\gamma \in \Gamma'$.

Note here we are studying the case where there is no interior marked point. For the case where there are interior marked points, we need to assume z is not a marked point in (35.59.1) either.

For $\Gamma' \subset \Gamma$ we define

$$C(F(\Gamma')) = \bigoplus_{z \in F(\Gamma')} \mathbb{C}[z]$$

the free vector space generated by $F(\Gamma')$.

In case Γ'' normalizes a subgroup $\Gamma'(\subset \Gamma)$, Γ' acts on $F(\Gamma'')$ in an obvious way and so induces an action on $C(F(\Gamma'))$. We put

$$C(F(\Gamma'))^{\Gamma''} = \{v \in C(F(\Gamma')) \mid \forall \gamma \in \Gamma'',\ \gamma v = v\}.$$

It is easy to see that

$$\dim_{\mathbb{C}} C(F(\Gamma'))^{\Gamma''} = \#\left(F(\Gamma')/\Gamma''\right)$$

and in particular

$$\dim_{\mathbb{C}} C(F(\Gamma))^{\Gamma} = \#\left(F(\Gamma)/\Gamma\right). \tag{35.60}$$

Definition 35.61. For each sphere component S^2_a, we define its *distance from the disc components* as the *minimal edge distance* of the vertex corresponding to the component S^2_a from the vertices corresponding to the disc components in the dual graph of Σ.

We recall that the minimal edge distance between two vertices in a graph is defined to be the minimum number of edges in all connected paths between the two in the graph. We denote by Σ_d the union of all disc components and the sphere components whose distance from the disc components are $\le d$. Σ_0 is by definition the union of all the disc components. See Figure 35.6.

Figure 35.6.

Let $(S_a^2, \vec{p}_a, o_a)$ be a sphere component of $((\Sigma, \vec{z}), v)$, whose distance from the disc components is d. Here o_a is the point where S_a^2 is attached to Σ_{d-1} and $\vec{p}_a$ is the set of other singular points, i.e., those to which some sphere components of distance $d+1$ from the disc components are attached. Put $n_a = \#\vec{p}_a$ and let Γ_a be the group of automorphisms on $(S_a^2, \vec{p}_a, o_a)$ consisting of the restrictions of some elements of Γ to S_a^2. Denote by $\mathcal{CF}_{m+1}(\mathbb{C}P^1)$ the moduli space (i.e., divided by the action of $PSL_2(\mathbb{C})$) of $m+1$ points on $\mathbb{C}P^1$. Denote by $\mathcal{CF}_{m+1}^{\Gamma_a}(\mathbb{C}P^1)$ the moduli space of distinct $m+1$ points on $\mathbb{C}P^1$ with the symmetry group Γ_a.

We define

$$\rho^{\Gamma_a}(S_a, \vec{p}_a, o_a) = \begin{cases} -\dim_{\mathbb{R}} \mathrm{Aut}(S_a, \vec{p}_a, o_a)^{\Gamma_a} & \text{if } n_a + 1 < 3 \\ \dim_{\mathbb{R}} \mathcal{CF}_{n_a+1}^{\Gamma_a}(\mathbb{C}P^1) & \text{if } n_a + 1 \geq 3. \end{cases}$$

Let $(\overline{S}_b, \vec{\overline{p}}_b, \overline{o}_b)$ be a sphere component of the reduced model $((\overline{\Sigma}, \vec{\overline{z}}), \overline{v})$. Here $\overline{o}_b$ is the point of $\overline{S}_b$ at which it is attached to $\overline{\Sigma}_{d-1}$, and $\vec{\overline{p}}_b$ are those at which some sphere components of distance $d+1$ are attached. Put $\overline{n}_b = \#\vec{\overline{p}}_b$. We define

$$\overline{\rho}(\overline{S}_b, \vec{\overline{p}}_b, \overline{o}_b) = \begin{cases} -\dim_{\mathbb{R}} \mathrm{Aut}(S_b, \vec{\overline{p}}_b, \overline{o}_b) & \text{if } \overline{n}_b + 1 < 3 \\ \dim_{\mathbb{R}} \mathcal{CF}_{\overline{n}_b+1}(\mathbb{C}P^1) & \text{if } \overline{n}_b + 1 \geq 3. \end{cases}$$

Let $(D_a^2; \vec{p}_a^{\,D}, \vec{p}_a^{\,S})$ be a disc component of Σ. Here $\vec{p}_a^{\,D} = (p_{a,1}^D, \cdots, p_{a,m_a}^D)$ be the boundary marked points and $\vec{p}_a^{\,S} = (p_{a,1}^S, \cdots, p_{a,n_a}^S)$ be the interior marked points. We put

$$\rho(D_a^2; \vec{p}_a^{\,D}, \vec{p}_a^{\,S}) = m_a + 2n_a - 3.$$

This number is the negative of the dimension of the automorphism group if $m_a + 2n_a \leq 3$ and is the dimension of appropriate moduli space (of $\mathbb{C}P^1$ with marked points) if $m_a + 2n_a \geq 3$. (We remark that the Γ-action is trivial on each disc component.)

We define $\overline{\rho}(\overline{D}^2_b; \vec{\overline{p}}_b^{\,D}, \vec{\overline{p}}_b^{\,S})$ by the same formula for the disc component $(\overline{D}^2_b; \vec{\overline{p}}_b^{\,D}, \vec{\overline{p}}_b^{\,S})$ of $\overline{\Sigma}$.

Let $\{S_{a(b)} \mid b \in \overline{I}\}$ be the complete set of representatives of the orbit space of the set of all sphere components of $((\Sigma, \vec{z}), v)$ under the action of Γ.

Let I^D and $\overline{I}^D$ be the set of disc components of Σ and $\overline{\Sigma}$, respectively. Note that there is a canonical identification between the two sets.

We define

$$\rho^{\Gamma}((\Sigma, \vec{z}), v) = \sum_{b \in \overline{I}} \rho^{\Gamma_{a(b)}}(S_{a(b)}, \vec{p}_{a(b)}, o_{a(b)}) + \sum_{a \in I^D} \rho(D^2_a; \vec{p}_a^{\,D}, \vec{p}_a^{\,S}).$$

For the reduced model, we define

$$\overline{\rho}((\overline{\Sigma}, \vec{\overline{z}}), \overline{v}) = \sum_{b \in \overline{I}} \rho(\overline{S}_b, \vec{\overline{p}}_b, \overline{o}_b) + \sum_{b \in \overline{I}^D} \overline{\rho}(\overline{D}^2_b; \vec{\overline{p}}_b^{\,D}, \vec{\overline{p}}_b^{\,S}).$$

Then we have the following :

Proposition 35.63.

$$\begin{aligned}
&\dim_{\mathbb{R}} \mathrm{Index}(\widetilde{C}((\Sigma, \vec{z}), v, \vec{x}))^{\Gamma} + \rho^{\Gamma}((\Sigma, \vec{z}), v) \\
&= \dim_{\mathbb{R}} \mathrm{Index}(\widetilde{C}((\overline{\Sigma}, \vec{\overline{z}}), \overline{v}, \vec{x})) + \overline{\rho}((\overline{\Sigma}, \vec{\overline{z}}), \overline{v}) + \dim_{\mathbb{R}} C(F(\Gamma))^{\Gamma}.
\end{aligned}$$

Proof. We begin with the following lemma.

Lemma 35.64.

$$\dim_{\mathbb{R}} \mathrm{Index}(\widetilde{C}((\Sigma, \vec{z}), v, \vec{x}))^{\Gamma} = \dim_{\mathbb{R}} \mathrm{Index}(\widetilde{C}((\overline{\Sigma}, \vec{\overline{z}}), \overline{v}, \vec{x})).$$

Proof. Let us consider the complex $\widetilde{C}((\Sigma, \vec{z}), v, \vec{x})^+$

$$D_v\overline{\partial} : \widetilde{C}_0((\Sigma, \vec{z}), v, \vec{x})^+ \to \widetilde{C}_1((\Sigma, \vec{z}), v, \vec{x})$$

where we replace $\widetilde{C}_0((\Sigma, \vec{z}), v, \vec{x})$ by $\widetilde{C}_0((\Sigma, \vec{z}), v, \vec{x})^+$. (See (35.56).)

We first prove

$$\mathrm{Index}(\widetilde{C}((\Sigma, \vec{z}), v, \vec{x})^+)^{\Gamma} = \mathrm{Index}(\widetilde{C}((\overline{\Sigma}, \vec{\overline{z}}), \overline{v}, \vec{\overline{x}})^+). \tag{35.65}$$

Note the index of $\widetilde{C}((\Sigma, \vec{z}), v, \vec{x})^+$ is the sum of indices of its components. Since the Γ-action is trivial on disc component, (35.65) is trivial for disc components. The part of sphere components of the left hand side is

$$\sum_{b \in \overline{I}} \mathrm{Index}(D_{v_{a(b)}}\overline{\partial})^{\Gamma_{a(b)}}. \tag{35.66}$$

Here $\{S_{a(b)} \mid b \in \overline{I}\}$ is the complete set of representatives of the Γ-orbit space of the set of all sphere components of $(\Sigma, \vec z, v)$ and the map $v_{a(b)}$ is the restriction of v to $S_{a(b)}$. The group $\Gamma_{a(b)}$ is a subgroup of Γ consisting of the elements which preserve $S_{a(b)}$. We remark that $\Gamma_{a(b)}$ is a cyclic group. Hence we can apply Lemma 35.55 to show that (35.66) is equal to the sum of $\text{Index}(D_{\overline{v}_b}\overline{\partial})$. Here $\overline{v}_b$ is the restriction of $\overline{v}$ to $S_{a(b)}/\Gamma_{a(b)} = \overline{S}_b \subset \overline{\Sigma}$. (35.65) follows.

We next remark that there exists an exact sequence

$$0 \to \widetilde{C}((\Sigma,\vec z),v,\vec x) \to \widetilde{C}((\Sigma,\vec z),v,\vec x)^+ \to \bigoplus_{x\in \text{Sing }\Sigma} T_{v(x)}M \to 0. \tag{35.67}$$

Here the $\bigoplus_{x\in\text{Sing }\Sigma} T_{v(x)}M$ is the sum over all singular points x of Σ.

We remark that Γ action on $\bigoplus_{x\in\text{Sing }\Sigma} T_{v(x)}M$ is by interchanging the factors. It follows that

$$\dim\left(\bigoplus_{x\in\text{Sing }\Sigma} T_{v(x)}M\right)^{\Gamma} = 2n\,\#((\text{Sing }\Sigma)/\Gamma).$$

We remark that $(\text{Sing }\Sigma)/\Gamma \cong \text{Sing }\overline{\Sigma}$. Therefore (35.65), (35.67) and a similar exact sequence for $\overline{\Sigma}$ imply the lemma. $\square$

We compare $\rho^{\Gamma_{a(b)}}(S_{a(b)},\vec p_{a(b)},o_{a(b)})$ and $\overline{\rho}(\overline{S}_b,\vec{\overline{p}}_b,\overline{o}_b)$. Note that $\Gamma_{a(b)}$ is isomorphic to $\mathbb{Z}_{m_{a(b)}}$. If $m_{a(b)} = 1$, $\Gamma_{a(b)} = \{e\}$ and so it is obvious that

$$\rho^{\Gamma_{a(b)}}(S_{a(b)},\vec p_{a(b)},o_{a(b)}) = \overline{\rho}(\overline{S}_b,\vec{\overline{p}}_b,\overline{o}_b). \tag{35.68}$$

Note that there are no free fixed points on $S_{a(b)}$.

From now on, we assume that $m_{a(b)} > 1$. The marked point $o_{a(b)}$ is a $\Gamma_{a(b)}$ fixed point. Denote by $q_{a(b)}$ the other fixed point on $S_{a(b)}$. Let $\overline{q}_b$ be its image of the reduction in $\overline{S}_b$. There are two cases: $q_{a(b)} \in \vec p_{a(b)}$ or $q_{a(b)} \notin \vec p_{a(b)}$.

First consider the case $q_{a(b)} \in \vec p_{a(b)}$, $\overline{q}_b \in \vec{\overline{p}}_b$. We claim that

$$\rho^{\Gamma_{a(b)}}(S_{a(b)},\vec p_{a(b)},o_{a(b)}) = \overline{\rho}(\overline{S}_b,\vec{\overline{p}}_b,\overline{o}_b). \tag{35.69}$$

If $n_{a(b)} = 1$, then $\overline{n}_b = 1$, $\vec p_{a(b)} = q_{a(b)}$, $\vec{\overline{p}}_b = \overline{q}_b$. We have

$$\text{Aut}(S_{a(b)},q_{a(b)},o_{a(b)})^{\Gamma_{a(b)}} \cong \text{Aut}(S_{a(b)},q_{a(b)},o_{a(b)}).$$

Both $\text{Aut}(S_{a(b)},q_{a(b)},o_{a(b)})^{\Gamma_{a(b)}}$ and $\text{Aut}(\overline{S}_b,\overline{q}_b,\overline{o}_b)$ are isomorphic to $\mathbb{C}^*$ and hence (35.69) follows.

In case $n_{a(b)} \ge 2$, we have $\overline{n}_b \ge 2$ and

$$\mathcal{CF}^{\Gamma_{a(b)}}_{n_{a(b)}+1}(S_{a(b)};\vec p_{a(b)},o_{a(b)}) \cong \mathcal{CF}_{\overline{n}_b+1}(\overline{S}_b;\vec{\overline{p}}_b,\overline{o}_b).$$

Hence, we obtain (35.69).

We next consider the case $q_{a(b)} \notin \vec{p}_{a(b)}$, $\overline{q}_b \notin \vec{p}_b$. In this case, the position of $\overline{q}_b$ in the reduced model is not a part of the data of the reduced model, $(\overline{S}_b, \vec{\overline{p}}_b, \overline{o}_b)$. We claim

$$\rho^{\Gamma_{a(b)}}(S_{a(b)}, \vec{p}_{a(b)}, o_{a(b)}) = \overline{\rho}(\overline{S}_b, \overline{o}_b) - 2. \tag{35.70}$$

If $n_{a(b)} = 0$, then $\overline{n}_b = 0$. Note that $q_{a(b)}$ is a fixed point on $S_{a(b)}$. Thus we find that the automorphism group of $(S_{a(b)}, o_{a(b)}, v|_{S_{a(b)}})$ is isomorphic to

$$\mathrm{Aut}(S_{a(b)}, \vec{p}_{a(b)}, o_{a(b)})^{\Gamma_{a(b)}} \cong \mathrm{Aut}(S_{a(b)}, q_{a(b)}, o_{a(b)}),$$

which is isomorphic to $\mathbb{C}^*$. On the other hand, $\mathrm{Aut}(\overline{S}_b, \overline{o}_b)$ is isomorphic to the semi-direct product $\mathbb{C}^* \times \mathbb{C}$ that is the group of all affine maps $z \mapsto Az + B$ with $A \neq 0$. (35.70) follows.

Suppose that $\overline{n}_b = 1$. Denote by $\overline{p}_b$ the unique point in $\vec{\overline{p}}_b$. In this case, we find that $\mathcal{CF}^{\Gamma_{a(b)}}_{n_{a(b)}+1}(\mathbb{C}P^1)$ is a point and $n_{a(b)} \geq 2$. On the other hand, $\mathrm{Aut}(\mathbb{C}P^1, \overline{p}_b, \overline{o}_b)$ is isomorphic to $\mathbb{C}^*$. (35.70) follows.

If $\overline{n}_b > 1$, then $\vec{p}_{a(b)}$ consists of $\overline{n}_b$ $\Gamma_{a(b)}$-orbits in $S_{a(b)} \setminus \{q_{a(b)}, o_{a(b)}\}$. Therefore we have

$$\dim_{\mathbb{R}} \mathcal{CF}^{\Gamma_{a(b)}}_{n_{a(b)}+1}(\mathbb{C}P^1) = 2\overline{n}_b.$$

On the other hand,

$$\dim_{\mathbb{R}} \mathcal{CF}_{\overline{n}_b+1}(\mathbb{C}P^1) = 2(\overline{n}_b + 1).$$

We have (35.70) also.

Note that $q_{a(b)}$ is a free fixed point if and only if $q_{a(b)} \notin \overline{p}_{a(b)}$. Therefore (35.70) is applied to the component S_a if it contains a free fixed point. Otherwise (35.68),(35.69) apply.

We remark that the contribution of disc component to $\rho^{\Gamma}((\Sigma, \vec{z}), v)$ coincides with that of disc components to $\overline{\rho}((\overline{\Sigma}, \vec{\overline{z}}), \overline{v})$.

Combining these, we obtain

$$\rho^{\Gamma}((\Sigma, \vec{z}), v) = \overline{\rho}((\overline{\Sigma}, \vec{\overline{z}}), \overline{v}) + \dim_{\mathbb{R}} C(F(\Gamma))^{\Gamma}.$$

This formula together with Lemma 35.64 implies Proposition 32.63. □

We next identify each side of Proposition 35.63 with the dimension of appropriate moduli space.

Definition 35.71. We say that $((\Sigma, \vec{z}), v, \vec{x})$ has the *same combinatorial type* as $((\Sigma', \vec{z}\,'), v', \vec{x}\,')$ if the following holds : There exists a homeomorphism $\Sigma \to \Sigma'$ preserving all the marked points together with the order. We also assume that the restriction of v to each component of Σ is homologous to the restriction of v' to the corresponding component of Σ'. We denote by $\mathbb{S}$ an equivalence class of this equivalence relation and call $\mathbb{S}$ the combinatorial type of $((\Sigma,\vec z),v,\vec x)$.

We denote by

$$\mathcal{M}^{\rm main}_{k+1}(\beta;P_1,\cdots,P_k)(\mathbb{S},\Gamma)\ (=\mathcal{M}^{\rm main}_{k+1}(\beta;P_1,\cdots,P_k)(\mathbb{S})^{\cong}(\Gamma))$$

the moduli space of $((\Sigma,\vec z),v,\vec x)$ with given combinatorial type $\mathbb{S}$ and isotropy group Γ.

Let $\mathbb{S}$ be the combinatorial type of $((\Sigma,\vec z),v,\vec x)$.

Lemma 35.72. *The left hand side of Proposition* 35.63 *is*

$$\dim_{\mathbb{R}}\mathcal{M}^{\rm main}_{k+1}(\beta;P_1,\cdots,P_k)(\mathbb{S},\Gamma)+\sum_i \deg P_i.$$

Proof. $\rho^{\Gamma}(\Sigma,\vec z,v)$ is the number of deformation parameters of $(\Sigma,\vec z,v)$ keeping the Γ-equivariance and the combinatorial type minus the dimension of the automorphism group of $(\Sigma,\vec z)$. (Note we did not include the parameter to resolve singularity of Σ in it. So this corresponds to the deformation keeping the combinatorial type.)

The condition that boundary marked points hit P_i reduces the dimension by $\sum_i \deg P_i$. Lemma 35.72 follows. □

By Proposition 35.52 we may take piecewise smooth perturbation $\mathfrak{s}$ so that each of $\mathcal{M}^{\rm main}_{k+1}(\beta;P_1,\cdots,P_k)^{\mathfrak{s}}(\Gamma)$ is a PL manifold whose dimension coincides with the virtual dimension. It is easy to see that we may choose $\mathfrak{s}$ so that it respects the stratification of the combinatorial type. Hence we may assume that

$$\begin{aligned}&\dim\mathcal{M}^{\rm main}_{k+1}(\beta;P_1,\cdots,P_k)^{\mathfrak{s}}(\mathbb{S},\Gamma)\\ &=\dim_{\mathbb{R}}\operatorname{Index}(\widetilde{C}((\Sigma,\vec z),v,\vec x))^{\Gamma}+\rho^{\Gamma}(\Sigma,\vec z,v)-\sum\deg P_i.\end{aligned}\tag{35.73}$$

To study the right hand side of Lemma 35.63 we need some notation. For each $((\Sigma,\vec z),v,\vec x)$ we consider its reduced model $((\overline{\Sigma},\vec{\overline{z}}),\overline{v},\vec x)$. We then add the image of elements in $F(\Gamma)$ to the reduced model as additional (interior) marked points. We denote them by $\vec z_+$ and the resulting stable map by $((\overline{\Sigma},\vec{\overline{z}},\vec z_+),\overline{v},\vec x)$.

Note

$$\#\vec z_+=\#F(\Gamma)/\Gamma.$$

We call $((\overline{\Sigma},\vec{\overline{z}},\vec z_+),\overline{v},\vec x)$ the *marked reduced model* of $((\Sigma,\vec z),v,\vec x)$. In this way, we have obtained a natural assignment

$$((\Sigma,\vec z),v,\vec x\,)\to((\overline{\Sigma},\vec{\overline{z}},\vec z_+),\overline{v},\vec x).$$

We remark that the reduced model $((\overline{\Sigma},\vec{\overline{z}}),\overline{v},\vec x)$ may be unstable. On the other hand the marked reduced model is always stable.

Let us denote by $\overline{\mathbb{S}}$ the combinatorial type of the marked reduced model. We consider $\mathcal{M}^{\rm main}_{k+1,\ell}(\overline{\beta};P_1,\cdots,P_k)(\overline{\mathbb{S}})$, the moduli space of marked reduced models with the combinatorial type $\overline{\mathbb{S}}$. Here ℓ is the order of $F(\Gamma)/\Gamma$. We now have :

Lemma 35.74. *The right hand side of Proposition* 32.63 *is equal to*

$$\dim_{\mathbb{R}} \mathcal{M}^{\text{main}}_{k+1,\ell}(\overline{\beta}; P_1, \cdots, P_k)(\overline{\mathbb{S}}) + \sum_i \deg P_i.$$

Proof. In the same way as the proof of Lemma 35.72, we find that

$$\dim_{\mathbb{R}} \text{Index}(\widetilde{C}((\overline{\Sigma}, \vec{\overline{z}}), \overline{v}, \vec{x})) + \overline{\rho}((\overline{\Sigma}, \vec{\overline{z}}), \overline{v}) - \sum \deg P_i$$

is the (virtual) dimension of the moduli space of reduced models. Adding marked points $\vec{z}_+$ increases the dimension by $2\#\vec{z}_+ = 2\#F(\Gamma)/\Gamma$, which is equal to $\dim_{\mathbb{R}} C(F(\Gamma))^{\Gamma}$. □

We next show the following.

Lemma 35.75. *Suppose that* (M, J) *is spherically positive and* $\mathbb{S}$ *contains at least one sphere component. Then we have*

$$\dim_{\mathbb{R}} \mathcal{M}^{\text{main}}_{k+1,\ell}(\overline{\beta}; P_1, \cdots, P_k)(\overline{\mathbb{S}}) \le \dim_{\mathbb{R}} \mathcal{M}^{\text{main}}_{k+1}(\beta; P_1, \cdots, P_k) - 2.$$

(Note the dimension here is the dimension in the sense of Kuranishi structure, that is the virtual dimension.)

Remark 35.76. We have not used spherical positivity of J up to this point. Namely the proof of Lemma 35.75 is the only place where we use these assumptions.

Proof. Let J be a spherically positive almost complex structure. We consider a component S_a of an element of $\mathcal{M}^{\text{main}}_{k+1}(\beta; P_1, \cdots, P_k)(\mathbb{S})$. Let $\vec{z}_a, \vec{w}_a$ be the sets of all marked or singular points on S_a, ∂S_a, respectively. Let $v_a = v|_{S_a}$. We put $k_a = \#\vec{z}_a + \#\vec{w}_a/2$. We put $s_a = 3$ if S_a is a disc component and $s_a = 6$ if S_a is a sphere component. Set

$$c(a) = 2c_1(M)[v_a]$$

for a sphere component and

$$c(a) = \mu_L(v_a)$$

for a disc component.

We claim that

$$c(a) + 2k_a - s_a \ge -2 \tag{35.77}$$

holds for the sphere components.

In fact, if the map is trivial on this component, then $2k_a \ge 6$ and $c(a) = 0$. (35.77) holds. If the map is nontrivial on this component, then we have $c(a) \ge 2$ by the choice of J made in the beginning of the proof using spherial positivity. Since $2k_a \ge 2$ and $s_a = 6$ for a sphere component, (35.77) also holds.

We put $\epsilon(a) = 2$ for a sphere component and $\epsilon(a) = 1$ for a disc component. It is easy to see from the index theorem that

$$\begin{aligned}&\dim_{\mathbb{R}} \mathcal{M}^{\text{main}}_{k+1}(\beta; P_1, \cdots, P_k)(\mathbb{S}) \\ &= \sum_{a\in I}(\epsilon(a)n + c(a) + 2k_a - s_a) \\ &\quad - (2n)\,\#\text{Sing}_{S^2}\mathbb{S} - n\,\#\text{Sing}_{D^2}\mathbb{S} - \sum \deg P_i,\end{aligned} \tag{35.78.1}$$

where I is the set of all components of $\mathbb{S}$, $\#\text{Sing}_{S^2}\mathbb{S}$ is the number of singular points which intersect with sphere components and $\#\text{Sing}_{D^2}\mathbb{S}$ is the number of singular points which do not intersect with sphere components. We recall $2n = \dim M$.

We have the similar identity for $\dim_{\mathbb{R}} \mathcal{M}^{\text{main}}_{k+1,\ell}(\overline{\beta}; P_1, \cdots, P_k)(\overline{\mathbb{S}})$. Namely for each component S_b of $\overline{\mathbb{S}}$ we define $\overline{k}_b$, $\overline{s}_b$, $\overline{c}(b)$, $\overline{\epsilon}(b)$ and obtain

$$\begin{aligned}&\dim_{\mathbb{R}} \mathcal{M}^{\text{main}}_{k+1,\ell}(\overline{\beta}; P_1, \cdots, P_k)(\overline{\mathbb{S}}) \\ &= \sum_{b\in \overline{I}}(\overline{\epsilon}(b)n + \overline{c}(b) + 2\overline{k}_b - \overline{s}_b) \\ &\quad - (2n)\,\#\text{Sing}_{S^2}\overline{\mathbb{S}} - n\,\#\text{Sing}_{D^2}\overline{\mathbb{S}} - \sum \deg P_i\end{aligned} \tag{35.78.2}$$

where $\overline{I}$ is the set of all components of $\overline{\mathbb{S}}$.

For each component $b \in \overline{I}$ we take $a(b) \in I$ such that $S_{a(b)}$ is a branched covering of S_b. (There may be several of them. We choose one of them.)

If S_b is a disc component, we have

$$\overline{c}(b) + 2\overline{k}_b - \overline{s}_b = c(a(b)) + 2k_{a(b)} - s_{a(b)}, \tag{35.79}$$

since the automorphism group is trivial on the disc components.

We next prove the following :

Sublemma 35.80.

$$\overline{c}(b) + 2\overline{k}_b - \overline{s}_b \le c(a(b)) + 2k_{a(b)} - s_{a(b)},$$

if S_b is a sphere component.

Proof. We have $\overline{s}_b = s_{a(b)} = 6$. By the spherial positivity we have $\overline{c}(b) \ge 0$. It also follows that $\overline{c}(b) \le c(a(b))$. If there is no free fixed point on $S_{a(b)}$, then $\overline{k}_b \le k_{a(b)}$ and we are done.

Now assume that there is a free fixed point on $S_{a(b)}$. In this case the degree, denoted by deg, of the map $S_{a(b)} \to S_b$ is greater than one.

We first consider the case when $[v_b] \ne 0$. Then by Condition 35.3 we have $c(a(b)) \ge 2$. Therefore

$$\overline{c}(b) = \frac{c(a(b))}{\deg} \le c(a(b)) - 2.$$

(Note $c(a), \overline{c}(b)$ are even numbers for the sphere components.) On the other hand, since there exists at most one free fixed point on $S_{a(b)}$, it follows that

$$\overline{k}_b \le k_{a(b)} + 1.$$

The sublemma follows in this case.

We next assume $[v_b] = 0$. Then $k_{a(b)} \ge 3$ by stability. Namely there exist at least 3 singular or marked points. The component S_b is identified with the quotient of $S_{a(b)}$ by the cyclic group $\Gamma_{a(b)}$ of order $\deg \in \{2, 3, \cdots\}$. Let d be the distance of $S_{a(b)}$ from the disc components. Let $o_{a(b)}$ be the singular point on $S_{a(b)}$ where $S_{a(b)}$ is attached with Σ_{d-1}. Clearly $o_{a(b)}$ is a fixed point of $\Gamma_{a(b)}$. Since there is a free fixed point on $S_{a(b)}$, no other singular points are fixed by $\Gamma_{a(b)}$. In other words the image in S_b of the singular points of $S_{a(b)}$ consists of $1 + (k_{a(b)} - 1)/\deg$ points. Therefore

$$\overline{k}_b = 2 + \frac{k_{a(b)} - 1}{\deg}.$$

Since $k_{a(b)} \ge 3$, we derive

$$\overline{k}_b \le k_{a(b)}.$$

The proof of Sublemma 35.80 is now complete. □

It follows from Sublemma 35.80 and (35.79) that

$$\sum_{b\in \overline{I}} (\overline{\epsilon}(b)n + \overline{c}(b) + 2\overline{k}_b - \overline{s}_b) \le \sum_{b\in\overline{I}} (\epsilon(a(b))n + c(a(b)) + 2k_{a(b)} - s_{a(b)}). \tag{35.81}$$

(35.77) implies that for each sphere component S_a ($a \in I$) we have

$$2n + c(a) + 2k_a - s_a \ge 2n - 2.$$

Note that the number of sphere components of $\mathbb{S}$ and of $\overline{\mathbb{S}}$ are equal to $\#\mathrm{Sing}_{S^2}\mathbb{S}$ and to $\#\mathrm{Sing}_{S^2}\overline{\mathbb{S}}$, respectively. Therefore we have

$$\begin{aligned}
&\sum_{a\in I} (\epsilon(a)n + c(a) + 2k_a - s_a) - (2n-2)\#\mathrm{Sing}_{S^2}\mathbb{S} \\
&\ge \sum_{b\in\overline{I}} (\epsilon(b(a))n + c(a(b)) + 2k_{a(b)} - s_{a(b)}) - (2n-2)\#\mathrm{Sing}_{S^2}\overline{\mathbb{S}}.
\end{aligned} \tag{35.82}$$

We remark that $\mathrm{Sing}_{D^2}\overline{\mathbb{S}} = \mathrm{Sing}_{D^2}\mathbb{S}$. The Formulas (35.40), (35.43) and (35.44) imply

$$\begin{aligned}
&\dim_{\mathbb{R}} \mathcal{M}^{\mathrm{main}}_{k+1}(\overline{\beta}; P_1, \cdots, P_k)(\overline{\mathbb{S}}) \\
&\le \dim_{\mathbb{R}} \mathcal{M}^{\mathrm{main}}_{k+1}(\beta; P_1, \cdots, P_k)(\mathbb{S}) + 2(\#\mathrm{Sing}_{S^2}\mathbb{S} - \#\mathrm{Sing}_{S^2}\overline{\mathbb{S}}).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\dim_{\mathbb{R}} \mathcal{M}^{\text{main}}_{k+1}(\beta; P_1, \cdots, P_k)\\ &= \dim_{\mathbb{R}} \mathcal{M}^{\text{main}}_{k+1}(\beta; P_1, \cdots, P_k)(\mathbb{S}) + 2\#\text{Sing}_{S^2}\mathbb{S} + \#\text{Sing}_{D^2}\mathbb{S}.\end{aligned}$$

Since $\#\text{Sing}_{S^2}\overline{\mathbb{S}} \geq 1$, we obtain the lemma. □

Let $\Gamma \neq \{1\}$ be an abstract group. If $\mathcal{M}^{\text{main}}_{k+1}(\beta; P_1, \cdots, P_k)(\mathbb{S}, \Gamma)$ is nonempty, then the combinatorial type $\mathbb{S}$ has at least one sphere bubble. It follows from Lemma 35.75 that

$$\dim_{\mathbb{R}} \mathcal{M}^{\text{main}}_{k+1,\ell}(\overline{\beta}; P_1, \cdots, P_k)(\overline{\mathbb{S}}) \leq \dim_{\mathbb{R}} \mathcal{M}^{\text{main}}_{k+1}(\beta; P_1, \cdots, P_k) - 2. \tag{35.83}$$

Therefore by Propositions 35.52, 35.63, Lemmas 35.72, 35.74 and (35.83), we have

$$\dim_{\mathbb{R}}(\mathcal{M}^{\text{main}}_{k+1}(\beta; P_1, \cdots, P_k)^{\mathfrak{s}})^{\cong}(\Gamma) \leq \dim_{\mathbb{R}} \mathcal{M}^{\text{main}}_{k+1}(\beta; P_1, \cdots, P_k)^{\mathfrak{s}} - 2. \tag{35.84}$$

Note the dimension in (35.83) is the virtual dimension. On the other hand, the dimension in (35.84) is an actual dimension of the simplicial complex. (This is a consequence of Proposition 35.52.)

The proof of Theorem 34.11 is now complete. □

Remark 35.85. Consider the case when $\mathbb{S}$ is a union of a disc D^2 and a sphere S^2 where $c_1(M)[v|_{S^2*}([S^2])] = 0$. We assume that $v|_{S^2}$ is a cyclic multiple cover of a map u. The reduced model consists of the same configuration where the map $v|_{S^2}$ is replaced by u. The virtual dimension of the reduced model is the same as the virtual dimension of $\mathcal{M}^{\text{main}}_{k+1}(\beta; P_1, \cdots, P_k)(\mathbb{S})$, the moduli space with combinatorial type $\mathbb{S}$. Since there is one sphere bubble, the virtual dimension of the moduli space $\mathcal{M}^{\text{main}}_{k+1}(\beta; P_1, \cdots, P_k)(\mathbb{S})$ is the virtual dimension of $\mathcal{M}^{\text{main}}_{k+1}(\beta; P_1, \cdots, P_k)$ minus 2.

However, since there is one free fixed point on the sphere bubble, it follows that the virtual dimension of *marked* reduced model $\mathcal{M}^{\text{main}}_{k+1,1}(\overline{\beta}; P_1, \cdots, P_k)(\overline{\mathbb{S}})$ is *equal to* the virtual dimension of $\mathcal{M}^{\text{main}}_{k+1}(\beta; P_1, \cdots, P_k)$. Namely (35.83) does not hold.

This is the reason why we assume spherical positivity in this book.

If we can use the reduced model in place of the marked reduced model, this condition would be removed. However it seems rather difficult to show the compatibility of the normally conical perturbation (which we constructed in Proposition 35.52) with the process of forgetting extra marked points in the marked reduced model.

§36. Filtered L_∞ algebra and symmetrization of filtered A_∞ algebra.

In the next section, we will study Lagrangian embeddings of a manifold in $\mathbb{C}^{n+1}$ that is homeomorphic to $S^1 \times S^n$ and illustrate calculations involving the Floer cohomology, spectral sequences and the filtered A_∞ algebra associated to the Lagrangian submanifold. It turns out that our calculation of the leading order contribution of holomorphic discs can be more conveniently described in the language of L_∞ algebra than that of A_∞ algebra.

In this section we review the basic definitions of L_∞ algebra and explain the symmetrization of filtered A_∞ algebra. The symmetrization is a canonical process of obtaining a (filtered) L_∞ algebra out of a (filtered) A_∞ algebra. (The symmetrization of A_∞ algebra is well established in the literature. See, for example, [LaMa95].)

Let C be a free graded module over $\Lambda_{0,nov}$. (In this section we use $\Lambda_{0,nov} = \Lambda_{0,nov}^{\mathbb{Q}}$.) Let $C[1]$ be the free graded $\Lambda_{0,nov}$ module obtained by shifting the grading. We consider $B_k(C[1])$ and an action of $\mathfrak{S}_k$ (the symmetric group of order $k!$) on it by

$$\sigma(x_1 \otimes \cdots \otimes x_k) = (-1)^{s(\sigma)} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}$$

where $\sigma : \{1, \cdots, k\} \to \{1, \cdots, k\}$ is a permutation and

$$s(\sigma) = \sum_{i<j, \sigma(i)>\sigma(j)} \deg' x_i \deg' x_j. \tag{36.1}$$

Here $\deg'$ is the shifted degree. Let $E_k(C[1]) \subset B_k(C[1])$ be the set of all elements of $B_k(C[1])$ fixed under the $\mathfrak{S}_k$ action. The energy filtration of $C[1]$ induces an energy filtration F^λ on $E_k(C[1])$. Let $\widehat{E}(C[1])$ be the completion of $\bigoplus_{k=0}^\infty E_k(C[1])$.

We put

$$[x_1, \cdots, x_k] = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} (-1)^{s(\sigma)} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}.$$

They generate $E_k(C[1])$. The coalgebra structure on $B(C[1])$ induces a (formal) coalgebra structure

$$\Delta : \widehat{E}(C[1]) \longrightarrow \widehat{E}(C[1]) \mathbin{\widehat{\otimes}} \widehat{E}(C[1]).$$

We remark that Δ is graded cocommutative.

We consider a sequence of operations

$$\mathfrak{l}_k : E_k(C[1]) \longrightarrow C[1]$$

$(k = 0, 1, 2, \cdots)$ of degree $+1$ such that

$$\mathfrak{l}_k(F^\lambda E_k(C[1])) \subset F^\lambda(C[1]), \tag{36.2.1}$$

$$\mathfrak{l}_0(1) \in \bigcup_{\lambda>0} F^\lambda(C[1]). \tag{36.2.2}$$

$\mathfrak{l}_k : E_k(C[1]) \to C[1]$ induces coderivation

$$\widehat{d}_k : \widehat{E}(C[1]) \longrightarrow \widehat{E}(C[1])$$

in the same way as (7.15), but replacing the tensor product by the bracketing $[x_1, \cdots, x_n]$. More precisely, it is defined by

$$\widehat{d}_k([x_1, \cdots, x_n]) = \sum (-1)^* \frac{k!(n-k)!}{n!} [x_{a_1}, \cdots, x_{a_{n-k}}, \mathfrak{l}_k[x_{b_1}, \cdots, x_{b_k}]].$$

Here the summation is taken over all the '$(n-k,k)$-shuffles' i.e., the permutations of length k satisfying

$$\{a_1, \cdots a_{n-k}\} \cup \{b_1, \cdots, b_k\} = \{1, \cdots, n\}; \quad a_i < a_{i+1}, \quad b_i < b_{i+1},$$

and $*$ is given by

$$* = \sum_{i=1}^{n-k} \deg' x_{a_i} + \sum_{a_i > b_j} \deg' x_{a_i} \deg' x_{b_j}.$$

We can prove that $\widehat{d} = \sum_k \widehat{d}_k$ converges and induces $\widehat{d} : \widehat{E}(C[1]) \to \widehat{E}(C[1])$ in the same way as Lemma 7.17.

Definition 36.3. We say that $\mathfrak{l} = \{\mathfrak{l}_k\}_{k\ge 0}$ defines a structure of *filtered L_∞ algebra* on C if $\widehat{d} \circ \widehat{d} = 0$. A filtered L_∞ algebra is said to be *strict* if $\mathfrak{l}_0 = 0$.

We can define the *center* of a filtered L_∞ algebra, in the similar way as the unit of a filtered A_∞ algebra. Namely $\mathbf{e}$ is said to be in the center if and only if

$$\mathfrak{l}_k([\mathbf{e}, x_1, \cdots, x_{k-1}]) = 0$$

for all k and $x_1, \cdots, x_{k-1}$.

We can define a filtered L_∞ algebra to be G-gapped in the same way as Definition 7.29. Hereafter we fix an isomorphism

$$C \cong \overline{C} \otimes_{\mathbb{Q}} \Lambda_{0,nov}^{\mathbb{Q}}$$

and $\overline{\mathfrak{l}}_k : E_k(\overline{C}[1]) \to \overline{C}[1]$ is the operations induced by $\mathfrak{l}_k$.

The restriction of the coefficient ring to $\mathbb{Q}$ gives rise to an (unfiltered) L_∞ algebra. (We assume $\overline{\mathfrak{l}}_0 = 0$ for the unfiltered L_∞ algebra.)

Example 36.4. Let $(\overline{C}, d, [\cdot,\cdot])$ be a differential graded Lie algebra over $\mathbb{Q}$. We put

$$\bar{\mathfrak{l}}_1(x) = (-1)^{\deg x} dx, \quad \bar{\mathfrak{l}}_2([x,y]) = (-1)^{\deg x(\deg y+1)}[x,y]$$

and set all other $\bar{\mathfrak{l}}_k = 0$. Then it is easy to check that $(\overline{C}, \bar{\mathfrak{l}}_k)$ is an L_∞ algebra.

We next define the symmetrization of a filtered A_∞ algebra. Let $(C, \mathfrak{m}_*)$ be a filtered A_∞ algebra. Hereafter we write $\mathfrak{l}_k(x_1,\cdots,x_k)$ in place of $\mathfrak{l}_k([x_1,\cdots,x_k])$ for simplicity. We put

$$\mathfrak{l}_k(x_1,\cdots,x_k) = \frac{1}{k!}\sum_\sigma (-1)^{s(\sigma)}\mathfrak{m}_k(x_{\sigma(1)},\cdots,x_{\sigma(k)}), \tag{36.5}$$

where $s(\sigma)$ is as in (36.1) and the the summation is taken over all σ in the permutation group of $\{1,\cdots,k\}$. Obviously $\mathfrak{l}_k$ induces a homomorphism $\mathfrak{l}_k : E_k(C[1]) \to C[1]$.

Proposition 36.6. $\mathfrak{l}_k : E_k(C[1]) \to C[1]$ *defines a structure of filtered* L_∞ *algebra.*

Proof. Let $\widehat{d} : \widehat{B}(C[1]) \to \widehat{B}(C[1])$ be the coderivation induced by $\mathfrak{m}$. It suffices to prove that $\widehat{d}(\widehat{E}(C[1])) \subset \widehat{E}(C[1])$. In fact it will then imply that $\widehat{d} : \widehat{E}(C[1]) \to \widehat{E}(C[1])$ coincides with the restriction of $\widehat{d} : \widehat{B}(C[1]) \to \widehat{B}(C[1])$.

We consider $\sigma_{ab} : \{1,\cdots,\ell\} \to \{1,\cdots,\ell\}$ ($\sigma_{ab} \in \mathfrak{S}_\ell$) such that $\sigma_{ab}(b) = a$, $\sigma_{ab}(a) = b$, $\sigma_{ab}(i) = i$ for $i \neq a,b$. Let $\pi_\ell : \widehat{B}(C[1]) \to B_\ell(C[1])$ be the projection. We calculate

$$\begin{aligned}&(\sigma_{ab}\circ\pi_\ell\circ\widehat{d})([x_1,\cdots,x_k])\\ &= \frac{1}{k!}\sigma_{ab}\sum_{\sigma\in\mathfrak{S}_k}(-1)^{s'(\sigma)}\sum_{i=1}^{\ell} x_{\sigma(1)}\otimes\cdots\otimes\mathfrak{m}_{k-\ell+1}(x_{\sigma(i)},\cdots,x_{\sigma(i+k-\ell)})\otimes\cdots\otimes x_{\sigma(k)}.\end{aligned}$$

Here

$$s'(\sigma) = s(\sigma) + \sum_{j=1}^{i-1}\deg' x_{\sigma(j)},$$

where $s(\sigma)$ is as in (36.1). We put

$$A_i = \sum_{\sigma\in\mathfrak{S}_k}(-1)^{s'(\sigma)}x_{\sigma(1)}\otimes\cdots\otimes\mathfrak{m}_{k-\ell+1}(x_{\sigma(i)},\cdots,x_{\sigma(i+k-\ell)})\otimes\cdots\otimes x_{\sigma(k)}.$$

It is easy to see that

$$\sigma_{ab}A_a = A_b, \quad \sigma_{ab}A_b = A_a, \quad \sigma_{ab}A_i = A_i \ (i \neq a,b).$$

Hence

$$(\sigma_{ab}\circ\pi_\ell\circ\widehat{d})([x_1,\cdots,x_k]) = (\pi_\ell\circ\widehat{d})([x_1,\cdots,x_k])$$

as required. □

Definition 36.7. We say $(C,\mathfrak{l})$ the *symmetrization* of $(C,\mathfrak{m})$. We can define a symmetrization of an (unfiltered) A_∞ algebra (which is an L_∞ algebra) in the same way.

Example 36.8. Let $(\overline{C},d,\wedge)$ be a differential graded algebra. By putting

$$\overline{\mathfrak{m}}_1(x) = (-1)^{\deg x} dx, \quad \overline{\mathfrak{m}}_2(x,y) = (-1)^{\deg x(\deg y+1)}(x\wedge y) \tag{36.9}$$

and $\overline{\mathfrak{m}}_k = 0$ for $k \ne 1,2$, we obtain an A_∞ algebra $(\overline{C},\overline{\mathfrak{m}})$. We define a differential graded Lie algebra $(\overline{C},d,[\cdot,\cdot])$ by

$$[x,y] = \frac{1}{2}(x\wedge y - (-1)^{\deg x\deg y} y\wedge x).$$

It is easy to check that the symmetrization of $(C,\mathfrak{m})$ is the L_∞ algebra obtained from $(\overline{C},d,[\cdot,\cdot])$ as in Example 36.4.

Proposition 36.10. *Let $(C,\mathfrak{m})$ be a filtered A_∞ algebra. Suppose that the reduction $(\overline{C},\overline{\mathfrak{m}})$ of its coefficient to $\mathbb{Q}$ is obtained from a differential graded algebra $(\overline{C},d,\wedge)$ by (36.9). We assume that $(\overline{C},d,\wedge)$ is graded commutative. Then the symmetrization $(C,\mathfrak{l})$ satisfies*

$$\mathfrak{l}_k \equiv 0 \mod \Lambda_{0,nov}^+$$

for $k\ne 1$.

Proof. It suffices to show $\bar{\mathfrak{l}}_2 = 0$, since $\bar{\mathfrak{l}}_k = 0$ for $k\ge 3$ obviously follows from $\overline{\mathfrak{m}}_k = 0$ for $k \ge 3$. We calculate

$$\begin{aligned}2\bar{\mathfrak{l}}_2(x,y) &= \overline{\mathfrak{m}}_2(x,y) + (-1)^{\deg' x\deg' y}\overline{\mathfrak{m}}_2(y,x)\\ &= (-1)^{\deg x(\deg y+1)}x\wedge y + (-1)^{\deg' x\deg' y+\deg y(\deg x+1)}y\wedge x.\end{aligned}$$

Since $(\overline{C},d,\wedge)$ is graded commutative, we find that the second term is

$$(-1)^{\deg' x\deg' y+\deg y}x\wedge y = (-1)^{\deg x(\deg y+1)+1}x\wedge y$$

and cancels with the first term. □

We will apply Proposition 36.10 to the symmetrization of the canonical model of $(C(L;\Lambda_{0,nov}),\mathfrak{m})$ later in this section.

Remark 36.11. In the year-2000 preprint version of this book the same calculation as the proof of Proposition 36.10 was mentioned (at page 105) as a trouble to define symmetrization of filtered A_∞ algebra. We now understand that it is not a trouble at all.

We next define a filtered L_∞ homomorphism. Let $(C_i,\mathfrak{l}^i)$, $i=1,2$ be filtered L_∞ algebras. For $k=0,1,2,\cdots$ we consider maps $\mathfrak{f}_k : E_k(C_1[1]) \to C_2[1]$ of degree 0 such that $\mathfrak{f}_k(F^\lambda E_k(C_1[1])) \subseteq F^\lambda C_2[1]$. It induces a coalgebra homomorphism $\widehat{\mathfrak{f}} : \widehat{E}(C_1[1]) \to \widehat{E}(C_2[1])$, in the same way as (7.28).

Definition 36.12. We call $\mathfrak{f} = \{\mathfrak{f}_k\}_{k\geq 0}$ a *filtered L_∞ homomorphism from C_1 to C_2* if $\widehat{\mathfrak{f}} \circ \widehat{d}^1 = \widehat{d}^2 \circ \widehat{\mathfrak{f}}$. A filtered L_∞ homomorphism $\mathfrak{f} = \{\mathfrak{f}_k\}_{k\geq 0}$ is said to be *strict* if $\mathfrak{f}_0 = 0$.

