ON SEMIPOSITIVITY, INJECTIVITY AND VANISHING THEOREMS

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Dedicated to Professor Steven Zucker on the occasion of his 65th birthday

Abstract. This is a survey article on the recent developments of semipositivity, injectivity, and vanishing theorems for higher-dimensional complex projective varieties.

Contents

1. Introduction

This paper is a survey article on the recent developments of semipositivity, injectivity, and vanishing theorems for higher-dimensional complex projective varieties (see, for example, [Fj7], [Fj9], [Fj11], [FF], [FFS], and so on).

We know that many important generalizations of Kodaira vanishing theorem, for example, Kawamata–Viehweg vanishing theorem, Kollár's

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injectivity, torsion-free, and vanishing theorems, Nadel vanishing theorem, and so on, were obtained in 1980s. They have already played crucial roles in the study of higher-dimensional complex projective varieties. We note that Fujita–Zucker–Kawamata semipositivity theorem for direct images of relative canonical bundles has also played important roles. One of my main motivations was to establish a more general cohomological package based on the theory of mixed Hodge structures on cohomology with compact support. Now I think that our new results are almost satisfactory (see Theorems 2.12, 2.13, and 3.6). They are waiting for applications. I hope that the reader would find various applications of our semipositivity, injectivity, and vanishing theorems.

Let us see the contents of this paper. In Section 2, we first discuss the Hodge theoretic aspect of Kodaira type vanishing theorems (see, for example, [EV], [Ko4, Part III], [Fj7], [Fj9], [Fj11], and so on). I emphasize the importance of Kollár's injectivity theorem and its generalizations. I think that one of the most important recent developments is the introduction of mixed Hodge structures on cohomology with compact support in order to generalize Kollár's injectivity theorem (see, for example, [Fj3], [Fj7], [Fj9], [Fj11], and so on). Next we discuss Enoki's injectivity theorem, which is an analytic counterpart of Kollár's injectivity theorem. I like Enoki's idea since it is very simple and powerful. Enoki's proof only uses the standard results of the theory of harmonic forms on compact Kähler manifolds. Although I obtained some generalizations of Enoki's injectivity theorem and their applications (see [Fj4] and [Fj5]), I think that they are not satisfactory for various geometric applications. In Section 3, we treat several semipositivity theorems for direct images of relative (log) canonical bundles and relative pluricanonical bundles. The (numerical) semipositivity of direct images of relative (log) canonical bundles discussed in this paper is more or less Hodge theoretic. Note that mixed Hodge structures on cohomology with compact support are also very useful for semipositivity theorems. By considering their variations, we can prove a powerful semipositivity theorem by the theory of gradedly polarizable admissible variation of mixed Hodge structure (see [FF] and [FFS]). Unfortunately, since I am not familiar with the recent developments of semipositivity theorems by L^2-method, I do not discuss the analytic aspect of semipositivity theorems in this paper. In Subsection 3.1, we explain new semipositivity theorems for direct images of relative pluricanonical bundles with the aid of the minimal model program (see [Fj12]). I think that it is highly desirable to recover them without using the minimal model program. In Section 4, we see that pluricanonical divisors sometimes behave much better than canonical divisors. We discuss two different topics. In Subsection 4.1, we explain Kollár's famous result on plurigenera in étale covers of smooth projective varieties of general type. We give Lazarsfeld's proof using the theory of asymptotic multiplier ideal sheaves for the reader's convenience and a proof based on the minimal model program. The proof based on the minimal model program is much harder than Lazarsfeld's proof but is interesting and natural from the minimal model theoretic viewpoint. In Subsection 4.2, we explain Viehweg's ampleness theorem for direct images of relative pluricanonical bundles, which is buried in Viehweg's papers. I think that these results may help the reader to understand the reason why we should consider pluricanonical divisors for the study of higher-dimensional algebraic varieties. In Section 5, we quickly review the finite generation of (log) canonical rings due to Birkar–Cascini–Hacon–M$^{\rm c}$Kernan. I want to emphasize that we need the semipositivity theorem discussed in Section 3 when we treat (log) canonical rings for varieties which are not of (log) general type (see [FM] and [F10]). We also explain the nonvanishing conjecture, which is one of the most important conjectures for higher-dimensional complex projective varieties. Section 6 is an appendix, where we collect some definitions, which help the reader to understand this paper. The reader can read each section separately.

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We will work over $\mathbb{C}$, the complex number filed, throughout this paper. A scheme means a separated scheme of finite type over $\mathbb{C}$ in this paper

2. On injectivity theorems and vanishing theorems

I think that one of the most fundamental results for complex projective varieties is Kollár's injectivity theorem (see Theorem 2.1). The importance of Kawamata–Viehweg (or Nadel) vanishing theorem for the study of higher-dimensional complex algebraic varieties is repeatedly emphasized in many papers and textbooks (see, for example, [KoM] and [Laz]). On the other hand, I think that the importance of Kollár's injectivity theorem has not been emphasized so far in the standard literature.

Let us recall Kollár's injectivity theorem.

Theorem 2.1 ([Ko1, Theorem 2.2]). *Let X be a smooth projective variety and let L be a semiample Cartier divisor on X, that is, the complete linear system $|mL|$ has no base points for some positive integer m. Let D be a member of $|kL|$ for some positive integer k. Then*

$$H^i(X, \mathcal{O}_X(K_X + lL)) \to H^i(X, \mathcal{O}_X(K_X + (l+k)L)),$$

which is induced by the natural inclusion $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(D) \simeq \mathcal{O}_X(kL)$, is injective for every i and every positive integer l.

Remark 2.2. If we assume that L is ample, $l = 1$, and k is sufficiently large in Theorem 2.1, then we obtain that

$$H^i(X, \mathcal{O}_X(K_X + L)) \hookrightarrow H^i(X, \mathcal{O}_X(K_X + (1+k)L)) = 0$$

for every $i > 0$ by Serre vanishing theorem. Therefore, Theorem 2.1 quickly recovers Kodaira vanishing theorem for projective varieties (see Theorem 2.3 below).

For the reader's convenience, we recall:

Theorem 2.3 (Kodaira vanishing theorem for projective varieties). *Let X be a smooth projective variety and let L be an ample Cartier divisor on X. Then we have*

$$H^i(X, \mathcal{O}_X(K_X + L)) = 0$$

for every $i > 0$.

We will give a proof of Theorem 2.3 after we discuss E_1-degenerations of Hodge to de Rham type spectral sequences.

Note that Theorem 2.1 is obviously a generalization of Tankeev's pioneering result.

Theorem 2.4 ([Tan, Proposition 1]). *Let X be a smooth projective variety with $\dim X \geq 2$. Assume that the complete linear system $|L|$ has no base points and determines a morphism $\Phi_{|L|} : X \to Y$ onto a variety Y with $\dim Y \geq 2$. Then*

$$H^0(X, \mathcal{O}_X(K_X + 2D)) \to H^0(D, \mathcal{O}_D((K_X + 2D)|_D))$$

is surjective for almost all divisors $D \in |L|$. Equivalently,

$$H^1(X, \mathcal{O}_X(K_X + D)) \to H^1(X, \mathcal{O}_X(K_X + 2D))$$

is injective for almost all divisors $D \in |L|$.

By Theorem 2.1, we can prove:

Theorem 2.5 ([Ko1, Theorem 2.1]). *Let X be a smooth projective variety, let Y be an arbitrary projective variety, and let $f : X \to Y$ be a surjective morphism. Then we have the following properties.*

(i) *$R^i f_*\mathcal{O}_X(K_X)$ is torsion-free for every i.*
(ii) *Let H be an ample Cartier divisor on Y, then*

$$H^j(Y, \mathcal{O}_Y(H) \otimes R^i f_*\mathcal{O}_X(K_X)) = 0$$

for every $j > 0$ and every i.

Theorem 2.5 (i) and (ii) are called Kollár's torsion-freeness and Kollár vanishing theorem respectively. We give a small remark on Theorem 2.5.

Remark 2.6. If $f = id_X : X \to X$ in Theorem 2.5 (ii), then we have $H^i(X, \mathcal{O}_X(K_X + H)) = 0$ for every $i > 0$ and every ample Cartier divisor H on X. This is nothing but Kodaira vanishing theorem for projective varieties (see Theorem 2.3). If f is birational in Theorem 2.5 (i), then $R^i f_*\mathcal{O}_X(K_X) = 0$ for every $i > 0$ since $R^i f_*\mathcal{O}_X(K_X)$ is a torsion sheaf for every $i > 0$. This is Grauert–Riemenschneider vanishing theorem for birational morphisms between projective varieties.