We can define composition of filtered L_∞ homomorphisms by

$$\widehat{\mathfrak{g} \circ \mathfrak{f}} = \widehat{\mathfrak{g}} \circ \widehat{\mathfrak{f}}.$$

Lemma 36.13. *Any filtered A_∞ homomorphism between two filtered A_∞ algebras induces a filtered L_∞ homomorphism by the symmetrization.*

The proof is easy and is omitted. We remark that there exist the unfiltered and/or the unital version of Definition 36.12 and Lemma 36.13.

We can also define the *gapped condition* for filtered L_∞ algebras and filtered L_∞ homomorphisms in the same way as the A_∞ case. Hereafter we assume that all filtered L_∞ algebra and filtered L_∞ homomorphisms are gapped.

We next discuss the notion of homotopy between filtered L_∞ homomorphisms. Let $(C, \mathfrak{l})$ be a filtered L_∞ algebra. We define a graded $\Lambda_{0,nov}$ module $Poly([0,1], C[1])$ by Definition 15.9. The maps $\mathrm{Incl}_k : E_k(C[1]) \to \mathfrak{C}$, $(\mathrm{Eval}_{s=s_0})_k : E_k(\mathfrak{C}[1]) \to C$ are defined in the same way as in the proof of Definition-Proposition 15.15.

Lemma 36.14. *There exists a structure of filtered L_∞ algebra on $Poly([0,1], C[1])$ such that* Incl *and* Eval *are filtered L_∞ homomorphisms. Moreover they induce an isomorphism on $\overline{\mathfrak{l}}_1$ cohomology.*

The proof is a straightforward analog to the proof of Lemma 15.13 and omitted.

Remark 36.15. The other construction $C^{[0,1]}$ of the model of $[0,1] \times C$ for filtered A_∞ algebra C, does not seem to work for the filtered L_∞ algebra, since it does not respect the symmetry. (See Remark 15.21.) So the authors do not know how to define homotopy between L_∞ homomorphisms over $\mathbb{Z}$ or $\mathbb{Z}_2$. The symmetrization of $C^{[0,1]}$ will be an L_∞ algebra. We need to work over $\mathbb{Q}$ for symmetrization.

Using the model $(Poly([0,1], C[1]), \mathfrak{l})$, we can define homotopy between filtered L_∞ homomorphisms, and prove that homotopy is an equivalence relation which is compatible to the composition. We can then define the notion of homotopy equivalence of filtered L_∞ algebras in an obvious way.

Theorem 36.16. *If $\mathfrak{f} = \{\mathfrak{f}_k\}_{k\geq 0}$ is a filtered L_∞ homomorphism from C_1 to C_2 and induces an isomorphism $H(\overline{C}_1; \overline{\mathfrak{l}}^1_1) \to H(\overline{C}_2; \overline{\mathfrak{l}}^2_1)$, then there exists a filtered L_∞ homomorphism $\mathfrak{g}$ from C_2 to C_1 such that $\mathfrak{f} \circ \mathfrak{g}$ and $\mathfrak{g} \circ \mathfrak{f}$ are homotopic to identity.*

The proof this theorem is parallel to that of the proof of Theorem 15.45 and omitted.

We next discuss the canonical models of filtered L_∞ algebras.

Definition 36.17. A filtered L_∞ algebra $(C, \mathfrak{l})$ is said to be *canonical* if $\bar{\mathfrak{l}}_1 = 0$.

We remark that the symmetrization of a canonical filtered A_∞ algebra is a canonical L_∞ algebra.

Theorem 36.18. *Any gapped filtered L_∞ algebra is homotopy equivalent to a canonical filtered L_∞ algebra.*

The proof is a straightforward analogue of that of Theorem 23.2 in §23.3 and so omitted.

Theorem 36.19. *Let L be a relatively spin Lagrangian submanifold of M and $(H(L;\Lambda_{0,nov}), \mathfrak{m})$ the canonical filtered A_∞ algebra obtained in Corollary 23.6. Let $(H(L;\Lambda_{0,nov}), \mathfrak{l})$ be its symmetrization. Then for all k we have*

$$\mathfrak{l}_k \equiv 0 \mod \Lambda_{0,nov}^+. \tag{36.20}$$

For the proof we need :

Proposition 36.21. *Let $\bar{\mathfrak{f}} : (\overline{C}_1, \overline{\mathfrak{m}}) \to (\overline{C}_2, \overline{\mathfrak{m}})$ be a homotopy equivalence of A_∞ algebra. Let $(C_1, \mathfrak{m})$ be a gapped filtered A_∞ algebra deforming $(\overline{C}_1, \overline{\mathfrak{m}})$. Then there exists a gapped filtered A_∞ algebra $(C_2, \mathfrak{m})$ deforming $(\overline{C}_2, \overline{\mathfrak{m}})$ and a homotopy equivalence of filtered A_∞ algebras $\mathfrak{f} : (C_1, \mathfrak{m}) \to (C_2, \mathfrak{m})$ such that $\mathfrak{f} \equiv \bar{\mathfrak{f}} \mod \Lambda_{0,nov}^+$.*

Proposition 36.21 is a direct consequence of Proposition 30.130.

Proof of Theorem 36.19. Let $(H_{DR}^*(L;\mathbb{R}), \bar{\mathfrak{l}}_{can,\mathbb{R}})$ be the canonical model of the L_∞ algebra obtained by symmetrization of the de Rham DGA $(\Omega(L), d, \wedge)$. Since the wedge product is graded commutative, it follows that $\bar{\mathfrak{l}}_{can,\mathbb{R}} = 0$.

On the other hand, if $(H^*(L;\mathbb{Q}), \bar{\mathfrak{l}}_{can})$ is the $\mathbb{Q}$ reduction of $(H^*(L;\Lambda_{0,nov}^{\mathbb{Q}}), \mathfrak{l}_{can})$, then $(H^*(L;\mathbb{Q}), \bar{\mathfrak{l}}_{can}) \otimes \mathbb{R}$ is homotopy equivalent to $(H_{DR}^*(L;\mathbb{R}), \bar{\mathfrak{l}}_{can,\mathbb{R}})$. Since both are canonical, they are isomorphic. It follows that $\bar{\mathfrak{l}}_{can} = 0$, as required. □

Theorem 36.19 implies that contributions of the classical cup product and (higher) Massey products become trivial under the symmetrization, of our filtered A_∞ structure of Lagrangian submanifold.

§37. Floer theory of Lagrangian embedding $S^1 \times S^n \to \mathbb{C}^{n+1}$.

In this section, we illustrate calculations involving Floer cohomology, spectral sequences and the filtered A_∞ algebras associated to Lagrangian submanifolds. We will particularly analyze the example of embeded Lagrangian submanifold L of

$\mathbb{C}^{n+1}$ diffeomorphic to $S^1 \times S^n$. We define a function $E : \pi_2(\mathbb{C}^{n+1}, L) \to \mathbb{R}$ by $E(\beta) = [\omega](\beta)$. Gromov [Grom85] proved that E is nontrivial for any compact Lagrangian embedding L in $\mathbb{C}^{n+1}$.

Unless otherwise stated, we will assume that $n \geq 2$ in this section. Then we have $\pi_2(\mathbb{C}^{n+1}, L) \cong \pi_1(L) = \mathbb{Z}$. We fix a generator β of $\pi_2(\mathbb{C}^{n+1}, L)$ so that

$$E(\beta) > 0.$$

We denote $\gamma = \delta\beta \in \pi_1(L)$.

By conformally scaling the embedding, we may assume $E(\beta) = 1$ without loss of any generality. We note the Maslov index $\mu_L(\gamma)$ of the class $\gamma \in \pi_1(L)$ lies in $2\mathbb{Z}$ since $S^1 \times S^n$ is orientable.

There are two kinds of Lagrangian embeddings $S^1 \times S^n \to \mathbb{C}^{n+1}$ known in the literature. One construction is based on the following two results.

(37.1) (Gromov [Grom85], Lees [Lee76]) If $TL \otimes \mathbb{C}$ is a trivial complex vector bundle, then there exists a Lagrangian immersion $L \to \mathbb{C}^n$. (Here $n = \dim L$.)
(37.2) (Audin-Lalonde-Polterovich [ALP94]) Let $i : S^n \to \mathbb{C}^n$ be a Lagrangian immersion, e.g., consider the so called Whitney Lagrangian immersion. (See [Wei77] for the precise definition of the Whitney Lagrangian immersion.) Consider any embedded circle $i' : S^1 \to \mathbb{C}$ which obviously becomes a Lagrangian embedding. Then the Lagrangian immersion $i \times i' : S^1 \times L \to \mathbb{C}^{n+1}$ is Lagrangian isotopic to a Lagrangian embedding.

Note the standard embedding $S^1 \to \mathbb{C}$ is a Lagrangian embedding with Maslov index 2. Thus (37.1),(37.2) imply that there exists a Lagrangian embedding $S^1 \times S^n \to \mathbb{C}^{n+1}$ such that $\mu_L(\gamma) = 2$.

The other example is given by Polterovich [Pol91II]. Let us consider $S^n = \{x \in \mathbb{R}^{n+1} | \|x\| = 1\}$ and put $L = \{\sigma x \in \mathbb{C}^{n+1} | x \in S^n, \sigma \in \mathbb{C}, |\sigma| = 1\}$.

Lemma 37.3. ([Pol91II]) *L is a Lagrangian submanifold. $\mu_L(\gamma) = n+1$. Moreover L is diffeomorphic to $S^1 \times S^n$ if n is odd .*

37.1. Floer cohomology and spectral sequence.

We start with proving a slight improvement of the result of second named author proven in [Oh96I]. Hereafter in this section, $L \sim S^1 \times S^n$ stands that L is homeomorphic to $S^1 \times S^n$. With some additional arguments, one can weaken this hypothesis to the condition that L is homotopy equivalent to $S^1 \times S^n$.

Theorem 37.4. (Compare with [Oh96I]) *Let L be an embedded Lagrangian submanifold of $\mathbb{C}^{n+1}$ that is homeomorphic to $S^1 \times S^n$. Then we have the following.*
(1) *If n is even, then $\mu_L(\gamma)$ is either 2 or is $(2 - n)/\ell$ for some $\ell \in \mathbb{Z}_{>0}$.*

(2) *If n is odd, then $\mu_L(\gamma) = (n+1)/\ell$ for some $\ell \in \mathbb{Z}_{>0}$.*

Proof. For the proof of Theorem 37.4 we use the Floer cohomology over $\mathbb{Q}$. Later on we will also use the Floer cohomology and the spectral sequence over $\mathbb{Z}$ in order to obtain additional information.

Case 1 : L is weakly unobstructed: More precisely, we consider the case when the cohomology groups containing all obstruction classes vanish. If L is weakly unobstructed (over $\mathbb{Q}$), then $HF(L, L; \Lambda^{\mathbb{Q}}_{0,nov})$ is defined and then $HF(L, L; \Lambda^{\mathbb{Q}}_{nov}) \cong 0$, since there exists a Hamiltonian diffeomorphism $\phi : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ of compact support such that $\phi(L) \cap L = \emptyset$. We now consider the spectral sequence in Theorem 24.5 or in §25.

We remark that the filtered A_∞ algebra associated to our L is G-gapped where $G \subset \mathbb{Z}_{\geq 0} \times 2\mathbb{Z}$ and $G \cong \mathbb{Z}_{\geq 0}$ as a monoid. This is because $E(\gamma) = 1$ and $\pi_2(\mathbb{C}^{n+1}, L) = \mathbb{Z}$. By (26.18) in §26, we defined a filtration on $C(L; \Lambda^{\mathbb{Q}}_{0,nov})$ which is used to construct the spectral sequence. We can take the number λ_0 appearing in (26.18) as $\lambda_0 = 1$, since $E(\gamma) = 1$.

When L is weakly unobstructed, we can take a bounding chain b of the form

$$b = \sum_{m \in \mathbb{Z}} T^m e^{m\mu(\gamma)/2} b_m. \tag{37.5}$$

This is because all the chains defining obstruction classes are of the monomial $T^m e^{m\mu_L(\gamma)/2}$ with some element from $C(L; \mathbb{Q})$ as its coefficient and b_n will be constructed based on the vanishing of the cohomology group of the relevant degree. If we choose b as in (37.5), the boundary operator $\mathfrak{m}_1^b$ is written as

$$\mathfrak{m}_1^b = \sum_k T^k e^{k\mu(\gamma)/2} \mathfrak{m}_{1,k\gamma}^b.$$

In particular, L is rationally unobstructed. (See Definition 25.1 (25.2.2).) This observation has led us to the following description of the differential of the spectral sequence.

The differential δ_2 of the spectral sequence is decomposed to

$$\delta_2 = \bigoplus_{m=0}^{n+1} \delta_2^{(m)}, \quad E_2 = H^*(L; \mathbb{Z}) \otimes \Lambda^{\mathbb{Q}}_{nov} = \bigoplus_{m=1}^{n+1} H^m(L; \mathbb{Z}) \otimes \Lambda^{\mathbb{Q}}_{nov},$$

where the homomorphism

$$\delta_2^{(m)} : H^m(L; \mathbb{Q}) \otimes \Lambda^{\mathbb{Q}}_{nov} \to H^{m+1-\mu_L(\gamma)}(L; \mathbb{Q}) \otimes \Lambda^{\mathbb{Q}}_{nov}$$

has the form

$$\delta_2^{(m)} = T e^{\frac{\mu_L(\gamma)}{2}} \overline{\delta}_2^{(m)}.$$

Moreover E_k and δ_k are inductively defined and has the decomposition

$$E_k = \bigoplus_{m=0}^{n+1} E_k^{(m)}, \qquad \delta_k^{(m)} : E_k^{(m)} \to E_k^{(m+1-(k-1)\mu_L(\gamma))},$$

where $E_k^{(m)} = H^m(L;\mathbb{Z}) \otimes \Lambda_{nov}^{\mathbb{Q}}$: Here δ_k satisfies $\delta_k^{(m+1-(k-1)\mu_L(\gamma))} \circ \delta_k^{(m)} = 0$ and $E_{k+1}^{(m)}$ is defined by the quotient

$$E_{k+1}^{(m)} = \frac{\operatorname{Ker} \delta_k^{(m)}}{\operatorname{Im} \delta_k^{(m-1+(k-1)\mu_L(\gamma))}}.$$

Furthermore $\delta_k^{(m)}$ also has the form

$$\delta_k^{(m)} = e^{(k-1)\frac{\mu_L(\gamma)}{2}} T^{k-1} \overline{\delta}_k^{(m)}$$

where $\overline{E}_k^{(m)}$ is the $\mathbb{Q}$-vector space such that

$$E_k^{(m)} = \overline{E}_k^{(m)} \otimes_{\mathbb{Q}} \Lambda_{nov}^{\mathbb{Q}}$$

and the map

$$\overline{\delta}_{k+1}^{(m)} : \overline{E}_{k+1}^{(m)} \to \overline{E}_{k+1}^{(m+1-k\mu_L(\gamma))}$$

is induced by $\mathfrak{m}_{1,k\gamma}^b$ appearing in the expression of $\mathfrak{m}_1^b$ above. We remark that δ_k is induced by the moduli space

$$\mathcal{M}((k-1)\beta).$$

Let $\mathfrak{a}_i \in H^i(S^1 \times S^n;\mathbb{Z})$ be the generators of $H^*(S^1 \times S^n;\mathbb{Z})$ which have degree $i = 0, 1, n, n+1$ respectively. We choose them so that they satisfy $\mathfrak{a}_0 = 1$ and

$$\mathfrak{a}_1 \cup \mathfrak{a}_n = \mathfrak{a}_{n+1} = \mathfrak{a}_0 \cup \mathfrak{a}_{n+1}.$$

Since $i_*([pt]) \in H_0(M;\mathbb{Q})$ is nonzero, we derive from (24.6.3) that $\mathfrak{a}_{n+1} = PD([pt])$ is not in the image of the differentials of the spectral sequence. On the other hand we have $HF(L,L;\Lambda_{nov}^{\mathbb{Q}}) = 0$. Therefore $\mathfrak{a}_{n+1}$ cannot be in the kernel of some differential. Since $H^i(S^1 \times S^n;\mathbb{Z})$ is generated by $\mathfrak{a}_i$ for $i \in \{0,1,n,n+1\}$ and zero otherwise, there must exist $i \in \{0,1,n,n+1\}$, $k \in \mathbb{Z}_{>0}$ and $0 \neq c \in \mathbb{Q}$ such that

$$\overline{\delta}_k^{(n+1)}(\mathfrak{a}_{n+1}) = c\,\mathfrak{a}_i. \tag{37.6}$$

Setting $m = n+1$, $m + 1 - (k-1)\mu_L(\gamma) = i$, we obtain

$$n + 1 - i = -1 + (k-1)\mu_L(\gamma).$$

Therefore $n+1-i$ must be odd in (37.6) and hence i cannot be $n+1$. This implies that i must be one among 0, 1, n.

Since $\delta_k^{(n+1)}(\mathfrak{a}_{n+1}) \in E_k^{(n+2-(k-1)\mu_L(\gamma))}$, (37.6) implies that $\mathfrak{a}_i \in E_k^{(n+2-(k-1)\mu_L(\gamma))}$. Therefore we obtain

$$(k-1)\mu_L(\gamma) \in \{2, n+1, n+2\}. \tag{37.7}$$

In particular, we have proved $\mu_L(\gamma) > 0$ and hence $S^1 \times S^n$ is monotone if and only if it is weakly unobstructed. Recall that any orientable monotone Lagrangian embedding is weakly unobstructed. See chapter 2.

Case 1.1 : n is odd. If n is odd, (37.7) implies that $\mu_L(\gamma)$ divides $n+1$ or 2. Since n is odd this means that $\mu_L(\gamma)$ divides $n+1$ as asserted.

Case 1.2 : n is even. Let us consider the case with even n. We will prove by contradiction that $\mu_L(\gamma) = 2$.

Suppose to the contrary that $\mu_L(\gamma) \neq 2$. Then we derive from (37.6) and (37.7) that there exists $k > 1$ such that $(k-1)\mu_L(\gamma) = n+2$ and $\overline{\delta}_k^{(n+1)}(\mathfrak{a}_{n+1}) = c\,\mathfrak{a}_0 \in H^0(L;\mathbb{Q})$, where $c \neq 0$. Since the generators $\mathfrak{a}_{n+1}, \mathfrak{a}_0$ are killed by each other, it follows that there must exist ℓ and $c' \neq 0$ such that

$$\overline{\delta}_\ell^{(n)}(\mathfrak{a}_n) = c'\,\mathfrak{a}_1 \in H^1(L;\mathbb{Q}),$$

because $HF(L, L; \Lambda_{nov}^{\mathbb{Q}}) = 0$. A degree counting gives rise to $(\ell-1)\mu_L(\gamma) = n$. Namely $\mu_L(\gamma)$ divides both n and $n+2$. Therefore it follows $\mu_L(\gamma) = 2$. This is a contradiction to the hypothesis.

Case 2 : L is not weakly unobstructed: We now consider the case where L is not weakly unobstructed. From Theorem C, which is the weakly unobstructed version of Theorem 11.43, we derive that the obstruction class

$$o_{k\gamma}^{2m_k}(L;\mathrm{weak}) \in H^{2m_k}(L;\mathbb{Q}) \tag{37.8}$$

is nonzero or some $k = k_0 > 0$. Here $k_0\gamma \in H_1(L) \cong H_2(\mathbb{C}^{n+1}, L)$, and

$$2m_{k_0} = 2 - \mu_L(k_0\gamma) = 2 - k_0\mu_L(\gamma). \tag{37.9.1}$$

By Corollary 13.16, the obstruction classes of top dimension must vanish. In other words, we have

$$2m_{k_0} \neq n+1 = \dim L, \tag{37.9.2}$$

if $o_{k\gamma}^{2m_k}(L;\mathrm{weak}) \neq 0$.

We remark that $o_{k\gamma}^{2m_k}(L;\mathrm{weak})$ is non-zero, only if $2m_k = 1, n, n+1$. Note also that the Maslov index $\mu_L(\gamma)$ is even. Therefore from (37.9.1) and (37.9.2), we derive that $2m_{k_0} = n$ and so n must be even. It follows that $k_0\mu_L(\gamma) = 2-n$. Namely $\mu_L(\gamma)$ divides $2-n$, as required. Combining Case 1 and Case 2, we have finished the proof of Theorem 37.4. □

We next use the Floer cohomology over $\mathbb{Z}$ to obtain some additional information, which we collect in Propositions 37.10 and 37.14.

Proposition 37.10. *Suppose that n is even and L is an embedded Lagrangian submanifold of $\mathbb{C}^{n+1}$ homeomorphic to $S^1 \times S^n$, then one of the following alternatives occurs :*

(37.11) $\mu_L(\gamma) = 2$. *The differential of the spectral sequence is trivial except*

$$\begin{aligned} \overline{\delta}_2^{(n+1)}(\mathfrak{a}_{n+1}) &= \pm\mathfrak{a}_n, \\ \overline{\delta}_2^{(1)}(\mathfrak{a}_1) &= \pm\mathfrak{a}_0. \end{aligned}$$

(37.12) $\mu_L(\gamma) = 2$, $\overline{\delta}_2^{(n+1)}(\mathfrak{a}_{n+1}) = \overline{\delta}_2^{(1)}(\mathfrak{a}_1) = 0$, *and*

$$\begin{aligned} \overline{\delta}_{(n+2)/2}^{(n)}(\mathfrak{a}_n) &= \pm\mathfrak{a}_1, \\ \overline{\delta}_{(n+4)/2}^{(n+1)}(\mathfrak{a}_{n+1}) &= \pm\mathfrak{a}_0. \end{aligned}$$

(37.13) $\mu_L(\gamma)$ *is negative and divides* $2 - n$ *with* $n \geq 3$ *or* $\mu_L(\gamma) = 0$ *with* $n = 2$.

Remark 37.14. We will sketch the argument to eliminate the possibility (37.12) in 'Proposition 37.83'. (The detail of it will appear in [Fuk07II].)

Proposition 37.15. *If n is odd and L is as in Proposition* 37.10, *then one of the following alternatives* (37.16), (37.17), (37.18) *occurs :*

(37.16) $\mu_L(\gamma) = (n + 1)/(k - 1)$ *is positive and even. The differential of the spectral sequence is trivial except*

$$\begin{aligned} \overline{\delta}_{k-1}^{(n+1)}(\mathfrak{a}_{n+1}) &= \pm\mathfrak{a}_1, \\ \overline{\delta}_{k-1}^{(n)}(\mathfrak{a}_n) &= \pm\mathfrak{a}_0. \end{aligned}$$

(37.17) $\mu_L(\gamma) = 2$ *and the differential of the spectral sequence is trivial except those appearing in* (37.12).
(37.18) $\mu_L(\gamma) = 2$ *and the differentials satisfy*

$$\overline{\delta}_2^{(n+1)}(\mathfrak{a}_{n+1}) = \lambda\mathfrak{a}_n, \tag{37.19.1}$$

$$\overline{\delta}_2^{(n)}(\mathfrak{a}_1) = \lambda\mathfrak{a}_0 \tag{37.19.2}$$

with $\lambda \notin \{0, 1, -1\}$. *Moreover*

$$\delta_k^{(n)}([\mathfrak{a}_n]) = \mu[\mathfrak{a}_0], \tag{37.19.3}$$

where λ and μ are relatively prime to each other. The differentials appearing in (37.19.1) – (37.19.3) *are all the nontrivial differentials of the spectral sequence.*

Remark 37.20. Lemma 37.3 gives an example of (37.17). We can construct an example of (37.16) with $k = 2$ by using (37.1), (37.2). The authors do not know whether (37.18) actually occurs or not. Also we do not know whether (37.16) with $k \neq 2$ occurs or not.

We remark that in the course of the proof of Theorem 37.4 we proved that L is monotone. Therefore $\mu_L(\gamma) > 0$ *as long as n is even and (37.13) does not occur.* In fact, the argument there shows that L is weakly unobstructed also over $\mathbb{Z}$, not just over $\mathbb{Q}$ in this case. Therefore we will assume the Floer cohomology $HF(L, L)$ is defined over $\Lambda^{\mathbb{Z}}_{0,nov}$ as long as n is even and (37.13) does not occur. (Note Theorem 34.3 does not apply if we deform Floer coboundary map by a bulk deformation (see §13.5). So it is essential here to assume that L is weakly unobstructed without performing any bulk deformation.)

We remark that L is rational since $\pi_1(L) \cong \mathbb{Z}$ and also L is rationally unobstructed as we showed before. So we can apply the construction of the spectral sequence for the rational Lagrangian submanifolds in §25. The spectral sequence constructed in §25 also converges for the coefficient $\Lambda^{\mathbb{Z}}_{nov}$. (We recall that it is not known yet whether the spectral sequence in Theorem 24.5 converges over the coefficient ring $\Lambda^{\mathbb{Z}}_{nov}$. We proved this convergence only over the Λ^{R}_{nov}-coefficient when R is a field. See the remark after Theorem 24.10.)

Proof of Proposition 37.10. Suppose (37.13) does not occur. Then we have shown that L is weakly unobstructed and $\mu_L(\gamma) = 2$ in the course of proving Theorem 37.4. As we mentioned, Floer cohomology $HF(L, L; \Lambda^{\mathbb{Z}}_{nov})$ of L is defined also over $\mathbb{Z}$ and trivial in that case.

Note that since $\mu_L(\gamma) = 2$, the differential $\overline{\delta}_2$ has degree -1. Therefore we have

$$\overline{\delta}_2^{(n+1)}(\mathfrak{a}_{n+1}) = \lambda \mathfrak{a}_n \tag{37.21.1}$$

for some $\lambda \in \mathbb{Z}$. We will prove in Corollary 37.29 that (37.21.1) also implies

$$\overline{\delta}_2^{(1)}(\mathfrak{a}_1) = \lambda \mathfrak{a}_0. \tag{37.21.2}$$

For the other components of $\overline{\delta}_2$, we have $\overline{\delta}_2^{(n)} = 0 = \delta_2^{(0)}$.

Now we consider the following three cases separately.

Case 1 : $\lambda = \pm 1$.
Case 2 : $\lambda = 0$.
Case 3 : $\lambda \notin \{0, \pm 1\}$.

In Case 1, $E_3 = 0$ and hence (37.11) holds.

In Case 2, we obtain $\overline{\delta}_2^{n+1}(\mathfrak{a}_{n+1}) = 0 = \overline{\delta}_2^{(1)}(\mathfrak{a}_1)$ and

$$\overline{\delta}^{(n)}_{(n+2)/2}(\mathfrak{a}_n) = \pm \mathfrak{a}_1, \tag{37.22.1}$$

$$\overline{\delta}^{(n+1)}_{(n+4)/2}(\mathfrak{a}_{n+1}) = \pm \mathfrak{a}_0 : \tag{37.22.2}$$

We recall that we have derived

$$\overline{\delta}^{(n)}_{(n+2)/2}(\mathfrak{a}_n) = c\mathfrak{a}_1, \quad \overline{\delta}^{(n+1)}_{(n+4)/2}(\mathfrak{a}_{n+1}) = c'\mathfrak{a}_0$$

for nonzero rational numbers c, c' from the vanishing of Floer cohomology $HF(L, L; \Lambda^{\mathbb{Q}}_{nov})$ (over the rational number) in the course of the proof of Theorem 37.4.

Here we use the vanishing of Floer cohomology $HF(L, L; \Lambda^{\mathbb{Z}}_{nov})$ (over the integer), which enables us to obtain the stronger statement (37.22). This gives rise to (37.12) in Case 2.

Next we consider Case 3. In this case, we have $E_3^{(m)} = (\mathbb{Z}/|\lambda|\mathbb{Z}) \otimes \Lambda^{\mathbb{Z}}_{nov}$ for $m = 0, n$ and $E_3^{(m)} = 0$ otherwise. Therefore, since we have $HF(L, L; \Lambda^{\mathbb{Z}}_{nov}) = 0$, there should be some k such that $\delta_k^{(n)}$ sends $E_k^{(n)}$ to $E_k^{(0)}$. This is impossible since n is even and $\delta_k^{(n)}$ has odd degree. This finishes the proof of Proposition 37.10, *except the statement that (37.21.1) implies (37.21.2).* □

Next we consider the case n is odd and prove Proposition 37.15.

Proof of Proposition 37.15. In this case, Theorem 37.4 implies that $\mu_L(\gamma)$ is positive, $\mu_L(\gamma)$ divides $n + 1$, and L is monotone (and so weakly unobstructed). In particular Floer cohomology $HF(L, L; \Lambda^{\mathbb{Z}}_{nov})$ is defined which must be trivial in $\mathbb{C}^n$. We consider the following two cases separately.

Case A : $\mu_L(\gamma) \neq 2$.
Case B : $\mu_L(\gamma) = 2$.

We first consider Case A. We put $(k - 1)\mu_L(\gamma) = n + 1$. Then we can prove

(37.23.1) $$\overline{\delta}^{(n+1)}_{k-1}(\mathfrak{a}_{n+1}) = \pm\mathfrak{a}_1,$$

(37.23.2) $$\overline{\delta}^{(n)}_{k-1}(\mathfrak{a}_n) = \pm\mathfrak{a}_0,$$

by the same way as the proof of Theorem 37.4. Here we derive that the coefficients of the right hand side become ± 1, again from the vanishing of Floer cohomology but this time *over* $\mathbb{Z}$. The identity (37.23) then implies that $E_k = 0$. Therefore Case A give rise to (37.16).

We next consider Case B. Then by the same argument as before for case of even n, we obtain the identities (37.21). Let λ be the number given in (37.21.1). Again we will see later in Corollary 37.29 that (37.21.1) implies (37.21.2) as well. We then consider the three cases Case 1 - Case 3 $\lambda = \pm 1$, $\lambda = 0$ and $\lambda \neq \{0, \pm 1\}$ separately as for the case of even n, Proposition 37.10.

For Case 1 ($\lambda = \pm 1$), we have $E_3 = 0$ which gives rise to (37.17).

For Case 2 ($\lambda = 0$), putting $(k - 1)\mu_L(\gamma) = 2(k - 1) = n + 1$, we again obtain (37.23) which corresponds to (37.16).

Now we consider Case 3 ($\lambda \notin \{0, \pm 1\}$). We then obtain

$$E_3^{(m)} = \begin{cases} \mathbb{Z}/|\lambda|\mathbb{Z} & \text{for } m = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

Putting $2(k-1) = n-1$, we can write

$$\delta_k^{(n)}([\mathfrak{a}_n]) = \mu[\mathfrak{a}_0] \tag{37.24}$$

for some integer μ such that $[\mu] \in \mathbb{Z}/|\lambda|\mathbb{Z}$ is a generator of $\mathbb{Z}/|\lambda|\mathbb{Z}$: Namely λ is relatively prime to μ. This gives rise to (37.18).

The proof of Proposition 37.15 is now complete, *modulo the statement that (37.21.1) implies (37.21.2).* □

Remark 37.25. We can prove that the sign in (37.12.1) coincides with the sign in (37.12.2). In fact this follows from Theorem 37.32. We can also prove that the sign in (37.17.1) coincides with the one in (37.17.2). In case $\mu_L(\gamma) = n+1$ this fact follows from Theorem 37.21. We will give a sketch of the proof of other cases in 'Proposition 37.78'.

37.2. Filtered L_∞ structure with inner product.

In this subsection, we study the L_∞ algebra associated to a Lagrangian submanifold L in $\mathbb{C}^{n+1}$ homeomorphic to $S^1 \times S^n$. *We start with a canonical model* $(H(L;\Lambda_{0,nov}), \mathfrak{m}_*)$ of the filtered A_∞ algebra obtained in Corollary 23.6. Symmetrizing it as in §36, we obtain a filtered L_∞ algebra $(H^*(L;\Lambda_{0,nov}), \mathfrak{l}_*)$. (Here $H^*(L;\Lambda_{0,nov}) \cong H^*(L;\mathbb{Q}) \otimes \Lambda_{0,nov}^{\mathbb{Q}}$.) We put

$$\mathfrak{l}_{k+1}^+(x_1, \cdots, x_{k+1}) = \langle x_1, \mathfrak{l}_k(x_2, \cdots, x_k, x_{k+1})\rangle. \tag{37.26}$$

Here $\langle \cdot, \cdot \rangle$ is a version of intersection pairing defined in Definition 47.16 in Chapter 9. We define $\mathfrak{m}_{k+1}^+$ in a similar way.

Proposition 37.27. *Let L be an embedded Lagrangian submanifold of $\mathbb{C}^{n+1}$ homeomorphic to $S^1 \times S^n$. Then $\mathfrak{l}_{k+1}$ is cyclically symmetric modulo T^2. Namely, we have*

$$\begin{aligned} &\mathfrak{l}_{k+1}^+(x_1, \cdots, x_{k+1}) \\ &\equiv (-1)^{\deg' x_{k+1}(\deg' x_1 + \cdots + \deg' x_k)} \mathfrak{l}_{k+1}^+(x_{k+1}, x_1, \cdots, x_k) \mod T^2. \end{aligned} \tag{37.28}$$

Postponing the proof of this proposition until the end of this subsection (and §47.3 about the sign), we first state several consequences thereof. We start with the following

Corollary 37.29. *Suppose $\mu_L(\gamma) = 2$. Then* (37.21.1) *implies* (37.21.2).

Proof. We remark that if $\overline{\delta}_2^{(n+1)}(\mathfrak{a}_{n+1}) = \lambda \mathfrak{a}_n$, we have $\delta_2^{(n+1)}(\mathfrak{a}_{n+1}) = \lambda \mathfrak{a}_n \cdot T e^{\frac{\mu_L(\gamma)}{2}}$. This implies

$$\lambda T \equiv \langle \mathfrak{a}_1, \delta_2^{(n+1)}(\mathfrak{a}_{n+1}) \rangle = \mathfrak{l}_2^+(\mathfrak{a}_1, \mathfrak{a}_{n+1}) \mod T^2.$$

(Note $\mathfrak{l}_1 = \mathfrak{m}_1$.) Also if $\overline{\delta}_2^{(1)}(\mathfrak{a}_1) = \mu \mathfrak{a}_0$ then $\mu T \equiv \mathfrak{l}_2^+(\mathfrak{a}_{n+1}, \mathfrak{a}_1) \mod T^2$. Therefore Corollary 37.29 follows from Proposition 37.27. □

We continue our analysis of our spectral sequence and the filtered L_∞ structure with some more details for the case of $L \cong S^1 \times S^n$.

Theorem 37.30. *Suppose $\mu_L(\gamma) = n + 1$, and n is odd. Then we have*

$$\mathfrak{l}_{k+2}^+(\mathfrak{a}_1^{\otimes k}, \mathfrak{a}_n, \mathfrak{a}_{n+1}) \equiv \pm \frac{1}{k!} e^{(n+1)/2} T \mod T^2. \tag{37.31}$$

Here the sign $\pm$ is independent of k. Moreover the left hand side is independent of the permutation of the variables modulo T^2. All the other operations are zero modulo T^2.

The result in the other case is less complete.

Theorem 37.32. *If $\mu_L(\gamma) = 2$, then there exists an integer λ such that*

$$\mathfrak{l}_{k+1}^+(\mathfrak{a}_1^{\otimes k}, \mathfrak{a}_{n+1}) \equiv \pm \frac{1}{k!} \lambda e T \mod T^2. \tag{37.33}$$

Here λ is independent of k. Moreover the left hand side is independent of the permutation of the variables modulo T^2. All the other operations are zero modulo T^2.

We note that we do not assume n is odd in Theorem 37.32 unlike in Theorem 37.30. We can say more about λ in Theorem 37.32, which we will discuss later in this section ('Proposition 37.84').

Remark 37.34. (1) For the case of $S^1 \times S^n$ constructed using small embeddings $S^1 \subset \mathbb{C}$ mentioned in (37.2), we can easily see that $\lambda = \pm 1$.

(2) We remark that the canonical model is well-defined only up to isomorphism. This means the following : Let $\mathfrak{l}'^+_{k+1}$ be the filtered L_∞ structure on $H(L; \Lambda_{0,nov})$ obtained by symmetrizing another canonical model. Then there exists a sequence of $\Lambda_{0,nov}$ linear maps $\varphi_k : E_k(H(L; \Lambda_{0,nov})) \to H(L; \Lambda_{0,nov})$ such that it induces a coalgebra isomorphism $\widehat{\varphi} : E(H(L; \Lambda_{0,nov})) \to E(H(L; \Lambda_{0,nov}))$ satisfying

$$\mathfrak{l}_*^+(\widehat{\varphi}(x_1 \otimes \cdots \otimes x_{k+1})) \equiv \mathfrak{l}_{k+1}^+(x_1 \otimes \cdots \otimes x_k \otimes x_{k+1}) \mod T^2.$$

Therefore, strictly speaking, we should state Theorems 37.30 and 37.32 as "there exists a canonical model such that ... ". We will explain this ambiguity in more detail later in §37.5.

Now we begin the proofs of Theorems 37.30 and 37.32. Let $\beta \in \pi_2(\mathbb{C}^{n+1}, L)$ be the class satisfying $\partial\beta = \gamma$ where $\partial : \pi_2(\mathbb{C}^{n+1}, L) \to \pi_1(L)$ is the natural boundary map. We consider the moduli spaces $\mathcal{M}_0(\beta)$ and $\mathcal{M}_1(\beta)$ of stable maps $(D^2, \partial D^2) \to (\mathbb{C}^{n+1}, L)$ in the class β and with zero (resp. one) marked point at the boundary. We recall that $\mathcal{M}_i(\beta)$ depends on the choice of almost complex structure J.

For our purpose of proving Theorems 37.30 and 37.32, it will be useful to give another description of $\mathcal{M}_0(\beta)$ and $\mathcal{M}_1(\beta)$ which is equivalent to that of stable maps and more close to Gromov's original description in [Grom85]. For given $J \in \mathcal{J}_{\omega_0}$ a compatible almost complex structure on $(\mathbb{C}^{n+1}, \omega_0)$, we recall the definition

$$\widetilde{\mathcal{M}}(J;\beta) = \{w : (D^2, \partial) \to (\mathbb{C}^{n+1}, L) \mid w \text{ is pseudo-holomorphic, } [w] = \beta\}.$$

The group $PSL(2;\mathbb{R})$ is identified with the group of biholomorphic maps $\varphi : D^2 \to D^2$, and then it acts on $\widetilde{\mathcal{M}}(\beta)$ by $\varphi \cdot w = w \circ \varphi^{-1}$. We put

$$G = \{\varphi \in PSL(2;\mathbb{R}) \mid \varphi(1) = 1\}.$$

It is easy to see that we have the isomorphism

$$G\backslash\widetilde{\mathcal{M}}(\beta) = \mathcal{M}_1(\beta)$$

and

$$\mathcal{M}_0(\beta) = PSL(2;\mathbb{R})\backslash\widetilde{\mathcal{M}}(\beta).$$

Now we state the following two lemmas concerning the structures of $\mathcal{M}_0(\beta)$ and $\mathcal{M}_1(\beta)$.

Lemma 37.35. *Let $L \sim S^1 \times S^n$ and J_0 be any compatible almost complex structure on $\mathbb{C}^{n+1}$. Then the moduli space $\mathcal{M}_0(J_0;\beta)$ is compact and any element $w \in \mathcal{M}_0(J_0;\beta)$ is somewhere injective.*

In particular, for a generic almost complex structure J, the moduli space $\mathcal{M}_0(J;\beta)$ is a smooth compact manifold without boundary of dimension given by $\mu_L(\gamma)+n-2$, as long as it is nonempty.

Proof. Assume $\widetilde{\mathcal{M}}(J_0;\beta) \neq \emptyset$. Then compactness of $\mathcal{M}_0(J_0;\beta)$ follows from the Gromov compactness theorem since β has the smallest possible positive symplectic area $E(\beta) = \omega([\beta])$. The latter also implies that any element w_0 of $\widetilde{\mathcal{M}}(J_0;\beta)$ is somewhere injective in the sense of McDuff [McD87] : Otherwise, according to the structure theorem of the image of pseudo-holomorphic *discs* from [KwOh00], w_0 is decomposed into

$$w_0 = \sum_{j=1}^{N} w_j$$

such that each w_j is a somewhere injective J-holomorphic disc with boundary lying on the same L, and their homotopy classes satisfy

$$[w_0] = \sum_{j=1}^{N} [w_j] \quad \text{in } \pi_2(\mathbb{C}^{n+1}, L), \quad \omega([w_j])0$$

which is impossible unless $N = 1$ because $\omega([\beta])$ has the smallest possible positive symplectic area.

Now once we have established the somewhere injectivity, we apply the standard argument [McD87] of the Sard-Smale theorem to elements of the pairs

$$(J, w) \in \mathcal{J}_\omega \times Map((D^2, \partial D^2), (\mathbb{C}^{n+1}, L); \beta)$$

near $\{J_0\} \times \widetilde{\mathcal{M}}(J_0; \beta)$ for each given J_0. Then we derive that for a generic choice of J, the moduli space $\widetilde{\mathcal{M}}(J; \beta)$ becomes a smooth manifold. The formula of dimension follows from Theorem 2.32 when it is nonempty. Since somewhere injectivity also implies that $PSL(2; \mathbb{R})$ acts freely on $\widetilde{\mathcal{M}}(J_0; \beta)$, the quotient space $\mathcal{M}_0(J_0; \beta) = PSL(2; \mathbb{R}) \backslash \widetilde{\mathcal{M}}(J_0; \beta)$ is also smooth. □

We also have

Lemma 37.36. *For a generic almost complex structure J on $\mathbb{C}^{n+1}$, the moduli space $\mathcal{M}_1(J; \beta)$ is a compact smooth manifold without boundary of dimension $\mu_L(\gamma) + n - 1$, when it is nonempty.*

We now take J as above and fix it throughout this section.

Since $PSL(2; \mathbb{R})/G \cong S^1$, there exists an S^1 fibration $\pi : \mathcal{M}_1(\beta) \to \mathcal{M}_0(\beta)$. We take a free S^1 action on $\mathcal{M}_1(\beta)$ so that π is identified with the natural projection $\mathcal{M}_1(\beta) \to \mathcal{M}_1(\beta)/S^1$. We embed $S^1 \subset PSL(2; \mathbb{R})$ so that $0 \in D^2$ is fixed by elements of S^1.

Lemma 37.37. *There exists an S^1 equivariant map $\mathcal{S} : \mathcal{M}_1(\beta) \to \widetilde{\mathcal{M}}(\beta)$ such that the composition $\pi \circ \mathcal{S}$ of $\mathcal{S}$ with natural projection $\pi : \widetilde{\mathcal{M}}(\beta) \to \mathcal{M}_1(\beta)$ is the identity.*

Proof. Let $\bigcup_{i=1}^{N} V_i = \mathcal{M}_0(\beta)$ be an open covering such that V_i are contractible. We put $U_i = \pi^{-1} V_i \subset \mathcal{M}_1(\beta)$ and $\bigcup_{i=1}^{k} U_i = U^k$. We will construct $\mathcal{S}$ on U^k by induction on k. Suppose we have defined $\mathcal{S}$ on U^{k-1}. Since V_k is contractible we have $s : V_k \to \mathcal{M}_1(\beta)$ such that $\pi \circ s$ is identity. We put $W_k = s^{-1}(U^{k-1}) \subset V_k$. The restriction of $\mathcal{S}$ to $s(V_k) \cap U^{k-1}$ can be identified with a map $S : W_k \to G$ since $\pi^{-1}(s(V_k)) \cong V_k \times G$. Therefore since G is contractible we can extend S to $S : V_k \to G$. Now let $x \in U_k$. There exists unique $\overline{x} \in V_k$ and $t \in S^1$ such that $x = t \cdot s(\overline{x})$. We put $\mathcal{S}(x) = t \cdot (\overline{x}, S(\overline{x}))$, where we identify $\pi^{-1}(s(V_k)) \cong V_k \times G$ and regard $(\overline{x}, S(\overline{x})) \in \pi^{-1}(s(V_k)) \subset \widetilde{\mathcal{M}}(\beta)$. □

Now we define

$$\widetilde{ev}^{(k)} = (ev_1, \cdots, ev_k, ev_0) : \widetilde{\mathcal{M}}(\beta) \times (S^1)^k \to L^{k+1}$$

by

$$\widetilde{ev}^{(k)}(\varphi, (t_1, \cdots, t_k)) = (\varphi(t_1), \cdots, \varphi(t_k), \varphi(1)).$$

Then we define $ev^{(k)} : \mathcal{M}_1(\beta) \times (S^1)^k \to L^{k+1}$ by

$$ev^{(k)} = \widetilde{ev}^k \circ (\mathcal{S} \times id).$$

Let $x_1, \cdots, x_k, x_0 \in H^*(L;\mathbb{Q})$.

We will give the proof of the following later in §37.4.

Proposition 37.38.

$$\begin{aligned}&\mathfrak{l}^+_{k+1}(x_1, \cdots, x_k, x_0)\\ &\equiv \pm \frac{1}{k!} e^{\mu_L(\gamma)/2} T \left(ev^{(k)*}(x_1 \times \cdots \times x_k \times x_0)([\mathcal{M}_1(\beta) \times (S^1)^k]) \right) \mod T^2.\end{aligned}$$

The intuitive picture behind Proposition 37.38 is quite clear from the definition. However the rigorous definition of $\mathfrak{l}^+_{k+1}$ is rather complicate partly due to the transversality problem : We first take perturbations in the fiber product, then go to a canonical model, and finally symmetrize the corresponding A_∞ operations. This sequence of general abstract constructions, none of which are not so transparent, makes the proof of Proposition 37.38 rather nontrivial. We postpone its proof until later in §37.4.

Assuming Proposition 37.38, we are ready to finish the proof of Proposition 37.27.

Proof of Proposition 37.27. Except the sign, the proof is immediate from Proposition 37.38. The sign in Proposition 37.27 is related to the sign in Proposition 37.38 and follows from the sign convention we will work out in detail in the next chapter. See §47.3. □

37.3. Free loop space of L and L_∞ structure.

In this subsection, we exploit some topological information of the loop space of the Lagrangian submanifold L in our study of the L_∞ structures on L. More extensive study on this aspect will be carried out in [Fuk07II] (see [Fuk05II] for some detailed outlines).

We denote by $\mathcal{L}(L)$ the set of all smooth loops on L. Namely

$$\mathcal{L}(L) = \{\ell : S^1 \to L \,|\, \ell \text{ is smooth}\}.$$

We denote by $\mathcal{L}(L;\gamma)$ the subset of $\mathcal{L}(L)$ consisting of the loops *homologous* to γ. We remark that $\mathcal{L}(L)$ and $\mathcal{L}(L;\gamma)$ have an S^1 actions induced by the reparametrization of the domain S^1. The S^1 action on $\mathcal{L}(L;\gamma)$ is free : Recall that γ is a nontrivial primitive class which rules out a possible finite isotropy group.

There exists an obvious S^1 equivariant map

$$\mathrm{res} : \widetilde{\mathcal{M}}(\beta) \to \mathcal{L}(L;\gamma)$$

whose image $\mathrm{res}(\varphi)$ is defined by $(\mathrm{res}(\varphi))(t) = \varphi(t)$ for $\varphi \in \widetilde{\mathcal{M}}(\beta)$. We denote

$$ev = \mathrm{res} \circ \mathcal{S} : \mathcal{M}_1(\beta) \to \mathcal{L}(L;\gamma).$$

We now study the homology class

$$ev_*([\mathcal{M}_1(\beta)]) \in H_*(\mathcal{L}(L;\gamma), \mathbb{Z}).$$

Let $\pi : \mathcal{L}(L;\gamma) \to \mathcal{L}(L;\gamma)/S^1$ be the natural projection. It is a projection of S^1 principal bundle and hence induces a homomorphism

$$\pi_!^* : H_k(\mathcal{L}(L;\gamma)/S^1, \mathbb{Z}) \to H_{k+1}(\mathcal{L}(L;\gamma), \mathbb{Z}) :$$

If $f : P \to \mathcal{L}(L;\gamma)/S^1$ represents a homology class $f_*([P])$, then $\pi_!^*(f_*([P]))$ is represented by $P \times_{\mathcal{L}(L;\gamma)/S^1} \mathcal{L}(L;\gamma) = \tilde{P}$ and a map $\tilde{f} : \tilde{P} \to \mathcal{L}(L;\gamma)$ induced by f. We have :

Lemma 37.39. *$ev_*([\mathcal{M}_1(\beta)])$ is in the image of $\pi_!^*$.*

Proof. Using the fact that ev is S^1 equivariant, we have $\overline{ev} : \mathcal{M}_0(\beta) \to \mathcal{L}(L;\gamma)/S^1$. It is immediate from definition that $\pi_!^*(\overline{ev}_*([\mathcal{M}_0(\beta)])) = ev_*([\mathcal{M}_1(\beta)])$. □

Recall that we assumed that L is homeomorphic to $S^1 \times S^n$. Identifying L with $S^1 \times S^n$, we define a map $\widetilde{I} : \mathcal{L}(S^n) \to \mathcal{L}(L;\gamma)$ by

$$\widetilde{I}(\lambda) = \tilde{\lambda}, \qquad \tilde{\lambda}(t) = (t, \lambda(t)) \in S^1 \times S^n \tag{37.40}$$

and $I : \mathcal{L}(S^n) \to \mathcal{L}(L;\gamma)/S^1$ by $I = \pi \circ \widetilde{I}$.

Lemma 37.41. *I is a homotopy equivalence.*

Proof. It is enough to construct a homotopy inverse to I. We remark $\mathcal{L}(L;\gamma) = \mathcal{L}(S^n) \times \mathcal{L}(S^1;\gamma)$. It is easy to see that $\mathcal{L}(S^1;\gamma)$ is homotopy equivalent to S^1 and $\mathcal{L}(S^1;\gamma)/S^1$ is contractible. Existence of a homotopy inverse to I easily follows from this fact. □

Now we recall some classical results on the cohomology of the loop space of a sphere. We fix a base point $p_0 \in S^n$ and let $\mathcal{L}_0(S^n) = \{\ell \in \mathcal{L}(S^n) \mid \ell(1) = p_0\}$ be the based-loop space. Let $\mathcal{P}(S^n) = \{\ell : [0,1] \to S^n \mid \ell(0) = p_0\}$. We have a path fibration $\mathcal{L}_0(S^n) \to \mathcal{P}(S^n) \to S^n$. Since $\mathcal{P}(S^n)$ is contractible, we can calculate the cohomology of $\mathcal{L}_0(S^n)$ by the Leray-Serre spectral sequence.

We denote by $E(\mathfrak{x}_1, \cdots, \mathfrak{x}_m)$ the free graded commutative algebra generated by the elements $\mathfrak{x}_1, \cdots, \mathfrak{x}_m$ of the degree $\deg \mathfrak{x}_i$. More precisely, if all of $\mathfrak{x}_1, \cdots, \mathfrak{x}_m$ are of even degree, $E(\mathfrak{x}_1, \cdots, \mathfrak{x}_m)$ is a polynomial algebra, and if all of them are of odd degree, it is an exterior algebra and etc. We quote the following result by Serre

Lemma 37.42. (Serre [Ser51]) *If n is odd, then there exists $\mathfrak{x}$ with $\deg \mathfrak{x} = n-1$ such that $H^*(\mathcal{L}_0(S^n);\mathbb{Q}) \cong E(\mathfrak{x})$. If n is even, then there exists $\mathfrak{x}$ with $\deg \mathfrak{x} = n-1$ and $\mathfrak{y}$ with $\deg \mathfrak{y} = 2n-2$ such that $H^*(\mathcal{L}_0(S^n);\mathbb{Q}) \cong E(\mathfrak{x},\mathfrak{y})$.*

We next consider the fibration $\mathcal{L}_0(S^n) \to \mathcal{L}(S^n) \to S^n$. (Here the projection $\mathcal{L}(S^n) \to S^n$ is $\ell \mapsto \ell(1)$.) Let $\mathfrak{z}_0, \mathfrak{z}_n$ be the generators of degree 0 and n in $H^*(S^n)$, respectively. We consider the Leray-Serre spectral sequence

$$H^*(\mathcal{L}_0(S^n);\mathbb{Q}) \otimes H^*(S^n;\mathbb{Q}) \Rightarrow H(\mathcal{L}(S^n);\mathbb{Q}). \tag{37.43}$$

Lemma 37.44. (Poirrier and Sullivan [PoSu76]) *If n is odd, then the differentials of the spectral sequence* (37.43) *are all zero. If n is even, then the differentials of the spectral sequence* (37.43) *are zero except :*

$$d(\mathfrak{y}^k \otimes \mathfrak{z}_0) = 2k(\mathfrak{x}\mathfrak{y}^{k-1} \otimes \mathfrak{z}_n), \quad k = 1, 2, \cdots .$$

With these preparations, we are now ready to give the proofs of Theorems 37.30 and 37.32. We start with Theorem 37.32.

Proof of Theorem 37.32. We recall the standing hypothesis $\mu_L(\gamma) = 2$. By Lemma 37.36, $\mathcal{M}_1(\beta)$ becomes a smooth manifold of dimension given by

$$\mu_L(\gamma) + n - 2 + 1 = n + 1.$$

Lemma 37.39 implies $ev_*[\mathcal{M}_1(\beta)] = \pi_!^*(\overline{ev}_*([\mathcal{M}_0(\beta)]))$ and $\deg[\mathcal{M}_0(\beta)] = n$. Note it follows from Lemma 37.42 and Lemma 37.44 that $H_n(\mathcal{L}(S^n);\mathbb{Q}) = \mathbb{Q}$ and is generated by the dual class $(1 \otimes \mathfrak{z}_n)^*$ of $1 \otimes \mathfrak{z}_n$. Lemma 37.41 induces an isomorphism

$$I : H_n(\mathcal{L}(S^n);\mathbb{Q}) \to H_n(\mathcal{L}(L;\gamma)/S^1;\mathbb{Q})$$

and hence $H_n(\mathcal{L}(L;\gamma)/S^1;\mathbb{Q}) \cong \mathbb{Q}$. Therefore we can write

$$\overline{ev}_*([\mathcal{M}_0(\beta)]) = \lambda I((1 \otimes \mathfrak{z}_n)^*) \tag{37.45}$$

for a constant λ where I is the map given in (37.40). We also recall

$$(1 \otimes \mathfrak{z}_n)^* = [S^n] \in H_n(\mathcal{L}(S^n);\mathbb{Q})$$

where $S^n \subset \mathcal{L}(S^n)$ is identified with the set of of constant loops.

We will now derive Theorem 37.32 from (37.45) and Proposition 37.38. First Proposition 37.38 implies that the operations $\mathfrak{l}^+_{k+1}$ depend only on the homology class of $[\mathcal{M}_0(\beta)]$. Since the right hand side of Proposition 37.38 is linear on $[\mathcal{M}_0(\beta)]$, we may assume $\lambda = 1$ without loss of generalities and write $\overline{ev}_*([\mathcal{M}_0(\beta)]) = I_*([S^n])$.

Then under the evaluation map $ev = \mathrm{res} \circ \mathcal{S} : \mathcal{M}_1(\beta) \to \mathcal{L}(L;\gamma)$, $ev_*([\mathcal{M}_1(\beta)])$ can be identified with the fundamental class of the set consisting of the loops $\ell \in \mathcal{L}(L;\gamma)$ defined by

$$\ell_{s,p}(t) = (s+t, p) \in S^1 \times S^n.$$

Obviously this set is diffeomorphic to $S^1 \times S^n$. Under this identification, the class $ev^{(k)}_*([\mathcal{M}_1(\beta) \times (S^1)^k]) \in H_{n+1}(L^{k+1};\mathbb{Q})$ can be represented by the map

$$f_k : (S^1 \times S^n) \times (S^1)^k \to L^{k+1}$$

defined by

$$f_k\left((s,p),(t_1,\cdots,t_k)\right) = ((s,p),(s+t_1,p),\cdots,(s+t_k,p)).$$

Now Theorem 37.32 easily follows from Proposition 37.38 by replacing the chain $[\mathcal{M}_1(\beta) \times (S^1)^k, ev^{(k)}]$ by this chain $[(S^1 \times S^n) \times (S^1)^k, f]$ and evaluating

$$\begin{aligned} & ev^{(k)*}(\mathfrak{a}_{n+1} \times \mathfrak{a}_1 \times \cdots \times \mathfrak{a}_1)([\mathcal{M}_1(\beta) \times (S^1)^k]) \\ &\quad = (\mathfrak{a}_{n+1} \times \mathfrak{a}_1 \times \cdots \times \mathfrak{a}_1)(ev^{(k)}_*([\mathcal{M}_1(\beta) \times (S^1)^k]) \\ &\quad = (\mathfrak{a}_{n+1} \times \mathfrak{a}_1 \times \cdots \times \mathfrak{a}_1)[f_{k,*}(S^1 \times S^n \times (S^1)^k)] = 1 \end{aligned}$$

where the last identity follows from the definition of f. □

We next turn to the case when $\mu_L(\gamma) = n+1$.

Proof of Theorem 37.30. In this case, we derive $\deg[\mathcal{M}_0(\beta)] = 2n-1$ from Lemma 37.35. We recall the standing assumption that n is odd in this case. By Lemmas 37.42 and 37.44, $H_{2n-1}(\mathcal{L}(S^n);\mathbb{Q}) = \mathbb{Q}$ and is generated by the dual class $(\mathfrak{x} \otimes \mathfrak{z}_n)^*$ to $\mathfrak{x} \otimes \mathfrak{z}_n$. Again applying Lemma 37.41, we obtain an isomorphism

$$I_* : H_{2n-1}(\mathcal{L}(S^n);\mathbb{Q}) \to H_{2n-1}(\mathcal{L}(L;\gamma)/S^1;\mathbb{Q}).$$

We put

$$\overline{ev}_*([\mathcal{M}_0(\beta)]) = \lambda I_*((\mathfrak{x} \otimes \mathfrak{z}_n)^*). \tag{37.46}$$

Lemma 37.47. *If* (37.46) *holds, then*

$$\mathfrak{l}^+_{k+2}(\mathfrak{a}_1^k, \mathfrak{a}_n, \mathfrak{a}_{n+1}) \equiv \pm\frac{1}{k!}\lambda e^{(n+1)/2}T \mod T^2. \tag{37.48}$$

Moreover the left hand side is independent of the permutation of the variables modulo T^2*.*

Proof. We remark that $\mathfrak{l}^+_{k+2}(\mathfrak{a}_{n+1}, \mathfrak{a}_1^k, \mathfrak{a}_n)$ is independent of the permutation of last $k+1$ variables from its definition. Then Proposition 37.27 implies that it is independent of all the permutations of variables modulo T^2.

To prove (37.48), it suffices to consider the case $\lambda = 1$. We need to examine the class $(\mathfrak{x} \otimes \mathfrak{z}_n)^*$ in more detail. Let $P \subset \mathcal{L}(S^n)$ be a chain representing the class $(\mathfrak{x} \otimes \mathfrak{z}_n)^*$ and consider the composition $\pi : P \to \mathcal{L}(S^n) \to S^n$. By perturbing the chain if necessary, we may assume $p_0 \in S^n$ is a regular value of π. We put $P_0 = \pi^{-1}(p_0) \subset \mathcal{L}_0(S^n)$. Then by definition, P_0 represents the class $\mathfrak{x}^* \in H_{n-1}(\mathcal{L}_0(S^n); \mathbb{Q})$ dual to $\mathfrak{x}$.

We then define a chain $[P_0 \times S^1, h]$ in S^n by the map $h(x,t) = \ell_x(t)$ where ℓ_x is a loop corresponding to $x \in P_0 \subset \mathcal{L}_0(S^n)$. Since P_0 represents the class $\mathfrak{x}^*$, it follows from the definition of $\mathfrak{x}$ that $[P_0 \times S^1, h]$ represents the fundamental class of S^n. Therefore, if we define $h_+ : P \times S^1 \to S^n \times S^n$ by $(x,t) \mapsto (\ell_x(t), \ell_x(1))$, we obtain

$$h_{+*}([P \times S^1]) = [S^n \times S^n]. \tag{37.49}$$

We recall the commutative diagram

$$\begin{array}{ccccc}
\mathcal{M}_1(\beta) & \xrightarrow{ev} & \mathcal{L}(L;\gamma) & & \\
\big\downarrow & & \big\downarrow{\scriptstyle \pi} & & \\
\mathcal{M}_0(\beta) & \xrightarrow{\overline{ev}} & \mathcal{L}(L;\gamma)/S^1 & \xleftarrow{I} & \mathcal{L}(S^n).
\end{array}$$

We have the identity

$$[\mathcal{M}_1(\beta), ev] = (\pi_!)^*(I_*[P]) = [P \times S^1, \mathrm{Rot}(\widetilde{I})] \quad \text{in } H_{2n}(\mathcal{L}(L;\gamma);\mathbb{Q})$$

where the map $\mathrm{Rot}(\widetilde{I}) : P \times S^1 \to \mathcal{L}(L;\gamma)$ is defined by the map

$$\mathrm{Rot}(\widetilde{I})(s,x)(t) = (t, \ell_x(t+s)).$$

Then the cycle $ev^{(k+1)} : \mathcal{M}_1(\beta) \times (S^1)^{k+1} \to L^{k+2} \cong (S^1 \times S^n)^{k+2}$ is homologous to the map

$$f_{k+1} : (P \times S^1) \times (S^1)^{k+1} \to L^{k+2}$$

defined by

(37.50)
$$f_{k+1}((x,s),(t_1,\cdots,t_{k+1})) = ((s+t_1,\ell_x(t_1)),\cdots,(s+t_{k+1},\ell_x(t_{k+1})),(s,\ell_x(1))\,.$$

Now we consider the map $pr : L^{k+2} \to (S^1)^k \times S^n \times (S^1 \times S^n)$, where

$$pr((t_1,p_1),\cdots,(t_{k+1},p_{k+1}),(t_0,p_0)) = (t_1,\cdots,t_k,p_{k+1},(t_0,p_0)).$$

Then (37.49) and (37.50) imply

$$\begin{aligned}(pr\circ ev^{(k+1)})_*([\mathcal{M}_1(\beta)\times(S^1)^{k+1}]) &= (pr\circ f_{k+1})_*([(S^1\times S^n)\times(S^1)^{k+1}]) \\ &= [(S^1)^k\times S^n\times(S^1\times S^n)].\end{aligned}$$

By a similar evaluation as in the end of the proof of Theorem 37.32, we obtain Lemma 37.47 from Proposition 37.38. □

Lemma 37.51. $\lambda = \pm 1$ *in* (37.46).

Proof. (37.48) implies $\mathfrak{l}_2^+(\mathfrak{a}_n,\mathfrak{a}_{n+1}) \equiv \lambda T \mod T^2$. Hence $\mathfrak{m}_1(\mathfrak{a}_n) = \mathfrak{l}_1(\mathfrak{a}_n) \equiv \lambda e^{(n+1)/2}T\mathfrak{e} \mod T^2$. Therefore $\lambda = \pm 1$ by Proposition 37.15. (Note we are in the case (37.16) and $\mu_L(\gamma) = n+1$.) (We also remark that the proof of Proposition 37.15 was completed when we proved Corollary 37.29.) □

We now have completed the proof of Theorems 37.31 except the proof of Proposition 37.38, which we will carry out in the next subsection. □

37.4. The de Rham version of the filtered L_∞ algebra : A simple case.

In this subsection we will prove Proposition 37.38. For this purpose, we need to perform some construction used in the de Rham version of the filtered A_∞ structure and to prove that this is isomorphic to the 'singular homology' version which we have worked out in detail in this book. Combining the results of this book with various additional ideas, especially those using the loop space homology, Chas-Sullivan bracket [ChSu99] and the iterated integrals [Chen73], we can construct the de Rham version in its full generality. This will be carried out elsewhere in the future. (See [Fuk07II].) In this subsection, we restrict ourselves only upto what we need for the proof of Proposition 37.38. The rest of this subsection will be occupied with this proof.

Proof of Proposition 37.38. Our proof below is based on a generalization of the argument used in the proof of Theorem 33.1. Let $(\Omega(L),d,\wedge)$ be the de Rham complex of L. It is a differential graded algebra which can be regarded as an A_∞

algebra $(\Omega(L), \overline{\mathfrak{m}})$ with the definitions of $\mathfrak{m}_1$ and $\mathfrak{m}_2$ and $\mathfrak{m}_k = 0$ for $k \geq 3$ as in Remark 7.4 (2).

We will deform this A_∞ algebra adding contributions by holomorphic discs. For the purpose of proving Proposition 37.38, we will need only its leading order terms. The description of the leading order terms is now in order.

We recall that $\mathcal{M}_1(\beta)$ is a smooth manifold for the case of our interest. However the evaluation map $ev : \mathcal{M}_1(\beta) \to L$ may not still be a submersion in general. Therefore we will increase the dimension of the moduli space by including an obstruction bundle and enlarge $\mathcal{M}_1(\beta)$ to $\mathcal{M}_1(\beta)^+$ equipped with the obstruction bundle $E \to \mathcal{M}_1(\beta)^+$ and a section s thereof so that $s^{-1}(0) = \mathcal{M}_1(\beta)$ and $ev^+ : \mathcal{M}_1(\beta)^+ \to L$ becomes a submersion. In other words, $\mathcal{M}_1(\beta)^+$ is nothing but a Kuranishi neighborhood constructed in Proposition 29.1.

We then take a manifold W_0 and a family $\{\mathfrak{s}_w\}_{w \in W_0}$ of perturbations of $\mathfrak{s}$ parameterized by w such that, as a section of the bundle $E \to \mathcal{M}_1^+(\beta) \times W_0$, $\mathfrak{s}$ is transversal to the zero section and so the zero set

$$\mathcal{M}_1(\beta)^{\mathfrak{s}} = \{(x, w) \in \mathcal{M}_1(\beta) \times W_0 \mid \mathfrak{s}_w(x) = 0\}$$

becomes a smooth manifold. By suitably choosing $\mathfrak{s}$, we can also assume that the restriction of the map given by $(x, w) \mapsto ev^+(x)$ to $\mathcal{M}_1(\beta)^{\mathfrak{s}}$ defines a submersion

$$ev_0^{\mathfrak{s}} : \mathcal{M}_1(\beta)^{\mathfrak{s}} \to L.$$

We take a smooth form ω_{W_0} with compact support in W_0 and of the top degree such that $\int_{W_0} \omega_{W_0} = 1$. We pull it back to $\mathcal{M}_1(\beta)^{\mathfrak{s}}$ and denote it by the same symbol. Now we explain how to define a k-linear map

$$\mathfrak{m}_{k,\beta}^{dR} : B_k(\Omega(L)[1]) \to \Omega(L)$$

for each integer $k \geq 1$. Let $u_1, \cdots, u_k \in \Omega(L)$ and

$$C_k = \{(t_1, \cdots, t_k) \in [0,1]^k \mid t_1 < \cdots < t_k\}.$$

We define

$$ev^{(k),\mathfrak{s}} = (ev_1^{\mathfrak{s}}, \cdots, ev_k^{\mathfrak{s}}, ev_0^{\mathfrak{s}}) : \mathcal{M}_1(\beta)^{\mathfrak{s}} \times C_k \to L^{k+1}$$

by

$$ev_i^{\mathfrak{s}}((x, w), (t_1, \cdots, t_k)) = \begin{cases} \ell_x(e^{2\pi\sqrt{-1}t_i}) & \text{for } i \neq 0, \\ \ell_x(1) & \text{for } i = 0. \end{cases}$$

Here $\ell_x : S^1 \to L$ be the boundary value of the map corresponding to x.

Now we put

$$\mathfrak{m}_{k,\beta}^{dR}(u_1, \cdots, u_k) = ev_{0!}^{\mathfrak{s}}\left((ev_1^{\mathfrak{s}}, \cdots, ev_k^{\mathfrak{s}})^*(u_1 \times \cdots \times u_k) \wedge \omega_{W_0}\right). \tag{37.52}$$

Here $ev_{0!}^{\mathfrak{s}}$ is the integration along the fiber. We remark that the right hand side of (37.52) is a smooth form since $ev_0^{\mathfrak{s}}$ is a submersion and ω_{W_0} has compact support.

We then take the sum

$$\mathfrak{m}_k^{dR(1)} = \overline{\mathfrak{m}}_k \pm \sum_{\beta} e^{\mu_L(\gamma)/2} T \mathfrak{m}_{k,\beta}^{dR} \tag{37.53}$$

where '(1)' stands for the 'first order'. (Here and hereafter in this section, we omit the detail of argument on signs. Actually, it can be handled in the same way as in §33 and §53.) We say that $(C, \mathfrak{m})$ is a *filtered A_∞ algebra modulo T^2* if the A_∞ relations hold modulo T^2.

Lemma 37.54. *The family of the operators $\mathfrak{m}_k^{dR(1)}$ defines a structure of filtered A_∞ algebra modulo T^2 on $\Omega(L) \otimes \Lambda_{0,nov}^{\mathbb{R}}$.*

Proof. Since $\mathcal{M}_1(\beta)$ is a manifold without boundary, $\mathfrak{m}_{k,\beta}^{dR} : B_k\Omega(L)[1] \to \Omega(L)$ can be easily shown to be a chain homomorphism. The lemma follows easily. □

We remark that Theorem 31.1 implies that there exists a homotopy equivalence $\overline{\mathfrak{f}}$ from $(\mathbb{R}\mathcal{X}_L, \overline{\mathfrak{m}})$ to $(\Omega(L), \overline{\mathfrak{m}})$. Here $(\mathbb{R}\mathcal{X}_L, \overline{\mathfrak{m}})$ is the tensor product

$$(\mathbb{R}\mathcal{X}_L, \overline{\mathfrak{m}}) = \mathbb{R} \otimes (\mathbb{Z}\mathcal{X}_L, \overline{\mathfrak{m}}^L)$$

where $(\mathbb{Z}\mathcal{X}_L, \overline{\mathfrak{m}}^L)$ is the A_∞ algebra constructed in Theorem 9.8.

We recall that in the course of the construction of the filtered A_∞ algebra in §30, we take a complex $C(L; \Lambda_{0,nov}^{\mathbb{Q}})$ containing $\mathbb{Q}\mathcal{X}_L$. By the same way as in the proof of Theorem 33.1 we can extend the A_∞-morphism $\overline{\mathfrak{f}}$ to $C(L; \Lambda_{0,nov}^{\mathbb{R}}) = C(L; \Lambda_{0,nov}^{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{R}$. We denote this extension by the same symbol $\overline{\mathfrak{f}}$.

The next lemma is the most essential part of the proof of Proposition 37.38.

Lemma 37.55. *There exists a strict A_∞ homomorphism $\mathfrak{f}$ modulo T^2 from $(C(L; \Lambda_{0,nov}^{\mathbb{R}}), \mathfrak{m})$ to $(\Omega(L) \otimes \Lambda_{0,nov}^{\mathbb{R}}, \mathfrak{m}^{dR(1)})$ which reduces to $\overline{\mathfrak{f}}$ after reducing the coefficient ring to $\mathbb{R}$.*

Since the proof of this lemma is rather lengthy, we postpone the proof until the end of this subsection and first complete the proof of Proposition 37.38 using this lemma.

Because $\overline{\mathfrak{f}}$ is a homotopy equivalence, Lemma 37.55 and the obvious analogs of Lemma 30.74, Lemma 30.128 and Proposition 36.21 imply that there exists a structure of filtered A_∞ algebra $\mathfrak{m}^{dR}$ on $\Omega(L) \otimes \Lambda_{0,nov}^{\mathbb{R}}$ which coincides with $(\Omega(L) \otimes \Lambda_{0,nov}^{\mathbb{R}}, \mathfrak{m}_k^{dR(1)})$ modulo T^2. Moreover there exists a homotopy equivalence $\mathfrak{f}$ from $(C(L; \mathbb{R}), \mathfrak{m})$ to $(\Omega(L) \otimes \Lambda_{0,nov}^{\mathbb{R}}, \mathfrak{m}^{dR})$ which is equal to $\mathfrak{f}^{(1)}$ modulo T^2. Therefore it suffices to study the canonical model of the symmetrization of $(\Omega(L) \otimes \Lambda_{0,nov}^{\mathbb{R}}, \mathfrak{m}^{dR})$.

Let $(\Omega(L) \otimes \Lambda_{0,nov}^{\mathbb{R}}, \mathfrak{l}^{dR})$ be the symmetrization of $(\Omega(L) \otimes \Lambda_{0,nov}^{\mathbb{R}}, \mathfrak{m}^{dR})$. Let $x_i \in H^*(L;\mathbb{R})$ be a cohomology class and $u_i \in \Omega(L)$ any smooth form representing the class x_i. Then it is easy to see from the definition of $\mathfrak{l}_{k+1}^{dR}$ that

$$
\begin{aligned}
&\mathfrak{l}_{k+1}^{dR}(u_1, \cdots, u_k, u_0) \equiv \\
&\pm \frac{1}{k!} e^{\mu_L(\gamma)/2} T \left(ev^{(k)*}(x_1 \times \cdots \times x_k \times x_0)([\mathcal{M}_1(\beta) \times (S^1)^k]) \right) \mod T^2.
\end{aligned} \tag{37.56}
$$

Thus we only need to show that (37.56) implies the same formula for the canonical model. To prove this, we need to recall the proof of Theorem 36.18 which is an L_∞ analog to the proof of Theorem 23.2 given in §23.4. The construction of the filtered L_∞ structure $\mathfrak{l}_{can}$ on the canonical model and that of the filtered L_∞ homomorphism from $(\Omega(L) \otimes \Lambda_{0,nov}^{\mathbb{R}}, \mathfrak{l}^{dR})$ to its canonical model can be done by the same induction argument using the versions of (23.34) and (23.36) if we replace $\mathfrak{m}$ by $\mathfrak{l}^{dR}$. Here we remark that $\mathfrak{l}_k^{dR} \equiv 0 \mod T$ for $k \neq 1$ by Proposition 36.10 and $\mathfrak{l}_1^{dR} \equiv \pm d \mod T^2$.

We denote the set of harmonic forms on L with respect to a metric on g by

$$\mathcal{H}(L) \subset \Omega(L) \otimes \Lambda_{0,nov}^{\mathbb{R}}.$$

Then it is easy to see from (23.34) and (23.36) that $\mathfrak{f}$ restricts to the identity on $\mathcal{H}(L)$ mod T and $\mathfrak{l}^{dR}$ restricts to $\mathfrak{l}_{cal}$ mod T^2 on $\mathcal{H}(L)$. The identity (37.56) then implies the same formula on $\mathcal{H}(L)$ which in turn implies Proposition 37.38 because $\mathcal{H}(L)$ is a canonical model of $\Omega(L) \otimes \Lambda_{0,nov}^{\mathbb{R}}$. The proof of Proposition 37.38 is now complete *modulo the proof of Lemma 37.55.* □

Now the proof of Lemma 37.55 occupies the rest of this subsection.

Proof of Lemma 37.55. We may regard $\bar{\mathfrak{f}}_k$ as a filtered A_∞ homomorphism modulo T from $(C(L;\Lambda_{0,nov}^{\mathbb{R}}), \mathfrak{m})$ to $(\Omega(L) \otimes \Lambda_{0,nov}^{\mathbb{R}})$. We will add the term $\mathfrak{f}_{k,\beta}$ to $\bar{\mathfrak{f}}$ to get a filtered A_∞ homomorphism $\bar{\mathfrak{f}} + q^{\mu_L(\gamma)/2} T \mathfrak{f}_\beta$ modulo T^2. For this purpose we use an argument similar to the one used in §30.9 and §33.5.

For the construction of A_∞ homomorphism, we need to use the moduli spaces

$$\mathcal{M}_{k+1}^{\text{main}}(\{J_{\rho,s_0}\}_\rho : \beta; \text{top}(\rho))$$

and

$$\mathcal{M}_{k+1}^{\text{main}}(\{J_{\rho,s_0}\}_\rho : \beta; \text{top}(\rho); \vec{P})$$

as in Definitions 19.8 and 19.10. In this section, we only need to consider the choices

$$\psi = id, \quad \{J_\rho\}_\rho \quad \text{with } J_\rho \equiv J$$

of the symplectic diffeomorphism ψ and of the family with $J_\rho \equiv J$ of almost complex structures, and β a generator of $\pi_1(\mathbb{C}^{n+1}, L)$. (We recall the standing hypothesis $L \sim S^1 \times S^n$.)

We recall that the moduli space $\mathcal{M}^{\rm main}_{k+1}(\{J_{\rho,s_0}\}_\rho : \beta; {\rm top}(\rho))$ given in Definition 19.8 is the set of equivalence classes of the bordered stable Riemann surfaces decorated by the time allocation, which was denoted by $((\Sigma,\vec z),(u_\alpha),(\rho_\alpha))$.

Let $\Sigma = \bigcup_\alpha \Sigma_\alpha$ be the decomposition as in Definition 19.6 : Namely each Σ_α contains a single irreducible disc component to which a finite set of bubble trees of sphere components are attached. The map $u_\alpha : (\Sigma_\alpha, \partial\Sigma_\alpha) \to (\mathbb{C}^{n+1}, L)$ is J_{ρ_α} holomorphic. Namely it is J holomorphic in the current context. Now we assign a special point $z_\alpha \in \partial\Sigma_\alpha$ to each α as follows : If Σ_α is the component containing the 0-th marked point, we put z_α to be the 0-th marked point. On the other hand, if Σ is not the component, we note that there exists a unique α' such that $\Sigma_{\alpha'}$ is adjacent to Σ_α and

$$\Sigma_{\alpha'} > \Sigma_\alpha$$

where $>$ is the order defined in Definition 19.6. This follows from the definition of the order $>$ and from the fact that Σ has genus zero. This allows us to define z_α to be the singular point $z_\alpha \in \Sigma_\alpha \cap \Sigma_{\alpha'}$ for such α'. (See Figure 19.4.) Then we derive the following from the fact that β is a primitive class :

(37.57.1) There exists one and only one α, say α_0, such that $((\Sigma_\alpha, z_\alpha), u_\alpha)$ is an element of $\mathcal{M}_1(\beta)$.
(37.57.2) If $\alpha \neq \alpha_0$, then u_α is constant.

Recall that in §30.8 we used the moduli space $\mathcal{M}^{\rm main}_{k+1}(\{J_\rho\}_\rho : \beta; {\rm top}(\rho); \vec P)$ in our definition of the filtered A_∞ homomorphism between the A_∞ algebras associated to two Lagrangian submanifolds. In the current context involving the de Rham model, we need to use a modification of the moduli space because our A_∞ algebra $\Omega(L)$ satisfies $\overline{\mathfrak m}_k = 0$ for $k \geq 3$. This modification is somewhat similar to the way we construct $\mathcal{N}'_{k+1}$ out of $\mathcal{N}_{k+1}$ in Theorem 29.51. (See the proof of Proposition 33.43.)

More precisely, we will cut-down $\mathcal{M}^{\rm main}_{k+1}(\{J_\rho\}_\rho : \beta; {\rm top}(\rho); \vec P)$ by an equivalence relation $\sim$ which similar to (33.44).

Definition 37.58. Let $((\Sigma,\vec z),(u_\alpha),(\rho_\alpha)) \in \mathcal{M}^{\rm main}_{k+1}(\{J_\rho\}_\rho : \beta; {\rm top}(\rho))$. We consider the union of all the components Σ_α with the following properties.

(37.59.1) $\rho_\alpha = 1$.
(37.59.2) $\Sigma_{\alpha_0} < \Sigma_\alpha, (\alpha \neq \alpha_0)$, where α_0 is as in (37.57.1) and $<$ is defined in Definition 19.6.

We denote its union by Σ_0.

Recalling all u_α are constant maps for $\alpha \neq \alpha_0$, we say

$$((\Sigma,\vec z),(u_\alpha),(\rho_\alpha)) \sim ((\Sigma',\vec z),(u'_\alpha),(\rho'_\alpha)),$$

if the complement $\Sigma \setminus \Sigma_0$ together with the restriction of $\vec z, (u_\alpha), (\rho_\alpha)$ there coincides with the restriction of $((\Sigma',\vec z),(u'_\alpha),(\rho'_\alpha))$ to $\Sigma' \setminus \Sigma'_0$. We denote the quotient space by

$$\mathcal{N}'_{k+1}(\beta) = \mathcal{M}^{\rm main}_{k+1}(\{J_\rho\}_\rho : \beta; {\rm top}(\rho))/\sim .$$

We alert the readers that $\mathcal{N}'_{k+1}(\beta)$ here has nothing to do with the moduli space that appeared in Definition 5.36, although we happen use the same calligraphic letter $\mathcal{N}$ for the notations.

Lemma 37.60. *$\mathcal{N}'_{k+1}(\beta)$ has a Kuranishi structure.*

Proof. For the proof, we can repeat the same arguments used in §29.5 using the moduli space $\mathcal{N}'_{k+1}(\beta)$ instead of $\mathcal{M}^{\rm main}_{k+1}(\{J_\rho\}_\rho : \beta; {\rm top}(\rho))$: This time $\mathcal{N}'_{k+1}$ replaces the role of $\mathcal{N}_{k+1}$ used in §29.5 and Proposition 33.43 replaces Theorem 29.51 in the proof. □

We next study the boundary of $\mathcal{N}'_{k+1}(\beta)$. We first remark that a compactification of $\mathcal{M}_1(\beta) \times C_k$, together with evaluation maps, can be identified with $\mathcal{M}^{\rm main}_{k+1}(\beta)$ as a space with Kuranishi structure. We recall that in §33.5 we constructed a family of spaces $\mathcal{N}'_{k+1}$ which we used in the definition of the homomorphisms $\bar{\mathfrak{f}}_k$. The boundary of the space $\mathcal{N}'_{k+1}(\beta)$ can be described in the following Lemma 37.61, which is similar to the descriptions in (33.42) and Lemma 33.45 of the boundaries of other somewhat similarly defined moduli spaces. Let $\underline{k}^* = \{1, \cdots, k\} \cup \{*\}$ and $A \in (\underline{k}^*)^\ell$. We denote by $\mathcal{N}'_A(\beta)$ a copy of $\mathcal{N}'_{\ell+1}(\beta)$. $\mathcal{M}^{\rm main}_A(\beta)$ and $\mathcal{N}'_A$ are as in (33.42).

Lemma 37.61. *$\partial\mathcal{N}'_{(1,\cdots,k)}(\beta)$ is a union of the following five types of components.*

(37.62.1) $\mathcal{M}^{\rm main}_{(i+1,\cdots,i+\ell)}(\beta) \times \mathcal{N}'_{(0,\cdots,i,*,i+\ell+1,\cdots,k)}$. $(0 \le i,\ 0 \le \ell,\ i+\ell \le k.)$

(37.62.2) $\mathcal{M}^{\rm main}_{(i+1,\cdots,i+\ell)}(\beta_0) \times \mathcal{N}'_{(1,\cdots,i,*,i+\ell+1,\cdots,k)}(\beta)$. $(0 \le i,\ 1 \le \ell,\ i+\ell \le k.)$

(37.62.3) $\mathcal{N}'_{(1,\cdots,\ell)}(\beta) \times \mathcal{N}'_{(\ell+1,\cdots,k)}$. $(1 \le \ell \le k-1.)$

(37.62.4) $\mathcal{N}'_{(1,\cdots,\ell)} \times \mathcal{N}'_{(\ell+1,\cdots,k)}(\beta)$. $(1 \le \ell \le k-1.)$

(37.62.5) $\left(\prod_{i=1}^m \mathcal{N}'_{(\ell_{i-1}+1,\cdots,\ell_i)}\right) \times \mathcal{M}^{\rm main}_{m+1}(\beta)$, *where* $0 = \ell_0 < \ell_1 < \cdots < \ell_m = k-1$.

See Figures 37.1 - 37.4.

Figure 37.1 (37.62.1)

Figure 37.2 (37.62.2)

Figure 37.3 (37.62.3)

Figure 37.4 (37.62.5)

Remark 37.63. When $k+1=2$, Lemma 37.61 implies

$$\partial \mathcal{N}'_{(1)}(\beta) = \left(\mathcal{M}^{\rm main}_{\emptyset}(\beta) \times (\mathcal{N}'_{(1,*)} \cup \mathcal{N}'_{(*,1)})\right) \cup \left(\mathcal{N}'_{(1)} \times \mathcal{M}^{\rm main}_{1+1}(\beta)\right).$$

It corresponds to the formula

$$(d \circ \mathfrak{f}_{1,\beta} - \mathfrak{f}_{1,\beta} \circ d)(x) = (-1)^{\deg' x}\bar{\mathfrak{f}}_2(x, \mathfrak{m}_{0,\beta}(1)) + \mathfrak{f}_2(\mathfrak{m}_{0,\beta}(1), x) - \mathfrak{m}^{dR}_{1,\beta}(\bar{\mathfrak{f}}_1(x)).$$

Proof of Lemma 37.61. We remark that an element $((\Sigma, \vec z), (u_\alpha), (\rho_\alpha))$ of

$$\mathcal{M}^{\rm main}_{k+1}(\{J_\rho\}_\rho : \beta; {\rm top}(\rho))$$

is on the boundary if $\rho_\alpha = 0$ or $\rho_\alpha = 1$ for some α. (Other boundary components cancel one another in the same way as (19.9.1) cancels (19.9.2).)

The case $\rho_\alpha = 1$ for some α are divided into two cases. One is the case $\rho_{\alpha_0} = 1$ where α_0 is as in (37.57). The other is $\rho_{\alpha_0} \neq 1$.

If we take the quotient of the totality of elements of $((\Sigma, \vec{z}), (u_\alpha), (\rho_\alpha))$ with $\rho_{\alpha_0} = 1$ by the equivalence relation $\sim$, then we obtain (37.62.5). From the other case $\rho_\alpha = 1$, $\rho_{\alpha_0} \neq 1$ we obtain (37.62.3) or (37.62.4) after taking the quotient by $\sim$.

Let us consider the case $\rho_\alpha = 0$ for some α. If $\rho_{\alpha_0} = 0$, we obtain (37.62.1). (We use the fact that the equivalence relation $\sim$ is a version of the one that appeared in (33.44).) If $\rho_\alpha = 0$ for some α but $\rho_{\alpha_0} \neq 0$, then we obtain (37.62.2). The proof of Lemma 37.61 is complete. $\square$

The rest of the proof of Lemma 37.55 is a straightforward analog of §33.5. Let P_i be the chains in $C(L; \Lambda_{0,nov}^{\mathbb{R}})$. We remark that there is an evaluation map

$$ev = (ev_1, \cdots, ev_k, ev_0) : \mathcal{N}'_{k+1}(\beta) \to L^{k+1}.$$

We put

$$\mathcal{N}'_{k+1}(\beta; P_1, \cdots, P_k) = \mathcal{N}'_{k+1}(\beta)\,{}_{(ev_1,\cdots,ev_k)} \times_{L^k} (P_1 \times \cdots \times P_k). \tag{37.64}$$

Then $\mathcal{N}'_{k+1}(\beta; P_1, \cdots, P_k)$ has a Kuranishi structure and carries the evaluation map

$$ev_0 : \mathcal{N}'_{k+1}(\beta; P_1, \cdots, P_k) \to L.$$

Now we have the boundary of $\mathcal{N}'_{k+1}(\beta; P_1, \cdots, P_k)$

$$\begin{aligned}\partial \mathcal{N}'_{k+1}(\beta; P_1, \cdots, P_k) &= (\partial \mathcal{N}'_{k+1}(\beta))_{(ev_1,\cdots,ev_k)} \times_{L^k} (P_1 \times \cdots \times P_k) \\ &\quad + \sum_{j=1}^{k} \pm \mathcal{N}'_{k+1}(P_1 \times \cdots \times \partial P_j \times \cdots \times P_k).\end{aligned}$$

Here the terms in the summation sign of this boundary correspond to

$$\left(\mathfrak{f}_1^{(1)} \circ d\right)(P_1, \cdots, P_k).$$

The first term of the boundary, which now is the same as

$$(d \circ \mathfrak{f}_1^{(1)} - \mathfrak{f}_1^{(1)} \circ d)(P_1, \cdots, P_k),$$

can be described using Lemma 37.61. More precisely, the five types (37.62.1)-(37.62.5) of the boundary components of $\mathcal{N}'_{k+1}(\beta)$ give rise to the corresponding boundary components (37.65.1)-(37.65.5) of $\mathcal{N}'_{k+1}(\beta; P_1,\cdots,P_k)$ given below, respectively.

For each given $A=(a_1,\cdots,a_m)\in(\underline{k}^*)^m$, we define $\mathcal{N}'_A(\beta; P_{a_1},\cdots,P_{a_m})$ by the same way as (37.64). Then we define

$$\text{(37.65.1)}\quad \mathcal{N}'_{(1,\cdots,i,*,i+\ell+1,\cdots,k)}\ {}_{ev}\times_{L^{k-\ell+1}} (P_1\times\cdots P_i\times Q_{i+1,\ell}\times P_{i+\ell+1}\cdots\times P_k)$$

where we define

$$Q_{i+1,\ell}=\mathfrak{m}_{\ell,\beta}(P_{i+1},\cdots,P_{i+\ell})=ev_{0,*}\left(\mathcal{M}_{\ell+1}(\beta)\times_{L^\ell}(P_{i+1},\cdots,P_{i+\ell})\right),$$

and

$$\text{(37.65.2)}\qquad \mathcal{N}'_{(1,\cdots,i,*,i+\ell+1,\cdots,k)}(\beta; P_1,\cdots,P_i,R_{i+1,\ell},P_{i+\ell+1},\cdots,P_k)$$

where we define

$$R_{i+1,\ell}=\overline{\mathfrak{m}}_\ell(P_{i+1},\cdots,P_{i+\ell}),$$

$$\text{(37.65.3)}\quad \mathcal{N}'_{(1,\cdots,\ell)}(\beta; P_1,\cdots,P_\ell)\times(\mathcal{N}'_{(\ell+1,\cdots,k)}\ {}_{ev}\times_{L^{k-\ell}}(P_{\ell+1}\times\cdots\times P_k)),$$

$$\text{(37.65.4)}\quad (\mathcal{N}'_{(1,\cdots,\ell)}\ {}_{ev}\times_{L^\ell}(P_1\times\cdots\times P_\ell))\times\mathcal{N}'_{(\ell+1,\cdots,k)}(\beta; P_{\ell+1},\cdots,P_k),$$

(37.65.5)

$$\mathcal{M}^{\rm main}_{m+1}(\beta)\times_{L^m}\prod_{i=1}^m\left(\mathcal{N}'_{(\ell_{i-1}+1,\cdots,\ell_i)}\ {}_{ev}\times_{L^{\ell_i-\ell_{i-1}}}(P_{\ell_{i-1}+1}\times\cdots\times P_{\ell_i})\right).$$

Now we take a family of sections $\mathfrak{t}_{\mathfrak{f}_\beta,k+1}$ of some Kuranishi neighborhoods of $\mathcal{N}'_{k+1}(\beta; P_1,\cdots,P_k)$ inductively over k so that they are compatible with the following (families of) sections :

(37.66.1) The family of sections constructed at the earlier stage of induction.
(37.66.2) The W_ℓ parameterized family $\mathfrak{t}^{(P_1,\cdots,P_\ell)}_{\mathfrak{f},\ell+1}$ of sections defined in Lemma 33.54 in §33.5. It was used to define $\bar{\mathfrak{f}}_\ell(P_1,\cdots,P_\ell)$.
(37.66.3) The W parameterized family $\mathfrak{s}$ of sections on $\mathcal{M}^{\rm main}_{m+1}(\beta)^+$ used to define $\mathfrak{m}^{dR}_{m,\beta}$.
(37.66.4) The multi-section $\mathfrak{s}^{(P_1,\cdots,P_m)}_{\mathfrak{m},m+1}$ of $\mathcal{M}^{\rm main}_{m+1}(\beta; P_1,\cdots,P_m)$ used to define $\mathfrak{m}_{m,\beta}$.

Existence of such sections $\mathfrak{t}^{(P_1,\cdots,P_k)}_{\mathfrak{f}_\beta,k+1}$ is a consequence of the standard transversality theorem.

In the way similar to the definition of $\bar{\mathfrak{f}}$, we can use this family of sections $\mathfrak{t}^{(P_1,\cdots,P_k)}_{\mathfrak{f}_\beta,k+1}$ on $\mathcal{N}'_{k+1}(\beta; P_1,\cdots,P_k)$ to obtain $\mathfrak{f}_\beta$, which we now explain.

Let our family $\mathfrak{t}_{\mathfrak{f}_\beta,k+1}$ be parameterized by $W_k^+ = W_k \times W_0 \times W$, where W_k is as in §33.5, W_0 is as in the beginning of this subsection, and the additional factor W is a manifold of sufficiently large dimension. On each Kuranishi neighborhood V_j of $\mathcal{N}_{k+1}(\beta; P_1, \cdots, P_k)$ we consider the zero set

$$\mathfrak{X}_j = \{(x,w) \in V_j \times W_k^+ \mid \mathfrak{t}_{\mathfrak{f}_\beta,k+1}(x,w) = 0\}.$$

We may choose $\mathfrak{t}_{\mathfrak{f}_\beta,k+1}$ so that $\mathfrak{X}_j$ is a smooth manifold. (Note in our case the automorphism group is trivial.) By gluing the families obtained for various j's, we obtain a smooth manifold $\mathfrak{X}$ with the projection map $\pi : \mathfrak{X} \to W_k^+$ and the evaluation map $ev_0 : \mathfrak{X} \to L$. We then define

$$\mathfrak{f}_{k,\beta}(\beta; P_1, \cdots, P_k) = ev_{0!}(\pi^*\omega_{W_k^+}) \tag{37.67}$$

where $ev_{0!}$ is the integration along the fiber and $\omega_{W_k^+}$ is a smooth probability measure on W_k^+. Then we put

$$\mathfrak{f}_k = \bar{\mathfrak{f}}_k + \sum_\beta e^{\mu_L(\gamma)/2} T \mathfrak{f}_{k,\beta}. \tag{37.68}$$

We now prove that (37.68) defines a filtered A_∞ homomorphism modulo T^2. We will use Lemma 33.10 and the description (37.65) of the boundary of $\mathcal{N}_{k+1}(\beta; P_1, \cdots, P_k)$ for this purpose. Observe that (37.65.1)-(37.65.5) give the following terms

(37.69.1) $\bar{\mathfrak{f}}_{k+\ell-1}(P_1, \cdots, \mathfrak{m}_{\ell,\beta}(P_{i+1}, \cdots, P_{i+\ell}), \cdots, P_k)$,
(37.69.2) $\mathfrak{f}_{k+\ell-1,\beta}(P_1, \cdots, \overline{\mathfrak{m}}_\ell(P_{i+1}, \cdots, P_{i+\ell}), \cdots, P_k)$,
(37.69.3) $\mathfrak{f}_{\ell,\beta}(P_1, \cdots, P_\ell) \wedge \bar{\mathfrak{f}}_{k-\ell}(P_{\ell+1}, \cdots, P_k)$,
(37.69.4) $\bar{\mathfrak{f}}_\ell(P_1, \cdots, P_\ell) \wedge \mathfrak{f}_{k-\ell,\beta}(P_{\ell+1}, \cdots, P_k)$,
(37.69.5) $\mathfrak{m}^{dR}_{m,\beta}\left(\bar{\mathfrak{f}}_{\ell_1}(P_1, \cdots, P_{\ell_1}), \cdots, \bar{\mathfrak{f}}_{k-\ell_{m-1}}(P_{\ell_{m-1}+1}, \cdots, P_k)\right)$,

respectively. Thus we conclude that the sum of the terms (37.69.1), ... ,(37.69.5) is equal to

$$(d \circ \mathfrak{f}_1^{(1)} - \mathfrak{f}_1^{(1)} \circ d)(P_1, \cdots, P_k).$$

This is implies that $\mathfrak{f}$ is a filtered A_∞ homomorphism modulo T^2. The proof of Lemma 37.55 is now complete. (We remark that the sign can be handled in the same way as in §53.) □

Remark 37.70. We remark that the proof of Proposition 37.38 works not only for $S^1 \times S^n$ but for the general semi-positive Lagrangian submanifolds with no essential change and can be used to study the leading order term of the filtered L_∞ structure.