In [Ko1], Kollár proved Theorem 2.1 and Theorem 2.5 simultaneously. Therefore, the relationship between Theorem 2.1 and Theorem 2.5 is not clear in [Ko1]. Now it is well known that Theorem 2.1 and Theorem 2.5 are equivalent by the works of Kollár himself and Esnault–Viehweg (see, for example, [Ko4, Chapter 9] and [EV]). We note that Theorem 2.1 follows from the E_1-degeneration of Hodge to de Rham spectral sequence.

2.7 (E_1-degeneration of Hodge to de Rham spectral sequence). Let V be a smooth projective variety. Then the spectral sequence

$$E_1^{p,q} = H^q(V, \Omega_V^p) \Rightarrow H^{p+q}(V, \mathbb{C})$$

degenerates at E_1. This is a direct consequence of the Hodge decomposition for compact Kähler manifolds.

Therefore, we can see that Theorem 2.1 is a result of the theory of *pure* Hodge structures. Thus, it is natural to consider *mixed* generalizations of Theorem 2.1.

We do not repeat the proof of Theorem 2.1 depending on the E_1-degeneration of Hodge to de Rham spectral sequence in 2.7 here. For the details, see, for example, [Ko4, Chapter 9] and [EV].

We note Deligne's famous generalization of the E_1-degeneration in 2.7.

2.8. Let V be a smooth projective variety and let Δ be a simple normal crossing divisor on V. Then the spectral sequence

$$E_1^{p,q} = H^q(V, \Omega_V^p(\log \Delta)) \Rightarrow H^{p+q}(V \setminus \Delta, \mathbb{C})$$

degenerates at E_1 by Deligne's theory of mixed Hodge structures for smooth noncompact algebraic varieties (see [Del]).

Unfortunately, the E_1-degeneration in 2.8 seems to produce no useful generalizations of Theorem 2.1. We think that the following E_1-degeneration is a correct ingredient for mixed generalizations of Theorem 2.1.

2.9. Let V and Δ be as in 2.8. Then the spectral sequence

$$E_1^{p,q} = H^q(V, \Omega_V^p(\log \Delta) \otimes \mathcal{O}_V(-\Delta)) \Rightarrow H_c^{p+q}(V \setminus \Delta, \mathbb{C})$$

degenerates at E_1. This is a consequence of mixed Hodge structures on cohomology with compact support $H_c^\bullet(V \setminus \Delta, \mathbb{C})$.

Remark 2.10. In 2.9, we see that $H^q(V, \Omega_V^p(\log \Delta) \otimes \mathcal{O}_V(-\Delta))$ is dual to $H^{n-q}(V, \Omega_V^{n-p}(\log \Delta))$ by Serre duality, where $n = \dim X$. Moreover, $H_c^{p+q}(V \setminus \Delta, \mathbb{C})$ is dual to $H^{2n-(p+q)}(V \setminus \Delta, \mathbb{C})$ by Poincaré duality. Therefore, we can check the E_1-degeneration in 2.9 by the E_1-degeneration in 2.8. However, it is better to discuss mixed Hodge structures on cohomology with compact support in order to treat more general situations (see Theorem 2.12, Theorem 2.13, Theorem 3.6, and so on).

We give a proof of Kodaira vanishing theorem for projective varieties by using the E_1-degeneration in 2.9 in order to make the reader grow familiar with the E_1-degeneration in 2.9.

Proof of Theorem 2.3. By the usual covering trick, we can reduce Theorem 2.3 to the case when the complete linear system $|L|$ has no base points. So, we assume that $|L|$ has no base points for simplicity. We take a smooth member D of $|L|$ by Bertini. We put $\iota : X \setminus D \hookrightarrow X$. By the E_1-degeneration of

$$E_1^{p,q} = H^q(X, \Omega_X^p(\log D) \otimes \mathcal{O}_X(-D)) \Rightarrow H_c^{p+q}(X \setminus D, \mathbb{C}),$$

we obtain that the natural map

$$\pi : H^j(X, \iota_! \mathbb{C}_{X \setminus D}) \to H^j(X, \mathcal{O}_X(-D))$$

induced by $\iota_! \mathbb{C}_{X \setminus D} \subset \mathcal{O}_X(-D)$ is surjective for every j. Since

$$\iota_! \mathbb{C}_{X \setminus D} \subset \mathcal{O}_X(-mD) \subset \mathcal{O}_X(-D)$$

for every $m \geq 1$, we obtain that

$$\pi : H^j(X, \iota_! \mathbb{C}_{X \setminus D}) \to H^j(X, \mathcal{O}_X(-mD)) \xrightarrow{p} H^j(X, \mathcal{O}_X(-D))$$

and that p is surjective for every j. Note that $H^j(X, \mathcal{O}_X(-mD)) = 0$ for $j < \dim X$ and for $m \gg 0$ by Serre vanishing theorem. Thus we obtain that $H^j(X, \mathcal{O}_X(-D)) = 0$ for $j < \dim X$. By Serre duality, we have $H^i(X, \mathcal{O}_X(K_X + D)) = 0$ for every $i > 0$. $\square$

We give a remark on [EV].

Remark 2.11. Let V be a smooth projective variety and let $A + B$ be a simple normal crossing divisor on V such that A and B have no common irreducible components. In [EV], Esnault–Viehweg discussed the E_1-degeneration of

$$\begin{aligned} E_1^{p,q} &= H^q(V, \Omega_V^p(\log(A+B)) \otimes \mathcal{O}_V(-B)) \\ &\Rightarrow \mathbb{H}^{p+q}(V, \Omega_V^\bullet(\log(A+B)) \otimes \mathcal{O}_V(-B)) \end{aligned}$$

(see also [DI]). This E_1-degeneration contains the E_1-degenerations in 2.8 and in 2.9 as special cases. However, they did not pursue geometric applications of the E_1-degeneration in 2.9, that is, in the case when $A = 0$.

By using the E_1-degeneration in 2.9 and some more general E_1-degenerations arising from mixed Hodge structures on cohomology with compact support, we can obtain various generalizations of Theorem 2.1 and Theorem 2.5. We write the following useful generalizations without explaining the precise definitions and the notation here (see 6.6, 6.7, 6.8, 6.9 in Section 6)

Theorem 2.12 (Injectivity theorem for simple normal crossing pairs)**.** *Let (X, Δ) be a simple normal crossing pair such that Δ is an $\mathbb{R}$-divisor on X whose coefficients are in $[0, 1]$, and let $\pi : X \to V$ be a proper morphism between schemes. Let L be a Cartier divisor on X and let D be an effective Cartier divisor that is permissible with respect to (X, Δ). Assume the following conditions.*

(i) *$L \sim_{\mathbb{R}, \pi} K_X + \Delta + H$,*
(ii) *H is a π-semiample $\mathbb{R}$-divisor, and*
(iii) *$tH \sim_{\mathbb{R}, \pi} D + D'$ for some positive real number t, where D' is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor that is permissible with respect to (X, Δ).*

Then the homomorphisms

$$R^q\pi_*\mathcal{O}_X(L) \to R^q\pi_*\mathcal{O}_X(L + D),$$

which are induced by the natural inclusion $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(D)$, are injective for all q.

Theorem 2.12 is a generalization of Theorem 2.1.

Theorem 2.13. *Let $f : (Y, \Delta) \to X$ be a proper morphism from an embedded simple normal crossing pair (Y, Δ) to a scheme X such that Δ is an $\mathbb{R}$-divisor whose coefficients are in $[0, 1]$. Let L be a Cartier divisor on Y and let q be an arbitrary nonnegative integer. Then we have the following properties.*

(i) *Assume that $L - (K_Y + \Delta)$ is f-semi-ample. Then every associated prime of $R^q f_*\mathcal{O}_Y(L)$ is the generic point of the f-image of some stratum of (Y, Δ).*

(ii) *Let $\pi : X \to V$ be a proper morphism between schemes. Assume that*
$$f^*H \sim_{\mathbb{R}} L - (K_Y + \Delta),$$
where H is nef and log big over V with respect to $f : (Y, \Delta) \to X$. Then we have
$$R^p\pi_*R^q f_*\mathcal{O}_Y(L) = 0$$
for every $p > 0$.