37.5. Dependence of the filtered A_∞ algebra on perturbations : An example.

In §37.3, we have used de Rham theory to simplify the discussion on the contribution from the (classical) rational homotopy type of L. Since our constructions in the previous chapters are based on the singular chains, it is not manifest how the two constructions are related to each other. Therefore we illustrate a direct use of the singular chains in our computation of (the leading term of) the A_∞ structure. Along the way, we also demonstrate, by an example, that our A_∞ structure itself really depends on perturbations, although its homotopy type does not. (See Chapter 4.)

Consider the Lagrangian submanifold $L \subset \mathbb{C}^n$

$$L \sim S^1 \times S^n \quad \text{with } \mu_L(\gamma) = 2.$$

Then $\mathcal{M}_0(\beta)$ is a compact smooth manifold with $\dim \mathcal{M}_0(\beta) = n$. We also *assume* that the evaluation map

$$ev_0 : \mathcal{M}_1(\beta) \to L \sim S^1 \times S^n$$

pushes down to the diffeomorphism on the quotient

$$\mathcal{M}_0(\beta) \cong \mathcal{M}_1(\beta)/S^1 \to (S^1 \times S^n)/S^1 \cong S^n,$$

where the S^1-actions on both sides are the obvious ones. Then each $x \in S^n$ has a unique correspondence with a holomorphic map $(D^2, \partial D^2) \to (\mathbb{C}^{n+1}, L)$ whose boundary value is the horizontal loop given by $\ell_x(t) = (t, x)$ (upto the reparameterization). (We remark that at least in the homology level such an example indeed can be constructed from the example mentioned in (37.2) which is constructed from the standard embedding $S^1 \to \mathbb{C}$ and the Whitney Lagrangian immersion $S^n \to \mathbb{C}^n$.)

Then, without perturbation, $\mathcal{M}^{\text{main}}_{k+1}(\beta)$ becomes a compact smooth manifold (with boundary and corners) for a generic choice of almost complex structure, and in fact we have

$$\mathcal{M}^{\text{main}}_{k+1}(\beta) \cong S^n \times S^1 \times C_k.$$

Under this identification, the the evaluation map $ev = (ev_1, \cdots, ev_k, ev_0)$ can be written as

$$ev((x,t_0),(t_1,\cdots,t_k)) = ((x, t_0+t_1), \cdots, (x, t_0+t_k), (x, t_0)). \tag{37.71}$$

Let $P(s) = \{s\} \times S^n$ be a cycle representing the cohomology class $\mathfrak{a}_1$. (Precisely speaking we fix a simplicial decomposition of S^n and regard $P(s)$ as a singular chain.) If $s_1, \cdots, s_k$ are all distinct, we have

$$\begin{aligned} &\mathfrak{m}_{k,\beta}(P(s_1), \cdots, P(s_k))) = ev_{0,*}(\mathcal{M}^{\text{main}}_{k+1}(\beta) \times_{L^k} (P(s_1) \times \cdots \times P(s_k))) \\ &= \begin{cases} [(s_k, s_1) \times S^n] & \text{if } (s_1, \cdots, s_k) \text{ respects the cyclic order of } S^1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{37.72}$$

Here (s_k, s_1) denotes the arc between s_k and s_1 i.e.,

$$(s_k, s_1) := \{s \in S^1 \mid (s_1, \cdots, s_k, s) \text{ respects the cyclic order}\}.$$

By symmetrizing $\mathfrak{m}_{k,\beta}$, we obtain

$$\mathfrak{l}_{k,\beta}(P(s_1), \cdots, P(s_k))) = \frac{1}{k!}[S^1 \times S^n]. \tag{37.73}$$

This is consistent with Theorem 37.32.

It is more difficult to study the case where $s_i = s_j$ for some $i \neq j$. This is because transversality breaks down and we need to perturb the moduli chain. For simplicity, we consider only the case $k = 2$ and $s_1 = s_2 = s$. In this case, the fiber product $\mathcal{M}^{\text{main}}_{2+1}(\beta) \times_{L^k} (P(s) \times P(s))$ is not transversal. So to define $\mathfrak{m}_{2,\beta}(P(s), P(s)))$ we need to perturb the moduli chain $(\mathcal{M}^{\text{main}}_{2+1}(\beta), ev)$. Since we know the space $\mathcal{M}^{\text{main}}_{2+1}(\beta)$ itself is smooth, we will perturb just the evaluation map $ev = (ev_1, ev_2, ev_0)$: We choose δ with its absolute value sufficiently small and perturb $ev = (ev_1, ev_2, ev_0)$ to

$$ev^\delta((x, t_0), (t_1, t_2)) := ((x, t_0 + t_1 + \delta), (x, t_0 + t_1 - \delta), (x, t_0))$$

near $t_1 = t_2$. We denote by $\mathfrak{m}^\delta_{2,\beta}$ the corresponding perturbed $\mathfrak{m}_{2,\beta}$. Then in the limit $\delta \to 0$, the chain $\mathfrak{m}^\delta_{2,\beta}(P(s), P(s)))$ has the formula

$$\mathfrak{m}^\delta_{2,\beta}(P(s), P(s))) = \begin{cases} [S^1 \times S^n] & \text{if } \delta > 0 \text{ and } \delta \to 0, \\ 0 & \text{if } \delta < 0 \text{ and } \delta \to 0. \end{cases} \tag{37.74}$$

This, in particular, demonstrates that different perturbations lead to different A_∞ operators $\mathfrak{m}$. For each of the two perturbations, the operator is defined over $\mathbb{Z}$. Its average, which however is not defied over $\mathbb{Z}$, gives a result consistent with Theorem 37.32.

Let $\mathfrak{m}^{>0}_*$ and $\mathfrak{m}^{<0}_*$ be the operations defined by the above perturbation with $\delta > 0$ and $\delta < 0$, respectively. As we already proved in Chapter 4, there exists a filtered A_∞ homomorphism $\mathfrak{f} : (C(L; \Lambda_{0,nov}), \mathfrak{m}^{>0}_*) \to (C(L; \Lambda_{0,nov}), \mathfrak{m}^{<0}_*)$ that is a homotopy equivalence. We now describe (a part of) $\mathfrak{f}$ in this case, following the construction of Chapters 4 and 7.

The key to the construction of an A_∞ homotopy equivalence is the construction of an A_∞ homomorphism between the classical parts, $(C(L; \mathbb{Q}), \overline{\mathfrak{m}}^{>0}_*)$ and $(C(L; \mathbb{Q}), \overline{\mathfrak{m}}^{<0}_*)$. For this purpose, we need to understand the classical parts $\overline{\mathfrak{m}}^{<0}_k$'s of the A_∞ homomorphism between them.

For $k = 2$, it turns out that we have

$$\overline{\mathfrak{m}}^{<0}_2(P(s), P(s)) = \overline{\mathfrak{m}}^{>0}_2(P(s), P(s)) = 0. \tag{37.75}$$

To see this, we recall the construction from §30 and the proof of (37.74). We remark that the intersection $P(s) \cap P(s) = P(s)$ is not transversal since $\dim P(s) < \dim L$. Therefore we need to choose a suitable perturbation of the moduli chains to define which led us to the perturbed chains $\overline{\mathfrak{m}}_2^{<0}(P(s), P(s))$ and $\overline{\mathfrak{m}}_2^{>0}(P(s), P(s))$. The perturbations should be consistent with that of $[\mathcal{M}_{2+1}^{\text{main}}(\beta), ev]$ given above : More precisely speaking, we first perturb $\mathcal{M}_{2+1}^{\text{main}}(\beta_0)$ to define $\overline{\mathfrak{m}}_2^{<0}$ and then choose a perturbation of $\mathcal{M}_{2+1}^{\text{main}}(\beta)$ so that it is consistent with the definition of $\mathfrak{m}_{2,\beta}^{<0}$. Following the recipe in §30, we consider a pair of diffeomorphisms

$$(\varphi_{0,\delta}, \varphi_{1,\delta}) : L \to L \times L$$

given by

$$(\varphi_{0,\delta}(t,x), \varphi_{1,\delta}(t,x)) = ((t+\delta, x), (t-\delta, x)).$$

Then according to §30.3 we have

$$\overline{\mathfrak{m}}_2^{>0}(P(s), P(s)) = \{p \in L \mid \varphi_{0,\delta}(p) \in P(s), \varphi_{1,\delta}(p) \in P(s)\}$$

for $\delta > 0$ and similarly for $\delta < 0$. It easily follows that

$$\overline{\mathfrak{m}}_2^{>0}(P(s), P(s)) = 0 = \overline{\mathfrak{m}}_2^{<0}(P(s), P(s)).$$

Now to construct an A_∞ homomorphism

$$(C(L;\mathbb{Q}), \overline{\mathfrak{m}}_*^{>0}) \to (C(L;\mathbb{Q}), \overline{\mathfrak{m}}_*^{<0}),$$

we need to take a homotopy between the two perturbations, one corresponding to $(\varphi_{0,\delta}, \varphi_{1,\delta})$ and the other corresponding to $(\varphi_{0,-\delta}, \varphi_{1,-\delta})$ for $\delta > 0$. The obvious choice of such a homotopy is $H : L \times [-1,1] \to L \times L$ defined by

$$(p, u) \mapsto (\varphi_{0,u\delta}(p), \varphi_{1,u\delta}(p)).$$

By the definition of the map $\bar{\mathfrak{f}}_2$ which is induced by H, we obtain

$$\bar{\mathfrak{f}}_2(P(s), P(s)) = \{p \in L \,|\, \exists u \in [-1,1],\ H(p,u) \in P(s) \times P(s)\}$$

which gives rise to

$$\bar{\mathfrak{f}}_2(P(s), P(s)) = [P(s)].$$

Note we do not need to perturb moduli chain $(\mathcal{M}_{1+1}^{\text{main}}(\beta), ev)$ to define the map $\mathfrak{m}_{1,\beta}$. This is because it follows from the formula (37.71) that the fiber product

$$\mathcal{M}_{1+1}^{\text{main}}(\beta) \times_L P$$

is always transversal for any chain $P \subset L$. In particular, this leads to the identity $\mathfrak{m}_{1,\beta}^{>0} = \mathfrak{m}_{1,\beta}^{<0}$ and allows us to choose $\bar{\mathfrak{f}}_1 = id$ for the map $\bar{\mathfrak{f}}_1$. We also derive

$$\mathfrak{m}_{1,\beta}(P(s)) = [S^1 \times S^n]$$

from (37.71).

Combining all these, we obtain the identities

$$\begin{aligned} &\overline{\mathfrak{m}}_{2,\beta}^{>0}(\bar{\mathfrak{f}}_1(P(s)), \bar{\mathfrak{f}}_1(P(s))) \\ &= \overline{\mathfrak{m}}_{2,\beta}^{>0}(P(s), P(s)) = [S^1 \times S^n] \\ &= \mathfrak{m}_{1,\beta}(\bar{\mathfrak{f}}_2(P(s), P(s))) + \overline{\mathfrak{m}}_{2,\beta}^{<0}(\bar{\mathfrak{f}}_1(P(s)), \bar{\mathfrak{f}}_1(P(s))). \end{aligned} \tag{37.76}$$

The identity between the first line and the last in (37.76) is precisely the one required for $\mathfrak{f}$ to be a filtered A_2 homomorphism modulo T^2.

37.6. Further study of Floer theory of $S^1 \times S^n \subset \mathbb{C}^{n+1}$.

We next explain some ideas how one can improve the proofs of Theorems 37.30 and 37.32 and obtain additional information on the integer λ given in (37.33) and others.

We recall the definition of $\mathfrak{l}_{k+1,\beta}^{+}$ from (37.26). The following information can be easily read off from Lemma 37.41, 37.42 and 37.44.

Proposition 37.77. *Let $L \subset \mathbb{C}^{n+1}$ be a Lagrangian submanifold which is homeomorphic to $S^1 \times S^n$. Then $\mathfrak{l}_{k+1,\beta}^{+} \equiv 0 \mod T^2$ unless one of the following alternatives holds :*

1. *$\mu_L(\gamma) = 2$,*
2. *n is odd and $\mu_L(\gamma) = n + 1$,*
3. *n is even and $\mu_L(\gamma) = 2 - n$.*

Proof. When $n = 1$, the assertion is obvious. So we may assume that $n \neq 1$. Let $\gamma = \partial\beta \in \pi_1(L)$ be the boundary class of β. Based on Proposition 37.38, it suffices to show that $\overline{ev}_*([\mathcal{M}_0(\beta)]) \in H_{\mu_L(\gamma)+n-2}(\mathcal{L}(L;\gamma)/S^1;\mathbb{Q})$ is zero otherwise.

Suppose to the contrary that $\overline{ev}_*([\mathcal{M}_0(\beta)])$ is non-zero. According to Lemmas 37.41, 37.42 and 37.44, the homology group $H_m(\mathcal{L}(L;\gamma)/S^1;\mathbb{Q})$ is nonzero only for

$$m = k(n-1), \text{ or } k(n-1) + n \quad \text{for } k \in \mathbb{Z}_{\geq 0} \text{ when } n \text{ is odd,}$$

and

$$m = 0, \ (2\ell+1)(n-1), \text{ or } 2\ell(n-1) + n \quad \text{for } \ell \in \mathbb{Z}_{\geq 0} \text{ when } n \text{ is even.}$$

Therefore it is easy to see that one of the three possibilities must occur if $\mu_L(\gamma) + n - 2 \le 2n-1$. On the other hand if $\mu_L(\gamma) + n - 2 > 2n-1$, we have $\mu_L(\gamma) \ge n+2$: This has been already ruled out in Theorem 37.4. This finishes the proof. □

Possibilities (1), (2) are the cases discussed in Theorems 37.21 and 37.22 and indeed occur as shown in (37.1), (37.2) and Lemma 37.3. (Compare Remark 37.5 also.)

The discussion in the rest of this section will be brief the details of which will be treated elsewhere. We will study the first nonzero term of the $\mathfrak{l}^+_{k+1}$ particularly for the case where the homology class $\overline{ev}_*([\mathcal{M}_0(\beta)]) \in H_{\mu_L(\gamma)+n-2}(\mathcal{L}(L;\gamma)/S^1;\mathbb{Q})$ is zero.

'Proposition 37.78'. *Suppose that n is odd and $\ell\mu_L(\gamma) = n+1$ with $\ell \ne 1, \frac{n+1}{2}$. Then, we have :*

$$\mathfrak{l}^+_{k+2}(\mathfrak{a}_1^k, \mathfrak{a}_n, \mathfrak{a}_{n+1}) \equiv \pm\frac{1}{k!}e^{(n+1)/2}\ell^k T^\ell \mod T^{\ell+1}. \tag{37.79}$$

Here the $\pm$ is independent of k. Moreover the left hand side is independent of the permutation of the variables modulo $T^{\ell+1}$. All the other operations are zero modulo $T^{\ell+1}$.

We have the same conclusion when $\mu_L(\gamma) = 2 = (n+1)/\ell$ and $\mathfrak{l}^+_{k+1}(\mathfrak{a}_1^k, \mathfrak{a}_{n+1}) \equiv 0 \mod T^2$.

Sketch of the proof. We proved that $\overline{ev}_*([\mathcal{M}_0(\beta)]) \in H_{\mu_L(\gamma)+n-2}(\mathcal{L}(L;\gamma)/S^1;\mathbb{Q})$ is zero in the course of the proof of Proposition 37.77 under the given hypotheses on $\mu_L(\gamma)$. We *assume* that $\overline{ev}_*([\mathcal{M}_0(\beta)])$ is zero not only as a homology class but also as a chain. Then by the same argument as the proof of Lemma 37.36, we can prove that $\mathcal{M}_0(2\beta)$ is a manifold without boundary for a generic almost complex structure. Its homology class lies in $H_{2\mu_L(\gamma)+n-2}(\mathcal{L}(L;\gamma)/S^1;\mathbb{Q})$ and hence is zero by the degree reason, unless $\ell = 2$. We *assume* that $\overline{ev}_*([\mathcal{M}_0(2\beta)])$ is zero as a *chain*. Using this assumption, we can continue. Then (under these assumptions) we find a manifold without boundary $\mathcal{M}_0(\ell\beta)$ which represents a homology class in $H_{2n-1}(\mathcal{L}(L;\ell\gamma)/S^1;\mathbb{Q})$. This is because $H_*(\mathcal{L}(L;\ell\gamma)/S^1;\mathbb{Q}) = 0$ for $* = n+1, n+2, \cdots, 2n-2$ and $m\mu_L(\gamma) + n - 2 \in \{n+2, n+4, \cdots, 2n-2\}$ for $m = 1, \cdots, \ell-1$.

Using $\mathcal{M}_0(\ell\beta)$ we can repeat the proof of Theorem 37.30 and obtain the following conclusion : Define λ by

$$\overline{ev}_*([\mathcal{M}_0(\ell\beta)]) = \lambda I((\mathfrak{x} \otimes \mathfrak{z}_n)^*). \tag{37.80}$$

(Compare this with (37.40) and (37.45).) We remark that in our case $I : \mathcal{L}(S^n) \to \mathcal{L}(L;\ell\gamma)/S^1$ is not a homotopy equivalence. (Compare Lemma 37.41). However it induces an isomorphism on homology group over $\mathbb{Q}$.

We first prove

$$\mathfrak{l}^+_{k+2}(\mathfrak{a}_1^k, \mathfrak{a}_n, \mathfrak{a}_{n+1}) \equiv \pm\frac{\lambda}{k!}e^{(n+1)/2}\ell^k T^\ell \mod T^{\ell+1}. \tag{37.81}$$

By the linearity, it suffices to consider the case $\lambda = 1$ in (37.80). Then we may assume in addition that $\overline{ev}_*([\mathcal{M}_0(\ell\beta)])$ is realized by $P \times S^1$ where P is a chain on $\mathcal{L}(S^n)$ as in the proof of Lemma 37.47. Then

$$ev^{k+1} : \mathcal{M}_1(\ell\beta) \times (S^1)^{k+1} \to (S^1 \times S^n)^{k+1}$$

is given as

$$\begin{aligned} &ev^{k+1}((x,s),(t_1,\cdots,t_{k+1})) \\ &= ((\ell_x(t_1), s+\ell t_1),\cdots,(\ell_x(t_{k+1}), s+\ell t_{k+1}),(\ell_x(1),s)). \end{aligned} \tag{37.82}$$

We remark that (37.82) is slightly different from (37.50). Namely we have $s + \ell t_i$ in place of $s + t_i$ in the right hand side. We use (37.82) to show (37.81) in the same way as the proof of Lemma 37.47.

We next show $\lambda = 1$. This follows from $\mathfrak{l}_1(\alpha_n) = \lambda e^{(n+1)/2} T^\ell \mathfrak{a}_0$ and Proposition 37.15.

Therefore we have proved (37.79).

Thus, the proof of Proposition 37.78 is finished under the assumptions mentioned above. However one of the chains $\mathcal{M}_0(m\beta)$ in $H_{m\mu_L(\gamma)+n-2}(\mathcal{L}(L; m\gamma)/S^1; \mathbb{Q})$, which is zero as a homology class, may be nonzero as a chain. Therefore to handle the fictitious assumptions in the above proof, we need to work everything in the chain level and to inductively find a sequence of chains on the loop space that bounds the cycles $\overline{ev}_*([\mathcal{M}_0(m\beta)])$ for various m's. This inductive argument is very similar to those presented in the earlier chapters of this book. The only difference is that we have constructed bounding chains as the chains *on the manifold L itself* in this book, while we now need to do a similar construction *on the loop space of L*. An outline of the details of this construction is given in [Fuk05II] and the full details thereof will be provided in [Fuk07II]. □

We remark that 'Proposition 37.78' also implies that the sign in (37.17.1) coincides with sign in (37.17.2).

We next consider the case when n is even.

'Proposition 37.83'. *The possibility* (37.12) *does not occur.*

Sketch of the proof. We prove this by contradiction. Suppose to the contrary that (37.12) occurs. Then, by the argument in the proof of Theorem 37.4, we have $\delta_2 = 0$. This means that $\overline{ev}_*([\mathcal{M}_0(\beta)])$ is zero as a homology class. We *assume* that it is zero as a chain. Now we repeat the process similar to the sketch of the proof of 'Proposition 37.78' until we arrive in the first nontrivial homology class $\overline{ev}_*([\mathcal{M}_0(m\beta)])$. On the other hand we have the formula (37.12),

$$\overline{\delta}^{(n)}_{(n+2)/2}(\mathfrak{a}_n) = \pm \mathfrak{a}_1$$

which we proved under the assumption (37.12). This implies that the homology class $\overline{ev}_*([\mathcal{M}_0(((n+2)/2-1)\beta)]) \in H_{2n-2}(\mathcal{L}(L;\gamma)/S^1;\mathbb{Q})$ is nonzero. This is however impossible since $H_{2n-2}(\mathcal{L}(L;\gamma)/S^1;\mathbb{Q}) = 0$ by Lemmas 37.41, 37.42 and 37.44 unless $n = 2$. (We remark that n is even in our situation. The nontriviality of the differential in Lemma 37.44 plays a crucial role here.) In case that $n = 2$, we find that $H_2(\mathcal{L}(S^1 \times S^2;\gamma)/S^1;\mathbb{Q}) \cong H_2(\mathcal{L}(S^2);\mathbb{Q})$ is generated by $1 \otimes \mathfrak{z}_2$. We can use it to show that $\mathfrak{m}_{1,\beta}(\mathfrak{a}_n) = 0$ again. The current proof is incomplete because of the same reason as that of 'Proposition 37.78' is. (But it is rigorous in case $n = 2$.) □

'Proposition 37.84'. *When n is even, λ in Theorem* 37.32 *satisfies* $\lambda = \pm 1$.

Proof. By 'Proposition 37.83' and Proposition 37.10, we have $\overline{\delta}_2^{(1)}(\mathfrak{a}_1) = \pm\mathfrak{a}_0$. This implies $\lambda = \pm 1$ in Theorem 37.32. □

§38. Anti-symplectic involutions.

In the rest of this chapter we work in the following situation. Let (M,ω) be a compact symplectic manifold. An involution τ of (M,ω) is assumed to be anti-symplectic. Namely we assume

$$\tau^*(\omega) = -\omega. \tag{38.1}$$

We also assume that the fixed point set $L =$ Fix τ of τ is non-empty. It is then easy to see that L is a Lagrangian submanifold.

Let $\mathcal{J}_\omega$ be the set of all ω compatible almost complex structures, and $\mathcal{J}_\omega^\tau$ be its subset consisting of almost complex structures J satisfying

$$\tau_* J = -J. \tag{38.2}$$

Such an almost complex structure J is said to be τ-*anti-invariant.*

Lemma 38.3. *The space $\mathcal{J}_\omega^\tau$ is non-empty and contractible. It becomes an infinite dimensional* (*Frechet*) *manifold.*

Proof. For given $J \in \mathcal{J}_\omega^\tau$, its tangent space $T_J\mathcal{J}_\omega^\tau$ consists of sections Y of the bundle $\mathrm{End}(TM)$ whose fiber at $p \in M$ is the space of linear maps $Y : T_pM \to T_pM$ such that

$$YJ + JY = 0, \quad \omega(Yv,w) + \omega(v,Yw) = 0, \quad \tau^*Y = -Y.$$

Note that the second condition means that JY is a symmetric endomorphism with respect to the metric $g_J = \omega(\cdot, J\cdot)$. It immediately follows that $\mathcal{J}_\omega^\tau$ becomes a manifold. The fact that $\mathcal{J}_\omega^\tau$ is non-empty (and contractible) follows from the polar decomposition theorem by choosing a τ-invariant Riemannian metric on M. □

We recall Definition 5.17 where we introduce the Γ-equivalence relation $\sim$ and the quotient group

$$\Pi(L) = \pi_2(M, L)/\sim$$

which is abelian. Then for each $\beta \in \Pi(L)$, we defined the moduli space $\mathcal{M}(J;\beta)$ as the union

$$\bigcup_{B\in\beta\subset\pi_2(M,L)} \mathcal{M}(J;B).$$

We also recall that $\mathcal{M}_{k+1,m}(J;\beta)$ is the set of all bordered stable maps of genus zero representing the class β with $k+1$ boundary and m interior marked points. Let $\mathcal{M}^{\rm reg}_{k+1,m}(J;\beta)$ be its subsets consisting of all elements $[((\Sigma, \vec z, \vec z^{+}), w)]$ such that $\Sigma = D^2$. Here $\vec z = (z_0, \cdots, z_k)$ are boundary marked points, and $z_1^+, \cdots, z_m^+$ are interior marked points. (See Definition 2.27.)

We now go back to our discussion. Let $J \in \mathcal{J}_\omega^\tau$. For a J holomorphic curve $w : (D^2, \partial D^2) \to (M, L)$, $u : S^2 \to M$, we define $\widetilde w$, $\widetilde u$ by

$$\widetilde w(z) = (\tau \circ w)(\overline z), \qquad \widetilde u(z) = (\tau \circ u)(\overline z). \tag{38.4}$$

For $[(D^2, w)] \in \mathcal{M}^{\rm reg}(J;\beta)$, $[((D^2, \vec z, \vec z^{+}), w)] \in \mathcal{M}^{\rm reg}_{k+1,m}(J;\beta)$ we define

$$\tau_*([(D^2, w)]) = [(D^2, \widetilde w)], \qquad \tau_*([((D^2, \vec z, \vec z^{+}), w)]) = [((D^2, \vec{\overline z}, \vec{\overline z}^{+}), \widetilde w)], \tag{38.5}$$

where

$$\vec{\overline z} = (\overline z_0, \cdots, \overline z_k), \qquad \vec{\overline z}^{+} = (\overline z_0^{+}, \cdots, \overline z_m^{+}).$$

For $\beta = [w]$, we put $\tau_*\beta = [\widetilde w]$. Note if $\tau_\sharp : \pi_2(M, L) \to \pi_2(M, L)$ is the homomorphism induced by τ then $\tau_*\beta = -\tau_\sharp\beta$. This is because $z \mapsto \overline z$ is of degree -1. In fact we have

$$\tau_*(\beta) = \beta$$

on $\Pi(L)$, since τ_* preserves both energy and Maslov index.

Lemma 38.6. *The definition* (38.5) *induces the maps*

$$\tau_* : \mathcal{M}^{\rm reg}(J;\beta) \to \mathcal{M}^{\rm reg}(J;\beta), \quad \tau_* : \mathcal{M}^{\rm reg}_{k+1,m}(J;\beta) \to \mathcal{M}^{\rm reg}_{k+1,m}(J;\beta),$$

which satisfy $\tau_* \circ \tau_* = \mathrm{id}$.

Proof. If $(w, (z_0, \cdots, z_k), (z_1^+, \cdots, z_m^+)) \sim (w', (z_0', \cdots, z_k'), (z_1'^{+}, \cdots, z_m'^{+}))$, then there exists $\varphi \in PSL(2, \mathbb{R}) = \mathrm{Aut}(D^2)$ such that $w' = w \circ \varphi^{-1}$, $z_i' = \varphi(z_i)$, $z_i'^{+} = \varphi(z_i^+)$.

We put $\overline{\varphi}(z) = \overline{(\varphi(\overline{z}))}$. Then $\overline{\varphi} \in PSL(2,\mathbb{R})$ and $\widetilde{w}' = \widetilde{w} \circ \overline{\varphi}^{-1}$, $\overline{z}'_i = \overline{\varphi}(\overline{z}_i)$, $\overline{z}'^+_i = \overline{\varphi}(\overline{z}^+_i)$. The property $\tau_* \circ \tau_* = \mathrm{id}$ is straightforward. □

We remark that the mapping $\varphi \mapsto \overline{\varphi}$, $PSL(2,\mathbb{R}) \to PSL(2,\mathbb{R})$ is orientation preserving.

In §44.1 Chapter 9, we will show that a choice of stable conjugacy class $[V,\sigma] \in \mathrm{Spin}(M,L)$ of relative spin structure on L induces an orientation on $\mathcal{M}_{k+1,m}(L;\beta)$ for any given $\beta \in \Pi(L)$. Hereafter we use this orientation to regard $\mathcal{M}_{k+1,m}(L;\beta)$ being oriented as a space with Kuranishi structure. We write as $\mathcal{M}_{k+1,m}(L;\beta)^{[V,\sigma]}$ when we specify the stable conjugacy class of relative spin structure.

For an anti-symplectic involution τ of (M,ω), we will define the pull back, which is denoted by $\tau^*[V,\sigma]$, of the stable conjugacy class of relative spin structure $[V,\sigma]$ in §44.5. Then from the definition of the map τ_* in Lemma 38.6 we have

$$\begin{aligned}
&\tau_* : \mathcal{M}^{\mathrm{reg}}(J;\beta)^{\tau^*[V,\sigma]} \to \mathcal{M}^{\mathrm{reg}}(J;\beta)^{[V,\sigma]},\\
&\tau_* : \mathcal{M}^{\mathrm{reg}}_{k+1,m}(J;\beta)^{\tau^*[V,\sigma]} \to \mathcal{M}^{\mathrm{reg}}_{k+1,m}(J;\beta)^{[V,\sigma]}.
\end{aligned}$$

Note that if $\tau^*[V,\sigma] = [V,\sigma]$ as stable conjugacy classes, it is called a *τ-relatively spin structure* (Definition 44.17). Thus if the relative spin structure is τ-relatively spin, τ_* defines an involution of the space with Kuranishi structure. (Sometimes, we will also use the terminology "involution" for the case that the orientation on the source space is opposite to one on the target, when no confusion can occur.) The notion of the group action on the space with Kuranishi structure is defined in §A1.3 Definition A1.45. It is easy to see that if L is spin, it is automatically τ-relatively spin. See Remark 44.18.

Proposition 38.7. *Let $J \in \mathcal{J}^\tau_\omega$. The map $\tau_* : \mathcal{M}^{\mathrm{reg}}(J;\beta)^{\tau^*[V,\sigma]} \to \mathcal{M}^{\mathrm{reg}}(J;\beta)^{[V,\sigma]}$ is orientation preserving if $\mu_L(\beta) \equiv 0 \mod 4$ and is orientation reversing if $\mu_L(\beta) \equiv 2 \mod 4$.*

We will prove this proposition in §47.2, Chapter 9 (Proposition 44.21). Here we give some examples.

Example 38.8. (1) Consider the case of $M = \mathbb{C}P^n$, $L = \mathbb{R}P^n$. In this case, each Maslov index $\mu_L(\beta)$ has the form

$$\mu_L(\beta) = \ell_\beta(n+1)$$

where $\beta = \ell_\beta$ times the generator. We know that when n is even L is not orientable, and so we consider only the case where n is odd. On the other hand, when n is odd, L is relatively spin. The class st will be the generator of $H^2(\mathbb{C}P^n;\mathbb{Z}_2)$. Moreover we will show in Proposition 44.19, §44.5 that $\mathbb{R}P^{4n+3}$ $(n \ge 0)$ is τ-relatively spin, (indeed, $\mathbb{R}P^{4n+3}$ is spin), but $\mathbb{R}P^{4n+1}$ $(n \ge 1)$ is *not* τ-relatively spin. Then using the above formula for the Maslov index, we can conclude from Proposition 38.7 that the map $\tau_* : \mathcal{M}^{\text{reg}}(J;\beta)^{[V,\sigma]} \to \mathcal{M}^{\text{reg}}(J;\beta)^{[V,\sigma]}$ is always an orientation preserving involution for any τ-relatively spin structure $[V,\sigma]$ of $\mathbb{R}P^{4n+3}$.

Of course, $\mathbb{R}P^1$ is spin and so τ-relatively spin. The map τ_* is an orientation preserving involution if ℓ_β is even, and an orientation reversing involution if ℓ_β is odd.

(2) Let M be a Calabi-Yau 3-fold and $L \subset M$ be the set of real points (i.e., the fixed point set of an anti-holomorphic involutive isometry). In this case, L is orientable (because it is a special Lagrangian) and spin (because any orientable 3-manifold is spin). Furthermore $\mu_L(\beta) = 0$ for any $\beta \in \pi_2(M,L)$. Therefore Proposition 38.7 implies that the map $\tau_* : \mathcal{M}^{\text{reg}}(J;\beta)^{[V,\sigma]} \to \mathcal{M}^{\text{reg}}(J;\beta)^{[V,\sigma]}$ is orientation preserving for any τ-relatively spin structure $[V,\sigma]$.

We next include markings. We consider the moduli space $\mathcal{M}^{\text{reg}}_{k+1,m}(J;\beta)$.

Lemma 38.9. *The map $\tau_* : \mathcal{M}^{\text{reg}}_{k+1,m}(J;\beta)^{\tau^*[V,\sigma]} \to \mathcal{M}^{\text{reg}}_{k+1,m}(J;\beta)^{[V,\sigma]}$ is orientation preserving if and only if $\mu_L(\beta)/2 + k + 1 + m$ is even.*

Proof. Admitting Proposition 38.7, we prove Lemma 38.9. Let us consider the diagram :

$$
\begin{array}{ccc}
(S^1)^{k+1} \times (D^2)^m & \xrightarrow{\ c\ } & (S^1)^{k+1} \times (D^2)^m \\
{\scriptstyle \text{inclusion}}\uparrow & & {\scriptstyle \text{inclusion}}\uparrow \\
((S^1)^{k+1} \times (D^2)^m)_0 & \xrightarrow{\ c\ } & ((S^1)^{k+1} \times (D^2)^m)_0 \\
\downarrow & & \downarrow \\
\widetilde{\mathcal{M}}^{\text{reg}}_{k+1,m}(J;\beta)^{\tau^*[V,\sigma]} & \xrightarrow{\text{Lemma 38.9}} & \widetilde{\mathcal{M}}^{\text{reg}}_{k+1,m}(J;\beta)^{[V,\sigma]} \\
{\scriptstyle \mathfrak{forget}}\downarrow & & {\scriptstyle \mathfrak{forget}}\downarrow \\
\widetilde{\mathcal{M}}^{\text{reg}}(J;\beta)^{\tau^*[V,\sigma]} & \xrightarrow{\text{Proposition 38.7}} & \widetilde{\mathcal{M}}^{\text{reg}}(J;\beta)^{[V,\sigma]}
\end{array}
$$

Diagram 38.1.

Here c is defined by $c(z_0, z_1 \cdots, z_k, z_1^+, \cdots, z_m^+) = (\overline{z}_0, \overline{z}_1, \cdots, \overline{z}_k, \overline{z}_1^+, \cdots, \overline{z}_m^+)$ and $\mathfrak{forget}$ are the forgetful maps of marked points. $((S^1)^{k+1} \times (D^2)^m)_0$ is the set of all $c(z_0, z_1 \cdots, z_k, z_1^+, \cdots, z_m^+)$ such that $z_i \neq z_j$, $z_i^+ \neq z_j^+$ for $i \neq j$.

Lemma 38.9 then follows from Proposition 38.7 and the fact that $\mathbb{Z}_2$ action of $PSL(2,\mathbb{R})$ is orientation preserving. □

We will next extend τ_* to the compactification and define a continuous map

$$\tau_* : \mathcal{M}_{k+1,m}(J;\beta)^{\tau^*[V,\sigma]} \to \mathcal{M}_{k+1,m}(J;\beta)^{[V,\sigma]}. \tag{38.10}$$

Proposition 38.11. *The map* $\tau_* : \mathcal{M}^{\rm reg}_{k+1,m}(J;\beta)^{\tau^*[V,\sigma]} \to \mathcal{M}^{\rm reg}_{k+1,m}(J;\beta)^{[V,\sigma]}$ *extends to a map* τ_**, denoted by the same symbol, as in* (38.10)*. It preserves orientation if and only if* $\mu_L(\beta)/2+k+1+m$ *is even. In particular, if* $[V,\sigma]$ *is a* τ*-relatively spin structure, it can be regarded as an involution on the space* $\mathcal{M}_{k+1,m}(J;\beta)^{[V,\sigma]}$ *with Kuranishi structure.*

Proof. We define τ_* by a double induction over $E(\beta)$ and k. Namely we define an order on the set of triples (β,k,m) by the relation

(38.12.1) $E(\beta') < E(\beta)$.
(38.12.2) $E(\beta') = E(\beta)$, $k' < k$.

With respect to this order, we will define the map (38.10) for (β,k,m) under the assumption that the map is already defined for all (β',k',m') smaller than (β,k,m).

Let $[((\Sigma,\vec z,\vec z^{\,+}),w)] \in \mathcal{M}_{k+1,m}(J;\beta)$. We first assume that Σ has a sphere bubble $S^2 \subset \Sigma$. We remove it from Σ to obtain Σ_0. We add one more marked point to Σ_0 at the location where the sphere bubble used to be attached. We then obtain an element

$$[((\Sigma_0,\vec z,\vec z^{(0)}),w_0)] \in \mathcal{M}_{k+1,m+1-\ell}(J;\beta').$$

Here ℓ is the number of marked points on S^2. By the induction hypothesis, τ_* is already defined on $\mathcal{M}_{k+1,m+1-\ell}(J;\beta')$ since $E(\beta) > E(\beta')$. We denote

$$\tau_*([((\Sigma_0,\vec z,\vec z^{(0)}),w_0)]) = [((\Sigma_0',\vec z^{\,\prime},\vec z^{(0)\prime}),w_0')].$$

We define $v|_{S^2} : S^2 \to M$ by

$$v(z) = w|_{S^2}(\overline{z}).$$

We assume that the singular point of S^2, i.e., the point in $\Sigma_0 \cap S^2$ corresponds to $0 \in \mathbb{C}\cup\{\infty\} \cong S^2$. We also map ℓ marked points on S^2 by $z \mapsto \overline{z}$ whose images we denote by $\vec z^{(1)} \in S^2$. We then glue $((S^2,\vec z^{(1)}),\widetilde w)$ to $((\Sigma_0',\vec z^{\,\prime},\vec z^{(0)\prime}),\widetilde w)$ at the point $0 \in S^2$ and at the last marked point of $(\Sigma_0,\vec z,\vec z^{(0)})$ and obtain a curve which is to be the definition of $\tau_*([((\Sigma,\vec z,\vec z^{\,+}),w)])$.

Next suppose that there is no sphere bubble on Σ. Let Σ_0 be the component containing the 0-th marked point. If there is only one irreducible component of Σ, then τ_* is already defined there. So we assume that there is at least one disc component other than Σ_0. Then Σ is a union of Σ_0 and Σ_i for $i=1,\cdots,m$ $(m \geq 1)$. We regard the unique point in $\Sigma_0\cap\Sigma_i$ for $i=1,\cdots,m$ as marked points of Σ_0. Here each of Σ_i itself is a union of disc components and is connected. We also regard the point in $\Sigma_0\cap\Sigma_i$ as $0 \in D^2 \cong \mathbb{H}\cup\{\infty\}$ where D^2 is the irreducible component of Σ_i joined to Σ_0, and also as one of the marked points of Σ_i. This defines an element $[((\Sigma_i,\vec z^{(i)},\vec z^{\,(i)+}),w_{(i)})]$ for each $i=0,\cdots,m$. By an easy combinatorics and the induction hypothesis, we can show that τ_* is already constructed on them. Now we define $\tau_*([((\Sigma,\vec z,\vec z^{\,+}),w)])$ by gluing $\tau_*([((\Sigma_i,\vec z^{(i)},\vec z^{\,(i)+}),w_{(i)})])$.

We next consider Kuranishi structure on a neighborhood around the element $\tau_*([((\Sigma, \vec{z}, \vec{z}^+), w)])$ in $\mathcal{M}_{k+1,m}(J;\beta)$. We first observe that the map $u \mapsto \widetilde{u}$ on the moduli space of spheres defined in (38.4) can be regarded as an involution on the space with Kuranishi structure, in the same way as the proof of Proposition 38.7. Then we can show that (38.10) induces a map of the space with Kuranishi structure by the same induction process as its construction. More precisely, we will prove existence of an involution for $\mathcal{M}_{k+1,m}(J;\beta)$ assuming that τ_* induces involutions on spaces with Kuranishi structures on $\mathcal{M}_{k+1,m+1-\ell}(J;\beta')$ for all (β', k', m') smaller than (β, k, m).

Let $[((\Sigma, \vec{z}, \vec{z}^+), w)] \in \mathcal{M}_{k+1,m}(J;\beta)$. If it is in $\mathcal{M}^{\rm reg}_{k+1,m}(J;\beta)$, we have defined an involution on its Kuranishi neighborhood in Proposition 38.7. If Σ is not irreducible, then $[((\Sigma, \vec{z}, \vec{z}^+), w)]$ is obtained by gluing some elements corresponding to $(\beta', k', m') < (\beta, k, m)$. For each irreducible component, the involution of its Kuranishi neighborhood is constructed by the induction hypothesis. A Kuranishi neighborhood of $[((\Sigma, \vec{z}, \vec{z}^+), w)]$ is a fiber product of the Kuranishi neighborhoods of the gluing pieces and the space of the smoothing parameters of the singular points. By definition, our involution obviously commutes with the process to take the fiber product. For the parameter space smoothing the interior singularities, the action of the involution is the complex conjugation. For the parameter space of smoothing boundary singularities, the action of involution is trivial. The analysis we worked out in §29 of the gluing is obviously compatible with the involution. Thus τ_* defines an involution on $\mathcal{M}_{k+1,m}(J;\beta)$ with Kuranishi structure.

The statement on the orientation follows from the corresponding statement on $\mathcal{M}^{\rm reg}_{k+1,m}(J;\beta)$ of Lemma 38.9. □

We next restrict our maps to the main component of $\mathcal{M}_{k+1,m}(J;\beta)$. We remark that $\tau_* : \mathcal{M}_{k+1,m}(J;\beta) \to \mathcal{M}_{k+1,m}(J;\beta)$ does *not* preserve the main component for $k > 1$. On the other hand the assignment given by

$$\begin{aligned} &(w, (z_0, z_1, z_2, \cdots, z_{k-1}, z_k), (z_1^+, \cdots, z_m^+)) \\ &\longmapsto (\widetilde{w}, (\overline{z}_0, \overline{z}_k, \overline{z}_{k-1}, \cdots, \overline{z}_2, \overline{z}_1), (\overline{z}_1^+, \cdots, \overline{z}_m^+)) \end{aligned} \tag{38.13}$$

respects the counter-clockwise cyclic order of $S^1 = \partial D^2$ and so preserves the main component, where $\widetilde{w}$ is as in (38.4). Therefore we consider this map instead which we denote by

$$\tau_*^{\rm main} : \mathcal{M}^{\rm main}_{k+1,m}(J;\beta)^{\tau^*[V,\sigma]} \to \mathcal{M}^{\rm main}_{k+1,m}(J;\beta)^{[V,\sigma]}.$$

Lemma 38.14. *The map $\tau_*^{\rm main}$ defines a map between the spaces with Kuranishi structures and satisfies $\tau^* \circ \tau^* = \mathrm{id}$. In particular, if $[V,\sigma]$ is τ-relatively spin, it defines an involution of the space with Kuranishi structure.*

The proof is the same as the proof of Propositions 38.7 and 38.11 and so omitted.

We now have the following commutative diagram,

$$\begin{array}{ccc}
(S^1)^{k+1} \times (D^2)^m & \xrightarrow{c'} & (S^1)^{k+1} \times (D^2)^m \\
\text{inclusion}\big\uparrow & & \text{inclusion}\big\uparrow \\
((S^1)^{k+1} \times (D^2)^m)_{00} & \xrightarrow{c'} & ((S^1)^{k+1} \times (D^2)^m)_{00} \\
\big\downarrow & & \big\downarrow \\
\widetilde{\mathcal{M}}^{\text{main}}_{k+1,m}(J;\beta)^{\tau^*[V,\sigma]} & \xrightarrow{\tau_*^{\text{main}}} & \widetilde{\mathcal{M}}^{\text{main}}_{k+1,m}(J;\beta)^{[V,\sigma]} \\
\mathfrak{forget}\big\downarrow & & \mathfrak{forget}\big\downarrow \\
\widetilde{\mathcal{M}}(J;\beta)^{\tau^*[V,\sigma]} & \xrightarrow{\tau_*} & \widetilde{\mathcal{M}}(J;\beta)^{[V,\sigma]}
\end{array}$$

Diagram 38.2.

where c' is defined by $c'(z_0, z_1 \cdots, z_k, z_1^+, \cdots, z_m^+) = (\overline{z}_0, \overline{z}_k, \cdots, \overline{z}_1, \overline{z}_1^+, \cdots, \overline{z}_m^+)$ and $\mathfrak{forget}$ are the forgetful maps of marked points. $((S^1)^{k+1} \times (D^2)^m)_{00}$ is the open subset of $(S^1)^{k+1} \times (D^2)^m$ consisting of the points such that all z_i's and z_j^+'s are distinct respectively.

Let $\text{Rev}_k : L^{k+1} \to L^{k+1}$ be the map defined by

$$\text{Rev}_k(x_0, x_1, \cdots, x_k) = (x_0, x_k, \cdots, x_1).$$

It is easy to see that

$$ev \circ \tau_* = \text{Rev}_k \circ ev. \tag{38.15}$$

We remark that $\text{Rev}_k = \text{id}$, and $\tau_*^{\text{main}} = \tau_*$ for $k = 0, 1$.

Let $P_1, \cdots, P_k$ be chains on L. By taking the fiber product and using (38.13), we obtain a map

$$\tau_*^{\text{main}} : \mathcal{M}^{\text{main}}_{k+1,m}(J;\beta;P_1,\cdots,P_k)^{\tau^*[V,\sigma]} \to \mathcal{M}^{\text{main}}_{k+1,m}(J;\beta;P_k,\cdots,P_1)^{[V,\sigma]} \tag{38.16}$$

which satisfies $\tau_*^{\text{main}} \circ \tau_*^{\text{main}} = \text{id}$. We put

$$\epsilon = \mu_L(\beta)/2 + k + 1 + m + \sum_{1 \le i < j \le k} \deg' P_i \deg' P_j.$$

Lemma 38.17. *The map* (38.16) *preserves orientation if ϵ is even, and reverses orientation if ϵ is odd.*

The proof of Lemma 38.17 is in §47.2, Chapter 9 (Proposition 44.23).

Now we are ready to give the proofs of Theorem 34.20 and Proposition 34.25.

Proof of Theorem 34.20. We can take a system of multi-sections $\mathfrak{s}$ on $\mathcal{M}^{\rm main}_{k+1}(J;\beta;P_1,\cdots,P_k)$ which is preserved by (38.16). This is possible since (38.16) is an automorphism of order 2. (Namely we take quotient by Lemma A1.49, take a perturbed multi-section of the quotient, and lift it.) Theorem 34.20 then is an immediate consequence of the definition of the operator $\mathfrak{m}_k$ and Lemma 38.17. (We remark that there may be a fixed point of (38.16). But this does not cause any problem as far as we work with *multi*-section and study virtual fundamental chain over $\mathbb{Q}$.) □

Proof of Proposition 34.25. Let (N,ω) be a symplectic manifold, $M = N\times N$, and $\omega_M = \omega_N \oplus(-\omega_N)$. We consider $\tau : M \to M$, $\tau(x,y) = (y,x)$. Then $L = N = \mathrm{Fix}\,\tau$. Let J_N be a compatible almost structure on N, and $J_M = J_N\otimes 1 - 1\otimes J_N$. The almost complex structure J_M is compatible with ω_M. Let $v : S^2 \to N$ be a J_N-holomorphic map. We fix 3 marked points $0,1,\infty \in S^2 = \mathbb{C}\cup\{\infty\}$. Then we consider the upper half plane $\mathbb{H}\subset\{\infty\}\subset\mathbb{C}\cup\{\infty\}$ and define the map $I(v) : \mathbb{H}\to M$ by

$$I(v)(z) = (v(z), v(\overline{z})).$$

Identifying $(\mathbb{H},(0,1,\infty))$ with $(D^2,(-1,1,\sqrt{-1}))$ with $(-1,1,\sqrt{-1})\in\partial D^2$, we obtain a map from $(D^2,\partial D^2)$ to (M,N) which we also denote by $I(v)$. One can easily check the converse : for any given map J_M-holomorphic $w : (D^2,\partial D^2)\cong(\mathbb{H},\mathbb{R}\cup\{\infty\})\to(M,N) = (N\times N,\Delta_N)$, the assignment

$$v(z) = \begin{cases} w(z) & \text{for } z\in\mathbb{H} \\ \tau\circ w(\overline{z}) & \text{for } z\in\mathbb{C} \text{ with } \overline{z}\in\mathbb{H}\end{cases}$$

defines a J_N-holomorphic sphere on N. Therefore the map $v\mapsto I(v)$ gives an isomorphism between the moduli spaces of J_N-holomorphic spheres and J_M-holomorphic discs with boundary in N. We can easily check that it is induced by the isomorphism of Kuranishi structures. Proposition 34.25 follows. □

Proof of Theorem N. Let L be as in Theorem N. By Theorem 34.20 we have

$$\mathfrak{m}_{0,\beta}(1) = -\mathfrak{m}_{0,\beta}(1) = 0$$

for all $\beta\in\Pi(L)$ and hence $\mathcal{M}(L;\mathbb{Q})\neq\emptyset$.

On the other hand, $c_1(M) = 0$ implies $\mu_L = 0$. (Recall that $L = \mathrm{Fix}\,\tau$.) Therefore the conclusion follows from Theorem 24.21. □

§39. Generic invariant almost complex structure.

In this section we show that for $L = \mathrm{Fix}\ \tau$ the transversality result in Chapter 7 and §35 can be proved using only the τ-anti-invariant J's :

$$\tau^* J = -J.$$

We recall that a symplectic manifold (M,ω) is said to be semi-positive if for each $\alpha \in \pi_2(M)$ we have either $\omega(\alpha) \le 0$ or $c_1(\beta) \notin [3-n, 0]$. We first prove the following proposition.

Proposition 39.1. *Let (M,ω) is a symplectic manifold and $\tau : M \to M$ is anti-symplectic involution.*

(1) *If M is semi-positive, then there exists $J \in \mathcal{J}_\omega^\tau$, such that every J-holomorphic sphere $v : S^2 \to M$ satisfies $c_1(M)[v] \ge 0$.*

(2) *If M is (positively spherically) monotone or* $\dim_{\mathbb{R}} M = 4$*, then there exists $J \in \mathcal{J}_\omega^\tau$, such that every non-constant J-holomorphic sphere $v : S^2 \to M$ satisfies $c_1(M)[v] > 0$.*

Proof. For the proof we need some notations. Let $J \in \mathcal{J}_\omega^\tau$.

Definition 39.2. A J-holomorphic sphere $v : S^2 \to M$ is said to be τ-*somewhere injective* if there exists $z \in S^2$ such that

(39.3.1) v is an immersion at z,
(39.3.2) $v(z) \notin v(S^2 \setminus \{z\})$,
(39.3.3) $v(z) \notin \tau(v(S^2))$.

Let $w : (D^2, \partial D^2) \to (M, L)$ a J-holomorphic disc. We say that it is τ-*somewhere injective* if there exists $z \in \mathrm{Int}\, D^2$ such that

(39.4.1) w is an immersion at z,
(39.4.2) $w(z) \notin w(D^2 \setminus \{z\})$,
(39.4.3) $w(z) \notin \tau(w(D^2))$.

Let $v : S^2 \to M$ be a J-holomorphic sphere and $w : (D^2, \partial D^2) \to (M, L)$ a J-holomorphic disc. Recalling the definition of the non-linear Cauchy-Riemann operator denoted by $\overline{\partial}$, we consider its linearizations along the map-direction

$$D_v\overline{\partial} : \Gamma(S^2, v^*TM) \to \Gamma(S^2, \Lambda^{0,1} \otimes v^*TM) \tag{39.5}$$

and

$$D_w\overline{\partial} : \Gamma((D^2, \partial D^2), (w^*TM, w|_{\partial D^2}^*(TL))) \to \Gamma(D^2, \Lambda^{0,1} \otimes w|_{\partial D^2}^* TM). \tag{39.6}$$

Lemma 39.7. *For generic $J \in \mathcal{J}_\omega^\tau$,* (39.5) *is surjective for any τ-somewhere injective sphere v and* (39.6) *is surjective for all τ-somewhere injective disc w.*

Proof. The proof is a minor modification of that of [McD87]. Let $v : S^2 \to M$ be a τ-somewhere injective J-holomorphic sphere. By considering the linearization of $\overline{\partial}$ along the J-direction, we obtain a linear map

$$D_J\overline{\partial} : T_J\mathcal{J}_\omega^\tau \to \Gamma(S^2, \Lambda^{0,1} \otimes v^*TM).$$

To prove the first half of the statement, it suffices to show

$$\Gamma(S^2, \Lambda^{0,1} \otimes v^*TM) = \operatorname{Im} D_v\overline{\partial} + \operatorname{Im} D_J\overline{\partial}$$

where the right hand side is the same as the image of the total linearization of $\overline{\partial}$ at (v, J).

Let $z \in S^2$ be a point as in (39.3). Then we can choose a small neighborhood W of $v(z)$ in M such that

$$W \cap \tau(W) = \emptyset$$

and a small neighborhood U of z in S^2 that satisfies

$$W \cap v(S^2 \setminus U) = \emptyset, \quad W \cap \tau(v(S^2)) = \emptyset.$$

Denote by $\mathcal{J}_\omega(W)$ the set of all compatible almost complex structures J' that coincide with J outside W. (Note that elements of $\mathcal{J}_\omega(W)$ may not be τ-anti-invariant.) Then it follows from [McD87] that

$$(D_J\overline{\partial})(T_J\mathcal{J}_\omega(W)) + \operatorname{Im}(D_v\overline{\partial}) = \Gamma(S^2, \Lambda^{0,1} \otimes v^*TM). \tag{39.8}$$

We define a map $S : \mathcal{J}_\omega(W) \to \mathcal{J}_\omega^\tau$ by putting

$$S(J') = \begin{cases} -\tau_*(J') & \text{on } \tau(W), \\ J' & \text{outside } \tau(W) \end{cases}$$

for $J' \in \mathcal{J}_\omega(W)$. Using the property $W \cap \tau(W) = \emptyset$ of W, we can easily check that $S(J')$ is τ-anti-invariant. On the other hand, it follows from the property $\tau(W) \cap v(S^2) = \emptyset$ that $D_J\overline{\partial} \circ DS = D_J\overline{\partial}$ at (v, J). Therefore

$$(D_J\overline{\partial})(T_J\mathcal{J}_\omega(W)) \subset (D_J\overline{\partial})(T_J\mathcal{J}_\omega^\tau).$$

Combining this with (39.8), we have proved

$$(D_J\overline{\partial})(T_J\mathcal{J}_\omega^\tau) + \operatorname{Im}(D_v\overline{\partial}) = \Gamma(S^2, \Lambda^{0,1} \otimes v^*TM)$$

which is precisely what we wanted to prove. This finishes the proof for the spheres v. The argument for τ-somewhere injective discs is similar and omitted. □

Remark 39.9. McDuff proved that every J-holomorphic sphere is a branched covering of somewhere injective sphere [McD87]. On the other hand, if we define somewhere injectivity of J-holomorphic disc by (39.4.1) and (39.4.2), then it is *not* the case in general that every J-holomorphic disc is branched covering of a somewhere injective disc. While analyzing the structure of the image of J-holomorphic discs requires more delicate study given as in [KwOh00] (see [Lazz00] also), we do not need the deep structure theorem in the proof of Proposition 39.1.

Now we go back to the proof of Proposition 39.1. We assume (M, ω) is semi-positive. Let us take a compatible almost complex structure satisfying the conclusion of Lemma 39.7. Then the semi-positivity of M implies that there exists no τ-somewhere injective J-holomorphic sphere $v : S^2 \to M$ with $c_1(M)[v] < 0$ because τ-somewhere injectivity implies somewhere injectivity. Note that in general J-holomorphic sphere is not necessarily a branched covering of τ-somewhere injective one. So we need some more argument to prove Proposition 39.1.

Let $v : S^2 \to M$ be a J-holomorphic sphere satisfying $c_1(M)[v] < 0$. Then v cannot be τ-somewhere injective by Lemma 39.7 and by the semi-positivity of M. (See [McD87].) If v is not somewhere injective either, we can find a somewhere injective sphere $\overline{v}$ such that v is a (branched) covering of $\overline{v}$. Evidently we have $c_1(M)[\overline{v}] < 0$. Therefore, replacing v by $\overline{v}$ if necessary, we may and will assume that $v : S^2 \to M$ is somewhere injective. However $\overline{v}$ itself may not be necessarily τ-somewhere injective.

Since v is somewhere injective, it is immersed away from a finite number of points in S^2. On the other hand, since it is not τ-somewhere injective, it follows that

$$\tau(v(S^2)) = v(S^2).$$

Therefore $\tau : M \to M$ induces an involution

$$\widetilde{\tau} : S^2 \to S^2$$

by the formula

$$v \circ \widetilde{\tau} = \tau \circ v.$$

Using the immersion property of v, one can check that $\widetilde{\tau}$ is smooth. Furthermore it also satisfies $T\widetilde{\tau} \circ j = -j \circ T\widetilde{\tau}$, i.e., $\widetilde{\tau}$ defines an anti-holomorphic involution on S^2. (In this section we denote by Tf the differential of a map f.)

We consider the linearization map along the v-direction

$$D_v\overline{\partial} : \Gamma(S^2, v^*TM) \to \Gamma(S^2, \Lambda^{0,1} \otimes v^*TM). \tag{39.10}$$

Using τ and $\widetilde{\tau}$, we define an involution I on $\Gamma(S^2, v^*TM)$ by

$$(I(u))(z) = (T\tau)^{-1}(u(\widetilde{\tau}(z))). \tag{39.11.1}$$

An involution I on $\Gamma(S^2, \Lambda^{0,1} \otimes v^*TM)$ is also defined by

$$(I(\alpha))(z) = (T\tau)^{-1} \circ \alpha(\widetilde{\tau}(z)) \circ T\widetilde{\tau}, \tag{39.11.2}$$

where $\alpha(z) \in (\Lambda^{0,1} \otimes v^*TM)_z$ is regarded as an anti-complex linear map : $T_z\Sigma \to T_{v(z)}M$.

It is easy to see that $D_v\overline{\partial}$ commutes with I. Let

$$\Gamma(S^2, v^*TM) = \Gamma(S^2, v^*TM)^+ \oplus \Gamma(S^2, v^*TM)^-$$

be the decomposition of $\Gamma(S^2, v^*TM)$ according to the I-invariant and I-anti-invariant parts. We define $\Gamma(S^2, \Lambda^{0,1} \otimes v^*TM)^{\pm}$ in a similar way. Then the complex (39.10) splits into

$$(D_v\overline{\partial})^+ : \Gamma(S^2, v^*TM)^+ \to \Gamma(S^2, \Lambda^{0,1} \otimes v^*TM)^+, \tag{39.12.1}$$

$$(D_v\overline{\partial})^- : \Gamma(S^2, v^*TM)^- \to \Gamma(S^2, \Lambda^{0,1} \otimes v^*TM)^-. \tag{39.12.2}$$

Lemma 39.13.

$$\mathrm{Index}(D_v\overline{\partial})^+ = \mathrm{Index}(D_v\overline{\partial})^-.$$

Here Index *stands for the real index.*

Proof. J induces isomorphisms on $\Gamma(S^2, v^*TM)$ and $\Gamma(S^2, \Lambda^{0,1} \otimes v^*TM)$ by $u \mapsto J \circ u$ and $\alpha \mapsto J \circ \alpha$, respectively. (Here we are using the notations of (39.11).) We denote them also by J. It is easy to see from (39.11) and the definition of J that $IJ = -JI$. Hence J exchanges $\Gamma(S^2, \Lambda^{0,1} \otimes v^*TM)^+$ and $\Gamma(S^2, \Lambda^{0,1} \otimes v^*TM)^-$.

Moreover J commutes with the symbol of the operator $D_v\overline{\partial}$. (Note J may not commutes with $D_v\overline{\partial}$.) The lemma then follows from the fact that the index depends only on the symbol. □

Now we prove the following lemma.

Lemma 39.14. *For a generic element J of $\mathcal{J}_\omega^\tau$, the operator* (39.12.1) *is surjective for any somewhere injective sphere v* (*which may not necessarily be τ-somewhere injective*).

Proof. The lemma follows from Lemmas 39.7 and 39.13, if v is τ-somewhere injective. Therefore it remains to consider the case where v is not τ-somewhere injective. Then there exists an anti-holomorphic involution $\widetilde{\tau} : S^2 \to S^2$ defined by $\tau \circ v = v \circ \widetilde{\tau}$ as before. This induces the splitting

$$\Gamma(S^2, \Lambda^{0,1} \otimes v^*TM) = \Gamma(S^2, \Lambda^{0,1} \otimes v^*TM)^+ \oplus \Gamma(S^2, \Lambda^{0,1} \otimes v^*TM)^-$$

corresponding to the eigenspaces of I with the eigenvalue ± 1. Using the same notation as in the proof of Lemma 39.7, we consider the linearization of $\overline{\partial}$ in the J-direction

$$D_J\overline{\partial} : T_J\mathcal{J}_\omega^\tau \to \Gamma(S^2, \Lambda^{0,1} \otimes v^*TM).$$

We denote by $(D_J\overline{\partial})^+$ the projection of $D_J\overline{\partial}$ with respect to the above splitting. It now suffices to show

$$(D_J\overline{\partial})^+(T_J\mathcal{J}_\omega^\tau(W)) + \mathrm{Im}((D_v\overline{\partial})^+) = \Gamma(S^2, \Lambda^{0,1} \otimes v^*TM)^+ \tag{39.15}$$

by a similar reason as for (39.8). By the somewhere injectivity of v, we have a point $z \in S^2$ that satisfies (39.3.1) and (39.3.2) (but not necessarily (39.3.3)). By slightly moving z, we may assume that z is not a fixed point of $\widetilde{\tau}$. Then we can choose an open neighborhood U of z such that $\widetilde{\tau}(U) \cap U = \emptyset$. We also choose an open subset W of M such that $v(U) \subset W$ and $\tau(W) \cap W = \emptyset$. Again we have (39.8).

We now recall the set $\mathcal{J}_\omega(W)$ and the map $S : \mathcal{J}_\omega(W) \to \mathcal{J}_\omega^\tau$ defined in the end of the proof of Lemma 39.7. We will now prove

$$(D_J\overline{\partial})^+ \circ DS = (D_J\overline{\partial})^+ \tag{39.16}$$

at (v, J). Let $\delta J \in T_J(\mathcal{J}_w(W)) \subseteq \Gamma(W, \mathrm{End}_{\mathbb{R}}(TM))$. Then we have

$$(DS)(\delta J)_{\tau(p)} = -(T\tau)^{-1} \circ \delta J(p) \circ T\tau \tag{39.17}$$

for $p \in W \subset M$. On the other hand, from the definition

$$\overline{\partial} v = \frac{1}{2}(Tv + J \circ Tv \circ j) \tag{39.18}$$

we have the formula

$$(D_J\overline{\partial})(\delta J) = \frac{1}{2}\delta J \circ Tv \circ j \tag{39.19}$$

for $\delta J \in T_J\mathcal{J}_w \subset \Gamma(M, \mathrm{End}_{\mathbb{R}}(TM))$. Combining $j \circ T\widetilde{\tau} = -T\widetilde{\tau} \circ j$, (39.11.2), (39.17) and (39.19), we derive

$$\begin{aligned}
&(I \circ D_J\overline{\partial} \circ DS)(\delta J)(z) \\
&= \frac{1}{2}(T\tau)^{-1} \circ DS(\delta J)_{\widetilde{\tau}(z)} \circ Tv \circ j \circ T\widetilde{\tau} \\
&= \frac{1}{2}DS(\delta J)_p \circ (T\tau)^{-1} \circ Tv \circ T\widetilde{\tau} \circ j.
\end{aligned}$$

By differentiating $\tau^{-1} \circ v \circ \widetilde{\tau} = v$, we show that the last term becomes $\frac{1}{2}DS(\delta J)_p \circ Tv \circ j$ from which we conclude

$$(I \circ D_J\overline{\partial} \circ DS)(\delta J) = (D_J\overline{\partial} \circ DS)(\delta J)$$

at (v, J). Projecting this identity to the eigenspace $\Gamma(S^2, \Lambda^{0,1} \otimes v^*TM)^+$ of I (with eigenvalue 1), the identity (39.16) follows.

Then (39.16) and (39.8) imply (39.15) which finishes the proof of Lemma 39.14. □

Now we assume that J satisfies the conclusion of Lemmas 39.14 and 39.7. We now wrap up the proof of Proposition 39.1 (1) by showing that for such J the conclusion of Proposition 39.1 (1) holds. Suppose to the contrary $c_1(M)[v] < 0$ for a J-holomorphic v. By the semi-positivity of M, this implies

$$c_1(M)[v] < 3 - n \quad \text{or} \quad c_1(M)[v] + n < 3.$$

By taking Lemma 39.13 into account, this implies that the *real* index of the operator (39.12.1), which is given by $c_1(M)[v] + n$, is less than 3.

On the other hand, since Lemma 39.14 shows that the operator $(D_v\overline{\partial})^+$ at v is surjective by the choice of J, $c_1(M)[v] + n$ is the actual real dimension of the τ-invariant part of the moduli space containing v. There are two cases for the restriction of τ to the domain of v. Namely, the reflection along the equator or the antipodal map. In the first case, the group of automorphisms of $\mathbb{C}P^1$ commuting with the reflection is isomorphic to $PSL(2;\mathbb{R})$. In the second case, the group of automorphisms commuting with the antipodal map is $SO(3)$. In both cases, the automorphism group is real 3-dimensional. Since v is non-constant, the actual dimension of the τ-invariant part of the moduli space must be at least greater than equal to 3, which gives rise to a contradiction. The proof of Proposition 39.1 (1) is now complete.

We next consider the situation of Proposition 39.1 (2). We remark that since (M, ω) is monotone or $\dim_{\mathbb{R}} M = 4$, it follows that (M, ω) is semi-positive. We take J which satisfies the conclusion of Lemmas 39.14 and 39.7. Let $v : S^2 \to M$ be a J holomorphic sphere. We already proved $c_1(M)[v] \geq 0$. If (M, ω) is positively monotone, we can prove $c_1(M)[v] = 0$ if and only if $\omega[v] = 0$. Such a v is necessarily constant. We next assume $\dim_{\mathbb{R}} M = 4$ and $c_1(M)[v] = 0$. We may replace v by somewhere injective disc. Then Lemma 39.12 imples that the *real* index of the operator (39.12.1), which is given by $c_1(M)[v] + 2$, is less than 3. By the same argument as (1) above, we obtain a contradiction. The proof of Proposition 39.1 (2) is now complete. □

Corollary 39.20. *Let (M, ω) be a symplectic manifold with anti symplectic involution τ. We put $L = \mathrm{Fix}\,\tau$.*

(1) *If (M, ω) is semi-positive, then there exists $J \in \mathcal{J}_\omega^\tau$, such that every J-holomorphic disc $w : (D^2, \partial D^2) \to (M, L)$ satisfies $\mu_L([w]) \geq 0$.*

(2) *If (M, ω) is (positively spherically) monotone or $\dim_{\mathbb{R}} M = 4$, then there exists $J \in \mathcal{J}_\omega^\tau$, such that every nonconstant J-holomorphic disc $w : (D^2, \partial D^2) \to (M, L)$ satisfies $\mu_L([w]) > 0$.*

Proof. We consider the case of (1). We choose J satisfying the conclusion of Proposition 39.1. Let $w : (D^2, \partial D^2) \to (M, L)$ be a J-holomorphic disc. We consider its double $v : S^2 \to M$, which is a J-holomorphic sphere. Proposition 39.1 implies $c_1(M)[v] \geq 0$. On the other hand, a simple topological calculation proves the gluing formula

$$2c_1(w_1 \# \overline{w}_2) = \mu_L(w_1) + \mu_L(w_2)$$

for two discs $w_i : (D^2, \partial D^2) \to (M, L)$, $i = 1, 2$ satisfying $w_1|_{\partial D^2} = w_2|_{\partial D^2}$. Here $\overline{w}_2$ denotes the obvious map induced by w_2 which is defined on the lower hemisphere by considering the reflection along the equator of S^2. (See [Vit87] for its proof.) On the other hand, when $L = \mathrm{Fix}\,\tau$ and $w_1 = w$ and $w_2 = \widetilde{w}$, this reduces to $c_1(M)[v] = \mu_L([w])$ because we have $\mu_L([w]) = \mu_L([\widetilde{w}])$. Therefore Proposition 39.1 (1) implies $\mu_L([w]) \geq 0$.

The proof of case (2) is the same by using Proposition 39.1 (2). □

The proof of Corollary 39.20 implies also the following.

Lemma 39.21. *Let $L = \mathrm{Fix}\tau$. If there exists a J holomorphic disc $w : (D^2, \partial D^2) \to (M, L)$ with $\mu[w] < 0$, $\omega[w] \leq E_0$, then there exists a J holomorphic sphere $v : S^2 \to M$ with $c_1[w] < 0$, $\omega[v] \leq 2E_0$.*

By the proof of Theorems 34.11, we can use the almost complex structure J satisfying the conclusion of Proposition 39.1 and Corollary 39.20, and an abstract perturbation (that is, a perturbation of the Kuranishi map) to obtain the filtered A_∞ algebras mentioned in Theorems 34.4 and 34.7. However to apply the cancellation argument, we still need to construct a τ_*-invariant perturbation $\mathfrak{s}^\epsilon$. When τ_* does not have a fixed point, such construction is easy to carry out. However, since τ_* may have a fixed point in general, it is not a trivial matter to construct a τ_*-invariant (abstract) perturbation $\mathfrak{s}^\epsilon$ so that the τ-invariant version of Theorem 34.11 holds. We will resolve this trouble in the next four sections.

Proof of Theorem 34.16 ⇒ Theorem 34.17. Proposition 39.1 (2) implies that $\mathcal{J}_\omega^{c_1>0} \cap \mathcal{J}_\omega^\tau$ is nonempty if (M, ω) is (positively spherically) monotone or $\dim_{\mathbb{R}} M = 4$. Clearly if (M, J) is Fano then $J \in \mathcal{J}_\omega^{c_1>0}$. Moreover if $J_i \in \mathcal{J}_{(M_i,\omega_i)}^{c_1>0}$ then $\prod J_i \in \mathcal{J}_{\prod(M_i,\omega_i)}^{c_1>0}$. Therefore under assumption of Theorem 34.17, $\mathcal{J}_\omega^{c_1>0} \cap \mathcal{J}_\omega^\tau$ is nonempty. Hence Theorem 34.16 implies that Conjecture 34.14 holds for (M, ω), as required. □

§40. Lantern.

In §40 - §42 we consider M, ω, J, and τ such that (M, ω) is semi-positive, J is compatible almost complex structure τ is (M, J) anti-holomorphic involution. We also assume that J satisfies the conclusion of Proposition 39.11 (1). Let $\beta \in \Pi(L)$. Hereafter we will write $\mathcal{M}_{k+1}^{\text{main}}(\beta)$ in place of $\mathcal{M}_{k+1}^{\text{main}}(J;\beta)$. We are going to study

$$\tau_* : \mathcal{M}_{k+1}^{\text{main}}(\beta) \to \mathcal{M}_{k+1}^{\text{main}}(\tau_*\beta) = \mathcal{M}_{k+1}^{\text{main}}(\beta), \qquad (k = 0, 1) \tag{40.1}$$

defined in (38.5) where we recall

$$\beta = \tau_*\beta \quad \text{in } \Pi(L).$$

In the rest of this chapter, we omit the symbol "main" since there is only one component for $k = 0, 1$. In this section we study the fixed point set of τ_*.

We first consider $\mathcal{M}_2(\beta)$. In this section we consider its subset $\mathcal{M}_2^{\text{reg}}(\beta)$. Here $\mathcal{M}_{k+1}^{\text{reg}}(\beta)$ is a subset of $\mathcal{M}_{k+1}(\beta)$ consisting of elements $[((\Sigma, \vec{z}), w)]$ such that $\Sigma = D^2$. (Namely it consists of the elements without bubble.)

To study the fixed point set of τ_*, we construct a map

$$\mathfrak{D} : \mathcal{M}_2^{\text{reg}}(\beta) \to \mathcal{M}_2^{\text{reg}}(\beta + \tau_*\beta) = \mathcal{M}_2^{\text{reg}}(2\beta).$$

Let $((D^2, (z_0, z_1)), w)$ represent an element $[((D^2, (z_0, z_1)), w)]$ of $\mathcal{M}_2^{\text{reg}}(\beta)$. By requiring $z_0 = 1$, $z_1 = -1$, the representative $((D^2, (1, -1)), w)$ is uniquely determined modulo the action of $\text{Aut}(D^2; 1, -1)$ which is isomorphic to $\mathbb{R}$.

Denote the upper and lower semi-discs by

$$D^2_\pm = \{z \in D^2 \mid \pm \operatorname{Im} z \geq 0\}.$$

We take a conformal isomorphism $\rho_+ : D^2_+ \to D^2$ such that $\rho_+(\pm 1) = \pm 1$ and define the conformal isomorphism $\rho_- : D^2_- \to D^2$ by $\rho_-(z) = \overline{(\rho_+(\overline{z}))}$.

Now for a given $w : (D^2, \partial D^2) \to (M, L)$ with $L = \operatorname{Fix} \tau$, we define another map $\widetilde{w} : (D^2, \partial D^2) \to (M, L)$

$$\widetilde{w}(z) = \begin{cases} w(\rho_-(z)) & \text{if } z \in D^2_-, \\ (\tau \circ w)\left(\overline{\rho_+(z)}\right) & \text{if } z \in D^2_+ \end{cases} \tag{40.2}$$

by gluing w and $\tau \circ w$. Using this gluing, we define the map $\mathfrak{D}$ by

$$\mathfrak{D}([((D^2, (1, -1)), w)]) = [((D^2, (1, -1)), \widetilde{w})]. \tag{40.3}$$

We like to remark that we have given the definition of $\mathfrak{D}$ only for the regular elements $[((\Sigma, (z_0, z_1)), w)]$ i.e., on $\mathcal{M}_2^{\text{reg}}(\beta)$.

Lemma 40.4. *The definitions* (40.2) *and* (40.3) *give a well-defined map*

$$\mathfrak{D} = \mathfrak{D}_\beta : \mathcal{M}_2^{\mathrm{reg}}(\beta) \to \mathcal{M}_2^{\mathrm{reg}}(2\beta).$$

$\mathfrak{D}$ *is independent of the choice of* ρ_+*. Moreover* $\mathfrak{D}$ *satisfies*

$$\tau_* \circ \mathfrak{D} = \mathfrak{D}. \tag{40.5}$$

Proof. If $z \in D^2_- \cap D^2_+ = \mathbb{R} \cap D^2$, $\rho_-(z) = \overline{\rho_+(z)}$ by definition of $\rho_\pm$ and so $w(\rho_-(z)) = w\left(\overline{\rho_+(z)}\right)$. Since both the upper and lower parts of $\widetilde{w}$ are pseudo-holomorphic, this matching condition along $D^2_- \cap D^2_+$ implies that $\widetilde{w}$ is smooth on D^2 and pseudo-holomorphic. Furthermore it follows $\widetilde{w}(\partial D^2) \subset L$ from the definition (40.2) of $\widetilde{w}$. Therefore $\widetilde{w}$ defines an element in $\mathcal{M}_2^{\mathrm{reg}}(2\beta)$.

If we take another choice ρ'_+ of a conformal diffeomorphism from D^2_+ to D^2 with $\rho_+(\pm 1) = \pm 1$, then we have $\rho'_\pm = \rho_+ \circ r$ where $r \in \mathrm{Aut}(D^2; 1, -1) \cong \mathbb{R}$. Therefore the map $\widetilde{w}'$ defined as in (40.2) this time using ρ'_+ will satisfy $\widetilde{w}' = \widetilde{w} \circ r$. This implies $[\widetilde{w}'] = [\widetilde{w}]$ as an element of $\mathcal{M}_2(2\beta)$. This proves that the definition (40.3) is independent to a representative $(D^2, (1,-1), w)$ but depends only on the stable map $[(D^2, (1,-1), w)]$ in $\mathcal{M}_2^{\mathrm{reg}}(\beta)$.

The identity (40.5) immediately follows from the definition (40.3) of $\mathfrak{D}$. $\square$

A fixed point of τ_* (that is an image of $\mathfrak{D}$) is a pseudo-holomorphic disc that can be pictured as in Figures 40.1 and 40.2 below.

Figure 40.1.

Figure 40.2.

Now we describe the fixed point set of $\tau_* : \mathcal{M}_2^{\rm reg}(\beta) \to \mathcal{M}_2^{\rm reg}(\beta)$ in more detail.

Lemma 40.6. *If $\mathfrak{w} \in \mathcal{M}_2^{\rm reg}(\beta)$ with $\tau_*(\mathfrak{w}) = \mathfrak{w}$, then there exist β' and $\mathfrak{w}' \in \mathcal{M}_2^{\rm reg}(\beta')$ satisfying*

$$\beta' + \tau_*\beta' = 2\beta' = \beta, \quad \mathfrak{w} = \mathfrak{D}(\mathfrak{w}'). \tag{40.6.1}$$

In other words, the fixed point of τ_* on $\mathcal{M}_2^{\rm reg}(\beta)$ coincides with the image of $\mathfrak{D}$ of $\mathcal{M}_2^{\rm reg}(\beta')$.

Moreover β' and $\mathfrak{w}'$ are uniquely determined by $\mathfrak{w}$.

Proof. We may assume that $\mathfrak{w}$ is represented by $((D^2,(1,-1)),w)$. By assumption, there exists $r \in \mathrm{Aut}(D^2;1,-1) \cong \mathbb{R}$ such that $(\tau\circ w)(\overline{z}) = (w\circ r)(z)$. Using the fact that r commutes with complex conjugation and τ is an involution, we have

$$w(\overline{z}) = (\tau\circ\tau\circ w)(\overline{z}) = (\tau\circ w\circ r)(z) = (w\circ r\circ r)(\overline{z}).$$

This implies that $r\circ r$ is the identity and so is r. Therefore we derive

$$(\tau\circ w)(\overline{z}) = w(z) \tag{40.7}$$

from the last identity of the above equation. It follows that $(\tau\circ w)(z) = w(z)$ and so $w(z) \in \mathrm{Fix}\,\tau$ for $z \in \mathbb{R}\cap D^2$. Namely we have $w(\mathbb{R}\cap D^2) \subset L$.

We consider the map w' defined by $w' = w\circ\rho_-$. It follows from $w(\mathbb{R}\cap D^2)\subset L$ that $w'(\partial D^2)\subset L$ and hence $w' \in \mathcal{M}_2^{\rm reg}(\beta')$ for $\beta' = [w'] \in \Pi(L)$. By the definition of $\mathfrak{D}$, (40.7) implies

$$\mathfrak{D}([((D^2,(1,-1)),w')]) = [((D^2,(1,-1)),w)].$$

Now we prove the uniqueness of β' and $\mathfrak{w}'$ among those satisfying (40.6.1). Suppose $w = \mathfrak{D}(w')$, $\beta' = [w']$ and

$$\mathfrak{D}([((D^2,(1,-1)),w')]) = [((D^2,(1,-1)),w)].$$

Then by the discussion as above, we prove that there exists $r \in \mathrm{Aut}(D^2;1,-1)$ such that $w \circ \rho_- = w' \circ r$, i.e., that w' is uniquely determined by w modulo the action of $\mathrm{Aut}(D^2;1,-1)$. Hence the equivalence class of w' depends only on w. □

Remark 40.8. One may be able to extend the definition of $\mathfrak{D}$ to the whole compactified moduli space $\mathcal{M}_2(\beta)$ but the analog to Lemma 40.6 more specifically the *uniqueness result will no longer hold.* See §43.1.

By iterating the construction in Lemma 40.6, we now obtain the following proposition.

Proposition 40.9. *For each $\mathfrak{w} \in \mathcal{M}_2^{\mathrm{reg}}(\beta)$ there exist unique $\beta^{(0)}$, $\mathfrak{w}_0 \in \mathcal{M}_2^{\mathrm{reg}}(\beta^{(0)})$ and ℓ respectively such that*

$$\mathfrak{w} = \mathfrak{D}^\ell(\mathfrak{w}_0), \qquad \beta = 2^\ell\beta^{(0)},$$

and

$$\mathfrak{w}_0 \neq \tau_*(\mathfrak{w}_0).$$

Here $\mathfrak{D}^\ell$ is the ℓ-th iteration of $\mathfrak{D}$. ($\ell = 0,1,2,\cdots$).

Proof. Let $\mathfrak{w} \in \mathcal{M}_2^{\mathrm{reg}}(\beta)$ be a fixed point of τ_* and let β', $\mathfrak{w}'$ be those obtained in Lemma 40.6. We consider the image of $\tau_*(\mathfrak{w}')$ again. If $\tau_*(\mathfrak{w}') \neq \mathfrak{w}'$, we are done with $\ell = 1$, $\mathfrak{w}_0 = \mathfrak{w}'$ and $\beta^{(0)} = \beta'$. If not, we apply Lemma 40.6 to $\mathfrak{w}'$ again. By repeatedly applying Lemma 40.6, we obtain a sequence of elements

$$\mathfrak{w}_{(i)} \in \mathcal{M}_2^{\mathrm{reg}}(\beta_{(i)}) \quad i = 0,1,\cdots$$

such that $\mathfrak{w}_{(i-1)} = \mathfrak{D}(\mathfrak{w}_{(i)})$ with $\mathfrak{w}_{(-1)} := \mathfrak{w}$. By construction, we have $\omega(\mathfrak{w}_{(i+1)}) = \frac{1}{2}\omega(\mathfrak{w}_{(i)})$. On the other hand, there exists $c > 0$ depending only on (M,J,ω) and L such that $E(w) \geq c > 0$ for any J-holomorphic disc $w : (D^2,\partial D^2) \to (M,L)$. Therefore the above iteration process should terminate in finite steps, say in the ℓ-th step. We put $\mathfrak{w}_0 = \mathfrak{w}_{(\ell)}$ and $\beta^{(0)} = [\mathfrak{w}_{(\ell)}]$. This finishes the proof. □

We can reduce the case of $\mathcal{M}_1^{\mathrm{reg}}(\beta)$ to the case of $\mathcal{M}_2^{\mathrm{reg}}(\beta)$ by using the following lemma.

Lemma 40.10. *If $\mathfrak{w} = [((D^2,z_0),w)] \in \mathcal{M}_1^{\mathrm{reg}}(\beta)$ is a fixed point of τ_*, then there exists $\mathfrak{w}^+ = [((D^2,(z_0,z_1)),w)] \in \mathcal{M}_2^{\mathrm{reg}}(\beta)$ such that $\tau_*\mathfrak{w}^+ = \mathfrak{w}^+$ and that $\mathfrak{forget}(\mathfrak{w}^+) = \mathfrak{w}$, where $\mathfrak{forget} : \mathcal{M}_2^{\mathrm{reg}}(\beta) \to \mathcal{M}_1^{\mathrm{reg}}(\beta)$ is the map forgetting the second marked point. Such $\mathfrak{w}^+$ is unique.*

Proof. Without loss of any generality, we may assume $z_0 = 1$. Let $h : \mathbb{H} \to D^2$ be a conformal diffeomorphism, where $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z \geq 0\}$ with $h(\infty) = 1$, $h(0) = -1$ and $\overline{h(z)} = h(-\overline{z})$. Existence of such a conformal diffeomorphism can be easily seen from the symmetry consideration of the Riemann mapping theorem, or by an explicit construction. We like to note that the imaginary axis $\sqrt{-1}\mathbb{R}_+ \subset \mathbb{H}$ is mapped to the diameter $\mathbb{R} \cap D^2 \subset D^2$ under any such h.

We denote $w' := w \circ h$. We remark that under the isomorphism h the complex conjugation $c : z \mapsto \overline{z}$ on D^2 is transformed to the anti-holomorphic involution

$$z \mapsto -\overline{z}$$

on $\mathbb{H}$. Therefore by the assumption $\mathfrak{w} \in \operatorname{Fix} \tau_*$ which in particular implies $\tau \circ w \circ c = w$ modulo the action of $\operatorname{Aut}(D^2; 1)$, it follows that there exists $\varphi \in \operatorname{Aut}(\mathbb{H}; \infty)$ such that

$$(\tau \circ w')(-\overline{z}) = (w' \circ \varphi)(z). \tag{40.11}$$

Since $\varphi(\infty) = \infty$, φ must be of the form

$$\varphi(z) = az + b$$

for some a, $b \in \mathbb{R}$ with $a > 0$. Now we obtain a chain of identities

$$w'(-\overline{z}) = (\tau \circ \tau \circ w')(-\overline{z}) = (\tau \circ w')(az+b) = w'(a(-\overline{(az+b)})+b) = w'(-a^2\overline{z}+b-ab).$$

Hence $a = 1$ and so $\varphi(z) = z + b$. To construct a fixed point of τ_* in $\mathcal{M}_2^{\mathrm{reg}}(\beta)$ out of $w \in \mathcal{M}_1^{\mathrm{reg}}(\beta)$, it is enough to find a point $z_1 \in \mathbb{R}$ that solves the equation

$$-\overline{z}_1 = -z_1 = \varphi(z_1) = z_1 + b$$

which results in $z_1 = -\frac{b}{2}$. Now translating these information on $\mathbb{H}$ back to D^2 via h, it follows that $\mathfrak{w}^+ = [((D^2, (1, h(-b/2))), w)]$ has all the required properties. The uniqueness automatically follows from this construction. $\square$

§41. Symmetry and invariant perturbation of the moduli space.

In this section and the next, we construct an appropriate perturbation of the Kuranishi map on $\mathcal{M}_2^{\mathrm{reg}}(\beta)$ that is invariant under the given symmetries. In §43, we will complete the proof of Theorems 34.16 using this perturbation.

For given $\beta \in \Pi(L)$, we define

$$\mathfrak{D}_*^{-m}(\beta) = \left\{\beta' \in \Pi(L) \mid 2^{m-1}\beta' + 2^{m-1}\tau_*\beta' = \beta,\, \mathcal{M}_2^{\mathrm{reg}}(\beta') \neq \emptyset\right\}.$$

By Lemma 40.6, Proposition 40.9 and the property $\tau_*\beta' = \beta'$, it follows that $\mathfrak{D}_*^{-m}(\beta)$ is either empty or consists of a single element. We write $\{2^{-m}\beta\} = \mathfrak{D}_*^{-m}(\beta)$ if it is nonempty. We obtain a map

$$\mathfrak{D}^m : \mathcal{M}_2^{\mathrm{reg}}(2^{-m}\beta) \to \mathcal{M}_2^{\mathrm{reg}}(\beta).$$

We denote

$$\mathcal{M}_2^{\mathrm{reg},(m)}(\beta) = \mathrm{Im}\left(\mathfrak{D}^m : \mathcal{M}_2^{\mathrm{reg}}(2^{-m}\beta) \to \mathcal{M}_2^{\mathrm{reg}}(\beta)\right). \tag{41.1}$$

It is easy to see that

$$\mathcal{M}_2^{\mathrm{reg},(m)}(\beta) \supset \mathcal{M}_2^{\mathrm{reg},(m+1)}(\beta). \tag{41.2}$$

Namely $\mathcal{M}_2^{\mathrm{reg},(m)}(\beta)$ defines a downward stratification of $\mathcal{M}_2^{\mathrm{reg}}(\beta)$. Moreover there exists an integer $m(\beta)$ such that

$$\mathcal{M}_2^{\mathrm{reg},(m(\beta))}(\beta) \neq \emptyset, \qquad \mathcal{M}_2^{\mathrm{reg},(m(\beta)+1)}(\beta) = \emptyset. \tag{41.3}$$

(See the proof of Proposition 40.9.) We note that τ_* induces an involution

$$\tau_* : \mathcal{M}_2^{\mathrm{reg}}(2^{-m}\beta) \to \mathcal{M}_2^{\mathrm{reg}}(2^{-m}\beta). \tag{41.4}$$

This involution induces an involution

$$\tau_*^{(m)} : \mathcal{M}_2^{\mathrm{reg},(m)}(\beta) \to \mathcal{M}_2^{\mathrm{reg},(m)}(\beta) \tag{41.5}$$

on $\mathcal{M}_2^{\mathrm{reg},(m)}(\beta)$, that is defined by the identity

$$\tau_*^{(m)} \circ \mathfrak{D}^m = \mathfrak{D}^m \circ \tau_*. \tag{41.6}$$

(41.5) is well-defined by Proposition 40.9. We remark that the well-definedness of $\tau_*^{(m)}$ strongly depends on the *uniqueness* statement on $\mathfrak{w}_0$ in Proposition 40.9.

Since $\mathcal{M}_2^{\mathrm{reg},(m+1)}(\beta)$ is the image under $\mathfrak{D}^m$ of the fixed point of (41.4), it follows from (41.3) that there exists an integer m such that $\tau_*^{(m)}\mathfrak{w}$ is well-defined and different from $\mathfrak{w}$.

The secondary involution $\tau_*^{(2)}$ looks like drawn as in Figures 41.1 and 41.2 below.

Figure 41.1.

Figure 41.2.

Now an idea to resolve the trouble coming from the presence of fixed points of τ_* can be summarized as follows. On the part where the involution $\tau_*^{(m)}$ is free, we can apply the proof of §35 and obtain a section $\mathfrak{s}^\epsilon$ that satisfies the conclusion of Theorem 34.11 and is invariant under $\tau_*^{(m)}$. Then we can use the cancellation argument of τ-symmetric pairs mentioned before to prove Theorems 34.16.

To carry out this plan, we first need to extend the involution $\tau_*^{(m)}$ to a (Kuranishi) neighborhood of $\mathcal{M}_2^{\mathrm{reg},(m)}(\beta)$.

Proposition 41.7. *$\mathcal{M}_2^{\mathrm{reg}}(\beta)$ has a Kuranishi structure,* (*which we call the ambient Kuranishi structure in this proposition to distinguish it from Kuranishi structures on various fixed point sets of involutions*)*, with the following property : For each*

m, there exists a Kuranishi neighborhood $U^{(m)}(\beta)$ of $\mathcal{M}_2^{\mathrm{reg},(m)}(\beta)$ in the ambient Kuranishi structure and an involution $\tau_*^{(m)}$ on it satisfying the following:

(41.8.1) $\tau_*^{(m)}$ *defines an involution of the space with Kuranishi structure.* (*See* §A1.3 *Definition* A1.45 *for the definition of the group action on a space with Kuranishi structure.*)
(41.8.2) *When we restrict $\tau_*^{(m)}$ to $\mathcal{M}_2^{\mathrm{reg},(m)}(\beta)$, it coincides with the one defined by* (41.6)*.*
(41.8.3) *For $m < \ell$, the set $U^{(m)}(\beta) \cap U^{(\ell)}(\beta)$ is invariant of both $\tau_*^{(m)}$ and $\tau_*^{(\ell)}$. On this set, we have :*

$$\tau_*^{(m)} \circ \tau_*^{(\ell)} = \tau_*^{(\ell)} \circ \tau_*^{(m)}.$$

(41.8.4) $ev \circ \tau_*^{(m)} = ev|_{\mathcal{M}_2^{\mathrm{reg},(m)}(\beta)}$. *Here $\tau_*^{(m)} : \mathcal{M}_2^{\mathrm{reg},(m)}(\beta) \to \mathcal{M}_2^{\mathrm{reg},(m)}(\beta)$ is the involution given in* (41.6) *by* (41.8.2) *and $ev : \mathcal{M}_2^{\mathrm{reg}}(\beta) \to L^2$ is the canonical evaluation map.*

We will prove Proposition 41.7 in the next section.

We next treat the case of $\mathcal{M}_1^{\mathrm{reg},(m)}(\beta)$. Consider the forgetful map

$$\mathfrak{forget} : \mathcal{M}_2^{\mathrm{reg}}(\beta) \to \mathcal{M}_1^{\mathrm{reg}}(\beta)$$

and put

$$\mathcal{M}_1^{\mathrm{reg},(m)}(\beta) = \mathfrak{forget}(\mathcal{M}_2^{\mathrm{reg},(m)}(\beta)). \tag{41.9}$$

We remark that $\mathfrak{forget} : \mathcal{M}_2^{\mathrm{reg},(m)}(\beta) \to \mathcal{M}_1^{\mathrm{reg},(m)}(\beta)$ is a homeomorphism by the uniqueness part of Lemma 40.10.

Proposition 41.10. *$\mathcal{M}_1^{\mathrm{reg}}(\beta)$ has a Kuranishi structure,* (*which we also call ambient Kuranishi structure as in Proposition* 41.7)*, with the following property : For each m, there exists a Kuranishi neighborhood $\mathcal{U}^{(m)}(\beta)$ of $\mathcal{M}_1^{\mathrm{reg},(m)}(\beta)$ in the ambient Kuranishi structure and an involution $\tau_*^{(m)}$ on it satisfying the following:*

(41.11.1) $\tau_*^{(m)}$ *can be regarded as an involution of the space with Kuranishi structure.*
(41.11.2) $\mathfrak{forget}(U^{(m)}(\beta)) \subseteq \mathcal{U}^{(m)}(\beta)$ *and* $\tau_*^{(m)} \circ \mathfrak{forget} = \mathfrak{forget} \circ \tau_*^{(m)}$.
(41.11.3) *For $m < \ell$, the set $\mathcal{U}^{(m)}(\beta) \cap \mathcal{U}^{(\ell)}(\beta)$ is invariant of both $\tau_*^{(m)}$ and $\tau_*^{(\ell)}$. On this set, we have :*

$$\tau_*^{(m)} \circ \tau_*^{(\ell)} = \tau_*^{(\ell)} \circ \tau_*^{(m)}.$$

(41.11.4) $ev \circ \tau_*^{(m)} = ev$. *Here $ev : \mathcal{M}_1^{\mathrm{reg}}(\beta) \to L$ is the evaluation map.*

The proof of Proposition 41.10 follows on the same line as that of Proposition 41.7 which will be given in the next section.

We now explain how we use Propositions 41.7 and 41.10 for the construction of a single-valued section of some Kuranishi structure on each of $\mathcal{M}_2^{\mathrm{reg},(m)}(\beta)$ and $\mathcal{M}_1^{\mathrm{reg},(m)}(\beta)$. In general, $\tau_*^{(m)}$ may have a fixed point in $U^{(m)}(\beta)$. We will take another Kuranishi neighborhood $U'^{(m)}(\beta)$ such that $\tau_*^{(m)}$ is fixed point free on $U'^{(m)}(\beta)$.

We will define the subset $U'^{(m)}(\beta) \subset U^{(m)}(\beta)$ by a downward induction on $m = 0, 1, 2, \cdots$ so that the following hold :

(41.12.1) $U'^{(m(\beta))}(\beta) = U^{(m(\beta))}(\beta)$.

(41.12.2) $U'^{(m)}(\beta)$ is a Kuranishi neighborhood of $\mathcal{M}_2^{\mathrm{reg},(m)}(\beta) \setminus \bigcup_{\ell > m} U'^{(\ell)}(\beta)$ and is invariant under $\tau_*^{(m)}$.

Obviously we have $\tau_*^{(0)} = \tau_*$ from definition and (41.12.2) implies that $\{U'^{(m)}(\beta)\}_{m=0}^{m(\beta)}$ covers $\mathcal{M}_2^{\mathrm{reg}}(\beta)$. By shrinking $U'^{(m)}(\beta)$ further if necessary, we can choose $U'^{(m)}(\beta)$ so that they satisfy the following additional conditions :

(41.12.3) $\tau_*^{(m)}$ is free on $U'^{(m)}(\beta)$.

(41.12.4) $U'^{(m)}(\beta) \cap U'^{(\ell)}(\beta)$ is invariant under $\tau_*^{(m)}$ and $\tau_*^{(\ell)}$.

We define $\mathcal{U}'^{(m)}(\beta) \subset \mathcal{U}^{(m)}(\beta)$ in a similar way.