Theorem 2.13 (i) and (ii) are generalizations of Theorem 2.5 (i) and (ii) respectively. For the details, see, for example, [Fj3, Sections 5 and 6], [Fj7, Theorem 1.1], [Fj9, Theorem 1.1], [Fj11, Chapter 5], and so on. Note that Theorem 2.12 and Theorem 2.13 have already played crucial roles in the proof of the fundamental theorems for log canonical pairs and semi log canonical pairs (see, for example, [Fj3], [Fj6], [Fj11], and so on).

Anyway, the formulation of Theorem 2.12 and Theorem 2.13 is natural and useful from the minimal model theoretic viewpoint although it may look technical and artificial.

Remark 2.14. Let V and Δ be as in 2.8. In the traditional framework, $\mathcal{O}_V(K_V + \Delta)$ was recognized to be $\det \Omega^1_V(\log \Delta)$. On the other hand, in our new framework for vanishing theorems, we see $\mathcal{O}_V(K_V + \Delta)$ as
$$\mathcal{H}om_{\mathcal{O}_V}(\mathcal{O}_V(-\Delta), \mathcal{O}_V(K_V))$$
and $\mathcal{O}_V(-\Delta)$ as the 0th term of $\Omega^\bullet_V(\log \Delta) \otimes \mathcal{O}_V(-\Delta)$.

I think that it is not so easy to understand the statements of Theorem 2.12 and Theorem 2.13. So we give a very special case of Theorem 2.12 to clarify the main difference between Theorem 2.1 and Theorem 2.12.

Theorem 2.15. *Let X be a smooth projective variety and let Δ be a simple normal crossing divisor on X. Let L be a semiample Cartier divisor on X and let D be a member of $|kL|$ for some positive integer k such that D contains no strata of Δ. Then the homomorphism*

$$H^i(X, \mathcal{O}_X(K_X + \Delta + lL)) \to H^i(X, \mathcal{O}_X(K_X + \Delta + (l+k)L))$$

induced by the natural inclusion $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(D)$ is injective for every positive integer l and every i.

If $\Delta = 0$ in Theorem 2.15, then Theorem 2.15 is nothing but Kollár's original injectivity theorem (see Theorem 2.1).

Remark 2.16. Let Δ be a simple normal crossing divisor on a smooth variety X. Let $\Delta = \sum_{i \in I} \Delta_i$ be the irreducible decomposition of Δ. Then a closed subset W of X is called a stratum of Δ if W is an irreducible component of $\Delta_{i_1} \cap \cdots \cap \Delta_{i_k}$ for some $\{i_1, \cdots, i_k\} \subset I$.

Remark 2.17. Let Δ be a simple normal crossing divisor on a smooth variety V. Then W is a stratum of Δ if and only if W is a log canonical center of (V, Δ) (see 6.5 in Section 6).

We have discussed the Hodge theoretic aspect of Kodaira type vanishing theorems. For the details and various related topics, see [EV],[Ko4, Part III], [Fj11], and references therein.

From now on, let us move to the analytic setting. After Kollár obtained Theorem 2.1, Enoki (see [Eno, Theorem 0.2]) proved:

Theorem 2.18 (Enoki's injectivity theorem)**.** *Let X be a compact Kähler manifold and let $\mathcal{L}$ be a semipositive line bundle on X. Then, for any nonzero holomorphic section s of $\mathcal{L}^{\otimes k}$ with some positive integer k, the multiplication homomorphism*

$$\times s : H^i(X, \omega_X \otimes \mathcal{L}^{\otimes l}) \longrightarrow H^i(X, \omega_X \otimes \mathcal{L}^{\otimes (l+k)}),$$

which is induced by $\otimes s$, is injective for every i and every positive integer l.

Remark 2.19. Let $\mathcal{L}$ be a holomorphic line bundle on a compact Kähler manifold X. We say that $\mathcal{L}$ is semipositive if there exists a smooth hermitian metric h on $\mathcal{L}$ such that $\sqrt{-1}\Theta_h(\mathcal{L})$ is a semipositive $(1,1)$-form on X, where $\Theta_h(\mathcal{L}) = D^2_{(\mathcal{L},h)}$ is the curvature form and $D_{(\mathcal{L},h)}$ is the Chern connection of $(\mathcal{L}, h)$.

Remark 2.20. Let X be a smooth projective variety and let $\mathcal{L}$ be a line bundle on X. If $\mathcal{L}$ is semiample, that is, $|\mathcal{L}^{\otimes k}|$ has no base points for some positive integer k, then $\mathcal{L}$ is semipositive in the sense of Remark 2.19.

Enoki's proof in [Eno] is arguably simpler than the proof of Theorem 2.1 based on Hodge theory. It only uses the standard results in the theory of harmonic forms on compact Kähler manifolds. Let us see Enoki's idea of the proof of Theorem 2.18.

Idea of Proof of Theorem 2.18. We put $n = \dim X$. Let $\mathcal{H}^{n,i}(X, \mathcal{L}^{\otimes l})$ (resp. $\mathcal{H}^{n,i}(X, \mathcal{L}^{\otimes (l+k)})$ be the space of $\mathcal{L}^{\otimes l}$-valued (resp. $\mathcal{L}^{\otimes (l+k)}$-valued) harmonic (n,i)-forms on X. By using the Nakano identity and the semipositivity of $\mathcal{L}$, we can easily check that $s \otimes \varphi$ is harmonic for every $\varphi \in \mathcal{H}^{n,i}(X, \mathcal{L}^{\otimes l})$. Therefore,

$$\times s : H^i(X, \omega_X \otimes \mathcal{L}^{\otimes l}) \longrightarrow H^i(X, \omega_X \otimes \mathcal{L}^{\otimes (l+k)}),$$

is nothing but $\otimes s : \mathcal{H}^{n,i}(X, \mathcal{L}^{\otimes l}) \to \mathcal{H}^{n,i}(X, \mathcal{L}^{\otimes (l+k)}) : \varphi \mapsto s \otimes \varphi$, which is obviously injective. $\square$

We note that Theorem 2.18 is better than Theorem 2.1 by Remark 2.20. Unfortunately, I do not know how to generalize Enoki's theorem appropriately for various geometric applications. Although I obtained some generalizations of Theorem 2.18 and their applications in [Fj4] and [Fj5], they are not so useful in the minimal model program compared with Theorem 2.12 and Theorem 2.13. Related to Theorem 2.15, we have:

Conjecture 2.21. *Let X be a compact Kähler manifold and let Δ be a simple normal crossing divisor on X. Let $\mathcal{L}$ be a semipositive line bundle on X and let s be a nonzero holomorphic section of $\mathcal{L}^{\otimes k}$ on X for some positive integer k. Assume that $(s = 0)$ contains no strata of Δ. Then the multiplication homomorphism*

$$\times s : H^i(X, \omega_X \otimes \mathcal{O}_X(\Delta) \otimes \mathcal{L}^{\otimes l}) \to H^i(X, \omega_X \otimes \mathcal{O}_X(\Delta) \otimes \mathcal{L}^{\otimes (l+k)}),$$

which is induced by $\otimes s$, is injective for every positive integer l and every i.

I do not know the true relationship between Kollár's injectivity theorem and Enoki's injectivity theorem.

Problem 2.22. Clarify the relationship between Kollár's injectivity theorem (see Theorem 2.1) and Enoki's injectivity theorem (see Theorem 2.18).

For almost all geometric applications, we use Theorem 2.5 (ii) for $i = 0$. Theorem 2.5 (ii) for $i = 0$ is sufficient for Viehweg's theory of weak positivity (see [Vie1], [Vie2], and [Fj13]). See also Subsection 4.2 below. Note that Theorem 2.5 (ii) for $i = 0$ is a special case of Ohsawa's vanishing theorem: Theorem 2.23.

Theorem 2.23 ([Oh1, Theorem 3.1])**.** *Let X be a compact Kähler manifold, let $f : X \to Y$ be a holomorphic map to an analytic space Y with a Kähler form σ, and let (E, h) be a holomorphic vector bundle on X with a smooth hermitian metric h. Assume that $\sqrt{-1}\Theta_h(E) \geq_{\mathrm{Nak}} \mathrm{Id}_E \otimes f^*\sigma$, that is, $\sqrt{-1}\Theta_h(E) - \mathrm{Id}_E \otimes f^*\sigma$ is semipositive in the sense of Nakano, where $\Theta_h(E)$ is the curvature form of (E, h). Then*

$$H^j(Y, f_*(\omega_X \otimes E)) = 0$$

for every $j > 0$.