Let P be a chain on L. By taking the fiber product $U'^{(m)}(\beta)\ {}_{ev_1}\times_L P$ we obtain $U'^{(m)}(\beta; P) \subset \mathcal{M}_2^{\mathrm{reg}}(\beta; P)$. It follows from (41.8.4) that $\tau_*^{(m)}$ induces an involution on $U'^{(m)}(\beta; P)$, which we also denote by $\tau_*^{(m)}$.

Proposition 41.13. *There exists a family of* **single valued** *sections* $\mathfrak{s}^\epsilon$ *of the obstruction bundles on* $\mathcal{M}_1^{\mathrm{reg}}(\beta)$ *and on* $\mathcal{M}_2^{\mathrm{reg}}(\beta)$. *The sections* $\mathfrak{s}^\epsilon$ *are transversal to zero and satisfy the following property :*

(41.14) *The sections* $\mathfrak{s}^\epsilon$ *are invariant under* $\tau_*^{(m)}$ *on* $U'^{(m)}(\beta; P)$ *and* $\mathcal{U}'^{(m)}(\beta)$ *respectively.*

Before proving Proposition 41.13, we explain how we will use this proposition in the proof of Theorem 34.17.

We use the sections $\mathfrak{s}^\epsilon$ provided in Proposition 41.13 to define a filtered A_∞ structure of L (over $\mathbb{Z}_2$). Let $\beta \neq \beta_0 = 0$. Then we prove

$$(ev_0)_*(\mathcal{M}_2^{\mathrm{reg}}(\beta; P)^{\mathfrak{s}^\epsilon}) = 0, \tag{41.15.1}$$

$$(ev_0)_*(\mathcal{M}_1^{\mathrm{reg}}(\beta)^{\mathfrak{s}^\epsilon}) = 0 \tag{41.15.2}$$

as $\mathbb{Z}_2$-chains. To prove (41.15.1), we use (41.14). We consider

$$(ev_0)_*(\mathcal{M}_2^{\mathrm{reg}}(\beta; P)^{\mathfrak{s}^\epsilon} \cap U'^{(m)}(\beta; P)), \quad m = 0, 1, 2, \cdots . \tag{41.16}$$

It follows from (41.14), (41.12.3) and (41.8.4) that (41.16) is zero as a $\mathbb{Z}_2$-chain. Since $\cup_m U'^{(m)}(\beta; P) \supset \mathcal{M}_2^{\mathrm{reg}}(\beta; P)$, (41.15.1) follows. The proof of (41.15.2) is similar. To finish the proof of Theorem 34.17, we need to study the *singular* part of the moduli spaces, i.e., the strata consisting of stable maps whose domains are not the smooth disc D^2. This will be carried out in §43.

Proof of Proposition 41.13. Take the open subsets (in the Kuranishi neighborhood)

$$U'^{(m)}(\beta;P) = U_0^{(m)}(\beta;P) \subset U_1^{(m)}(\beta;P) \subset \cdots \subset U_{2m(\beta)+2}^{(m)}(\beta;P) \subset U^{(m)}(\beta;P)$$

so that

(41.17.1) The closure of $U_i^{(m)}(\beta;P)$ is in the interior of $U_{i+1}^{(m)}(\beta;P)$.
(41.17.2) $U_i^{(m)}(\beta;P) \cap U_i^{(\ell)}(\beta;P)$ is invariant under the actions of $\tau_*^{(m)}$ and $\tau_*^{(\ell)}$.
(41.17.3) The action of $\tau_*^{(m)}$ is free on $U_{2m(\beta)+2}^{(m)}(\beta;P)$.

Let $\mathfrak{I}$ be the set of all $\vec{m} = (m_1, \cdots, m_\ell)$ with $0 < m_1 < \cdots < m_\ell \le m(\beta)$. We put $\ell = |\vec{m}|$. For $\vec{m} \in \mathfrak{I}$, we define

$$W_{\vec{m}}(\beta;P) = \bigcap_{j=1,\cdots,|\vec{m}|} U_{2+2|\vec{m}|}^{(m_j)}(\beta;P). \tag{41.18}$$

Note that $\{W_{\vec{m}}(\beta;P) \mid \vec{m} \in \mathfrak{I}\}$ forms a covering of $\mathcal{M}_2^{\text{reg}}(\beta;P)$.

Lemma 41.19. *For each $\vec{m} \in \mathfrak{I}$, there exists a section $\mathfrak{s}_{\vec{m}}^{\epsilon}$ of the obstruction bundle on $W_{\vec{m}}$ with the following properties.*

(41.20.1) $\mathfrak{s}_{\vec{m}}^{\epsilon}$ *is invariant under* $\tau_*^{(m_j)}$ $(j = 1, \cdots, |\vec{m}|)$.
(41.20.2) $\mathfrak{s}_{\vec{m}}^{\epsilon}$ *is transversal to zero.*

Proof. We first observe that, on $\mathcal{M}_{k+1}^{\text{reg}}(\beta)$, the automorphism groups (that is the group Γ_p appeared is the definition of Kuranishi structure (Definition A1.1)) is always trivial.

We take the quotient by the (free) $\mathbb{Z}_2^{|\vec{m}|}$ action generated by $\tau_*^{(m_j)}$ $(j = 1, \cdots, |\vec{m}|)$ on $W_{\vec{m}}$ and obtain a Kuranishi structure on the quotient space. (See §A1.3 Lemma A1.49.) The automorphism group is still trivial. Hence we obtain a perturbation of the Kuranishi map transversal to zero. We then lift it to $W_{\vec{m}}$ to obtain desired $\mathfrak{s}_{\vec{m}}^{\epsilon}$. □

We next glue the sections $\mathfrak{s}_{\vec{m}}^{\epsilon}$ to obtain the section $\mathfrak{s}^{\epsilon}$ required in Proposition 41.13. For $\vec{m} = (m_1, \cdots, m_\ell)$, $\vec{m}' = (m_1', \cdots, m_{\ell'}')\in \mathfrak{I}$, we say

$$\vec{m} < \vec{m}' \quad \text{if } \{m_1, \cdots, m_\ell\} \subset \{m_1', \cdots, m_{\ell'}'\}.$$

We denote

$$U_{\vec{m}}(\beta;P) = W_{\vec{m}}(\beta;P) \setminus \bigcup_{\vec{m}'>\vec{m}} \bigcap_{j=1,\cdots,|\vec{m}'|} \overline{U}_{1+2|\vec{m}'|}^{(m_j')}(\beta;P), \tag{41.21}$$

where $\overline{U}_{1+2|\vec{m}'|}^{(m_j')}(\beta;P)$ is the closure of $U_{1+2|\vec{m}'|}^{(m_j')}(\beta;P)$.

Lemma 41.22. *The collection* $\{U_{\vec m}(\beta;P) \mid \vec m \in \mathfrak{I}\}$ *covers* $\mathcal{M}_2^{\rm reg}(\beta;P)$.

Proof. We first remark

$$W_{\vec m}(\beta;P) \subseteq U_{\vec m}(\beta;P) \cup \bigcup_{\vec m' > \vec m} W_{\vec m'}(\beta;P). \tag{41.23}$$

In fact, by (41.17.1) and (41.18), we have

$$\bigcap_{j=1,\cdots,|\vec m'|} \overline{U}^{(m'_j)}_{1+2|\vec m'|}(\beta;P) \subset \bigcap_{j=1,\cdots,|\vec m'|} U^{(m'_j)}_{2+2|\vec m'|}(\beta;P) = W_{\vec m'}(\beta;P).$$

The inclusion (41.23) then follows from (41.21). Using (41.23) and a downward induction on $|\vec m|$, we can prove

$$\bigcup_{\vec m' \ge \vec m} W_{\vec m'}(\beta;P) \subseteq \bigcup_{\vec m' \ge \vec m} U_{\vec m'}(\beta;P).$$

Since $\{W_{\vec m}(\beta;P) \mid \vec m \in \mathfrak{I}\}$ covers $\mathcal{M}_2^{\rm reg}(\beta;P)$, this proves the lemma. □

Lemma 41.24. *If* $U_{\vec m}(\beta;P) \cap U'^{(m)}(\beta;P) \neq \emptyset$, *then* $\vec m$ *contains* m.

Proof. We assume $m \notin \vec m$. We put $\vec m^+ = \vec m \cup \{m\}$. Then $\vec m^+ > \vec m$. Therefore (41.21) implies

$$U_{\vec m}(\beta;P) \cap \bigcap_{j=1,\cdots,|\vec m^+|} \overline{U}^{(m^+_j)}_{1+2|\vec m^+|}(\beta;P) = \emptyset. \tag{41.25}$$

We remark that

$$\begin{aligned}\bigcap_{j=1,\cdots,|\vec m^+|} \overline{U}^{(m^+_j)}_{1+2|\vec m^+|}(\beta;P) &= \bigcap_{j=1,\cdots,|\vec m|} \overline{U}^{(m_j)}_{1+2|\vec m|+2}(\beta;P) \cap U^{(m)}_{1+2|\vec m|+2}(\beta;P) \\ &\supseteq W_{\vec m}(\beta;P) \cap U'^{(m)}(\beta;P) \\ &\supseteq U_{\vec m}(\beta;P) \cap U'^{(m)}(\beta;P).\end{aligned}$$

Then using (41.25), we have obtained $U_{\vec m}(\beta;P) \cap U'^{(m)}(\beta;P) = \emptyset$. This contradicts the hypothesis and hence the proof. □

Now, we take a partitions of unity $\{(\chi_{\vec m} : U_{\vec m}(\beta;P)) \mid \vec m \in \mathfrak{I}\}$ subordinate to the covering $\{U_{\vec m}(\beta;P) \mid \vec m \in \mathfrak{I}\}$ of $\mathcal{M}_2^{\rm reg}(\beta;P)$ so that $\chi_{\vec m}$ is invariant under a series of involutions $\tau_*^{(m_j)}$ and define

$$\mathfrak{s}^\epsilon = \sum \chi_{\vec m} \mathfrak{s}^\epsilon_{\vec m}.$$

Lemma 41.26. *$\mathfrak{s}^\epsilon$ is invariant under $\tau_*^{(m)}$ on $U'^{(m)}(\beta; P)$.*

Proof. Lemma 41.24 implies that if $\chi_{\vec m}$ is nonzero at some point of $U'^{(m)}(\beta; P)$ then $\mathfrak{s}^\epsilon_{\vec m}$ is $\tau_*^{(m)}$ invariant. The lemma follows immediately. □

By the argument used in the proof of Lemma 41.19, we can also prove that for a generic choice of $\mathfrak{s}^\epsilon_{\vec m}$, the section $\mathfrak{s}^\epsilon$ is transversal to zero. Thus $\mathfrak{s}^\epsilon$ has the required properties. The proof for the case $\mathcal{M}_1^{\mathrm{reg},(m)}(\beta)$ is similar, and so omitted. Now the proof of Proposition 41.13 is complete. □

§42. Family index and extension of symmetry to a Kuranishi neighborhood.

The purpose of this section is to prove Propositions 41.7 and 41.10. We first reduce the problem to a problem of family index. To make more transparent the problem for us to deal with, we describe the problem in the general context of spaces with Kuranishi structure.

We study a space X with Kuranishi structure on which the group $\mathbb{Z}_2$ acts by the involution τ. Let $F \subset X$ be the fixed point set of τ. For each $x \in F \subset X$ we take its Kuranishi neighborhood (V, E, Γ, ψ, s) in X. Let $V_F \subset V$, $E_F \subset E$, $\Gamma_F \subset \Gamma$ be the fixed point set of $\mathbb{Z}_2$ actions respectively. By the definition of the action of a group on a space with Kuranishi structure (see Definition A1.45), $(V_F, E_F, \Gamma_F, \psi, s)$ defines a Kuranishi structure on F. Let τ' be another $\mathbb{Z}_2$ action on this space F with Kuranishi structure. We are looking for the condition under which this second $\mathbb{Z}_2$ action can be extended to a Kuranishi neighborhood of F in X.

For $x \in F$, we consider the *fiber derivative* of s, which we denote by $d_x s : T_x V \to E_x$. If E is the trivial bundle $E = V \times E_0$ as in Definition A1.1 where E_0 denotes a typical fiber, the fiber derivative is nothing but the ordinary derivative of the E_0-component of $s : V \to V \times E_0$. With an abuse of notation, we will write the corresponding map $T_x V \to E_0$ by the same notation $d_x s : T_x V \to E_0$.

The fiber derivative $d_x s$ gives rise to the following diagram

$$\begin{array}{ccc} T_x V_F & \xrightarrow{d_x s} & (E_F)_x \\ \downarrow & & \downarrow \\ T_x V & \xrightarrow{d_x s} & E_x \end{array}$$

Diagram 42.1.

Here the vertical arrows are the obvious inclusions. When we move x, both of $d_x s : T_x V_F \to (E_F)_x$ and $d_x s : T_x V \to E_x$ can be regarded as bundle systems. (See Definition 5.10 [FuOn99II] or §A1.1 Definition A1.33 for the definition of bundle system). Diagram 42.1 may be regarded as an inclusion between bundle systems. We remark that we have the action of τ' on the bundle system $d_x s : T_x V_F \to (E_F)_x$ and that of τ on $d_x s : T_x V \to E_x$. Moreover restriction of the τ-action to the bundle system $d_x s : T_x V_F \to (E_F)_x$ is trivial.

Proposition 42.1. *We assume that the τ'-action is lifted to the bundle system $\{d_x s : T_x V \to E_x\}_{x\in V}$ so that the lifted action covers the τ' action on the base F and that the lifted τ'-action commutes with the τ-action on $d_x s : T_x V \to E_x$, $x \in F$. Then the τ'-action on F extends to its Kuranishi neighborhood so that the extension commutes with the τ-action on it.*

We prove a more general statement (Proposition 42.5) later.

In Proposition 42.1, we consider the case where we have only two actions. To deal with the case of Proposition 41.7, we need to consider an arbitrary finite number of actions. We now describe the general setting which enables us to study the general case in a systematic way.

First, let X be a space with Kuranishi structure acted by an involution τ_0. We call the Kuranishi structure on X the *ambient Kuranishi structure* in this section to distinguish it from Kuranishi structures on various fixed point sets of involutions. Denote by F_0 the fixed point set of τ_0,

$$F_0 = \operatorname{Fix} \tau_0.$$

At each $x \in F_0$, we consider the τ_0-invariant part of its Kuranishi neighborhood $(V_x, E_x, \Gamma_x, s_x)$ which we denote by

$$(V_x^{\tau_0}, E_x^{\tau_0}, \Gamma_x^{\tau_0}, s_x).$$

This defines a Kuranishi structure on F_0.

Next, suppose that we are given an involution τ_1 on F_0 in the sense of Definition A1.45. We denote by F_1 be the fixed point set of τ_1,

$$F_1 = \operatorname{Fix} \tau_1.$$

At each $x \in F_1$, the above defined Kuranishi structure on F_0 provides a Kuranishi neighborhood $(V_x^{\tau_0}, E_x^{\tau_0}, \Gamma_x^{\tau_0}, s_x)$ of $x \in F_0$. Since τ_1 is an involution on F_0 regarded as a space with Kuranishi structure, it induces an involution on $(V_x^{\tau_0}, E_x^{\tau_0}, \Gamma_x^{\tau_0}, s_x)$. We denote by

$$(V_x^{\tau_0,\tau_1}, E_x^{\tau_0,\tau_1}, \Gamma_x^{\tau_0,\tau_1}, s_x),$$

the fixed point part thereof under the τ_1-action, which then define a Kuranishi structure on F_1.

Next we assume that there exists an involution τ_2 on F_1 as above, and put $F_2 = \mathrm{Fix}\,\tau_2$. Then we similarly obtain a Kuranishi neighborhood $(V_x^{\tau_0,\tau_1}, E_x^{\tau_0,\tau_1}, \Gamma_x^{\tau_0,\tau_1}, s_x)$ of F_1 at each $x \in F_2 \subset F_1$. By considering the τ_2-fixed point set of this Kuranishi neighborhood, which we denote by

$$(V_x^{\tau_0,\tau_1,\tau_2}, E_x^{\tau_0,\tau_1,\tau_2}, \Gamma_x^{\tau_0,\tau_1,\tau_2}, s_x),$$

we obtain a Kuranishi structure on F_2. We inductively impose the hypotheses that there exist a sequence of involutions τ_m defined on F_{m-1} for $m = 0, \cdots, m_0$ where $F_{-1} = X$ and $F_{m-1} = \mathrm{Fix}\,\tau_{m-1}$ for $m \geq 1$.

In summary, we have defined a Kuranishi structure on F_m assuming the existence of an involution τ_m on F_{m-1} in the sense of Definition A1.45. Furthermore we assume that the action of τ_{m_0} on F_{m_0-1} is free.

In this situation, we would like to extend the involution τ_m on F_{m-1} to its Kuranishi neighborhood in X. Proposition 42.5 below provides a sufficient condition for this extension to be possible.

To describe this condition we need to prepare with some more notations. Let $x \in F_m$ and assume that for each $m' \leq m$ and $x \in F_m$ a Kuranishi neighborhood $(V_x^{\tau_0,\cdots,\tau_{m'}}, E_x^{\tau_0,\cdots,\tau_{m'}}, \Gamma_x^{\tau_0,\cdots,\tau_{m'}}, s_x)$ of x on $F_{m'}$ has been defined. We consider

$$d_x s_x : T_x V_x^{\tau_0,\cdots,\tau_{m'}} \to (E_x^{\tau_0,\cdots,\tau_{m'}})_x. \tag{42.2}$$

Varying $x \in F_m$, the maps (42.2) over $x \in F_m$ define a bundle system on F_m. We denote this bundle system by $\mathrm{Tan}_{F_m}^{(m')}$ for each $m' \in \{-1, 0, \cdots, m\}$. Here Tan stands for tangential complex and $\mathrm{Tan}_{F_m}^{(m')}$ for $m' = -1$ stands for nothing but the restriction of the initial bundle system $d_x s : T_x V_x \to (E_x)_x$ on X.

For given $m'' < m'$ with $m'' < m' \leq m$, we have the obvious inclusion of bundle systems $\mathrm{Tan}_{F_m}^{(m')} \to \mathrm{Tan}_{F_m}^{(m'')}$ i.e., the commutative diagram

$$\begin{array}{ccc} T_x V_x^{\tau_0,\cdots,\tau_{m'}} & \xrightarrow{d_x s} & (E_x^{\tau_0,\cdots,\tau_{m'}})_x \\ \downarrow & & \downarrow \\ T_x V_x^{\tau_0,\cdots,\tau_{m''}} & \xrightarrow{d_x s} & (E_x^{\tau_0,\cdots,\tau_{m''}})_x \end{array}$$

Diagram 42.2.

We consider the group $\mathbb{Z}_2^{m+2}$ and write its generators $\tau_0, \tau_1, \cdots, \tau_{m+1}$, i.e.,

$$\tau_i = (1, \cdots, 1, \overset{i+1}{-1}, 1, \cdots, 1).$$

The group $\mathbb{Z}_2^{m+2}$ acts on F_m in the way that the action of τ_i is trivial for $i = 0, 1, \cdots, m$. Recall that τ_{m+1} is an involution defined on F_m and that we put

$$F_{m+1} := \mathrm{Fix}\,\tau_{m+1}.$$

Assumption 42.3. The action of $\mathbb{Z}_2^{m+2}$ on F_m lifts to an action to $\mathrm{Tan}_{F_m}^{(m')}$ for each $m' = 0, \cdots, m$. This action satisfies the following properties :

(42.4.1) If $m'' \le m'$, then the action of $\tau_{m''}$ on $\mathrm{Tan}_{F_m}^{(m')}$ is trivial.

(42.4.2) The action of $\tau_{m'+1}$ on $\mathrm{Tan}_{F_m}^{(m')}$ coincides with one induced by $\tau_{m'+1}$ action on $F_{m'}$.

(42.4.3) The embedding $\mathrm{Tan}_{F_m}^{(m')} \to \mathrm{Tan}_{F_m}^{(m'')}$ is $\mathbb{Z}_2^{m+2}$ equivariant for $m'' \le m'$.

Now we are ready to state a generalization of Proposition 42.1 we need.

Proposition 42.5. *Let $\tau_0, \cdots, \tau_{m+1}$ be a sequence of involutions given as above that satisfy Assumption* 42.3. *Then, we can perturb the Kuranishi map $s : V \to E$, so that, for each m, there exists a Kuranishi neighborhood $U_m = (V_m, E_m, \Gamma_m, s_m)$ of F_m in the ambient Kuranishi structure with the following properties :*

(1) *The $\mathbb{Z}_2$-action on F_m via τ_{m+1} extends to U_m as an action on a space with Kuranishi structure.*

(2) *For $m', m'' \le m$, the intersection $U_{m'} \cap U_{m''}$ is invariant under the actions of both $\tau_{m'}$ and $\tau_{m''}$. Moreover $\tau_{m'}$ commutes with $\tau_{m''}$ on $U_{m'} \cap U_{m''}$.*

Proof. At $x \in F_m$, we have the decomposition

$$T_x V \cong \bigoplus_{i=0}^{m} \frac{T_x V_x^{\tau_0, \cdots, \tau_{i-1}}}{T_x V_x^{\tau_0, \cdots, \tau_i}} \oplus T_x V_x^{\tau_0, \cdots, \tau_m}.$$

Therefore we have the isomorphism

$$(N_{V_x^{\tau_0, \cdots, \tau_m}} V_x)_x \cong \bigoplus_{i=0}^{m} \frac{T_x V_x^{\tau_0, \cdots, \tau_{i-1}}}{T_x V_x^{\tau_0, \cdots, \tau_i}} \tag{42.6}$$

where $N_{V_x^{\tau_0, \cdots, \tau_m}} V_x$ is the normal bundle of $V_x^{\tau_0, \cdots, \tau_m}$ in V_x with respect to the Kuranishi structure of F_m. It follows from Definition A1.45 that we are given a τ_{m+1}-action on $V_x^{\tau_0, \cdots, \tau_m}$ since τ_{m+1} acts on F_m and $V_x^{\tau_0, \cdots, \tau_m}$ is a Kuranishi neighborhood of x.

The $\mathbb{Z}_2^{m+2}$-action on the tangential complex in Assumption 42.3 canonically induces a $\mathbb{Z}_2^{m+2}$-action on $N_{V_x^{\tau_0, \cdots, \tau_m}} V_x$ via (42.6).

Therefore an identification of a neighborhood of $V_x^{\tau_0, \cdots, \tau_{m'}}$ in V_x with that of the zero section of the normal bundle $N_{V_x^{\tau_0, \cdots, \tau_m}} V_x$, induces a $\mathbb{Z}_2^{m+2}$-action on a neighborhood of $V_x^{\tau_0, \cdots, \tau_m}$ in V_x. Then (42.4) implies that this action can be glued and the glued action is one that has the required properties. Moreover we can lift the $\mathbb{Z}_2^{m+2}$-action to the obstruction bundle E since existence of such a lift is assumed in Assumption 42.3. Finally we perturb the Kuranishi map s. Take a small open neighborhood $\mathfrak{U}_m$ of F_m for each m. By choosing another neighborhood $\mathfrak{V}_m$ if necessary, we may assume that

$$\bigcup_{m=0}^{m_0} \mathfrak{V}_m \supset X,$$

$$\mathfrak{V}_m \cap F_{m'} = \emptyset \quad \text{if } m' > m,$$

and that there exists a $\mathbb{Z}_2^{m+2}$ action on $\mathfrak{V}_m$. Let χ_m be a $\mathbb{Z}_2^{m+2}$-invariant partition of unity of X subordinate to the covering $\{\mathfrak{V}_m\}$. We then define

$$s^{(m)} = 2^{-(m+2)} \sum_{\vec\rho \in \mathbb{Z}_2^{m+2}} \rho \circ s \circ \rho, \quad s' = \sum \chi_m s^{(m)}.$$

It is easy to see that s' defines a Kuranishi structure on its zero point set and also it is $\mathbb{Z}_2^{m+2}$ invariant in the neighborhood of F_m. □

Now we restrict our study to the case considered in Proposition 41.7. We first explain how the case considered in Proposition 41.7 can be put into the context of Assumption 42.3 and its proof is reduced to that of Proposition 42.5 : The space $\mathcal{M}_2^{\rm reg}(\beta)$ is our X. The first involution on it is $\tau_0 = \tau_*$. Its fixed point set F_0 is $\mathcal{M}_2^{{\rm reg},(1)}(\beta)$ which is the image of $\mathfrak{D}$ on $\mathcal{M}_2^{\rm reg}(\beta')$ with $\beta' + \tau_*\beta' = \beta$. Let $\mathbf{p} \in \mathcal{M}_2^{\rm reg}(\beta')$ and let

$$(U_{\mathfrak{D}(\mathbf{p})}, E_{\mathfrak{D}(\mathbf{p})}, \Gamma_{\mathfrak{D}(\mathbf{p})}, s_{\mathfrak{D}(\mathbf{p})})$$

be a Kuranishi neighborhood of $\mathfrak{D}(\mathbf{p})$ in $\mathcal{M}_2^{\rm reg}(\beta)$. Denote by $U^{\tau_0}_{\mathfrak{D}(\mathbf{p})}$, $E^{\tau_0}_{\mathfrak{D}(\mathbf{p})}$, $\Gamma^{\tau_0}_{\mathfrak{D}(\mathbf{p})}$ the fixed point set of τ_0 acting on the Kuranishi neighborhood. Then the Kuranishi structure of F_0 at $\mathfrak{D}(\mathbf{p})$ is defined by the collection of Kuranishi neighborhoods

$$(U^{\tau_0}_{\mathfrak{D}(\mathbf{p})}, E^{\tau_0}_{\mathfrak{D}(\mathbf{p})}, \Gamma^{\tau_0}_{\mathfrak{D}(\mathbf{p})}, s_{\mathfrak{D}(\mathbf{p})})$$

of $\mathfrak{D}(\mathbf{p})$ for $\mathbf{p} \in F_0$.

Lemma 42.7. *The Kuranishi structure on $F_0 = \mathcal{M}_2^{{\rm reg},(1)}(\beta)$ defined above is identified with the Kuranishi structure on $\mathcal{M}_2^{\rm reg}(2^{-1}\beta)$ by $\mathfrak{D}$.*

Proof. The proof is immediate from construction. □

Now the involution τ_* on $\mathcal{M}_2^{\rm reg}(2^{-1}\beta)$ defines an involution $\tau_1 = \tau_*^{(1)}$ on $F_0 = \mathcal{M}_2^{{\rm reg},(1)}(\beta)$. Its fixed point set is $F_1 = \mathcal{M}_2^{{\rm reg},(2)}(\beta)$. We can continue and identify $F_{m-1} = \mathcal{M}_2^{{\rm reg},(m)}(\beta)$. We also have :

Lemma 42.8. *The Kuranishi structure on $F_{m-1} = \mathcal{M}_2^{\mathrm{reg},(m)}(\beta)$ defined above is identified with the Kuranishi structure on $\mathcal{M}_2^{\mathrm{reg}}(2^{-m}\beta)$ by $\mathfrak{D}^m$.*

The proof is immediate from Lemma 42.7.

Thus we are in the situation of Proposition 42.5. Therefore for the proof of Proposition 41.7, it suffices to check Assumption 42.3 for the case considered. Namely we need to define a lift of $\mathbb{Z}_2^{m+1}$ action to the tangential complex $\mathrm{Tan}^{(m')}_{F_{m-1}}$ for $m' \le m-1$ satisfying (42.4). (*Note that since*

$$\mathcal{M}_2^{\mathrm{reg},(m)}(\beta) = F_{m-1},$$

we adopt the situation in Assumption 42.3 for the case of $m-1$.)

Consider the strip

$$\Sigma = \{z \in \mathbb{C} \mid |\mathrm{Im} z| \le 1\}$$

which we know is conformally isomorphic to $D^2 \setminus \{\pm 1\}$. We use the standard coordinate $z = s + \sqrt{-1}t$, $|t| \le 1$ on $\Sigma \subset \mathbb{C}$. Note that $(z_0, z_1) = (1, -1)$ corresponds to $(\infty, -\infty)$. (In this and the next sections we use s in place of τ as the coordinate of the first factor in order to avoid a confusion with the involution $\tau : M \to M$.)

Let $w \in \mathcal{M}_2^{\mathrm{reg}}(\beta)$ and regard it as a map from Σ. Now we will construct a $\mathbb{Z}_2^{m+1}$-action in a Kuranishi neighborhood of

$$\mathfrak{D}^m([((\Sigma, (\infty, -\infty)), w)]).$$

Let

$$\Sigma_\pm = \{z \in \partial\Sigma \mid \pm\mathrm{Im} z \ge 0\}, \quad \partial_\pm\Sigma = \partial\Sigma \cap \Sigma_\pm.$$

We define $I_\pm : \Sigma \to \Sigma_\pm$ as follows

$$I_-(z) = \frac{z - \sqrt{-1}}{2}, \qquad I_+(z) = \overline{I_-(z)}. \tag{42.9.1}$$

I_- is holomorphic and I_+ is anti-holomorphic.

Let $w : \Sigma \to M$ be a pseudo-holomorphic map with $w(\partial\Sigma) \subset L$. We define $\mathfrak{D}(w) : \Sigma \to M$ by

$$\mathfrak{D}(w)(z) = \begin{cases} w(I_-^{-1}(z)) = w(2z + \sqrt{-1}) & \text{if } \mathrm{Im}\ z \le 0, \\ \tau(w(I_+^{-1}(z))) = \tau(w(2\bar{z} + \sqrt{-1})) & \text{if } \mathrm{Im}\ z \ge 0. \end{cases} \tag{42.9.2}$$

It is easy to see that

$$[((\Sigma, (\infty, -\infty)), \mathfrak{D}(w))] = \mathfrak{D}([((\Sigma, (\infty, -\infty)), w)]).$$

We consider the deformation complex of w in our moduli space. It is induced by the operator defined in (29.9). We recall its definition in the current context. See Lemma 29.5. Let $\delta > 0$ and let $|\cdot|'$ be a smooth function $: \mathbb{R} \to [0, \infty)$ such that $|s|' = |s|$ outside a compact set.

Definition 42.10. We consider triples $(V,(v_{+\infty},v_{-\infty}))$ such that

(42.11.1) V is a section of $w^*(TM)$ of $W^{1,p}_{loc}$-class.
(42.11.2) $v_{\pm\infty} \in T_{w(\pm\infty)}L$ respectively.
(42.11.3)

$$\int_0^{+\infty}\int_{-1}^{1} e^{\delta|s|'}(|\nabla(V-\mathrm{Pal}\,v_{+\infty})|^p + |V-\mathrm{Pal}\,v_{+\infty}|^p)\;dsdt < \infty$$
$$\int_{-\infty}^{0}\int_{-1}^{1} e^{\delta|s|'}(|\nabla(V-\mathrm{Pal}\,v_{-\infty})|^p + |V-\mathrm{Pal}\,v_{-\infty}|^p)\;dsdt < \infty.$$

Here $\mathrm{Pal}: T_{w(\pm\infty)}M \to T_{w(s,t)}M$ stands for the parallel transport.
(42.11.4) $V(s,-1) \in T_{w(s,-1)}L$.
(42.11.5) $V(s,1) \in T_{w(s,1)}L$.

We remark that $v_{\pm\infty} \in T_{w(\pm\infty)}L$ satisfying (42.11.3) are determined by V above. We denote by

$$C^0_{\delta,p}(\Sigma,w)$$

the set of such $(V,(v_{+\infty},v_{-\infty}))$'s. The p-th power $|(V,(v_{+\infty},v_{-\infty}))|^p$ of the $L^{1,p}$ norm of $(V,(v_{+\infty},v_{-\infty}))$ is, by definition, the sum of the two terms in left hand side of (42.11.3) and $|v_{-\infty}|^p + |v_{+\infty}|^p$. Equipped with this norm, $C^0_{\delta,p}(\Sigma,w)$ then becomes a Banach space.

We define the Banach space

$$W^{1,p}_{\delta}(\Sigma,\partial\Sigma;\,w^*TM, w|^*_{\partial\Sigma}TL) = \{V \mid (V,(0,0)) \in C^0_{\delta,p}(\Sigma,w)\}.$$

Next we define the Banach space

$$W^{1,p}_{\delta}(\Sigma,\partial_-\Sigma,\partial_+\Sigma;\,w^*TM, w|^*_{\partial_-\Sigma}TL, Jw|^*_{\partial_+\Sigma}TL)$$

to be the set of all V such that $(V,(0,0))$ satisfies (42.11.1)-(42.11.4) with (42.11.5) replaced by the following condition

(42.11.5') $V(s,1) \in JT_{w(s,1)}L$.

Finally we denote

$$C^1_{\delta,p}(\Sigma,w) = W^{0,p}_{\delta}(\Sigma;\Lambda^{0,1}\otimes w^*TM)$$

which is the set of all sections V of $\Lambda^{0,1}(\Sigma)\otimes w^*TM$ of L^p_{loc}-class that satisfy

$$\int_{-\infty}^{+\infty}\int_{-1}^{1} e^{\delta|s|'}|V|^p\;dsdt < \infty.$$

The formal linearization $D_w\overline{\partial}$ induces the following Fredholm operators

$$\begin{aligned}
&D_w\overline{\partial} : C^0_{\delta,p}(\Sigma, w) \to C^1_{\delta,p}(\Sigma, w),\\
&D^0_w\overline{\partial} : W^{1,p}_\delta(\Sigma, \partial\Sigma;\, w^*TM, w|^*_{\partial\Sigma}TL) \to C^1_{\delta,p}(\Sigma, w),\\
&D^J_w\overline{\partial} : W^{1,p}_\delta(\Sigma, \partial_-\Sigma, \partial_+\Sigma;\, w^*TM, w|^*_{\partial_-\Sigma}TL, J\cdot w|^*_{\partial_+\Sigma}TL) \to C^1_{\delta,p}(\Sigma, w).
\end{aligned}$$

As we discussed in §29, the operator $D_w\overline{\partial}$ defines the tangential complex

$$(C^*_{\delta,p}(\Sigma, w), D_w\overline{\partial})$$

of the deformation of $w \in \mathcal{M}^{\mathrm{reg}}_2(\beta)$ whose index of $D_w\overline{\partial}$ is given by $\mu_L(\beta) + n$.

Now we firstly define a $\mathbb{Z}_2^k$-action on the tangential complex

$$(C^*_{\delta,p}(\Sigma, \mathfrak{D}^k(w)), D_{\mathfrak{D}^k(w)}\overline{\partial}).$$

After that we will define one more $\mathbb{Z}_2$-action which commutes with the $\mathbb{Z}_2^k$-action.

For each section V of w^*TM satisfying (42.11.4), (42.11.5), we define

$$(\mathfrak{I}'^{,0}_+(V))(z) = \begin{cases} V(I_-^{-1}(z)) = V(2z+\sqrt{-1}) & z \in \Sigma_- \\ (T\tau)^{-1}(V(I_+^{-1}(z)) = (T\tau)^{-1}(V(2\bar{z}+\sqrt{-1}) & z \in \Sigma_+, \end{cases} \tag{42.12.1}$$

and for one satisfying (42.11.4), (42.11.5'), we define

$$(\mathfrak{I}'^{,0}_-(V))(z) = \begin{cases} V(I_-^{-1}(z)) & z \in \Sigma_- \\ -(T\tau)^{-1}(V(I_+^{-1}(z)) & z \in \Sigma_+. \end{cases} \tag{42.12.2}$$

Here we denote by $T\tau$ the differential of τ and see (42.9.1) for the definition of the maps $I_\pm$.

These give rise to the linear maps

$$\begin{aligned}
&\mathfrak{I}'^{,0}_+ : C^0_{\delta,p}(\Sigma, w) \to C^0_{\delta,p}(\Sigma, \mathfrak{D}(w))\\
&\mathfrak{I}'^{,0}_- : W^{1,p}_\delta(\Sigma, \partial_-\Sigma, \partial_+\Sigma;\, w^*TM, w|^*_{\partial_-\Sigma}TL, Jw|^*_{\partial_+\Sigma}TL) \to C^0_{\delta,p}(\Sigma, \mathfrak{D}(w))
\end{aligned}$$

defined by

$$\begin{aligned}
\mathfrak{I}'^{,0}_+(V, (v_{+\infty}, v_{-\infty})) &= (\mathfrak{I}'^{,0}_+(V), (v_{+\infty}, v_{-\infty})),\\
\mathfrak{I}'^{,0}_-(V, (0,0)) &= (\mathfrak{I}'^{,0}_-(V), (0,0)).
\end{aligned}$$

Similarly we also define the maps $\mathfrak{I}'^{,1}_\pm : C^1_{\delta,p}(\Sigma, w) \to C^1_{\delta,p}(\Sigma, \mathfrak{D}(w))$ by the formula

$$(\mathfrak{I}'^{,1}_+(V))(z) = \begin{cases} V(I_-^{-1}(z)) & z \in \Sigma_- \\ (T\tau)^{-1}(V(I_+^{-1}(z)) & z \in \Sigma_+, \end{cases} \tag{42.12.3}$$

and then $(\mathfrak{I}'^{,1}_{-}(V))$ by

$$(\mathfrak{I}'^{,1}_{-}(V))(z) = \begin{cases} V(I_-^{-1}(z)) \circ (TI_-)^{-1} & z \in \Sigma_- \\ -(T\tau)^{-1} \circ (V(I_+^{-1}(z)) \circ (TI_+)^{-1} & z \in \Sigma_+. \end{cases} \tag{42.12.4}$$

($TI_\pm$ is the differential of $I_\pm$ respectively.)

Let $\widetilde{\tau} : \Sigma \to \Sigma$ be the anti-holomorphic involution $\widetilde{\tau}(z) = \overline{z}$. We define the involution τ_* on $C^i_{\delta,p}(\Sigma, w)$ for each $i = 0$, 1 by

$$\tau_*(V, (v_{+\infty}, v_{-\infty})) = \begin{cases} (\tau_*(V), (v_{+\infty}, v_{-\infty})) & i = 0 \\ \tau_*(V) & i = 1, \end{cases}$$

where we set

$$(\tau_*(V))(z) = (T\tau)^{-1}(V(\widetilde{\tau}(z)) \qquad i = 0 \tag{42.13.1}$$

$$(\tau_*(V))(z) = (T\tau)^{-1} \circ V(\widetilde{\tau}(z)) \circ T\widetilde{\tau} \qquad i = 1. \tag{42.13.1}$$

(Compare these definitions with those in (39.11). Note in (39.11) we wrote I in place of τ_*.)

Hereafter we simply write $\mathfrak{I}'_\pm$ for both of $\mathfrak{I}'^{;0}_{\pm}$ and $\mathfrak{I}'^{;1}_{\pm}$ when no confusion can occur.

Lemma 42.14. (1) *$\mathfrak{I}'_+$ and $\mathfrak{I}'_-$ are well-defined. $\mathfrak{I}'_+$ is an isomorphism onto*

$$\{\mathbf{V} = (V, (v_{+\infty}, v_{-\infty})) \in C^0_{\delta,p}(\Sigma, \mathfrak{D}(w)) \mid \tau_*\mathbf{V} = \mathbf{V}\}.$$

(2) *$\mathfrak{I}'_-$ is an isomorphism onto*

$$\{(V, (0,0)) \in C^0_{\delta,p}(\Sigma, \mathfrak{D}(w)) \mid \tau_* V = -V\}.$$

(3) *And $\mathfrak{I}'_\pm$ restricts to an isomorphism between*

$$W^{1,p}_\delta(\Sigma, \partial\Sigma;\, w^*TM, w|^*_{\partial\Sigma}TL)$$

resp.

$$W^{1,p}_\delta(\Sigma, \partial_-\Sigma, \partial_+\Sigma;\, w^*TM, w|^*_{\partial_-\Sigma}TL, Jw|^*_{\partial_+\Sigma}TL)$$

and

$$\left\{V \in W^{1,p}_\delta(\Sigma, \partial_-\Sigma, \partial_+\Sigma;\, \mathfrak{D}(w)^*TM, \mathfrak{D}(w)|^*_{\partial_-\Sigma}TL, \mathfrak{D}(w)|^*_{\partial_+\Sigma}TL) \;\middle|\; \tau_*V = \pm V\right\},$$

respectively.

Proof. Using the fact that $d\tau = id$ on TL and $d\tau = -id$ on $J(TL)$, we can show that $\mathfrak{I}'_{\pm}(V)$ are continuous along $\Sigma_+ \cap \Sigma_-$ and so are well-defined. The rest of the proof is straightforward and omitted. In the claim (2) we remark that $\tau(\mathbf{V}) = -\mathbf{V}$ implies $\tau_*(v_{+\infty}) = -v_{+\infty}$ and $\tau_*(v_{-\infty}) = -v_{-\infty}$. Thus by (42.11.2) we have $(v_{+\infty}, v_{-\infty}) = (0,0)$. □

Consider a family of the bundle maps $K_t : TM \to TM$ defined by

$$K_t(V) = \left(\cos \frac{\pi(t+1)}{4}\right) V + \left(\sin \frac{\pi(t+1)}{4}\right) J(V). \tag{42.15}$$

This induces a linear map

$$\begin{aligned}\mathcal{K} : W^{1,p}_{\delta}&(\Sigma, \partial\Sigma;\, w^*TM, (w|_{\partial\Sigma})^*TL) \\ &\to W^{1,p}_{\delta}(\Sigma, \partial_-\Sigma, \partial_+\Sigma;\, w^*TM, (w|_{\partial_-\Sigma})^*TL, J(w|_{\partial_+\Sigma})^*TL)\end{aligned}$$

by

$$(\mathcal{K}(V))(s,t) = K_t(V(s,t)). \tag{42.16}$$

It follows that $\mathcal{K}$ is an isomorphism. We define another linear map, which we denote by the same letter,

$$\mathcal{K} : C^1_{\delta,p}(\Sigma, w) \to C^1_{\delta,p}(\Sigma, w)$$

by the same formula.

We put

$$\mathfrak{I}_+ = \mathfrak{I}'_+ \quad \text{and} \quad \mathfrak{I}_- = \mathfrak{I}'_- \circ \mathcal{K} \tag{42.17}$$

and $\mathfrak{I}_{(1)} = (\mathfrak{I}_+, \mathfrak{I}_-)$. Then $\mathfrak{I}_{(1)}$ defines isomorphisms

$$\mathfrak{I}_{(1)} :\ C^0_{\delta,p}(\Sigma, w) \oplus W^{1,p}_{\delta}(\Sigma, \partial\Sigma;\, w^*TM, w|^*_{\partial\Sigma}TL) \to C^0_{\delta,p}(\Sigma, \mathfrak{D}(w)) \tag{42.18.1}$$

and

$$\mathfrak{I}_{(1)} : C^1_{\delta,p}(\Sigma, w) \oplus C^1_{\delta,p}(\Sigma, w) \to C^1_{\delta,p}(\Sigma, \mathfrak{D}(w)) \tag{42.18.2}$$

respectively.

Lemma 42.19. *Let $w : \Sigma \to M$ be a pseudo-holomorphic map with $w(\partial\Sigma) \subset L$ with finite energy, and $c \in \mathbb{R}$ be any given constant. Then there exists $\delta = \delta(w) > 0$ depending only on w, such that we have the identity*

$$\mathfrak{I}^{-1}_{(1)} \circ (D_{\mathfrak{D}(w)}\overline{\partial} + c) \circ \mathfrak{I}_{(1)} \equiv \left(2D_w\overline{\partial} + c, 2D_w\overline{\partial} + c - \frac{\pi}{4}\right)$$
$$\text{modulo a compact operator}$$

as a Fredholm linear map from

$$C^0_{\delta,p}(\Sigma, w) \oplus W^{1,p}_\delta(\Sigma, \partial\Sigma; w^*TM, w|^*_{\partial\Sigma}TL)$$

to

$$C^1_{\delta,p}(\Sigma, w) \oplus C^1_{\delta,p}(\Sigma, w).$$

Proof. Let ∇ is an almost complex connection of J. Then we have the linearization

$$D_w\overline{\partial} = \frac{1}{2}(\nabla_{\frac{\partial}{\partial s}} + J\nabla_{\frac{\partial}{\partial t}}) + \left(T\left(\frac{\partial w}{\partial s}, V\right) + JT\left(\frac{\partial w}{\partial t}, V\right)\right)$$

where T is the torsion tensor of the connection ∇. Since the terms involving T are of the zero-th order, it follows from a straightforward calculation that we have

$$(\mathcal{K}^{-1} \circ D_w\overline{\partial} \circ \mathcal{K})(V) = (D_w\overline{\partial})(V) - \frac{\pi}{8}V + \mathrm{Err}_-(V), \tag{42.20}$$

for any J-holomorphic map w, where Err_- is the term involving the torsion T and $\frac{DJ}{dt}(w(s,t))$.

Now from the definition (42.9.2) of $\mathfrak{D}(w)$ and $\mathfrak{I}_{(1)}$, we can calculate $\mathfrak{I}_{(1)}^{-1} \circ D_{\mathfrak{D}(w)}\overline{\partial} \circ \mathfrak{I}_{(1)}$ separately on $\Sigma_+ = \mathbb{R} \times [0,1]$ and $\Sigma_- = \mathbb{R} \times [-1,0]$. We start with Σ_-. On $\mathbb{R} \times [-1,0]$, we have

$$\begin{aligned}(\mathfrak{I}_{(1)}^{-1} \circ D_{\mathfrak{D}(w)}\overline{\partial} \circ \mathfrak{I}_{(1)})(V)(z) &= (\mathfrak{I}'^{-1}_- \circ \mathcal{K})^{-1} \circ D_{\mathfrak{D}(w)} \circ (\mathfrak{I}'_- \circ \mathcal{K})(V)(z) \\ &= \mathcal{K}^{-1}\left((\mathfrak{I}'^{-1}_-)^{-1} \circ D_{\mathfrak{D}(w)} \circ \mathfrak{I}'_-\right)(\mathcal{K}(V))(z).\end{aligned}$$

Now, recalling that I_- is a conformal map given by

$$I_-(z) = \frac{z - \sqrt{-1}}{2},$$

it is straightforward to check that (42.20) implies

$$(\mathfrak{I}_{(1)}^{-1} \circ D_{\mathfrak{D}(w)}\overline{\partial} \circ \mathfrak{I}_{(1)})(V)(z) = 2(D_w\overline{\partial})(V)(z) - \frac{\pi}{4}V(z) + \mathrm{Err}_-(V)(z)$$

for $z \in \mathbb{R} \times [-1,0]$. Here the term $\mathrm{Err}_-(V)(z)$ does not involve derivatives of V and decays as fast as $e^{-\delta'|s|}$ for some $\delta' > 0$ which depends only on the exponential order with which $w(s,t)$ converges to $w(\pm\infty)$ as $s \to \infty$. On the other hand, such δ' always exists : For the finite energy condition of $w : \Sigma = \mathbb{R} \times [-1,1] \to M$ together with the Lagrangian boundary condition $w(\partial\Sigma) \subset L$ enables one to apply the removable singularity theorem and to get the Hölder estimates around $\pm\infty$.

(See [Proposition 3.6, Oh92].) This then is translated back into the exponential decay estimate of w on Σ.

Next we consider $\mathfrak{I}_{(1)}^{-1} \circ D_{\mathfrak{D}(w)}\overline{\partial} \circ \mathfrak{I}_{(1)}$ on the domain $\Sigma_+ = \mathbb{R} \times [0,1]$. This time it is much easier to verify that

$$\left(\mathfrak{I}_{(1)}^{-1} \circ D_{\mathfrak{D}(w)}\overline{\partial} \circ \mathfrak{I}_{(1)}\right)(V)(z) = 2D_w\overline{\partial}(V)(z) + \mathrm{Err}_+(V)(z)$$

where $\mathrm{Err}_+(V)$ is a term that has the decay estimates similar to $\mathrm{Err}_-(V)$.