For the proof of Theorem 2.23, see Ohsawa's original paper [Oh1]. I am not so familiar with Theorem 2.23 and do not know if the formulation of Theorem 2.23 is natural or not.

Remark 2.24. For the details of σ and $f^*\sigma$ in Theorem 2.23, see [Oh1, §3]. Note that Y may have singularities in Theorem 2.23.

By comparing Theorem 2.23 with Theorem 2.5 (ii), it is natural to consider:

Conjecture 2.25. *On the same assumption as in Theorem 2.23, we have*

$$H^j(Y, R^i f_*(\omega_X \otimes E)) = 0$$

for every i and every positive integer j.

We close this section with:

Problem 2.26. Clarify the relationship between Kollár's vanishing theorem (see Theorem 2.5 (ii)) and Ohsawa's vanishing theorem (see Theorem 2.23).

For Enoki type injectivity theorems, see, for example, [Eno], [Take], [Oh2], [Fj4], [Fj5], [Ma1], [Ma2], [Ma3], [Ma4], [Ma5], [GM], and so on.

3. On local freeness and semipositivity theorems

Let us start with Fujita's semipositivity theorem in [Ft].

Theorem 3.1 ([Ft, (0.6) Main Theorem])**.** *Let $f : M \to C$ be a surjective morphism from a compact Kähler manifold onto a smooth projective curve C with connected fibers. Then $f_*\omega_{M/C}$ is nef.*

Before we go further, let us recall the definition of nef locally free sheaves.

Definition 3.2 (Nef locally free sheaves)**.** Let $\mathcal{E}$ be a locally free sheaf of finite rank on a complete algebraic variety V. Then $\mathcal{E}$ is called nef if $\mathcal{E} = 0$ or $\mathcal{O}_{\mathbb{P}_V(\mathcal{E})}(1)$ is nef on $\mathbb{P}_V(\mathcal{E})$. This means that $\mathcal{O}_{\mathbb{P}_V(\mathcal{E})}(1) \cdot C \geq 0$

for every curve C on $\mathbb{P}_V(\mathcal{E})$. A nef locally free sheaf $\mathcal{E}$ was originally called a (numerically) semipositive locally free sheaf in the literature.

Remark 3.3. Assume that X is a smooth projective variety for simplicity. Let $\mathcal{L}$ be a line bundle on X. Then $\mathcal{L}$ is nef in the sense of Definition 3.2 if and only if $\mathcal{L}$ is nef in the usual sense. If $\mathcal{L}$ is semipositive in the sense of Remark 2.19, then $\mathcal{L}$ is nef. However, a nef line bundle $\mathcal{L}$ is not necessarily semipositive in the sense of Remark 2.19.

Remark 3.4. Note that f is not necessarily smooth in Theorem 3.1. If f is smooth in Theorem 3.1, then the nefness of $f_*\omega_{M/C}$ follows from Griffiths's calculations of connections and curvatures in [Gri].

Although Fujita's theorem was inspired by Griffiths's paper [Gri] (see Remark 3.4), Fujita's original proof of Theorem 3.1 in [Ft] is not so Hodge theoretic. In [Ft, Introduction], Fujita wrote:

> The method looks rather elementary and purely computational, but it depends deeply (often implicitly) on the theory of variation of Hodge structures.

Professor Steven Zucker informed me that he read Fujita's article [Ft] at Rutgers University in 1978 and reproved Fujita's theorem from rather basic Hodge theory that appeals to Steenbrink's work [St]. It is not surprising that he had already been very familiar with Schmid's result (see [Sc]) on asymptotic behaviors of Hodge metrics (see [Zuc1] and [Zuc2]). I think that he could write [Zuc3] without any difficulties. He is probably the first one who directly applies Hodge theory to obtain semipositivity results like Theorem 3.1, that is, semipositivity results for nonsmooth morphisms.

Independently, Kawamata obtained the following semipositivity theorem in [Kaw1] by using Schmid's paper [Sc]. His result is:

Theorem 3.5 ([Kaw1, Theorem 5])**.** *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers which satisfies the following conditions:*

(i) *There is a Zariski open dense subset Y_0 of Y such that $D = Y \setminus Y_0$ is a simple normal crossing divisor on Y.*
(ii) *Put $X_0 = f^{-1}(Y_0)$ and $f_0 = f|_{X_0}$. Then f_0 is smooth.*
(iii) *The local monodromies of $R^n f_{0*}\mathbb{C}_{X_0}$ around D are unipotent, where $n = \dim X - \dim Y$.*

Then $f_*\omega_{X/Y}$ is a locally free sheaf and nef.

However, the proof of Theorem 3.5 in [Kaw1] seems to be insufficient when $\dim Y \geq 2$ (see Morihiko Saito's comments in [FFS, 4.6. Remarks]). Fortunately, we have some generalizations of Theorem 3.5

in [Fj2], [FF], and [FFS] (see, for example, Theorem 3.6 below). The proofs in [FF] and [FFS] are independent of Kawamata's arguments in [Kaw1]. Note that our arguments in [FF] and [FFS] need some results on Hodge theory obtained after the publication of Kawamata's paper [Kaw1] (see, for example, [CK], [CKS], and so on). Kawamata could and did use only [Del], [Gri], and [Sc] on Hodge theory when he wrote [Kaw1]. Although I sometimes called Theorem 3.5 Fujita–Kawamata semipositivity theorem (see, for example, [FF]), it is probably not appropriate. It may be better to call it Fujita–Zucker–Kawamata semipositivity theorem. I apology for ignoring Zucker's contribution [Zuc3] and misleading the readers.

Theorem 3.5 follows from the theory of polarizable variation of *pure* Hodge structure. It is natural to consider *mixed* generalizations of Theorem 3.5. We have already known that mixed Hodge structures on cohomology with compact support are very useful (see Section 2). So, we consider their variations and prove some powerful generalizations of Theorem 3.5, which depend on the theory of gradedly polarizable admissible variation of mixed Hodge structure (see, for example, [SZ], [Kas], and so on). We have:

Theorem 3.6 (Semipositivity theorem)**.** *Let (X, D) be a simple normal crossing pair such that D is reduced and let $f : X \to Y$ be a projective surjective morphism onto a smooth complete algebraic variety Y. Assume that every stratum of (X, D) is dominant onto Y. Let Σ be a simple normal crossing divisor on Y such that every stratum of (X, D) is smooth over $Y^* = Y \setminus \Sigma$. Then $R^p f_*\omega_{X/Y}(D)$ is locally free for every p. We put $X^* = f^{-1}(Y^*)$, $D^* = D|_{X^*}$, and $d = \dim X - \dim Y$. We further assume that all the local monodromies on $R^{d-i}(f|_{X^*\setminus D^*})_!\mathbb{Q}_{X^*\setminus D^*}$ around Σ are unipotent. Then we obtain that $R^i f_*\omega_{X/Y}(D)$ is a nef locally free sheaf on Y.*

For the definitions and the notation used in Theorem 3.6, see 6.6 and 6.7 in Section 6. Theorem 3.6 was first obtained in [FF]. Then, we gave an alternative proof of Theorem 3.6 based on Saito's theory of mixed Hodge modules (see [Sa1], [Sa2], and [Sa3]) in [FFS]. As an application of Theorem 3.6, we establish the projectivity of various moduli spaces (for the details, see [Fj6], [Fj8], [KvP], and so on). In [Ft, Introduction], Fujita wrote:

> Perhaps our result is closely related with the problem about the (quasi-)projectivity of moduli spaces. Of course, however, the relation will not be simple.

Now we know that generalizations of Fujita's semipositivity theorem (see Theorem 3.1 and Theorem 3.6) with Viehweg's mysterious covering arguments are useful for the projectivity of coarse moduli spaces of stable (log-)varieties (see, for example, [Ko2], [Fj8], [KvP], and so on).