Combining the two, if we choose a constant $\delta > 0$ even smaller than δ', the operator

$$\mathrm{Err} = (\mathrm{Err}_+, \mathrm{Err}_-)$$

becomes a compact operator. This finishes the proof. $\square$

Now we consider $\mathfrak{D}^m(w)$ and will promote the isomorphisms (42.18) to the isomorphisms

(42.21.1)
$$\mathfrak{I}_{(m)} : C^0_{\delta,p}(\Sigma, w) \oplus (W^{1,p}_\delta(\Sigma, \partial\Sigma;\ w^*TM, w|^*_{\partial\Sigma}TL))^{\oplus(2^m-1)} \to C^0_{\delta,p}(\Sigma, \mathfrak{D}^m(w))$$

and

$$\mathfrak{I}_{(m)} :\ C^1_{\delta,p}(\Sigma, w)^{\oplus(2^m)} \to C^1_{\delta,p}(\Sigma, \mathfrak{D}^m(w)) \tag{42.21.2}$$

for general $m \geq 2$.

Let $\vec{\epsilon} = (\epsilon_0, \cdots, \epsilon_{m-1}) \in \{\pm 1\}^m$. To define the isomorphism (42.21.1) we write an element of

$$C^0_{\delta,p}(\Sigma, w) \oplus (W^{1,p}_\delta(\Sigma, \partial\Sigma;\ w^*TM, w|^*_{\partial\Sigma}TL))^{\oplus(2^m-1)}$$

as $(V_{\vec{\epsilon}})_{\vec{\epsilon}\in\{\pm1\}^m}$, where $V_{(1,\cdots,1)}$ denotes the $C^0_{\delta,p}(\Sigma, w)$-component. We denote

$$\mathfrak{I}_{\epsilon_0,\cdots,\epsilon_{m-1}} = \mathfrak{I}_{\epsilon_{m-1}} \circ \cdots \circ \mathfrak{I}_{\epsilon_0} \tag{42.22.1}$$

where

$$\mathfrak{I}_\epsilon = \begin{cases} \mathfrak{I}_+ & \text{if } \epsilon = +1 \\ \mathfrak{I}_- & \text{if } \epsilon = -1, \end{cases}$$

(see (42.17) for the definition of $\mathfrak{I}_\pm$) and then define

$$\mathfrak{I}_{(m)} = (\mathfrak{I}_{\epsilon_0,\cdots,\epsilon_{m-1}})_{(\epsilon_0,\cdots,\epsilon_{m-1})\in\{\pm1\}^m}. \tag{42.22.2}$$

The definition of (42.21.2) is the same. Lemma 42.14 implies that (42.21) are isomorphisms.

Now using the isomorphisms (42.21), we are ready to define a $\mathbb{Z}_2^m$-action on the spaces

$$C^0_{\delta,p}(\Sigma, w) \oplus (W^{1,p}_\delta(\Sigma, \partial\Sigma;\, w^*TM, w|^*_{\partial\Sigma}TL))^{\oplus(2^m-1)}$$

and $C^1_{\delta,p}(\Sigma, w)^{\oplus(2^m)}$. Let $(V_{\epsilon_0,\cdots,\epsilon_{m-1}})_{(\epsilon_0,\cdots,\epsilon_{m-1})\in\{\pm1\}^m}$ be an element of either of the two spaces, and $\vec\rho = (\rho_0, \cdots, \rho_{m-1}) \in \{\pm 1\}^m = \mathbb{Z}_2^m$. We define an action of $\vec\rho$ by

$$(\vec\rho V)_{\epsilon_0,\cdots,\epsilon_{m-1}} = \left(\prod_{i;\rho_i=-1} \epsilon_i\right) V_{\epsilon_0,\cdots,\epsilon_{m-1}}.$$

This action induces a $\mathbb{Z}_2^m$-action on $C^i_{\delta,p}(\mathfrak{D}^m(w))$ via the isomorphisms (42.21) for each $i = 0$, 1 respectively.

We next define one more $\mathbb{Z}_2$ action on the bundle $C^i_{\delta,p}(\mathrm{Im}\,\mathfrak{D}^m)$:

$$C^i_{\delta,p}(\mathrm{Im}\,\mathfrak{D}^m) \;:\; \bigcup_w \{w\} \times C^i_{\delta,p}(\mathfrak{D}^m(w)) \to \mathrm{Im}\,\mathfrak{D}^m$$

which lifts $\tau_*^{(m)}$ by

$$\begin{aligned}&\Big(\mathfrak{I}_{(m)}((V_{\epsilon_0,\cdots,\epsilon_{m-1}})_{(\epsilon_0,\cdots,\epsilon_{m-1})}), \mathfrak{D}^m(w)\Big)\\ &\mapsto \Big(\mathfrak{I}_{(m)}((\tau_* V_{\epsilon_0,\cdots,\epsilon_{m-1}})_{(\epsilon_0,\cdots,\epsilon_{m-1})}), \mathfrak{D}^m(\tau_* w)\Big).\end{aligned}$$

We remark that

$$\mathfrak{D}^m(\tau_* w) = \tau_*^{(m)} \mathfrak{D}^m(w).$$

It is easy to see that this $\mathbb{Z}_2$ action commutes with the $\mathbb{Z}_2^m$ action and hence defines a $\mathbb{Z}_2^{m+1}$ action. By abuse of notation, we use the same notation $(\rho_0, \cdots, \rho_{m-1}, \rho_m)$ to denote an element of this group $\mathbb{Z}_2^{m+1}$.

We next verify that this action satisfies the properties analogous to (42.4). (We remark again that we adopt the situation of (42.4) for the case of $m-1$, because $\mathcal{M}_2^{\mathrm{reg},(m)} = F_{m-1}$ in our notation.) We have an embedding

$$\mathfrak{I}^k_+ : C^*_{\delta,p}(\mathfrak{D}^m(w)) \to C^*_{\delta,p}(\mathfrak{D}^{m+k}(w)), \tag{42.23}$$

which is the k-th iteration of $\mathfrak{I}_+$. We define a homomorphism $I_k : \mathbb{Z}_2^{m+1} \to \mathbb{Z}_2^{m+k+1}$ by $I_k(\rho_0, \cdots, \rho_m) = (\rho_0, \cdots, \rho_m, 1, \cdots, 1)$.

Lemma 42.24.

(42.25.1) *Consider the action of $\mathbb{Z}_2^{m+1}$ on the bundle $C^*_{\delta,p}(\mathrm{Im}\,\mathfrak{D}^{m+k})$ induced by I_k. Then the embedding* (42.23) *is equivariant under this $\mathbb{Z}_2^{m+1}$-action.*
(42.25.2) *Let $(1, \cdots, 1, -1) \in \mathbb{Z}_2^{m+1}$. The $(1, \cdots, 1, -1)$-invariant part of the bundle $C^*_{\delta,p}(\mathrm{Im}\,\mathfrak{D}^m)$ coincides with the image of $\mathfrak{I}_+ : C^*_{\delta,p}(\mathrm{Im}\,\mathfrak{D}^{m-1}) \to C^*_{\delta,p}(\mathrm{Im}\,\mathfrak{D}^m)$.*

(42.25.3) *The natural action of* $\tau_* = \tau_*^{(0)}$ *on* $C^*_{\delta,p}(\mathrm{Im}\,\mathfrak{D}^m)$ *coincides with the action of* $(-1, 1, \cdots, 1)$.

The proof is obvious from definition.

We now verify Assumption 42.3 in our present context of $\mathcal{M}_2^{\mathrm{reg}}$. For this purpose, we use Lemma 42.24 and the fact that our Kuranishi structure is constructed from the deformation complex

$$D_{\mathfrak{D}^m(w)}\overline{\partial} : C^0_{\delta,p}(\mathfrak{D}^m(w)) \to C^1_{\delta,p}(\mathfrak{D}^m(w))$$

by a finite dimensional reduction. The next lemma will be also used in an essential way.

Lemma 42.26. *For* $\vec{\rho} \in \mathbb{Z}_2^{m+1}$, *the operator*

$$D_{\mathfrak{D}^m(w)}\overline{\partial} - \vec{\rho} \circ D_{\mathfrak{D}^m(w)}\overline{\partial} \circ \vec{\rho} \ : \ C^0_{\delta,p}(\mathfrak{D}^m(w)) \to C^1_{\delta,p}(\mathfrak{D}^m(w))$$

is a compact operator.

Proof. We first express the conjugate of the linearization $D_{\mathfrak{D}^m(w)}\overline{\partial}$ by the isomorphism (42.21). We denote this conjugate by

$$\mathcal{D} : C^0_{\delta,p}(\Sigma, w) \oplus (W^{1,p}_\delta(\Sigma, \partial\Sigma; w^*TM, w|^*_{\partial\Sigma}TL))^{\oplus(2^m-1)} \to C^1_{\delta,p}(\Sigma, w)^{\oplus(2^m)}$$

with $\mathcal{D} = (\mathcal{D}_{\vec{\epsilon}})_{\vec{\epsilon}\in\{-1,+1\}^m}$ where its $\vec{\epsilon}$-component is given by

$$\mathcal{D}_{\vec{\epsilon}} = 2^m \left(D_w\overline{\partial} - \sum_{i=1}^m 2^{-3-i}(1-\epsilon_i)\pi \right). \tag{42.27}$$

Applying Lemma 42.19 inductively over m, we can prove that

$$\mathcal{D} \circ \mathfrak{I}_{(m)} - \mathfrak{I}_{(m)} \circ D_{\mathfrak{D}(w)}\overline{\partial}$$

is a compact operator. Obviously the $\mathbb{Z}_2^{m+1}$-action commutes with $\mathcal{D}$. Now this implies Lemma 42.26 via (42.21) and (42.25.2). This finishes the proof. We remark that the first eigen value of the operator $\frac{1}{2}J\frac{\partial}{\partial\tau}$ on $W^{1,p}([-1,1], \partial[-1,1]; \mathbb{C}^n, \mathbb{R}^n)$ is equal to $\pi/2$ which satisfies

$$\frac{\pi}{2} > \sum_{i=1}^m 2^{-3-i}(1-\epsilon_i)\pi.$$

□

To construct a Kuranishi structure from the tangent complex, we need to choose a finite dimensional subspace $\mathcal{E}_{\mathbf{p}}$ of $C^1_{\delta,p}(\mathbf{p})$ for each of $\mathbf{p} \in \mathcal{M}_2^{\mathrm{reg}}(\beta)$. (See §29.) In case $\mathbf{p} = \mathfrak{D}^m(\mathbf{p}')$ with $\mathbf{p}' = [((\Sigma, (\infty, -\infty)), w)] \in \mathcal{M}_2^{\mathrm{reg}}(2^{-m}\beta)$, we choose $\mathcal{E}_{\mathbf{p}} \subseteq C^1_{\delta,p}(\mathfrak{D}^m(w))$ so that the following condition is satisfied.

Condition 42.28. Let $\chi : \{\pm 1\}^{m+1} \to [0,1]$ be a function such that

$$\sum_{\vec\rho \in \{\pm 1\}^{m+1}} \chi(\vec\rho) = 1.$$

Consider the operator $D_\chi : C^0_{\delta,p}(\mathfrak{D}^m(w)) \to C^1_{\delta,p}(\mathfrak{D}^m(w))$ defined by

$$D_\chi = \sum_{\vec\rho \in \{\pm 1\}^{m+1}} \chi(\vec\rho)(\vec\rho \circ D_{\mathfrak{D}^m w}\overline{\partial} \circ \vec\rho).$$

Then $\mathcal{E}_{\mathbf{p}}$ satisfies

$$\mathrm{Im} D_\chi + \mathcal{E}_{\mathbf{p}} = C^1_{\delta,p}(\mathfrak{D}^m(w))$$

for any χ. We also assume that $\mathcal{E}_{\mathbf{p}}$ is invariant under the above $\mathbb{Z}_2^{m+1}$-action.

We can find such $\mathcal{E}_{\mathbf{p}}$ by Lemma 42.26, since the set of all such χ is a compact subset of $[0,1]^{2^{m+1}}$.

We want to use this choice of $\mathcal{E}_{\mathbf{p}}$'s to construct a Kuranishi structure in the same way as in §29. We need to handle one more trouble. Namely the map $(\Sigma, w') \mapsto \overline{\partial} w' = s(w')$ may not be invariant under the $\mathbb{Z}_2^{m+1}$ action when we vary $[(\Sigma, (\infty, -\infty)), w']$ in a neighborhood of $\mathbf{p} = \mathfrak{D}^m(\mathbf{p}')$ in the Banach manifold $W^{1,p}_2((\Sigma, \partial\Sigma), (M, L))/\mathbb{R}$. ($W^{1,p}_2((\Sigma, \partial\Sigma), (M, L))$ is defined in the same way as Definition 29.3.)

Now we explain how we take care of this trouble. For each $m \le m(\beta)$, we consider a small neighborhood $\mathfrak{V}_m$ of

$$F_{m-1} = \mathcal{M}_2^{\mathrm{reg},(m)}(\beta)$$

in $W^{1,p}_2((\Sigma, \partial\Sigma), (M, L))/\mathbb{R}$ for $m = 0, 1, \cdots$. (Here $F_{-1} = \mathcal{M}(\beta)^{\mathrm{reg}}$.) Shrinking $\mathfrak{V}_m$ if necessary, we may assume that

$$\bigcup_{m=0}^{m(\beta)} \mathfrak{V}_m \supset \mathcal{M}_2^{\mathrm{reg}}(\beta), \tag{42.29.1}$$

$$\mathfrak{V}_{m_1} \cap F_{m_2} = \emptyset \quad \text{if } m_2 > m_1. \tag{42.29.2}$$

We can choose $\mathfrak{V}_m$ so small that the infinitesimal $\mathbb{Z}_2^{m+1}$-action on $C^0_{\delta,p}(\mathfrak{D}^m(w))$ induces a $\mathbb{Z}_2^{m+1}$ action on $\mathfrak{V}_m$ by exponentiation, and this action can be lifted to an action of the bundle $\mathbf{p} \mapsto C^1_{\delta,p}(\mathbf{p})$ where $\mathbf{p} \in \mathfrak{V}_m$. We also assume that the actions on various $\mathfrak{V}_m$'s are compatible on the overlapped parts. Then we define $s^{(m)}$ by

$$s^{(m)}(\mathbf{p}) = 2^{-(m+1)} \sum_{\vec\rho \in \mathbb{Z}_2^{m+1}} (\vec\rho \circ s)(\vec\rho\,\mathbf{p}) \in C^1_{\delta,p}(\mathbf{p})$$

on each $\mathfrak{V}_m$. Here s denotes the original Kuranishi map given by

$$s(\mathbf{p}) = \overline{\partial} w, \quad \mathbf{p} = ((\Sigma, (\infty, -\infty)), w) \in \mathfrak{V}_m.$$

We observe that $s^{(m)}$ is $\mathbb{Z}_2^{m+1}$-equivariant.

Choosing a partition of unity χ_m subordinate to $\mathfrak{V}_m$, we define

$$s'(\mathbf{p}) = \sum_m \chi_m(\mathbf{p}) s^{(m)}(\mathbf{p}).$$

Now we use Condition 42.28 and choose $\mathfrak{V}_m$ sufficiently close to $F_{m-1} = \mathcal{M}_2^{\mathrm{reg},(m)}(\beta)$ so that the composition

$$\pi \circ d_{\mathbf{p}} s' : C^0_{\delta,p}(\mathbf{p}) \to C^1_{\delta,p}(\mathbf{p})/\mathcal{E}_{\mathbf{p}}$$

is surjective. Therefore we replace the equation (29.16) by

$$s'(\mathbf{p}) \equiv 0 \mod \mathcal{E}_{\mathbf{p}} \tag{42.30}$$

to define a space with Kuranishi structure. Then we derive from (42.29.2) that (42.30) is invariant under the $\mathbb{Z}_2^{m+1}$-action on the neighborhood $\mathfrak{V}_m$ of $F_{m-1} = \mathcal{M}_2^{\mathrm{reg},(m)}(\beta)$. This implies that the Kuranishi structure thus constructed carries a $\mathbb{Z}_2^{m+1}$-action on $\mathfrak{V}_m$. Then it follows from Lemma 42.24 that this action satisfies Assumption 42.3. Applying Proposition 42.5, we now finish the proof of Proposition 41.7. The proof of Proposition 41.10 is similar and omitted. □

§43. Completion of the proof of Theorem 34.16.

The main purpose of this section is to complete the proof of Theorem 34.16. We remark that we did not use spherical positivity yet in the previous sections. It is in the discussion of this section we need to use this unpleasant assumption.

Because we will often use the moduli space of pseudo-holomorphic spheres of M in this section which we denote by $\mathcal{M}(M;\alpha)$, we write $\mathcal{M}(L;\beta)$ in place of $\mathcal{M}(\beta)$ for the moduli space of pseudo-holomorphic discs to avoid causing possible confusion from the readers.

43.1. A problem to extend $\tau_*^{(i)}$ to the compactification.

The principal reason why we need to assume Condition 34.15 lies in the fact that the perturbation $\mathfrak{s}^\epsilon$ we produced in §41 and §42 may not extend to the compactified moduli space.

First, we provide an example which illustrates how this continuous extension is obstructed. Let $S = \mathbb{R} \times [-1, 1] \subset \mathbb{C}$, where $z = s + \sqrt{-1}t$ is the standard complex coordinate of $\mathbb{C}$, and a compatible almost complex structure J on M be given. Consider a pseudo-holomorphic map $u : (S, \partial S) \to (M, L)$ and denote by $[((S, (+\infty, -\infty)), u)]$ the corresponding element in $\mathcal{M}_2^{\rm reg}(L; \beta)$. Here $\pm\infty$ are the marked points on ∂S which is the limit of $s \to \pm\infty$. We will often denote by the same letter S for the compactified disc including the two 'infinities' to S. This should not cause any confusion. Suppose u satisfies

$$\tau_*((S, (+\infty, -\infty)), u) = ((S, (+\infty, -\infty)), u). \tag{43.1.1}$$

By definition, we have

$$u(\mathbb{R} \times \{0\}) \subset \operatorname{Fix} \tau = L, \quad \tau(u(s, -t)) = u(s, t).$$

In addition, we suppose

$$\tau_*^{(1)}((S, (+\infty, -\infty)), u) \neq ((S, (+\infty, -\infty)), u). \tag{43.1.2}$$

Let $v : S^2 \to M$ be a pseudo-holomorphic sphere such that $\tau(v(z)) = v(\overline{z})$. Here we regard $S^2 = \mathbb{C} \cup \{\infty\}$. Then $((S^2, 0), v)$ is an element of $\mathcal{M}_1(M; \widetilde{\beta})$, the moduli space of pseudo-holomorphic sphere with one marked point. It follows that $v(\mathbb{R}) \subset L$ where $\mathbb{R} \subset \mathbb{C} \subset S^2$. We restrict to the case of v's such that v satisfies $v(0) = u(0, 0)$ and its group of automorphisms is trivial.

We glue S^2 to S by identifying $0 \in S^2$ with $(0, 0) \in S$ and denote by Σ the resulting semi-stable curve.

We now define a stable map $w : (\Sigma, \partial\Sigma) \to (M, L)$ by setting

$$w = u \text{ on } S \text{ and } w = v \text{ on } S^2.$$

Obviously we have

$$\tau_*((\Sigma, (+\infty, -\infty)), w) = ((\Sigma, (+\infty, -\infty)), w)$$

and $(\Sigma, (+\infty, -\infty), w) \in \mathcal{M}_2(L; \beta + \widetilde{\beta})$. We are going to observe that $\tau_*^{(1)}$ does *not* extend to $((\Sigma, (+\infty, -\infty)), w)$.

Recall that the set of smoothing parameters of the double point of Σ is $\mathbb{C}$ and for each $\mathfrak{z}$ in a neighborhood of 0 we obtain a smooth curve which we denote by $\Sigma_{\mathfrak{z}}$. If the obstruction bundle of $\mathcal{M}_2(L; \beta + \widetilde{\beta})$ at $((\Sigma, (+\infty, -\infty)), w)$ is zero, for example, then $((\Sigma_{\mathfrak{z}},(+\infty,-\infty)),w_{\mathfrak{z}}) \in \mathcal{M}_2(L;\beta+\widetilde{\beta})$ would converge to $((\Sigma,(+\infty,-\infty)),w)$ as $\mathfrak{z} \to 0$ and satisfies

$$\tau_*((\Sigma_{\mathfrak{z}},(+\infty,-\infty)),w_{\mathfrak{z}}) = ((\Sigma_{\overline{\mathfrak{z}}},(+\infty,-\infty)),w_{\overline{\mathfrak{z}}}).$$

In particular for the real parameter $r \in (-\epsilon,+\epsilon) \setminus \{0\}$, we obtain an element $((\Sigma_r,(+\infty,-\infty)),w_r) \in \mathcal{M}_2^{\text{reg}}(L;\beta+\widetilde{\beta})$, which is fixed by τ_*. Therefore we can define $\tau_*^{(1)}((\Sigma_r,(+\infty,-\infty)),w_r)$ as in §41. We claim

$$\lim_{r\uparrow 0}\tau_*^{(1)}((\Sigma_r,(+\infty,-\infty)),w_r) \neq \lim_{r\downarrow 0}\tau_*^{(1)}((\Sigma_r,(+\infty,-\infty)),w_r) \tag{43.2}$$

which will show that the map $\tau_*^{(1)} : \mathcal{M}_2^{\text{reg}}(L;\beta+\widetilde{\beta}) \to \mathcal{M}_2^{\text{reg}}(L;\beta+\widetilde{\beta})$ cannot be continuously extended to the compactification $\mathcal{M}_2(L;\beta+\widetilde{\beta})$.

To show (43.2) we give the descriptions of the right and left hand sides thereof. We regard the restriction of $v : \mathbb{C}\cup\{\infty\} \to M$ to $\mathbb{H}$ as a map $v_1 : (D^2,\partial D^2) \to (M,L)$ where $0 \in \partial\mathbb{H}$ is identified with $1 \in \partial D^2$. Then $(((D^2,\partial D^2),1),v_1) \in \mathcal{M}_1(L;\beta_1)$. Let $\tau_*(((D^2,\partial D^2),1),v_1) = (((D^2,\partial D^2),1),v_2) \in \mathcal{M}_1(L;\beta_2)$. (In fact, if we regard $\beta_i \in \pi_2(M,L)$, then $\beta_2 = \tau_*(\beta_1)$.) Then v is obtained by gluing v_1 and v_2 along their boundaries.

We take two copies of D^2 and denote them by D^2_-, D^2_+ respectively. We glue S with D^2_- and D^2_+ by identifying $(0,-1) \in S$ with $1 \in D^2_-$ and also identifying $(0,+1) \in S$ with $1 \in D^2_+$. We denote by Σ' the bordered semi-stable curve obtained by it. We define $w_i : (\Sigma_{(0,0)},\partial\Sigma_{(0,0)}) \to (M,L)$ for $i=1,2$ as follows. See Figures 43.1 and 43.2.

$$w_1(z) = \begin{cases} (\tau_*^{(1)}(u))(z) & z \in S \\ v_1(z) & z \in D^2_- \\ v_2(z) & z \in D^2_+, \end{cases} \tag{43.3.1}$$

$$w_2(z) = \begin{cases} (\tau_*^{(1)}(u))(z) & z \in S \\ v_2(z) & z \in D^2_- \\ v_1(z) & z \in D^2_+. \end{cases} \tag{43.3.2}$$

Figure 43.1

Figure 43.2

It is easy to prove

$$
\left\{
\begin{aligned}
&\lim_{r\uparrow 0}\tau_*^{(1)}((\Sigma_r,(+\infty,-\infty)),w_r) = ((\Sigma_{(0,0)},(+\infty,-\infty)),w_1)\\
&\lim_{r\downarrow 0}\tau_*^{(1)}((\Sigma_r,(+\infty,-\infty)),w_r) = ((\Sigma_{(0,0)},(+\infty,-\infty)),w_2).
\end{aligned}
\right.
\tag{43.4}
$$

Thus the argument of §41 and §42 to cancel the effect of pseudo-holomorphic discs by secondary and higher involutions does not directly apply at the singular strata. In fact the trouble becomes more serious if the sphere bubble v as above is a multiple (say 2^k-th) cover. In that case there is a non trivial automorphism (cyclic group of order 2^k) in the sphere bubble side. On the other hand, in the disc bubble side, we have an element of the image of $\mathfrak{D}^{k-1}$ as the disc bubble. The symmetry we find for such element is $\mathbb{Z}_2^{k-1}$. It appears that these two symmetries are not compatible to each other as, for example, the groups are different.

Here enters the additional assumption the spherical positivity which will make the image of the evaluation map of the bubble have smaller dimension than that of $\mathcal{M}_2^{\mathrm{reg}}(\beta + \widetilde{\beta})$. This will enable us to derive vanishing of the obstruction of Floer cohomology and and collapsing of the associated spectral sequence in the E_2-level, which will lead to the proof of Theorem 34.16.

43.2. Estimate of the dimension of the 'infinity' of the moduli space.

In this subsection, we use spherical positivity of J to show that the evaluation image of the singular strata of the moduli space of J-holomorphic discs has the dimension strictly smaller than that of the image of the regular strata. (See Proposition 43.9.)

Definition 43.5. Let $E > 0$. We denote by $\mathcal{J}_{\omega,E}^{c_1>0}$ or $\mathcal{J}_{(M,\omega),E}^{c_1>0}$ the set of all ω compatible almost complex structures on M such that every J-holomorphic sphere $v : S^2 \to M$ with $c_1(M)[v] \le 0$, $\omega[v] \le E$ is constant. Clearly $\bigcap_E \mathcal{J}_{\omega,E}^{c_1>0} = \mathcal{J}_{\omega}^{c_1>0}$.

The following lemma will be needed for the proof of Theorem 34.16.

Lemma 43.6. *If $J \in \mathcal{J}_{\omega}^{c_1>0} \neq \emptyset$, for each given $E > 0$ there exists an open neighborhood $\mathcal{V}_{\omega\le E}(J)$ of J in $\mathcal{J}_\omega$ such that $\mathcal{V}_{\omega\le E}(J) \subset \mathcal{J}_{\omega,E}^{c_1>0}$.*

Proof. Define

$$\Gamma'_{\omega,J} = \{\omega(u) \mid u : S^2 \to M \text{ non-constant } J\text{-holomorphic}\}$$

and denote by $\lambda_{\omega,J}$ the minimum of $\Gamma'_{\omega,J}$ which is positive by ϵ-regulaity. Then Gromov's compactness gives rise to upper semi-continuity of the function $J \mapsto \lambda_{\omega,J}$ and so we can choose a C^∞-neighborhood $\mathcal{V}_0(J)$ of J in $\mathcal{J}_\omega$ such that

$$\lambda_{\omega,J'} \ge \lambda_{\omega,J}$$

for all $J' \in \mathcal{V}_0(J)$.

Gromov's compactness also implies that $\Gamma'_{\omega,J}$ is a discrete subset of $\mathbb{R}_+$. We enumerate

$$\Gamma'_{\omega,J} = \{\lambda_{\omega,J} = \lambda_1,\, \lambda_2, \cdots, \lambda_j \to \infty\}.$$

We denote by $\Gamma_{\omega,J}$ the monoid generated by $\Gamma'_{\omega,J}$ and set

$$\Gamma_{\omega,J;j} := \{\lambda \in \Gamma_{\omega,J} \mid \lambda \le \lambda_j\}.$$

We denote

$$\delta_j = \min\{|\lambda - \mu| \mid \lambda,\, \mu \in \Gamma_{\omega,J;j}\}.$$

We fix the sequence E_N so that

$$E_N = \lambda_N + \frac{\delta_N}{2}.$$

We now inductively prove the lemma over $E = E_j$. We start with $E = E_1$. Suppose to the contrary that there exists a sequence $J_i \in \mathcal{J}_\omega$ be a sequence with $J_i \to J$ such that J_i carries a non-constant J_i-holomorphic sphere u_i satisfying

$$c_1([u_i]) \leq 0,\, \omega([u_i]) \leq E_1.$$

Since u_i is non-constant, we have

$$\omega([u_i]) \geq \lambda_{\omega,J_i} \geq \lambda_{\omega,J}$$

where the first inequality follows from the definition of λ_{ω,J_i} and the second from the upper semi-continuity of the function $J \mapsto \lambda_{\omega,J}$.

On the other hand by the energy bound and the convergence $J_i \to J$, we can find a subsequence still denoted by u_i such that u_i converges to a J-holomorphic stable map u_∞. Since $\omega(u_\infty) = \lim_{i\to\infty}\omega([u_i])$, we have

$$\lambda_{\omega,J} \leq \omega(u_\infty) \leq E_1.$$

In particular $u_i \to u_\infty$ in C^∞ (modulo reparameterization) by the choice of E_1 above and u_∞ must be a non-constant smooth J-holomorphic sphere and $[u_i] = [u_\infty]$ in $H_2(M)$. Since J does not carry non-constant J-holomorphic sphere with negative c_1, we have $c_1([u_i]) = c_1([u_\infty]) > 0$, a contradiction to the hypothesis $c_1([u_i]) \leq 0$. Therefore there exists an open neighborhood $\mathcal{V}_{\omega\leq E_1}(J) \subset \mathcal{V}_0(J)$ that satisfies the property stated in the lemma.

Now suppose we have produced such neighborhoods $\mathcal{V}_{\omega\leq E_j}(J)$ for all $j \leq N$. Without loss of generality, by taking intersections, we may assume

$$\mathcal{V}_{\omega\leq E_{j+1}}(J) \subset \mathcal{V}_{\omega\leq E_j}(J)$$

for all $1 \leq j \leq N$. Again suppose to the contrary that we have a sequence u_i of J_i-holomorphic with $J_i \to J$ and $J_i \in \mathcal{J}_\omega$ such that $c_1(u_i) \leq 0$ and $\omega([u_i]) \leq E_{N+1}$. By the induction hypothesis, it must hold $\omega([u_i]) > E_N$ for all sufficiently large i. By the energy bound, we have the stable map convergence $u_i \to u_\infty$ after taking a subsequence and so $[u_i] = [u_\infty]$ in $H_2(M)$. The inequality $\omega([u_i]) > E_N$ implies u_∞ cannot be constant and so has positive c_1 and so $c_1([u_i]) > 0$, a contradiction. This finishes the proof. □

Before launching the main goal of analyzing the structure of singular part of the moduli space $\mathcal{M}_2(L;\beta)$, we first need to equip ourselves with a variety of definitions.

Let $\mathcal{M}^{\rm sing}_{k+1}(L;\beta) = \mathcal{M}_{k+1}(L;\beta) \setminus \mathcal{M}^{\rm reg}_{k+1}(L;\beta)$. We take a set of smooth singular cycles $P_j = (|P_j|, f_j)$ with $j = 1, \cdots, \dim H(L;\mathbb{Z}_2)$ whose homology classes generate $H_*(L;\mathbb{Z}_2)$. Note a singular cycle P_j of degree d_j is written as

$$P_j = \sum_{\ell=1}^{k_j} (\Delta^{d_j}, f_{j,\ell})$$

where $f_{j,\ell}$ is a smooth map from the simplex Δ^{d_j} to L. Since $\partial P_j = 0$ as a singular chain we can glue simplicies to obtain a simplicial complex $|P_j|$ and a piecewise smooth map $f_j : |P_j| \to M$.

We denote

$$\mathcal{M}^{\rm sing}_2(L;\beta, P_j) = \mathcal{M}^{\rm sing}_2(L;\beta)\, {}_{ev_1}\times_{f_j} |P_j|. \tag{43.7}$$

Definition 43.8. Let $J_0 \in \mathcal{J}^{c_1>0}_\omega \neq \emptyset$ and $E_0 > 0$. Let $\mathcal{V}_{\omega \le E_0}(J_0)$ be as in Lemma 43.6. We put

$$\Pi(M, E_0, J) = \{\beta \in \Pi(L) \mid E(\beta) \le 2E_0, \mathcal{M}_{k+1}(L, J;\beta) \neq \emptyset\}.$$

and

$$\Pi(M, E_0, \mathcal{V}_{\omega\le E_0}(J_0)) = \bigcup_{J \in \mathcal{V}_{\omega \le E_0}(J_0)} \Pi(M, E_0, J).$$

It follows from Gromov's compactness theorem that $\Pi(M, E_0, J)$ has a finite order. In the same way as the proof of Lemma 43.6, we may choose $\mathcal{V}_{\omega\le E_0}(J_0)$ small enough such that $\Pi(M, E_0, \mathcal{V}_{\omega\le E_0}(J_0))$ is also of finite orer. We put

$$\Pi(M, E_0, \mathcal{V}_{\omega\le E_0}(J_0)) = \{\beta_0, \cdots, \beta_N\}$$

so that $E(\beta_i) \le E(\beta_{i+1})$.

For $\alpha \in \pi_2(M)$, we denote by $\mathcal{M}^{\rm reg}_1(M;\alpha)$ the moduli space of pseudo-holomorphic spheres with one marked point and of homotopy class α. We denote by $\mathcal{M}^{\rm reg}_1(M;\alpha)^{\rm inj}$ (resp. $\mathcal{M}^{\rm reg}(M;\alpha)^{\rm inj}$) the set of elements of $\mathcal{M}^{\rm reg}_1(M;\alpha)$ (resp. $\mathcal{M}^{\rm reg}(M;\alpha)$) represented by somewhere injective spheres, and we put

$$\mathcal{M}^{\rm reg}_1(M;\alpha)^{{\rm inj},\tau} := \mathfrak{forget}^{-1}(\mathcal{M}^{\rm reg}(M;\alpha)^{{\rm inj},\tau})).$$

Here $\mathcal{M}^{\rm reg}(M;\alpha)^{{\rm inj},\tau}$ is the τ-fixed point set of $\mathcal{M}^{\rm reg}(M;\alpha)^{\rm inj}$ and

$$\mathfrak{forget} : \mathcal{M}^{\rm reg}_1(M;\alpha)^{\rm inj} \to \mathcal{M}^{\rm reg}(M;\alpha)^{\rm inj}$$

is the forgetful map of the marked point. The map $ev_{\rm int} : \mathcal{M}_1(M;\alpha) \to M$ denotes the evaluation map at the (unique) interior marked point.

Let $\mathcal{M}^{\rm reg}_{k+1,1}(L;\beta)$ be the moduli space of pseudo-holomorphic discs with one interior marked point and $k+1$ boundary marked points and of the class in $\beta \in \Pi(L)$. We denote by $ev_{\rm int} : \mathcal{M}^{\rm reg}_{k+1,1}(L;\beta) \to M$ the evaluation map at the interior marked point, and $ev_i : \mathcal{M}^{\rm reg}_{k+1,1}(L;\beta) \to L$ is the evaluation map at the i-th boundary marked point for $i = 0, 1, 2, \cdots, k$ respectively.

Proposition 43.9. *Let $J_0 \in \mathcal{J}_\omega^{c_1>0} \neq \emptyset$, $E_0 > 0$, $\mathcal{V}_{\omega\le E_0}(J_0)$ and β_i are as in Definition* 43.8*. Then the following holds for J in a dense subset of $\mathcal{V}_{\omega\le E_0}(J_0)$.*

There exist a finite number of (paracompact) smooth manifolds $S(\beta_i,\ell)$, $S(\beta_i,\ell,j)$ and $f_{i,\ell} : S(\beta_i,\ell) \to L$, $f_{i,\ell,j} : S(\beta_i,\ell,j) \to L$ such that the following holds.

(43.10.1) $\dim S(\beta_i,\ell) \le \dim \mathcal{M}_1(L;\beta_i) - 1$.

(43.10.2) $\dim S(\beta_i,\ell,j) \le \dim \mathcal{M}_2(L;\beta_i,P_j) - 1$.

(43.10.3) $\bigcup_\ell f_{i,\ell}(S(\beta_i,\ell)) = ev_0(\mathcal{M}_1^{\rm sing}(L;\beta_i))$.

(43.10.4) $\bigcup_\ell f_{i,\ell,j}(S(\beta_i,\ell,j)) = ev_0(\mathcal{M}_2^{\rm sing}(L;\beta_i,P_j))$.

(43.10.5) *For each given β_i and ℓ, we have*

$$\left(\bigcap_{K\subseteq S(\beta_i,\ell) \text{ is compact}} \overline{f_{i,\ell}(S(\beta_i,\ell)\setminus K)}\right) \subset \bigcup_{\ell'<\ell} f_{i,\ell'}(S(\beta_i,\ell')).$$

(43.10.6) *For each given β_i, ℓ and j, we have*

$$\left(\bigcap_{K\subseteq S(\beta_i,\ell,j) \text{ is compact}} \overline{f_{i,\ell,j}(S(\beta_i,\ell,j)\setminus K)}\right) \subset \bigcup_{\ell'<\ell} f_{i,\ell',j}(S(\beta_i,\ell',j)).$$

Note we write $\mathcal{M}_1(L;\beta)$ etc. in place of $\mathcal{M}_1(L,J;\beta)$ in (43.10) and also in the proof of Proposition 43.9.

We remark that the conclusion of Proposition 43.9 is related to the notion of pseudo-cycle. (See [McSa94, §7.1].)

We note that we will not use abstract perturbations but only use perturbations of J for the proof of Proposition 43.9. After we have proved Proposition 43.9, we use abstract perturbations in §43.4 and §43.5. We need to use both perturbations of J and abstract perturbations to prove Theorem 34.16.

In the proof below we use the non-existence of J-holomorphic *disk* of Maslov index 0 but will not use yet the non-existence of J-holomorphic sphere of Chern number 0.

Proof. Let $\beta \in \Pi(L)$.

We consider the following four kinds of moduli spaces :

(43.11.1) $\mathcal{M}_1^{\rm reg}(L;\beta')$. Here $E(\beta') < E(\beta)$ and $\mu_L(\beta') < \mu_L(\beta)$.

(43.11.2) $\mathcal{M}_1^{\rm reg}(M;\alpha)^{\rm inj}\ {}_{ev_{\rm int}}\times_M L$. Here the element $\alpha \in \pi_2(M)$ is assumed to satisfy $E(\alpha) \le E(\beta)$, $2c_1(M)(\alpha) \le \mu_L(\beta)$.

(43.11.3) $\mathcal{M}_1^{\rm reg}(M;\alpha)^{\rm inj}\ {}_{ev_{\rm int}}\times_{ev_{\rm int}} \mathcal{M}_{1,1}^{\rm reg}(L;\beta')$. Here the elements $\alpha \in \pi_2(M)$, $\beta' \in \Pi(L)$ are assumed to satisfy

$$E(\alpha) + E(\beta') \le E(\beta), \quad 2c_1(M)(\alpha) + \mu_L(\beta') \le \mu_L(\beta).$$

(43.11.4) $\quad \mathcal{M}_1^{\mathrm{reg}}(M;\alpha)^{\mathrm{inj}}\ {}_{ev_{\mathrm{int}}}\times_{ev_1} \mathcal{M}_{2,0}^{\mathrm{reg}}(L;\beta')$. Here α, β' are as in (43.11.3).

See Figures 43.3–43.5.

Figure 43.3. (43.11.2)

Figure 43.4. (43.11.3)

Figure 43.5. (43.11.4)

The following lemma will be crucial for the proof of $\mathbb{Z}_2$-unobstructedness of $L = \mathrm{Fix}\,\tau$ in §43.4.

Lemma 43.12. *The image* $ev_0(\mathcal{M}_1^{\mathrm{sing}}(L;\beta))$ *is contained in the union of images of the evaluation maps of the moduli spaces* (43.11.1) – (43.11.4)*.*

Proof. Let $((\Sigma, z_0), w) \in \mathcal{M}_1^{\mathrm{sing}}(L;\beta)$ be a singular element of $\mathcal{M}_1(L;\beta)$, i.e., assume that Σ has at least two irreducible components. We denote by Σ_0 the component of Σ that contains the 0-th marked point z_0.

(Case 1) : We first consider the case where w is non-constant on Σ_0. Let $\beta' = [w|_{\Sigma_0}]$. It follows from stability of $((\Sigma, z_0), w)$ that there exists at least one other component Σ_1 on which w is non-constant.

If Σ_1 is a disc, then $J \in \mathcal{J}_{\omega,E_0}^{c_1>0}$ and Lemma 39.21 imply that

$$E(\beta') < E(\beta), \quad \mu_L(\beta') < \mu_L(\beta).$$

Therefore the images $ev_0((\Sigma, z_0), w) = w(z_0)$ of such w is contained in the image of the evaluation map of the moduli space in (43.11.1).

Next assume Σ_0 is the only non-constant disc component of Σ. Then Σ_1 must be a sphere component. By re-choosing Σ_1 if necessary, again by the stability of $((\Sigma, z_0), w)$, we may assume that there exists a point $z_1 \in \Sigma_1$ such that $w(z_1) = w(z_1')$ for $z_1' \in \Sigma_0$. Replacing w by its reduced curve, we may assume that the restriction of w on Σ_1 is somewhere injective.

If $z_1' \in \operatorname{Int}\Sigma_0$, we glue Σ_0 and Σ_1 at z_1' and z_1 and obtain a connected Σ'. We denote the restriction of w to Σ' by $((\Sigma', z_0), w')$. Then this defines an element of the moduli space (43.11.3) and hence the image $ev_0((\Sigma, z_0), w) = ev_0((\Sigma', z_0), w')$ lies in the image of the moduli space (43.11.3). If $z_1' \in \partial\Sigma_0$, then we find that $ev_0((\Sigma, z_0), w)$ lies in the image of the moduli space (43.11.4).

(Case 2) : We next consider the case where w is constant on Σ_0, but there is another disc component Σ_0' on which w is non-constant. Then by re-choosing Σ_0' if necessary we can find a connected chain of disc components $\{\Sigma_i\}$ joining Σ_0 and Σ_0' such that w is constant on each Σ_i. Then we have $z_0' \in \partial\Sigma_0'$ at which w satisfies $w(z_0) = w(z_0')$. We put $w' = w|_{\Sigma_0'}$ and $\beta' = [w|_{\Sigma_0'}]$. Then replacing $((\Sigma_0, z_0), w)$ by $((\Sigma_0', z_0'), w')$, we can apply the same argument as in (Case 1) and prove that the image $w(z_0')(= w(z_0))$ is contained in the image of the evaluation map of the moduli space in (43.11.1), (43.11.3) or (43.11.4).

(Case 3) : Finally we assume that w is constant on all the disc components of Σ'. We take a sphere component Σ_1 on which w is non-constant. By the same argument as in (Case 2), we may assume that Σ_1 has attached to Σ_0 and so there is a point $z_1 \in \Sigma_1$ with $w(z_1) = w(z_0)$. Denote $\alpha = [w|_{\Sigma_1}]$. Then $ev_0((\Sigma, z_0), w) = w(z_1)$ is contained in the image of the evaluation map of the moduli space in (43.11.2). As before we can replace w by somewhere injective one.

The proof of Lemma 43.12 is then complete. □

Now we order the moduli spaces appearing in (43.11.1)–(43.11.4) by the energy $E(\beta')$, $E(\alpha)$ or $E(\beta') + E(\alpha)$ respectively. We denote them as $S(\beta, \ell)$ which are precisely those appearing in Proposition 43.9.

The identity (43.10.3) immediately follows from Lemma 43.12 and (43.10.5) follows from the proof of Lemma 43.12.

We now check (43.10.1). If $S(\beta, \ell)$ is the moduli space from (43.11.1), then (43.10.1) is obvious by the inequality $\mu_L(\beta') < \mu_L(\beta)$.

Next consider the case where $S(\beta, \ell)$ is one of the moduli spaces of the type

$$(43.11.2.1) \qquad (\mathcal{M}_1^{\mathrm{reg}}(M; \alpha)^{\mathrm{inj}} \setminus \mathcal{M}_1^{\mathrm{reg}}(M; \alpha)^{\mathrm{inj},\tau})_{ev_{\mathrm{int}}} \times_M L$$

from (43.11.2). We note that spheres in $\mathcal{M}_1^{\rm reg}(M;\alpha)^{\rm inj} \setminus \mathcal{M}_1^{\rm reg}(M;\alpha)^{{\rm inj},\tau}$ are τ-somewhere injective by definition and so Lemma 39.7 implies that for a generic choice of $J \in \mathcal{J}_\omega^\tau$, $ev_{\rm int} : (\mathcal{M}_1^{\rm reg}(M;\alpha)^{\rm inj} \setminus \mathcal{M}_1^{\rm reg}(M;\alpha)^{{\rm inj},\tau}) \to M$ is transversal to L. We remark that it is essential here to remove $\mathcal{M}_1^{\rm reg}(M;\alpha)^{{\rm inj},\tau}$ since we need to take $J \in \mathcal{J}_\omega^\tau$ for our purpose. (See Remark 43.15. There we will show an example that this transversality breaks down without removing $\mathcal{M}_1^{\rm reg}(M;\alpha)^{{\rm inj},\tau}$.) Then we have

$$\begin{aligned}
&\dim \big((\mathcal{M}_1^{\rm reg}(M;\alpha)^{\rm inj} \setminus \mathcal{M}_1^{\rm reg}(M;\alpha)^{{\rm inj},\tau}) \;{}_{ev_{\rm int}}\times_M L\big)\\
&\quad = 2 + (2n + 2c_1(M)(\alpha) - 6) + n - 2n \\
&\quad \le n + \mu_L(\beta) - 4 < n + \mu_L(\beta) - 2 = \dim \mathcal{M}_1(L;\beta),
\end{aligned}$$

as required for the type (43.11.2.1). On the other hand, we find that the removed part

$$\mathcal{M}_1^{\rm reg}(M;\alpha)^{{\rm inj},\tau}{}_{ev_{\rm int}}\times_M L \tag{43.11.2.2}$$

in (43.11.2) has smaller dimension than (43.11.2.1) because we have

$$\begin{aligned}
&\dim \big(\mathcal{M}_1^{\rm reg}(M;\alpha)^{{\rm inj},\tau} \;{}_{ev_{\rm int}}\times_M L\big)\\
&\quad = 2 + \frac{1}{2}(2n + 2c_1(M)(\alpha) - 6) + n - 2n \\
&\quad \le 2 + (2n + 2c_1(M)(\alpha) - 6) + n - 2n \\
&\quad < \dim \mathcal{M}_1(L;\beta).
\end{aligned}$$

We recall that $\mathcal{M}_1^{\rm reg}(M;\alpha)^{{\rm inj},\tau}$ is not the τ-fixed point set of $\mathcal{M}_1^{\rm reg}(M;\alpha)^{\rm inj}$ but is the inverse image of the τ-fixed point set $\mathcal{M}^{\rm reg}(M;\alpha)^{{\rm inj},\tau}$ in $\mathcal{M}^{\rm reg}(M;\alpha)^{\rm inj}$ under the forgetful map of the marked point. So $ev_{\rm int}$ may not take values in $L = M^\tau$. Here we used the following slight modification of Lemma 39.14 for the calculation of the dimension of (43.11.2.2). Hence we have checked (43.10.1) for the case (43.11.2).

Lemma 39.14bis. *Let Q be a smooth singular chain in M. For a generic $J \in \mathcal{J}_\omega^\tau$ (depending on Q), the evaluation map $ev_{\rm int} : \mathcal{M}_1^{\rm reg}(M;\alpha)^{{\rm inj},\tau} \to M$ is transversal to Q in addition to the condition that the operator* (39.12.1) *is surjective.*

We next consider the case of (43.11.3). We also separate the argument into two cases as above. We observe that the fiber product

$$(\mathcal{M}_1^{\rm reg}(M;\alpha)^{\rm inj} \setminus \mathcal{M}_1^{\rm reg}(M;\alpha)^{{\rm inj},\tau}) \;{}_{ev_{\rm int}}\times_{ev_{\rm int}} \mathcal{M}_{1,1}^{\rm reg}(L;\beta') \tag{43.11.3.1}$$

is transversal for a generic $J \in \mathcal{J}_\omega^\tau$. (We again use the fact that the τ-invariant part $\mathcal{M}_1^{\rm reg}(M;\alpha)^{{\rm inj},\tau}$ was removed from $\mathcal{M}_1^{\rm reg}(M;\alpha)^{\rm inj}$.) Hence

$$\begin{aligned}
&\dim \big((\mathcal{M}_1^{\rm reg}(M;\alpha)^{\rm inj} \setminus \mathcal{M}_1^{\rm reg}(M;\alpha)^{{\rm inj},\tau})_{ev_{\rm int}} \times_{ev_{\rm int}} \mathcal{M}_{1,1}^{\rm reg}(L;\beta')\big)\\
&\quad = (2n + 2c_1(M)(\alpha) - 4) + (n + \mu_L(\beta')) - 2n \\
&\quad \le n + \mu_L(\beta) - 4 < n + \mu_L(\beta) - 2 = \dim \mathcal{M}_1(L;\beta),
\end{aligned}$$

as required. For the case

$$\mathcal{M}_1^{\rm reg}(M;\alpha)^{{\rm inj},\tau} \ {}_{ev_{\rm int}}\times_{ev_{\rm int}} \mathcal{M}_{1,1}^{\rm reg}(L;\beta'), \tag{43.11.3.2}$$

we can show, by using Lemma 39.14bis, that it has smaller dimension than (43.11.3.1) in a way similar to the case of (43.11.2).