Anyway, by Theorem 3.5, we have:

Theorem 3.7 (Fujita, Zucker, Kawamata, $\cdots$)**.** *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Then there exists a generically finite morphism $\tau : Y' \to Y$ from a smooth projective variety Y' with the following property. Let X' be any resolution of the main component of $X \times_Y Y'$. Then $f'_*\omega_{X'/Y'}$ is a nef locally free sheaf, where $f' : X' \to X \times_Y Y' \to Y'$.*

Theorem 3.7 has already played crucial roles in the study of higher-dimensional algebraic varieties. For some geometric applications, we have to treat $f_*\omega^{\otimes m}_{X/Y}$ or $f'_*\omega^{\otimes m}_{X'/Y'}$ with $m \geq 2$ (see Section 4). Thus we have:

Conjecture 3.8 (Semipositivity of direct images of relative pluricanonical bundles)**.** *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Then there exists a generically finite morphism $\tau : Y' \to Y$ from a smooth projective variety Y' with the following property. Let X' be any resolution of the main component of $X \times_Y Y'$ sitting in the following commutative diagram:*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\tau]{} & Y. \end{array}$$

Then $f'_*\omega^{\otimes m}_{X'/Y'}$ is a nef locally free sheaf for every positive integer m.

Note that the local freeness of $f'_*\omega^{\otimes m}_{X'/Y'}$ for $m \geq 2$ in Conjecture 3.8 is highly nontrivial even when f' is a smooth projective morphism. The following theorem by Siu (see [Siu2]) is nontrivial for $m \geq 2$ and can be proved only by using L^2-method. For a simpler proof, see [Pă1].

Theorem 3.9 (Siu)**.** *Let $f : X \to Y$ be a smooth projective morphism between smooth quasiprojective varieties with connected fibers. Then $f_*\omega^{\otimes m}_{X/Y}$ is locally free for every nonnegative integer m.*

Theorem 3.9 is a clever application of Ohsawa–Takeboshi L^2 extension theorem. We have no Hodge theoretic proofs of Theorem 3.9. Therefore, we have:

Problem 3.10. Find a Hodge theoretic proof or an algebraic proof of Theorem 3.9.

We note:

Remark 3.11. If Y is projective in Theorem 3.9, then $f_*\omega_{X/Y}^{\otimes m}$ is nef for every positive integer m by Theorem 3.17 below. Therefore, Conjecture 3.8 holds true when $f : X \to Y$ is smooth.

We recommend the reader to see [FF] and [FFS] for the Hodge theoretic aspect of semipositivity theorems discussed in this section. Note that the style of [FF] is the same as my other papers. On the other hand, [FFS] is written in the language of Saito's theory of mixed Hodge modules.

3.1. **New semipositivity theorems using MMP.** In this subsection, we discuss new semipositivity theorems with the help of the minimal model program following [Fj12].

Let us start with the definition of (good) minimal models. We recommend the reader to see 6.2 and 6.5 in Section 6 if he is not familiar with the minimal model program.

Definition 3.12 (Good minimal models)**.** Let $f : X \to Y$ be a projective morphism between normal quasiprojective varieties. Let Δ be an effective $\mathbb{Q}$-divisor on X such that (X, Δ) is kawamata log terminal. A pair (X', Δ') sitting in a diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & Y & \end{tikzcd}$$

is called a minimal model of (X, Δ) over Y if

(i) X' is $\mathbb{Q}$-factorial,
(ii) f' is projective,
(iii) ϕ is birational and ϕ^{-1} has no exceptional divisors,
(iv) $\phi_*\Delta = \Delta'$,
(v) $K_{X'} + \Delta'$ is f'-nef, and
(vi) $a(E, X, \Delta) < a(E, X', \Delta')$ for every ϕ-exceptional divisor $E \subset X$.

Furthermore, if $K_{X'} + \Delta'$ is f'-semiample, then (X', Δ') is called a good minimal model of (X, Δ) over Y. When Y is a point, we usually omit "over Y" in the above definitions. We sometimes simply say that (X', Δ') is a relative (good) minimal model of (X, Δ).

We also need the notion of weakly semistable morphisms due to Abramovich–Karu (see [AK]).

Definition 3.13 (Weakly semistable morphisms)**.** Let $f : X \to Y$ be a projective surjective morphism between quasiprojective varieties. Then $f : X \to Y$ is called weakly semistable if

(i) the varieties X and Y admit toroidal structures $(U_X \subset X)$ and $(U_Y \subset Y)$ with $U_X = f^{-1}(U_Y)$,
(ii) with this structure, the morphism f is toroidal,
(iii) the morphism f is equidimensional,
(iv) all the fibers of the morphism f are reduced, and
(v) Y is smooth.

Note that $(U_X \subset X)$ and $(U_Y \subset Y)$ are toroidal embeddings without self-intersection in the sense of [KKMS, Chapter II, §1]. We also note that X has only rational Gorenstein singularities (see [AK, Lemma 6.1]). For the details, see [AK].

We propose the following conjecture.

Conjecture 3.14. *Let $f : X \to Y$ be a weakly semistable morphism with connected fibers. Then $f_*\omega_{X/Y}^{\otimes m}$ is locally free for every $m \geq 1$.*

By the argument in [Fj12, Section 4], we have:

Theorem 3.15 (Local freeness)**.** *Let $f : X \to Y$ be a weakly semistable morphism with connected fibers. Assume that the geometric generic fiber $X_{\overline{\eta}}$ of $f : X \to Y$ has a good minimal model. Then $f_*\omega_{X/Y}^{\otimes m}$ is locally free for every $m \geq 1$.*

Idea of Proof of Theorem 3.15. Let us consider the following commutative diagram:

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & \widetilde{X} \arrow[dl, "\widetilde{f}"] \\ & Y & \end{tikzcd}$$

where $\widetilde{f} : \widetilde{X} \to Y$ is a relative good minimal model of $f : X \to Y$. We can always construct a relative good minimal model by the assumption that the geometric generic fiber of f has a good minimal model. Then we have

$$(\spadesuit) \qquad f_*\omega_{X/Y}^{\otimes m} \simeq \widetilde{f}_*\mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}/Y})$$

for every positive integer m. We note that X has only rational Gorenstein singularities.

The following lemma due to Nakayama is a variant of Kollár's torsion-freeness: Theorem 2.5 (i). This is a key ingredient of the proof of Theorem 3.15.

Lemma 3.16 (cf. [Nak, Corollary 3]). *Let $g : V \to C$ be a projective surjective morphism from a normal quasiprojective variety V to a smooth quasiprojective curve C. Assume that V has only canonical singularities and that K_V is g-semiample. Then $R^i g_* \mathcal{O}_V(mK_V)$ is locally free for every i and every positive integer m.*

By the above isomorphism (♠), it is sufficient to prove the local freeness of $\widetilde{f}_* \mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}/Y})$. Since $f : X \to Y$ is weakly semistable, we can prove that the diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & \widetilde{X} \arrow[dl, "\widetilde{f}"] \\ & Y & \end{tikzcd}$$

behaves well by the base change by $H \hookrightarrow Y$, where H is a general smooth Cartier divisor on Y. Roughly speaking, by this observation, we can reduce the problem to the case when Y is a smooth projective curve. Note that f and $\widetilde{f}$ are both flat. By Lemma 3.16, we see that $\dim H^0(\widetilde{X}_y, \mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}/Y})|_{\widetilde{X}_y})$ is independent of $y \in Y$. Therefore, $\widetilde{f}_* \mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}/Y})$ is locally free by the flat base change theorem. Thus, we obtain that $f_* \omega_{X/Y}^{\otimes m}$ is locally free. For the details, see [Fj12, Section 4]. $\square$

By the argument in [Fj12, Section 5], we can prove:

Theorem 3.17 (Semipositivity). *Let $f : X \to Y$ be a weakly semistable morphism between projective varieties with connected fibers. Assume that $f_* \omega_{X/Y}^{\otimes m}$ is locally free for some $m \geq 1$. Then $f_* \omega_{X/Y}^{\otimes m}$ is nef.*

Idea of Proof of Theorem 3.17. The following theorem by Popa–Schnell is a clever and interesting application of Kollár vanishing theorem: Theorem 2.5 (ii).

Theorem 3.18 ([PoSc, Theorem 1.4]). *Let $f : V \to W$ be a surjective morphism from a smooth projective variety V onto a projective variety W with $\dim W = n$. Let $\mathcal{L}$ be an ample line bundle on W such that $|\mathcal{L}|$ has no base points. Let k be a positive integer. Then*

$$f_* \omega_V^{\otimes k} \otimes \mathcal{L}^{\otimes l}$$

is generated by global sections for every $l \geq k(n+1)$.