We finally consider the case (43.11.4). In the same way as above, we have

$$\begin{aligned}
&\dim\big((\mathcal{M}_1^{\rm reg}(M;\alpha)^{\rm inj}\setminus \mathcal{M}_1^{\rm reg}(M;\alpha)^{{\rm inj},\tau})_{ev_{\rm int}}\times_{ev_1}\mathcal{M}_{2,0}^{\rm reg}(L;\beta')\big)\\
&\quad = (2n+2c_1(M)(\alpha)-4)+(n+\mu_L(\beta'))-1-2n\\
&\quad < \dim\mathcal{M}_1(L;\beta),
\end{aligned}$$

as required. For the part $\mathcal{M}_1^{\rm reg}(M;\alpha)^{{\rm inj},\tau}{}_{ev_{\rm int}}\times_{ev_1}\mathcal{M}_{2,0}^{\rm reg}(L;\beta')$, the argument is similar.

Therefore we completed the proof of Proposition 43.9 for the part $S(\beta_i,\ell)$ or $\mathcal{M}_1^{\rm sing}(L;\beta)$.

Now we consider the case of $S(\beta_i,\ell,j)$. This will involve the moduli spaces $\mathcal{M}_2(L;\beta,P_j)$. We will show that the singular strata thereof consist of the following types, which replaces (43.11) for this case :

(43.13.1) $\mathcal{M}_2^{\rm reg}(L;\beta_1)\times_L\mathcal{M}_2^{\rm reg}(L;\beta_2)\cdots\times_L\mathcal{M}_2^{\rm reg}(L;\beta_m)\times_L P_j$. Here $m>1$ and $\sum_{k=1}^m E(\beta_k)\le E(\beta)$, $\sum_{k=1}^m\mu_L(\beta_i)\le\mu_L(\beta)$. (Figure 43.6).
(43.13.2) $\mathcal{M}_2^{\rm reg}(L;\beta',P_j)$. Here $E(\beta')<E(\beta)$ and $\mu_L(\beta')<\mu_L(\beta)$.
(43.13.3) $\mathcal{M}_1^{\rm reg}(M;\alpha)^{\rm inj}{}_{ev_{\rm int}}\times_M P_j$. Here the element $\alpha\in\pi_2(M)$ is assumed to satisfy $E(\alpha)\le E(\beta)$, $2c_1(M)(\alpha)\le\mu_L(\beta)$. (Figure 43.7).
(43.13.4.1) $\mathcal{M}_1^{\rm reg}(M;\alpha)^{\rm inj}\ {}_{ev_{\rm int}}\times_{ev_{\rm int}}\mathcal{M}_{2,1}^{\rm reg}(L;\beta',P_j)$. Here

$$\mathcal{M}_{2,1}^{\rm reg}(L;\beta',P_j)=\mathcal{M}_{2,1}^{\rm reg}(L;\beta')\,{}_{ev_1}\times_L P_j.$$

The elements $\alpha\in\pi_2(M)$, $\beta'\in\Pi(L)$ are assumed to satisfy $E(\alpha)+E(\beta')\le E(\beta)$ and $2c_1(M)(\alpha)+\mu_L(\beta')\le\mu_L(\beta)$. (Figure 43.8).
(43.13.4.2) $\mathcal{M}_1^{\rm reg}(M;\alpha)^{\rm inj}\ {}_{ev_{\rm int}}\times_{ev_1}\mathcal{M}_{2,0}^{\rm reg}(L;\beta',P_j)$. Here

$$\mathcal{M}_{2,0}^{\rm reg}(L;\beta',P_j)=\mathcal{M}_{2,0}^{\rm reg}(L;\beta')\,{}_{ev_1}\times_L P_j.$$

The elements $\alpha\in\pi_2(M)$, $\beta'\in\Pi(L)$ are assumed to satisfy $E(\alpha)+E(\beta')\le E(\beta)$ and $2c_1(M)(\alpha)+\mu_L(\beta')\le\mu_L(\beta)$. (Figure 43.9).
(43.13.4.3) $(ev_{\rm int}(\mathcal{M}_1^{\rm reg}(M;\alpha)^{\rm inj})\cap P_j)\times_{ev_1}\mathcal{M}_{2,0}^{\rm reg}(L;\beta')$. Here the elements $\alpha\in\pi_2(M)$, $\beta'\in\Pi(L)$ are assumed to satisfy $E(\alpha)+E(\beta')\le E(\beta)$ and $2c_1(M)(\alpha)+\mu_L(\beta')\le\mu_L(\beta)$. (Figure 43.10).
(43.13.5.1) $(P_j\times\mathcal{M}_1^{\rm reg}(M;\alpha)^{\rm inj})\ {}_{(f_j,ev_{\rm int})}\times_{(ev_1,ev_2)}\mathcal{M}_{3,0}^{\rm reg}(L;\beta')$. Here the elements $\alpha\in\pi_2(M)$, $\beta'\in\Pi(L)$ are assumed to satisfy $E(\alpha)+E(\beta')\le E(\beta)$ and $2c_1(M)(\alpha)+\mu_L(\beta')\le\mu_L(\beta)$. (Figure 43.11).

(43.13.5.2) $(\mathcal{M}_1^{\text{reg}}(M;\alpha)^{\text{inj}} \times P_j)\ {}_{(ev_{\text{int}},f_j)} \times_{(ev_1,ev_2)} \mathcal{M}_{3,0}^{\text{reg}}(L;\beta')$. Here the elements $\alpha \in \pi_2(M)$, $\beta' \in \Pi(L)$ are assumed to satisfy $E(\alpha) + E(\beta') \le E(\beta)$ and $2c_1(M)(\alpha) + \mu_L(\beta') \le \mu_L(\beta)$. (Figure 43.12).

(43.13.6) $\mathcal{M}_1^{\text{reg}}(L;\beta')\ {}_{ev_0} \times_L P_j$. Here $E(\beta') \le E(\beta)$ and $\mu_L(\beta') \le \mu_L(\beta)$. (Figure 43.13).

Figure 43.6. (43.13.1)

Figure 43.7. (43.13.3)

Figure 43.8. (43.13.4.1)

Figure 43.9. (43.13.4.2)

Figure 43.10. (43.13.4.3)

Figure 43.11. (43.13.5.1)

Figure 43.12. (43.13.5.2)

Figure 43.13. (43.13.6)

The following lemma will be essential for the proof of collapsing of the spectral sequence associated to the Floer coboundary map in the E_2-level. See §43.5.

Lemma 43.14. *The image* $ev_0(\mathcal{M}_2^{\rm sing}(L,\beta,P_j))$ *is contained in the union of the image of evaluation maps of the moduli spaces in* (43.13.1) – (43.13.6).

Proof. Let $((\Sigma,(z_0,z_1)),w,x)$ be an element of $\mathcal{M}_2^{\rm sing}(L;\beta,P_j)$. (Namely we have $((\Sigma,(z_0,z_1)),w) \in \mathcal{M}_2^{\rm sing}(L;\beta)$ and $x \in P_j$ such that $w(z_1) = f_j(x)$.)

We consider the minimum union Σ' of (disc) components of Σ such that Σ' is connected and contains both z_0 and z_1. (Figure 43.14).

Figure 43.14.

(Case 1) : If Σ' contains at least two components on which w is nontrivial, then

$$ev_0((\Sigma,(z_0,z_1)),w,x) = w(z_0)$$

lies in the image of ev_0 of the moduli space of the form (43.13.1). (See Figure 43.6.)

(Case 2) : We next assume that Σ' contains exactly one component Σ'_0 on which w is nonconstant. Let $\beta' = [w|_{\Sigma'_0}]$. ($\Sigma'_0$ may or may not coincide with Σ_0.) There exist $z'_0, z'_1 \in \Sigma'_0$ such that $w(z_0) = w(z'_0)$, $w(z_1) = w(z'_1)$.

If there is at least one disc component other than Σ'_0 on which w is nonconstant, then Condition 43.5 implies that $\mu_L(\beta') < \mu_L(\beta)$ and $E(\beta') < E(\beta)$. Therefore $w(z_0) = w(z'_0)$ is contained in $\mathcal{M}_2(L;\beta',P_j)$ satisfying (43.13.2).

Let us assume Σ'_0 is the only disc component of Σ on which w is nonconstant. Then there exists a sphere component Σ_2 on which w is nontrivial. We may assume that there exist $z'_2 \in \Sigma'_0$ and $z_2 \in \Sigma_2$ such that $w(z'_2) = w(z_2)$.

Then $w(z'_0)$ is contained in the ev_0 image of the moduli space in either (43.13.4.1) or (43.13.4.2),(43.13.4.3),(43.13.5.1),(43.13.5.2), according to whether z'_2 is in the interior or the exterior of the disc Σ_0'. (See Figures 43.15–43.18 below.)

Figure 43.15. (43.13.4.1)

Figure 43.16. (43.13.4.3)

Figure 43.17. (43.13.5.1)

Figure 43.18. (43.13.4.2)

(Case 3) : We next consider the case when w is constant on Σ'.

We first consider the case when there is a disc component Σ_2 (outside Σ') on which w is nonconstant. We may assume that there is a union Σ'' of disc components joining Σ' to Σ_2 such that w is constant on Σ'' and that Σ'' is connected. It follows that there exists $z_2 \in \Sigma_2$ such that $w(z_0) = w(z_1) = w(z_2)$. Therefore $w(z_0)$ is contained in the ev_0 image of the moduli spaces in (43.13.6). (See Figure 43.19 below.)

Finally we consider the case when w is constant on all of the disc components of Σ. We then can find a sphere component Σ_2 and $z_2 \in \Sigma_2$ such that w is nonconstant on Σ_2 and $w(z_0) = w(z_1) = w(z_2)$. Thus $w(z_0)$ is contained in the image of (43.13.3). (See Figure 43.20 below.) We may replace $w|_{\Sigma_2}$ by somewhere injective one.

The proof of Lemma 43.14 is now complete. □

Figure 43.19. (43.13.6)

Figure 43.20. (43.13.3)

We define $S(\beta_i,\ell,j)$ by ordering the moduli spaces in (43.13.1) $\sim$ (43.13.6) according to its energy.

The properties (43.10.4) and (43.10.6) are easy to check from Lemma 43.14 and its proof. It is also easy to check (43.10.2) in case (43.13.1),(43.13.2). In case (43.13.3) we have

$$\begin{aligned}&\dim\big((\mathcal{M}_1^{\rm reg}(M;\alpha)^{\rm inj}\setminus\mathcal{M}_1^{\rm reg}(M;\alpha)^{{\rm inj},\tau})\ {}_{ev_{\rm int}}\times_M P_j\big)\\ &=(2n+2c_1(M)(\alpha)-4)+\dim P_j-2n\\ &\le \dim P_j-4+\mu_L(\beta)<\dim P_j-1+\mu_L(\beta)=\dim\mathcal{M}_2(L;\beta,P_j),\end{aligned}$$

as required. Thanks to Lemma 39.14bis again, we can estimate the dimension of the removed part $\mathcal{M}_1^{\rm reg}(M;\alpha)^{{\rm inj},\tau}\ {}_{ev_{\rm int}}\times_M P_j$ in a way similar to the case $S(\beta,\ell)$ as before and get also the required inequality for this part. So we do not repeat the argument. As for the other cases below, the treatment of the τ-fixed part is similar. Thus we only consider the case where the fiber product of the τ-fixed part $\mathcal{M}_1^{\rm reg}(M;\alpha)^{{\rm inj},\tau}$ is removed below.

In case (43.13.4.1), we have

$$\begin{aligned}&\dim\big((\mathcal{M}_1^{\rm reg}(M;\alpha)^{\rm inj}\setminus\mathcal{M}_1^{\rm reg}(M;\alpha)^{{\rm inj},\tau})\ {}_{ev_{\rm int}}\times_{ev_{\rm int}}\mathcal{M}_{2,1}^{\rm reg}(L;\beta',P_j)\big)\\ &=(2n+2c_1(M)(\alpha)-4)+(\dim P_j+1+\mu_L(\beta'))-2n\\ &\le \dim P_j-3+\mu_L(\beta)<\dim P_j-1+\mu_L(\beta)=\dim\mathcal{M}_2(L;\beta,P_j),\end{aligned}$$

as required. In the cases (43.13.4.2), we have

$$\begin{aligned}&\dim\big((\mathcal{M}_1^{\rm reg}(M;\alpha)^{\rm inj}\setminus\mathcal{M}_1^{\rm reg}(M;\alpha)^{{\rm inj},\tau})\ {}_{ev_{\rm int}}\times_{ev_1}\mathcal{M}_{2,0}^{\rm reg}(L;\beta',P_j)\big)\\ &=(2n+2c_1(M)(\alpha)-4)+(\dim P_j-1+\mu_L(\beta'))-2n\\ &\le \dim P_j-5+\mu_L(\beta)<\dim P_j-1+\mu_L(\beta)=\dim\mathcal{M}_2(L;\beta,P_j),\end{aligned}$$

as required. In the cases (43.13.4.3), we have

$$\begin{aligned}&\dim\big((ev_{\rm int}(\mathcal{M}_1^{\rm reg}(M;\alpha)^{\rm inj}\setminus\mathcal{M}_1^{\rm reg}(M;\alpha)^{{\rm inj},\tau})\cap P_j)\times_{ev_1}\mathcal{M}_{2,0}^{\rm reg}(L;\beta')\big)\\ &=((2n+2c_1(M)(\alpha)-4)+\dim P_j-2n)+(n-1+\mu_L(\beta'))-n\\ &\le \dim P_j-5+\mu_L(\beta)<\dim P_j-1+\mu_L(\beta)=\dim\mathcal{M}_2(L;\beta,P_j),\end{aligned}$$

as required. In case (43.13.5.1)

$$\begin{aligned}&\dim\big((P_j\times(\mathcal{M}_1^{\rm reg}(M;\alpha)^{\rm inj}\setminus\mathcal{M}_1^{\rm reg}(M;\alpha)^{{\rm inj},\tau}))\ {}_{(f_j,ev_{\rm int})}\times_{(ev_1,ev_2)}\mathcal{M}_{3,0}^{\rm reg}(L;\beta')\big)\\ &=\dim P_j+(2n+2c_1(M)(\alpha)-4)+(n+\mu_L(\beta'))-3n\\ &\le \dim P_j-4+\mu_L(\beta)<\dim P_j-1+\mu_L(\beta)=\dim\mathcal{M}_2(L;\beta,P_j),\end{aligned}$$

as required. Note we are taking fiber product over $L\times M$ whose dimension is $3n$.

The case (43.13.5.2) is similar. In case (43.13.6), we have

$$\begin{aligned}
&\dim \mathcal{M}_1(L,\beta')_{ev_0} \times_L P_j \\
&= (n+\mu_L(\beta')-2) + \dim P_j - n = \mu_L(\beta') - 2 + \dim P_j \\
&< \dim P_j - 1 + \mu_L(\beta) = \dim \mathcal{M}_2(L;\beta,P_j),
\end{aligned}$$

as required.

The proof of Proposition 43.9 is now complete. □

Remark 43.15. The conclusion of Proposition 43.9 does not hold without the assumption that $J \in \mathcal{J}_{\omega,E_0}^{c_1>0}$ (or the spherical positivity of J_0) as the following example illustrates.

Suppose that $\mathcal{M}(L,J;\beta) \neq \emptyset$ and $\mu_L(\beta) = 0$. Let $w : (D^2,\partial D^2) \to (M,L)$ be a J-holomorphic disc representing an element of $\mathcal{M}(L;\beta)$. We obtain a map $u : S^2 \to M$ by gluing w and $\tau_* w$ along their boundaries. We glue u with a trivial disc (at its interior) to obtain an element of $\mathcal{M}_1^{\rm sing}(L;\beta+\tau_*\beta)$. By varying the element (D^2,w) in $\mathcal{M}^{\rm reg}(L;\beta)$ we obtain a family X of elements of $\mathcal{M}_1^{\rm sing}(L;\beta+\tau_*\beta)$. The image by ev_0 of this family X is an $n-2$ dimensional chain in L. We remark that $n-2 = \dim \mathcal{M}_1^{\rm reg}(L;\beta+\tau_*\beta)$. Therefore we can not find $S(\beta+\tau_*\beta,\ell)$ that satisfies (43.10.1) and (43.10.3).

We remark that this phenomenon always occurs as far as there exists a pseudo-holomorphic disc with Maslov index zero.

We note that this example is also the example for which we can not extend $\tau_*^{(1)}$. (See §43.1.) Because of the presence of this phenomenon, the authors have not been able to prove the Arnold-Givental conjecture for semi-positive Lagrangian submanifold in general yet at the time of writing this book.

43.3. A topological lemma.

The condition below is the conclusion of Proposition 43.9.

Condition 43.16. We consider a sequence of smooth manifolds $S_{i,\ell}$ and smooth maps $f_{i,\ell} : S_{i,\ell} \to L$ and positive integers $d_{i,\ell}$. We assume that

(43.17.1) $\dim S_{i,\ell} < d_{i,\ell}$.

(43.17.2) For each given i and ℓ, we have

$$\left(\bigcap_{K \subseteq S_{i,\ell} \text{ is compact}} \overline{f_{i,\ell}(S_{i,\ell} \setminus K)} \right) \subset \bigcup_{\ell'<\ell, d_{i,\ell'} \le d_{i,\ell}} f_{i,\ell'}(S_{i,\ell'}).$$

(43.17.3) The union $S_i^d = \bigcup_{\ell, d_{i,\ell} \le d} f_{i,\ell}(S_{i,\ell})$ is compact.

We fix a metric on L. For each subset $X \subset L$ and $\epsilon > 0$, we consider the neighborhood given by

$$B_\epsilon(X) = \{x \in L \mid d(x, X) < \epsilon\}.$$

Lemma 43.18. *Suppose that $S_{i,\ell}$, $f_{i,\ell}$ and $d_{i,\ell}$ satisfy Condition 43.16. Then for each $\delta > 0$ there exists $\epsilon > 0$ such that the inclusion $B_\epsilon(S_i^d) \to B_\delta(S_i^d)$ induces zero on $H_d(B_\epsilon(S_i^d);\mathbb{Z}_2) \to H_d(B_\delta(S_i^d);\mathbb{Z}_2)$.*

Proof. Choose a closed domain $U \subset B_\delta(S_i^d)$ that has smooth boundary and contains $B_{\delta/2}(S_i^d)$. Let $(Q_k, \partial Q_k) \in C_*(U, \partial U)$ ($k = 1, 2, \cdots$) be the $n-d$ dimensional relative (singular) cycles that generate $H_{n-d}(U, \partial U;\mathbb{Z}_2)$. By the standard transversality theorem and a dimension counting argument, we can perturb Q_k so that $Q_k \cap f_{i,\ell}(S_{i,\ell}) = \emptyset$ for all i, ℓ with $d_{i,\ell} \le d$. This can be carried out inductively over ℓ using the conditions (43.17). By compactness of Q_i and by the finiteness of the sets $S_{i,\ell}$, there exists $\epsilon > 0$ such that $Q_k \cap B_\epsilon(S_{i,\ell}) = \emptyset$. The conclusion now follows from the Poincaré duality. $\square$

For each given $\delta > 0$, let

$$\epsilon(S_i^d, \delta)$$

be the supremum of $\epsilon > 0$ satisfying the conclusion of Lemma 43.18. We define the function of t

$$h(t; S_i^d) = \inf\{\delta \mid \epsilon(S_i^d, \delta/2) > t\}.$$

The following corollary is easy to prove by definition and Lemma 43.18, whose proof is left to readers.

Corollary 43.19. *Let $h(t; S_i^d)$ be as above. Then the followings hold :*

(43.19.1) *The natural homomorphism*

$$H_d(B_\epsilon(S_i^d);\mathbb{Z}_2) \to H_d(B_{h(\epsilon;S_i^d)}(S_i^d);\mathbb{Z}_2)$$

induced by the inclusion $B_\epsilon(S_i^d) \to B_{h(\epsilon;S_i^d)}(S_i^d)$ becomes zero for each integer d.

(43.19.2) *We have $\lim_{\epsilon\to 0} h(\epsilon; S_i^d) = 0$.*

(43.19.3) *The function $t \mapsto h(t; S_i^d)$ is defined on a neighborhood of 0 in $[0, 1)$.*

In the next two subsections we will consider another functions h'_i defined on a neighborhood of 0 in $[0, 1)$ such that $\lim_{\epsilon\to 0} h'_i(\epsilon) = 0$. Using these functions, we will inductively define $h^i(\epsilon; S)$ by the formula

$$\begin{aligned} h^1(\epsilon; S) = h(\epsilon; S), \quad h^i(\epsilon; S) = h(h'_i(h^{i-1}(\epsilon; S)); S), \\ h(\epsilon; S) = \max_{i,d} h(\epsilon; S_i^d) \quad \text{or} \quad \max_{i,d,j} h(\epsilon; S_{i,j}^d), \end{aligned} \tag{43.20}$$

where $S_i^d = \bigcup_{\ell, d_{i,\ell}<d} f_{i,\ell}(S_{i,\ell})$ is given by (43.17.3) with $S_{i,\ell} = S(\beta_i, \ell)$ or $S_{i,j}^d = \bigcup_{\ell, d_{i,\ell,j}<d} f_{i,\ell,j}(S_{i,\ell,j})$ with $S_{i,\ell,j} = S(\beta_i, \ell, j)$, respectively, as in Proposition 43.9. The former case will be used in §43.4 and the latter will be used in §43.5. Note that in our situation the possibility of i, d, j is finite.

43.4. L is unobstructed over $\mathbb{Z}_2$.

In the rest of this section, we assume that J_0 is spherically positive and $\tau : (M, J_0) \to (M, J_0)$ is an anti-holomorphic involution. By Theorem 34.7, we have a filtered A_∞ algebra $(C(L, J_0; \Lambda_{0,nov}^{\mathbb{Z}_2}), \mathfrak{m}^{J_0})$ over $\Lambda_{0,nov}^{\mathbb{Z}_2}$ coefficient. Our goal of this subsection is to prove that $(C(L, J_0; \Lambda_{0,nov}^{\mathbb{Z}_2}), \mathfrak{m}^{J_0})$ is unobstructed. For given $E > 0$, we take $\mathcal{V}_{\omega \le E}(J_0)$ as in Definition 43.8 and Proposition 43.9. We may assume that $\mathcal{V}_{\omega \le E}(J_0)$ is connected.

We enumerate the set $\Pi(M, E, \mathcal{V}_{\omega \le E}(J_0))$ which we defined in Definition 43.8, as

$$0 = \beta_0,\, \beta_1, \cdots, \beta_{m'} \in \Pi(L) = \pi_2(M, L)/\sim$$

so that $E(\beta_i) \le E(\beta_{i+1})$. We list the energies in the set $\{E(\beta_i) \mid i = 0, \cdots m'\}$ as

$$E_0 < E_1 < \cdots < E_m.$$

For simplicity of notation, we assume that

$$E(\beta_0) < E(\beta_1) < \cdots < E(\beta_m),$$

that is,

$$m = m'.$$

We will prove the following by induction on i.

Proposition 43.21. *Let $J \in \mathcal{V}_{\omega \le E}(J_0)$ satisfy the conclusion of Proposition 43.9. There is a choice of*

1. *a sequence of $b(\beta_i) \in C^{1-\mu_L(\beta_i)}(L, J)$,*
2. *a sequence of functions $h_i' : [0, \epsilon) \to \mathbb{R}_+$ satisfying $h_i'(\epsilon) \to 0$ as $\epsilon \to 0$,*
3. *a compatible system of single valued sections $\mathfrak{s}^\epsilon$ of obstruction bundles for $\mathcal{M}_k(L, J; \beta_i)$,*
4. *the countable set of chains $\mathcal{X}_g(L, J)$ as in §30,*

so that they satisfy the following properties :

(43.22.1) *For each i, we have*

$$\mathfrak{m}_{1,0}^J(b(\beta_i)) + \sum_k \sum_* \mathfrak{m}_{k,\beta_{i(0)}}^J(b(\beta_{i(1)}), \cdots, b(\beta_{i(k)})) = 0.$$

Here summation $*$ is taken over all $i(0), \cdots, i(k)$ such that

$$\sum_j E(\beta_{i(j)}) = E(\beta_i), \quad \sum_j \mu_L(\beta_{i(j)}) = \mu_L(\beta_i).$$

(43.22.2) *The support of cochain $b(\beta_i)$ is contained in $\bigcup_\ell B_{h^i(\epsilon;S)}(f_{i,\ell}(S(\beta_i,\ell)))$. Here h^i is inductively defined by* (43.20)*, using h'_i given in* (2) *above.*
(43.22.3) *$\mathfrak{s}^\epsilon$ satisfies the conclusion of Theorem* 34.11*.*

Proof. The proof is by induction on i. For the first step $\beta_0 = 0$ there is nothing to show. We assume the proposition for all i' with $i' < i$ and prove it for i. We consider

$$\mathfrak{m}^J_{0,\beta_i}(1) = (ev_0)_*\left(\mathcal{M}_1(L,J;\beta_i)^{\mathfrak{s}^\epsilon}\right) \in C^{2-\mu_L(\beta_i)}(L,J;\mathbb{Z}_2),$$

($\mathfrak{s}^\epsilon$ will be determined later). We denote

$$o(\beta_i) = \mathfrak{m}^J_{0,\beta_i}(1) + \sum_k \sum_* \mathfrak{m}^J_{k,\beta_{i(0)}}(b(\beta_{i(1)}), \cdots, b(\beta_{i(k)})).$$

Here the summation $*$ is taken over all $i(0), \cdots, i(k)$ satisfying $\sum E(\beta_{i(j)}) = E(\beta_i)$, $\sum \mu_L(\beta_{i(j)}) = \mu_L(\beta_i)$. As in the proof of Theorem 11.8, we find that

$$\delta(o(\beta_i)) = 0. \tag{43.23}$$

We now prove :

Lemma 43.24. *There exist functions h'_i such that $h'_i(\epsilon) \to 0$ as $\epsilon \to 0$ and we can choose $\mathfrak{s}^\epsilon$ so that the support of the cochain $o(\beta_i)$ defined above is contained in*

$$B_{h'_i(h^{i-1}(\epsilon;S))}\left(\bigcup_\ell f_{i,\ell}(S(\beta_i,\ell))\right).$$

Proof. Proposition 41.13 implies that we may choose $\mathfrak{s}^\epsilon$ so that the chain $ev_{0*}\left(\mathcal{M}_1(L,J;\beta_i)^{\mathfrak{s}^\epsilon}\right)$ cancels each other to give zero outside an arbitrary small neighborhood of $ev_{0*}(\mathcal{M}_1^{\rm sing}(L,J;\beta_i))$. Therefore we may choose $\mathfrak{s}^\epsilon$ so that the support of $\mathfrak{m}_{0,\beta_i}(1)$ is contained in the set $B_{h'_i(h^{i-1}(\epsilon;S))}(\bigcup_\ell f_{i,\ell}(S(\beta_i,\ell)))$. More precisely, in a neighborhood of $\bigcup_\ell f_{i,\ell}(S(\beta_i,\ell))$, we take a normally polynomial perturbation as in §35 and use Proposition 43.13 outside a neighborhood of $\bigcup_\ell f_{i,\ell}(S(\beta_i,\ell))$ to obtain $\mathfrak{s}^\epsilon$. Here it is essential to use normally polynomial perturbation in order the perturbed moduli space to have a smooth triangulation.

Let us consider the other terms

$$\mathfrak{m}^J_{k,\beta_{i(0)}}(b(\beta_{i(1)}), \cdots, b(\beta_{i(k)})). \tag{43.25}$$

By the induction hypothesis, the support of (43.25) is contained in the union of

$$ev_0\left(\mathcal{M}_{k+1}(L,J;\beta_{i(0)}) \times_{L^k} \prod_{j=1}^k B_{h^{i-1}(\epsilon;S)}\left(\bigcup_\ell f_{i(j),\ell}(S(\beta_{i(j)},\ell))\right)\right), \tag{43.26}$$

where we have $\sum_j E(\beta_{i(j)}) = E(\beta_i)$, $\sum_j \mu_L(\beta_{i(j)}) = \mu_L(\beta_i)$.

Sublemma 43.27.

$$ev_0\left(\mathcal{M}_{k+1}(L,J;\beta_{i(0)})\times_{L^k}\prod_{j=1}^{k}\bigcup_{\ell} f_{i(j),\ell}(S(\beta_{i(j)},\ell))\right)\subseteq ev_0\left(\bigcup_{\ell} f_{i,\ell}(S(\beta_i,\ell))\right).$$

Proof. We recall that $S(\beta_{i(j)},\ell)$ is one of the moduli spaces of the form (43.11.1), (43.11.2), (43.11.3) or (43.11.4). After taking a fiber product, we obtain an element of $\mathcal{M}_1^{\rm sing}(L,J;\beta')$ with $E(\beta')\le E(\beta_i)$, $\mu_L(\beta')\le\mu_L(\beta_i)$. Sublemma 43.27 then follows from Lemma 43.12. □

Sublemma 43.27 implies that we can choose $h_i':[0,\epsilon_i)\to\mathbb{R}_+$ for some $\epsilon_i>0$ so that (43.26) is contained in the neighborhood

$$B_{h_i'(h^{i-1}(\epsilon;S))}\left(\bigcup_{\ell} f_{i,\ell}(ev_0(S(\beta_i,\ell)))\right)$$

of $\bigcup_\ell f_{i,\ell}(ev_0(S(\beta_i,\ell)))$. The proof of Lemma 43.24 is complete. □

Lemma 43.24, (43.23) and Lemma 43.18 imply that we can find $b(\beta_i)$ satisfying (43.22.1) and (43.22.2). We may choose $\mathcal{X}_g(L,J)$ so that $C(L,J)$ contains $b(\beta_i)$. The proof of Proposition 43.21 is now complete. □

We choose $J(E)\in\mathcal{V}_{\omega\le E}(J_0)$ such that the conclusion of Proposition 43.9 holds. Since $J(E)\in\mathcal{J}^{c_1>0}_{\omega,E}$ it follows from the proof of Theorem 34.7 (and the argument of §30) that filtered $A_{\overline{n}(E),\overline{K}(E)}$ algebra $(C(L,J(E);\Lambda^{\mathbb{Z}_2}_{0,nov}),\mathfrak{m}^{J(E)})$ is defined with $\lim_{E\to\infty}\overline{n}(E)=\lim_{E\to\infty}\overline{K}(E)=\infty$. Moreover since $\mathcal{V}_{\omega\le E}(J_0)$ is connected it follows from $A_{\overline{n}(E),\overline{K}(E)}$ version of Theorem 34.7 that $(C(L,J(E);\Lambda^{\mathbb{Z}_2}_{0,nov}),\mathfrak{m}^{J(E)})$ is $A_{\overline{n}(E),\overline{K}(E)}$ homotopy equivalent to $(C(L,J_0;\Lambda^{\mathbb{Z}_2}_{0,nov}),\mathfrak{m}^{J_0})$. Therefore, Proposition 43.21 implies that we can make the choice so that the obstruction of the filtered A_∞ algebra $(C(L,J_0;\Lambda^{\mathbb{Z}_2}_{0,nov}),\mathfrak{m}^{J_0})$ vanishes up to the m-stage for any given m. We then prove the following :

Corollary 43.28. $(C(L,J_0;\Lambda^{\mathbb{Z}_2}_{0,nov}),\mathfrak{m}^{J_0})$ *is unobstructed.*

Proof. We have already proved that $(C(L,J_0;\Lambda^{\mathbb{Z}_2}_{0,nov}),\mathfrak{m}^{J_0})$ is unobstructed modulo T^E for any $E>0$. To complete the proof, we proceed in the same way as in the proof of Lemma 30.156. We replace $C(L,J_0;\Lambda^{\mathbb{Z}_2}_{0,nov})$ by the canonical model $(H^*(L;\Lambda^{\mathbb{Z}_2}_{0,nov}),\mathfrak{m})$. Then the existence of $b(\beta_i)$ up to $E(\beta_i)\le E$ is equivalent to the existence of a solution of finitely many algebraic equations with finitely many variables. In particular (since we are working over the $\mathbb{Z}_2$-coefficient), the set

$$B(E)=\{b\in H^1(L;\Lambda^{\mathbb{Z}_2}_{0,nov})/T^EH^1(L;\Lambda^{\mathbb{Z}_2}_{0,nov})\mid\widehat{d}(e^b)\equiv 0\mod T^E\}$$

is a finite set. For $E < E'$, there exists a natural map $B(E') \to B(E)$. It follows that this defines an inverse system $\{B(E)\}$. The projective limit

$$\varprojlim B(E)$$

coincides with $\widehat{\mathcal{M}}(H^*(L;\Lambda_{0,nov}^{\mathbb{Z}_2}),\mathfrak{m})$ (that is the set of all bounding cochains). Proposition 43.21 implies that $B(E)$ for each given $E > 0$ is nonempty. Since they are finite sets, it follows that the projective limit is also nonempty. This implies that L is unobstructed over $\mathbb{Z}_2$. $\square$

43.5. Degeneration of spectral sequence for $HF(L,L)$ over $\mathbb{Z}_2$.

In this subsection we prove existence of a bounding cochain b for which we have the isomorphism

$$HF((L,b),(L,b);\Lambda_{0,nov}^{\mathbb{Z}_2}) \cong H(L;\mathbb{Z}_2) \otimes \Lambda_{0,nov}^{\mathbb{Z}_2}$$

and complete the proof of Theorem 34.16. We use the same notation as §43.4.

We recall that the cohomology classes dual to $\{P_j \mid j = 1,\cdots,\dim H(L;\mathbb{Z}_2)\}$ form a basis of $H(L;\mathbb{Z}_2)$. We will construct $Q(\beta_i,P_j)$ so that the chain

$$\tilde P_j = P_j + \sum_{i\,;\,E(\beta_i)<E} T^{E(\beta_i)} e^{\mu_L(\beta_i)/2} Q(\beta_i,P_j) \tag{43.29}$$

will satisfy

$$\mathfrak{m}^J(e^b,\tilde P_j,e^b) \equiv 0 \mod T^E. \tag{43.30}$$

This then implies that $\delta_r[P_j] = 0$ for r satisfying $r\lambda_0 < E$.

For this purpose, we will prove the following by the induction.

Proposition 43.31. *There exist a choice of b (a bounding cochain modulo T^E), a sequence of chains $Q(\beta_i,P_j) \in C^{\deg P_j+1-\mu_L(\beta_i)}(L,J;\mathbb{Z}_2)$, a compatible system of single valued sections $\mathfrak{s}^\epsilon$ of the obstruction bundles, the countable set of chains $\mathcal{X}_g(L,J)$ as in §30, and the functions h'_i, with the following properties.*

(43.32.1) *We have :*

$$\begin{aligned}&\sum_{i'<i}\mathfrak{m}^J(e^b,T^{E(\beta_{i'})}e^{\mu_L(\beta_{i'})/2}Q(\beta_{i'},P_j),e^b)\\&+\mathfrak{m}^J(e^b,P_j,e^b)+\mathfrak{m}^J_{1,\beta_0}(T^{E(\beta_i)}e^{\mu_L(\beta_i)/2}Q(\beta_i,P_j)) \equiv 0 \mod T^{E_{i+1}}.\end{aligned}$$

(43.32.2) *The support of* $Q(\beta_i, P_j)$ *is in* $\bigcup_\ell B_{h^i(\epsilon;S)}(f_{i,\ell,j}(S(\beta_i,\ell,j)))$. *Here* h^i *is inductively defined by* (43.20), *using* h'_i, *as in Proposition* 43.21.
(43.32.3) $\mathfrak{s}^\epsilon$ *satisfies the conclusion of Theorem* 34.11.

We remark that (43.32.1) is equivalent to (43.30).

Proof. We first choose b's for which Lemma 43.24 holds. The lemma will then be proved by induction on i. Suppose that $Q(\beta_{i'}, P_j)$ have been constructed for $\beta_{i'}$ with $E(\beta_{i'}) < E(\beta_i)$. We then define

$$\begin{aligned} o(\beta_i, P_j) = &\\ &\sum_{k_1,k_2,i'<i,j}\sum_{(1)} \mathfrak{m}^J_{k,\beta_{i(0)}}(b(\beta_{i(1,1)}),\cdots,b(\beta_{i(1,k_1)}),\\ &\qquad\qquad Q(\beta_{i'},P_j), b(\beta_{i(2,1)}),\cdots,b(\beta_{i(2,k_2)}))\\ &+\sum_{k_1,k_2}\sum_{(2)} \mathfrak{m}^J_{k,\beta_{i(0)}}(b(\beta_{i(1,1)}),\cdots,b(\beta_{i(1,k_1)}),\\ &\qquad\qquad P_j, b(\beta_{i(2,1)}),\cdots,b(\beta_{i(2,k_2)})). \end{aligned}$$

Here the first summation over (1) stands for the summation over $i(0)$, $i(1,*)$, $i(2,*)$ and i' that satisfy

$$\begin{cases} E(\beta_{i(0)}) + E(\beta_{i(1,1)}) + \cdots + E(\beta_{i(1,k_1)}) + E(\beta_{i'}) \\ \qquad\qquad + E(\beta_{i(2,1)}) + \cdots + E(\beta_{i(2,k_2)}) = E(\beta_i), \\ \mu_L(\beta_{i(0)}) + \mu_L(\beta_{i(1,1)}) + \cdots + \mu_L(\beta_{i(1,k_1)}) + \mu_L(\beta_{i'}) \\ \qquad\qquad + \mu_L(\beta_{i(2,1)}) + \cdots + \mu_L(\beta_{i(2,k_2)}) = \mu_L(\beta_i). \end{cases} \tag{43.33}$$

The second summation over (2) is taken over those $i(0)$, $i(1,*)$ and $i(2,*)$ that satisfy

$$\begin{cases} E(\beta_{i(0)}) + E(\beta_{i(1,1)}) + \cdots + E(\beta_{i(1,k_1)}) + \\ \qquad E(\beta_{i(2,1)}) + \cdots + E(\beta_{i(2,k_2)}) = E(\beta_i), \\ \mu_L(\beta_{i(0)}) + \mu_L(\beta_{i(1,1)}) + \cdots + \mu_L(\beta_{i(1,k_1)}) + \\ \qquad \mu_L(\beta_{i(2,1)}) + \cdots + \mu_L(\beta_{i(2,k_2)}) = \mu_L(\beta_i). \end{cases} \tag{43.34}$$

(43.32.1) then is equivalent to

$$o(\beta_i, P_j) + \mathfrak{m}^J_{1,\beta_0}(Q(\beta_i, P_j)) = 0. \tag{43.35}$$

Lemma 43.36. $\mathfrak{m}^J_{1,\beta_0}(o(\beta_i, P_j)) = 0.$

Proof. By the induction hypothesis, we have

$$\sum_{i'<i} \mathfrak{m}^J((e^b, e^{\mu_L(\beta_{i'})/2} T^{E(\beta_{i'})} Q(\beta_{i'}, P_j), e^b) + \mathfrak{m}^J(e^b, P_j, e^b) \equiv 0 \mod T^{E_i}. \tag{43.37}$$

Then, by definition,

$$\begin{aligned}&\sum_{i'<i} \mathfrak{m}^J((e^b, e^{\mu_L(\beta_{i'})/2} T^{E(\beta_{i'})} Q(\beta_{i'}, P_j), e^b) + \mathfrak{m}^J(e^b, P_j, e^b)\\ &\equiv T^{E(\beta_i)} e^{\mu_L(\beta_i)/2} o(\beta_i, P_j) \mod T^{E_{i+1}}.\end{aligned} \tag{43.38}$$

The A_∞ formula and $\widehat{d}(e^b) = 0$ imply

$$\sum_{i'<i} \mathfrak{m}^J(e^b, \mathfrak{m}^J(e^b, e^{\mu_L(\beta_{i'})/2} T^{E_{i'}} Q(\beta_{i'}, P_j), e^b), e^b) + \mathfrak{m}^J(e^b, \mathfrak{m}^J(e^b, P_j, e^b), e^b) = 0. \tag{43.39}$$

By (43.38) the coefficient of $T^{E(\beta_i)} e^{\mu_L(\beta_i)/2}$ of (43.39) is $\mathfrak{m}^J_{1,\beta_0}(o(\beta_i, P_j)) = 0.$ □

Lemma 43.40. *We can choose h'_i with $\lim_{\epsilon\to 0} h'_i(\epsilon) = 0$ and then $\mathfrak{s}^\epsilon$ so that the support of $o(\beta_i, P_j)$ defined above is contained in*

$$B_{h'_i(h^{i-1}(\epsilon;S))}(f_{i,\ell,j}(S(\beta_i, \ell, j))). \tag{43.41}$$

Proof. The term $\mathfrak{m}^J_{1,\beta_i}(P_j)$ is

$$ev_{0,*}\left(\mathcal{M}_2(L; \beta_i, P_j)^{\mathfrak{s}^\epsilon}\right). \tag{43.42}$$

By Proposition 41.13 and Proposition 43.9, we may choose $\mathfrak{s}^\epsilon$ satisfying (43.32.3) such that the support of (43.42) is in (43.41). By induction hypothesis the support of the sum of the other terms are contained in the $ev_{0,*}$ images of the unions of

$$\begin{aligned}\mathcal{M}_{k_1+k_2+2}(L, J; \beta_{i(0)}) \times_{L^{k_1+k_2}} &\left(\prod_{j=1}^{k_1} B_{h^{i-1}(\epsilon;S)}\left(\bigcup_{\ell_j} f_{i(1,j),\ell_j}(S(\beta_{i(1,j)}, \ell_j))\right)\right.\\ &\times \bigcup_{\ell} B_{h^{i-1}(\epsilon;S)}\left(f_{i',\ell,j}(S(\beta_{i'}, \ell, j))\right)\\ &\left.\times \prod_{j=1}^{k_2} B_{h^{i-1}(\epsilon;S)}\left(\bigcup_{\ell'_j} f_{i(2,j),\ell'_j}(S(\beta_{i(2,j)}, \ell'_j))\right)\right)\end{aligned} \tag{43.43}$$

with (43.33), $k_1 + k_2 \geq 0$ and

(43.44)

$$\mathcal{M}_{k_1+k_2+2}(L, J; \beta_{i(0)}) \times_{L^{k_1+k_2}} \left(\prod_{j=1}^{k_1} B_{h^{i-1}(\epsilon;S)} \left(\bigcup_{\ell_j} f_{i(1,j),\ell_j}(S(\beta_{i(1,j)}, \ell_j)) \right) \right.$$
$$\left. \times P_j \times \prod_{j=1}^{k_2} B_{h^{i-1}(\epsilon;S)} \left(\bigcup_{\ell'_j} f_{i(2,j),\ell'_j}(S(\beta_{i(2,j)}, \ell'_j)) \right) \right)$$

with (43.34), $k_1 + k_2 \geq 1$. We can use Lemma 43.9 and Sublemma 43.27 to show that the $ev_{0,*}$ image of (43.43) is contained in (43.41). In the same way as the proof of Lemma 43.14, we can show that the $ev_{0,*}$ image of (43.44) is contained in (43.41). The proof of Lemma 43.40 is complete. □

We go back to the proof of Proposition 43.31. Combination of Lemmas 43.18, 43.36 and 43.40 now give rise to the construction of $Q(\beta_i, P_j)$'s. The proof of Proposition 43.31 is complete. □

We now study the spectral sequence associated to $(C(L, J_0, \Lambda_{0,nov}^{\mathbb{Z}_2}), \mathfrak{m}_1^b)$ for an appropriate $b \in \widehat{\mathcal{M}}(L, J_0, \Lambda_{0,nov}^{\mathbb{Z}_2})$.

Corollary 43.45. *For each m, we can choose b so that the differential δ_r of the spectral sequence to calculate $HF((L,b),(L,b);\Lambda_{0,nov}^{\mathbb{Z}_2})$ is zero for $2 \leq i \leq m$.*

Proof. We use homotopy equivalence

$$C(L, J_0, \Lambda_{0,nov}^{\mathbb{Z}_2}) \to C(L, J(E), \Lambda_{0,nov}^{\mathbb{Z}_2})$$

of filtered $A_{\overline{n}(E),\overline{\lambda}(E)}$ algebras. It preserves the filtrations

$$\mathfrak{F}^q C(L, J_0, \Lambda_{0,nov}^{\mathbb{Z}_2}) = F^{q\lambda_0} C(L, J_0, \Lambda_{0,nov}^{\mathbb{Z}_2})$$
$$\mathfrak{F}^q C(L, J, \Lambda_{0,nov}^{\mathbb{Z}_2}) = F^{q\lambda_0} C(L, J, \Lambda_{0,nov}^{\mathbb{Z}_2}).$$

Hence it induces an isomorphism of spectral sequence after E_2 term. (See §24.1.) We define $\tilde{P}_j$ by (43.29). Then Proposition 43.31 implies (43.30). Therefore, if $m\lambda_0 < E_0$, then the differential $\delta_r[P_j]$ is zero for $2 \leq r \leq m$. (This is a consequence of (43.30).) Corollary 43.45 follows. □

Remark 43.46. In the above argument we consider the moduli spaces $\mathcal{M}(L, J; \beta)$ for $E(\beta) < E$ for some fixed E only. This is the reason why it suffices to assume $J \in \mathcal{J}_{\omega,E}^{c_1>0}$ so that we have virual fundamental chains over $\mathbb{Z}_2$.

Corollary 43.47. *We may choose b so that the spectral sequence calculating the Floer cohomology $HF((L,b),(L,b);\Lambda_{0,nov}^{\mathbb{Z}_2})$ degenerates at the E_2-level.*

Proof. We replace $C(L;\Lambda_{0,nov}^{\mathbb{Z}_2})$ by the canonical model $(H^*(L;\Lambda_{0,nov}^{\mathbb{Z}_2}),\mathfrak{m})$. Let $P_1,\cdots,P_m$ be a basis of $H^*(L;\mathbb{Z}_2)$. We put $d_j = \deg P_j$. We consider the set of $(\tilde b,\tilde P_1,\cdots,\tilde P_m)$ such that

(43.48.1) $\tilde b \in H^1(L;\Lambda_{0,nov}^{\mathbb{Z}_2})/T^E H^1(L;\Lambda_{0,nov}^{\mathbb{Z}_2})$.

(43.48.2) $\tilde P_j \in H^{d_j}(L;\Lambda_{0,nov}^{\mathbb{Z}_2})/T^E H^{d_j}(L;\Lambda_{0,nov}^{\mathbb{Z}_2})$.

(43.48.3) $\widehat{d}(e^{\tilde b}) \equiv 0 \mod T^E$.

(43.48.4) $\tilde P_j \equiv P_j \mod \Lambda_{0,nov}^{+,\mathbb{Z}_2}$.

(43.48.5) $\mathfrak{m}(e^{\tilde b},\tilde P_j,e^{\tilde b}) \equiv 0 \mod T^E$.

Let $BDeg(E)$ be the set of such $(\tilde b,\tilde P_1,\cdots,\tilde P_m)$'s. It follows that this is a finite set. We also have a natural map $BDeg(E') \to BDeg(E)$ for $E' > E$ which defines an inverse system $\{BDeg(E) \mid E > 0\}$. It follows from Corollary 43.45 that the $BDeg(E)$ is nonempty for each E. Hence the projective limit

$$\varprojlim BDeg(E)$$

as $E \to \infty$ becomes nonempty. This proves Corollary 43.47. □

Finally, the proof of Theorem 34.16 has been completed by Corollaries 43.28 and 43.47. □