By Viehweg's fiber product trick and the local freeness of $f_*\omega_{X/Y}^{\otimes m}$, we can prove that there exists an ample line bundle $\mathcal{A}$ on Y such that

$$\left(\bigotimes^{s} f_*\omega_{X/Y}^{\otimes m}\right)\otimes \mathcal{A}$$

is generated by global sections for every positive integer s. Here, we used the fact that weakly semistable morphisms behave well by taking fiber products. This implies that $f_*\omega_{X/Y}^{\otimes m}$ is nef. For the details, see [Fj12, Section 5]. □

As we saw above, a key ingredient of Theorem 3.15 (resp. Theorem 3.17) is Kollár's torsion-freeness (resp. Kollár vanishing theorem (see Theorem 2.5)). Of course, the existence of relative good minimal models plays a crucial role in the proof of Theorem 3.15.

Remark 3.19. In the proof of Theorem 3.15, we need the finite generation of relative canonical ring

$$R(X/Y)=\bigoplus_{m}^{\infty} f_*\mathcal{O}_X(mK_X)$$

by [BCHM] to construct a relative good minimal model of $f: X\to Y$. Note that the finite generation of $R(X/Y)$ is more or less Hodge theoretic when $X_{\overline{\eta}}$ is not of general type. This is because the reduction argument due to Fujino–Mori (see Theorem 5.4 and [FM]) uses Theorem 3.5.

Remark 3.20. Let V be a smooth projective variety. It is well known that V has a good minimal model when $\dim V-\kappa(V)\leq 3$.

By combining Theorem 3.15 with Theorem 3.17, we obtain:

Theorem 3.21. *Let $f: X\to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Assume that*

$$f: X\xrightarrow{\delta} X^{\dagger}\xrightarrow{f^{\dagger}} Y$$

such that $f^{\dagger}: X^{\dagger}\to Y$ is weakly semistable and that δ is a resolution of singularities. We further assume that the geometric generic fiber $X_{\overline{\eta}}$ of f has a good minimal model. Then $f_*\omega_{X/Y}^{\otimes m}$ is a nef locally free sheaf for every positive integer m.

By the weak semistable reduction theorem due to Abramovich–Karu (see [AK]) and Theorem 3.21, we have:

Theorem 3.22. *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Assume that the geometric generic fiber $X_{\overline{\eta}}$ of $f : X \to Y$ has a good minimal model. Then there exists a generically finite morphism $\tau : Y' \to Y$ from a smooth projective variety Y' with the following property. Let X' be any resolution of the main component of $X \times_Y Y'$ sitting in the following commutative diagram:*

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "\tau"'] & Y. \end{tikzcd}$$

Then $f'_*\omega^{\otimes m}_{X'/Y'}$ is a nef locally free sheaf for every positive integer m.

This means that Conjecture 3.8 holds true under the assumption that the geometric generic fiber of f has a good minimal model. More precisely, Conjecture 3.8 follows from Conjecture 3.14 by the weak semistable reduction theorem due to Abramovich–Karu (see [AK]) and Theorem 3.17. Moreover, Conjecture 3.14 holds under the assumption that the geometric generic fiber has a good minimal model (see Theorem 3.15).

We close this section with Takayama's result. In [Taka], Takayama strengthened Theorem 3.21 as follows.

Theorem 3.23 (Takayama)**.** *In Theorem 3.21, for every positive integer m, the m-th Narasimhan–Simha Hermitian metric g_m on the locally free sheaf $\mathcal{E}_m = f_*\omega^{\otimes m}_{X/Y}$ has Griffiths semipositive curvature, the induced singular Hermitian metric $h = e^{-\varphi}$ on $\mathcal{O}_{\mathbb{P}_X(\mathcal{E}_m)}(1)$ of $\mathbb{P}_X(\mathcal{E}_m)$ has semipositive curvature, and the Lelong number of the local weight φ is zero everywhere on $\mathbb{P}_X(\mathcal{E}_m)$. In particular, $\mathcal{O}_{\mathbb{P}_X(\mathcal{E}_m)}(1)$ is nef.*

For the definition of Narasimhan–Simha Hermitian metric and the details of Theorem 3.23, see the original paper [Taka]. Note that Theorem 3.23 is based on the arguments in [Fj12].

I am not familiar with the analytic aspect of semipositivity theorems. For the details, see [Ber], [BP1], [BP2], [Mou], [MT1], [MT2], [MT3], [PăT], [Taka], and so on.

4. Canonical divisors versus pluricanonical divisors

In this section, let us see that mK_X with $m \geq 2$ sometimes behaves much better than K_X. We will discuss two different topics: Kollár's result on plurigenera in étale covers of smooth projective varieties of general type and Viehweg's ampleness theorem on direct images of relative pluricanonical bundles of semistable families of projective varieties. I was impressed by these results.

4.1. **Plurigenera in étale covers.** Let us recall Kollár's famous result on plurigenera in étale covers of smooth projective varieties of general type (see [Ko3]). For the details and some related topics, see also [Ko5, 2. Vanishing Theorems] and [Ko4, Chapter 15].

Theorem 4.1 (Kollár)**.** *Let X be a smooth projective variety of general type. Let $f : Y \to X$ be an étale morphism from a smooth projective variety Y. Then we have*

$$h^0(Y, \mathcal{O}_Y(mK_Y)) = \deg f \cdot h^0(X, \mathcal{O}_X(mK_X))$$

for every positive integer $m \geq 2$.

Here, we will explain Lazarsfeld's proof of Theorem 4.1 following [Laz, Theorem 11.2.23]. It is an easy application of the theory of asymptotic multiplier ideal sheaves. We will give an alternative proof of Theorem 4.1 after we discuss canonical models of smooth projective varieties of general type in Theorem 4.5.

Proof. Let D be a big Cartier divisor on X. Then $\mathcal{J}(X, \|D\|)$ is the asymptotic multiplier ideal sheaf associated to the complete linear systems $|mD|$ for all $m \gg 0$. For the details of $\mathcal{J}(X, \|D\|)$, see [Laz, Chapter 11]. By Nadel vanishing theorem (see [Laz, Theorem 11.2.12 (ii)]),

$$H^i(X, \mathcal{O}_X(mK_X) \otimes \mathcal{J}(X, \|(m-1)K_X\|)) = 0$$

for every $i > 0$ and every $m \geq 2$. Therefore, we have

$$\begin{aligned} &h^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{J}(X, \|(m-1)K_X\|)) \\ &= \chi(X, \mathcal{O}_X(mK_X) \otimes \mathcal{J}(X, \|(m-1)K_X\|)) \end{aligned}$$

for every $m \geq 2$. Since $\mathcal{J}(X, \|mK_X\|) \subset \mathcal{J}(X, \|(m-1)K_X\|)$ (see [Laz, Theorem 11.1.8 (ii)]), we have

$$\begin{aligned} &H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{J}(X, \|mK_X\|)) \\ &= H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{J}(X, \|(m-1)K_X\|)) \\ &= H^0(X, \mathcal{O}_X(mK_X)) \end{aligned}$$

for every $m \geq 1$ by [Laz, Proposition 11.2.10]. Thus, we obtain

$$h^0(X, \mathcal{O}_X(mK_X)) = \chi(X, \mathcal{O}_X(mK_X) \otimes \mathcal{J}(X, \|(m-1)K_X\|))$$

for every $m \geq 2$. Similarly, we have

$$h^0(Y, \mathcal{O}_Y(mK_Y)) = \chi(Y, \mathcal{O}_Y(mK_Y) \otimes \mathcal{J}(Y, \|(m-1)K_Y\|))$$

for $m \geq 2$. Since f is étale, $K_Y = f^*K_X$ and

$$\mathcal{J}(Y, \|(m-1)K_Y\|) = f^*\mathcal{J}(X, \|(m-1)K_X\|)$$

by [Laz, Theorem 11.2.16]. Thus we have

$$\begin{aligned}
&\chi(Y, \mathcal{O}_Y(mK_Y) \otimes \mathcal{J}(Y, \|(m-1)K_Y\|)) \\
&= \chi(Y, f^*(\mathcal{O}_X(mK_X) \otimes \mathcal{J}(X, \|(m-1)K_X\|))) \\
&= \deg f \cdot \chi(X, \mathcal{O}_X(mK_X) \otimes \mathcal{J}(X, \|(m-1)K_X\|))
\end{aligned}$$

for $m \geq 2$. Therefore, we obtain the desired equality $h^0(Y, \mathcal{O}_Y(mK_Y)) = \deg f \cdot h^0(X, \mathcal{O}_X(mK_X))$ for every $m \geq 2$. $\square$

The proof of Theorem 4.1 says that mK_X with $m \geq 2$ should be seen as $K_X + (m-1)K_X$. Since $m \geq 2$, $(m-1)K_X$ is big. Therefore, we can apply Nadel vanishing theorem to

$$\begin{aligned}
&\mathcal{O}_X(mK_X) \otimes \mathcal{J}(X, \|(m-1)K_X\|) \\
&= \mathcal{O}_X(K_X + (m-1)K_X) \otimes \mathcal{J}(X, \|(m-1)K_X\|).
\end{aligned}$$

Obviously, the equality in Theorem 4.1 does not hold for $m = 1$.

Example 4.2. Let C be a smooth projective curve with the genus $g(C) \geq 2$. Let $f : \widetilde{C} \to C$ be an étale cover with $\deg f = n \geq 2$. Then we have

$$2g(\widetilde{C}) - 2 = n(2g(C) - 2)$$

by Hurwitz. This implies that $g(\widetilde{C}) = n(g(C) - 1) + 1$. Thus we have

$$h^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(K_{\widetilde{C}})) \neq n \cdot h^0(C, \mathcal{O}_C(K_C)).$$

The following example also shows that mK_X with $m \geq 2$ sometimes has much more informations than K_X.

Example 4.3 (Godeaux surface)**.** We put

$$Y = (Z_0^5 + Z_1^5 + Z_2^5 + Z_3^5 = 0) \subset \mathbb{P}^3.$$

Then Y is a smooth projective surface such that

$$\mathcal{O}_Y(K_Y) = \mathcal{O}_{\mathbb{P}^3}(-4+5)|_Y = \mathcal{O}_Y(1)$$

is very ample. Therefore, Y is of general type,

$$h^0(Y, \mathcal{O}_Y(K_Y)) = h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) = 4,$$

and $q(Y) = h^1(Y, \mathcal{O}_Y) = 0$. We put $G = \mathbb{Z}/5\mathbb{Z}$. Then G acts freely on Y by

$$[Z_0 : Z_1 : Z_2 : Z_3] \mapsto [Z_0 : \zeta Z_1 : \zeta^2 Z_2 : \zeta^3 Z_3]$$

where $\zeta = \exp\left(\frac{2\pi\sqrt{-1}}{5}\right)$. We put $X = Y/G$. Then X is a smooth projective surface with ample canonical divisor. Let $f : Y \to X$ be the natural map. Then f is a finite étale morphism. We can directly check that $h^0(X, \mathcal{O}_X(K_X)) = 0$ by $H^0(X, \mathcal{O}_X(K_X)) = H^0(Y, \mathcal{O}_Y(K_Y))^G$. Note that $q(X) = h^1(X, \mathcal{O}_X) = 0$, $K_Y^2 = 5$, and $K_X^2 = 1$. We also note that X is known as a Godeaux surface. It is well known that the linear system $|mK_X|$ gives an embedding into a projective space for every $m \geq 5$. Note that

$$4 = h^0(Y, \mathcal{O}_Y(K_Y)) \neq \deg f \cdot h^0(X, \mathcal{O}_X(K_X)) = 0.$$

Let us discuss canonical models of finite étale covers of smooth projective varieties of general type. The existence of canonical models was unkonwn when [Ko3] was written.

4.4 (Canonical models)**.** Let $\pi : V \to W$ be a projective surjective morphism from a smooth quasiprojective variety V onto a quasiprojective variety W. Assume that K_V is π-big. Then, by [BCHM], the (relative) canonical ring

$$R(V/W) = \bigoplus_{m=0}^{\infty} \pi_*\mathcal{O}_V(mK_V)$$

is a finitely generated $\mathcal{O}_W$-algebra. We put

$$V_c = \mathrm{Proj}_W R(V/W)$$

and call it the canonical model of V over W or the relative canonical model of $\pi : V \to W$. It is well known that V_c is birationally equivalent to V over W, V_c has only canonical singularities, and K_{V_c} is ample over W.

A finite étale morphism between smooth projective varieties of general type induces a natural finite étale morphism between their canonical models.

Theorem 4.5. *Let $f : Y \to X$ be a finite étale morphism between smooth projective varieties of general type. Let X_c be the canonical model of X and let Y_c be the canonical model of Y. Then there exists a finite étale morphism $f_c : Y_c \to X_c$ such that*

$$\begin{tikzcd} Y \arrow[r, dashed] \arrow[d, "f"'] & Y_c \arrow[d, "f_c"] \\ X \arrow[r, dashed] & X_c \end{tikzcd}$$

is commutative.

The following proof was suggested by Yoshinori Gongyo.

Proof. By taking an elimination of indeterminacy of $X \dashrightarrow X_c$ and the base change of Y, we may assume that $g : X \to X_c$ is a morphism. Let $Y' \to X_c$ be the relative canonical model of $g \circ f : Y \to X_c$ (see [BCHM] and 4.4). Then, by using the negativity lemma (see, for example, [KoM, Lemma 3.39]), we see that $K_{Y'} = f'^* K_{X_c}$ and that $f' : Y' \to X_c$ is finite since $K_{Y'}$ is f'-ample. Therefore, $K_{Y'}$ is ample since K_{X_c} is ample. This implies that $Y' = Y_c$, that is, Y' is the canonical model of Y. Note that Y is smooth and is finite over X. Therefore, Y is the normalization of the main component of $X \times_{X_c} Y_c$. Thus we obtain the following commutative diagram:

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f"'] & Y_c \arrow[d, "f_c"] \\ X \arrow[r, "g"'] & X_c. \end{tikzcd}$$

By Lemma 4.8 below, we obtain that f_c is a finite étale morphism. □

By the proof of Theorem 4.5, we have:

Theorem 4.6. *Let $f : Y \to X$ be a finite étale morphism between smooth projective varieties. Let X_m be a minimal model of X. Then we can construct a commutative diagram:*

$$\begin{tikzcd} Y \arrow[r, dashed, "\varphi"] \arrow[d, "f"'] & \widetilde{Y} \arrow[d, "\widetilde{f}"] \\ X \arrow[r, dashed] & X_m \end{tikzcd}$$

such that $\widetilde{f} : \widetilde{Y} \to X_m$ is a finite étale morphism and that φ is birational.

For the definition of minimal models, see Definition 3.12.

Proof. As in the proof of Theorem 4.5, we may assume that $g : X \to X_m$ is a morphism. Let $\widetilde{f} : \widetilde{Y} \to X_m$ be the relative canonical model of $g \circ f : Y \to X_m$. Then, by the proof of Theorem 4.5, $\varphi : Y \dashrightarrow \widetilde{Y}$ and $\widetilde{f} : \widetilde{Y} \to X_m$ satisfy the desired properties. □

Related to Theorem 4.5 and Theorem 4.6, we have:

Problem 4.7. Find a projective variety X such that X has only $\mathbb{Q}$-factorial terminal (or canonical) singularities with nef (or ample) canonical divisor and a finite étale morphism $f : Y \to X$ such that Y is not $\mathbb{Q}$-factorial.

The following lemma is a special case of [NZ, Lemma 3.9].

Lemma 4.8. *We consider the following commutative diagram of quasiprojective varieties:*

$$\begin{tikzcd} Y \arrow[r, "q"] \arrow[d, "f"'] & W \arrow[d, "h"] \\ X \arrow[r, "p"'] & V \end{tikzcd}$$

such that

(i) *X and Y are smooth,*
(ii) *p and q are projective birational morphisms,*
(iii) *V and W are normal and have only rational singularities,*
(iv) *f is a finite étale morphism, and*
(v) *h is finite.*

Then h is an étale morphism.

We give a proof for the reader's convenience. It is interesting for me that the proof of Lemma 4.8 below uses the E_1-degeneration of Hodge to de Rham type spectral sequences for projective simple normal crossing varieties.

Proof. We take an arbitrary point $P \in V$. By taking a birational modification of X and the base change of Y, we may assume that $E = \mathrm{Supp}(p^{-1}(P))$ is a simple normal crossing divisor on X. Since f is étale, $f^{-1}(E)$ is a simple normal crossing divisor on Y. Note that $f^{-1}(E)$ is the disjoint union of $E_Q = \mathrm{Supp}(q^{-1}(Q))$ for points $Q \in h^{-1}(P)$. Since V has only rational singularities, $R^ip_*\mathcal{O}_X = 0$ for every $i > 0$.

Claim. *The natural map*

$$\pi : H^i(E, \mathbb{C}) \to H^i(E, \mathcal{O}_E)$$

induced by the inclusion $\mathbb{C}_E \hookrightarrow \mathcal{O}_E$ is surjective for every i.

Proof of Claim. By using the Mayer–Vietoris simplicial resolution of a projective simple normal crossing variety E, we can construct a cohomological mixed Hodge complex $(K_{\mathbb{Z}}, (K_{\mathbb{Q}}, W_{\mathbb{Q}}), (K_{\mathbb{C}}, W, F))$ which induces a natural mixed Hodge structure on $H^\bullet(E, \mathbb{Z})$. By the theory of mixed Hodge structures, we see that

$$E_1^{p,q} = \mathbb{H}^{p+q}(E, \mathrm{Gr}_F^p K_{\mathbb{C}}) \Rightarrow H^{p+q}(E, \mathbb{C})$$

degenerates at E_1. We can directly check that $\mathrm{Gr}_F^0 K_{\mathbb{C}}$ is quasi-isomorphic to $\mathcal{O}_E$ by using the Mayer–Vietoris simplicial resolution of E. Thus, we obtain that

$$\pi : H^i(E, \mathbb{C}) \to H^i(E, \mathcal{O}_E)$$

is surjective for every i. $\square$

By the following commutative diagram:

$$\begin{tikzcd}
(R^ip_*\mathbb{C}_X)_P \arrow[r, "\simeq"] \arrow[d] & H^i(E, \mathbb{C}) \arrow[d, "\pi"] \\
(R^ip_*\mathcal{O}_X)_P \arrow[r] & H^i(E, \mathcal{O}_E),
\end{tikzcd}$$

we obtain that $H^i(E, \mathcal{O}_E) = 0$ for every $i > 0$ by $R^ip_*\mathcal{O}_X = 0$ for every $i > 0$. Thus, we obtain $\chi(E, \mathcal{O}_E) = 1$. By the same argument, we have $\chi(E_Q, \mathcal{O}_{E_Q}) = 1$. On the other hand,

$$\chi(f^{-1}(E), \mathcal{O}_{f^{-1}(E)}) = \deg f \cdot \chi(E, \mathcal{O}_E) = \deg f$$

since f is étale. Therefore, we have

$$\sharp h^{-1}(P) = \sum_{Q \in h^{-1}(P)} \chi(E_Q, \mathcal{O}_{E_Q}) = \chi(f^{-1}(E), \mathcal{O}_{f^{-1}(E)}) = \deg f.$$

This implies that $f : E_Q \to E$ is an isomorphism for every $Q \in h^{-1}(P)$. Thus, by the theorem on formal functions, we have

$$\widehat{\mathcal{O}}_{V,P} = (p_*\mathcal{O}_X)^{\wedge}_P \simeq (q_*\mathcal{O}_Y)^{\wedge}_Q = \widehat{\mathcal{O}}_{W,Q}$$

for every $Q \in h^{-1}(P)$. Thus, h is an étale morphism. $\square$

We give an alternative proof of Theorem 4.1, which is natural from the minimal model theoretic viewpoint but unfortunately may be much more difficult than Lazarsfeld's proof given before.

Proof of Theorem 4.1. By Theorem 4.5, we may replace $f : Y \to X$ with $f_c : Y_c \to X_c$. Note that

$$h^0(X_c, \mathcal{O}_{X_c}(mK_{X_c})) = h^0(X, \mathcal{O}_X(mK_X))$$

and

$$h^0(Y_c, \mathcal{O}_{Y_c}(mK_{Y_c})) = h^0(Y, \mathcal{O}_Y(mK_Y))$$

for every $m \geq 1$. By Kawamata–Viehweg vanishing theorem for singular varieties (see, for example, [Fj11, Corollary 5.7.7]), we have

$$H^i(X_c, \mathcal{O}_{X_c}(mK_{X_c})) = H^i(Y_c, \mathcal{O}_{Y_c}(mK_{Y_c})) = 0$$

for every $i > 0$ and every $m \geq 2$. We also note that $f_c^*\mathcal{O}_{X_c}(mK_{X_c}) = \mathcal{O}_{Y_c}(mK_{Y_c})$ for every m since f_c is étale. Therefore, we obtain

$$\begin{aligned}
h^0(Y_c, \mathcal{O}_{Y_c}(mK_{Y_c})) &= \chi(Y_c, \mathcal{O}_{Y_c}(mK_{Y_c})) \\
&= \deg f_c \cdot \chi(X_c, \mathcal{O}_{X_c}(mK_{X_c})) \\
&= \deg f_c \cdot h^0(X_c, \mathcal{O}_{X_c}(mK_{X_c}))
\end{aligned}$$

for every $m \geq 2$. This implies the desired equality. $\square$

Remark 4.9. In Lemma 4.8, the assumption that V and W have only rational singularities (see (iii) in Lemma 4.8) is indispensable.

Let $C \subset \mathbb{P}^2$ be an elliptic curve and let $V \subset \mathbb{P}^3$ be a cone over $C \subset \mathbb{P}^2$. Let $p : X \to V$ be the blow-up at the vertex P of V and let E be the p-exceptional divisor on X. Note that there is a natural $\mathbb{P}^1$-bundle structure $\pi : X \to C$ and E is a section of π. We take a nontrivial finite étale cover $D \to C$. We put $Y = X \times_C D$ and $F = E \times_C D$. Let H be an ample Cartier divisor on V. We consider $q = \Phi_{|mf^*p^*H|} : Y \to W$ for a sufficiently large positive integer m. Note that q contracts F to an isolated normal singular point Q of W. Then we have the following commutative diagram

$$\begin{tikzcd} Y \arrow[r, "q"] \arrow[d, "f"'] & W \arrow[d, "h"] \\ X \arrow[r, "p"'] & V \end{tikzcd}$$

such that f is étale, h is finite, but h is not étale. Note that $h^{-1}(P) = Q$ since $f^{-1}(E) = F$. We also note that the singularities of V and W are not rational.

4.2. **Viehweg's ampleness theorem.** We treat direct images of relative pluricanonical bundles. The following theorem is buried in Viehweg's papers (see [Vie1] and [Vie2]). The statement seems to be mysterious.

Theorem 4.10 (Viehweg)**.** *Let $f : X \to Y$ be a surjective morphism from a smooth projective variety X onto a smooth projective curve Y with connected fibers. Then $f_*\omega_{X/Y}^{\otimes m}$ is nef for every positive integer m. In particular, we have $\deg \det f_*\omega_{X/Y}^{\otimes m} \geq 0$ for every positive integer m. Assume that f is semistable. If $\deg \det f_*\omega_{X/Y}^{\otimes k} > 0$, that is, $\det f_*\omega_{X/Y}^{\otimes k}$ is ample, then $f_*\omega_{X/Y}^{\otimes k'}$ is ample, where k' is any multiple of k with $k' \geq 2$.*

Roughly speaking, Theorem 4.10 says if $f_*\omega_{X/Y}$ is a little bit positive then $f_*\omega_{X/Y}^{\otimes m}$ is very positive for $m \geq 2$.

Proof. By Kawamata (see [Kaw2, Theorem 1]), $f_*\omega_{X/Y}^{\otimes m}$ is nef for every $m \geq 1$. Or, by Viehweg's weak positivity: [Vie1, Theorem III] (see also [Fj13, Theorem 4.3 and Theorem 5.5] and Remark 4.14 below), $f_*\omega_{X/Y}^{\otimes m}$ is weakly positive for every $m \geq 1$. Since Y is a smooth projective curve, the weak positivity implies that $f_*\omega_{X/Y}^{\otimes m}$ is nef for every $m \geq 1$. By [Vie2, Theorem 3.5] (see also [Fj13, Theorem 5.11]), $\deg \det f_*\omega_{X/Y}^{\otimes k} > 0$ implies that $f_*\omega_{X/Y}^{\otimes k'}$ is big in the sense of Viehweg (see [Fj13, Definition 3.1]) when f is semistable. Then, by [Vie2, Lemma 3.6] (see also [Fj13, Lemma 3.7]), we have a generically isomorphic injection

$$\bigoplus_r \mathcal{A} \to S^{\nu}(f_*\omega_{X/Y}^{\otimes k'})$$

for some ample invertible sheaf $\mathcal{A}$ on Y and some positive integer ν, where $r = \mathrm{rank} S^{\nu}(f_*\omega_{X/Y}^{\otimes k'})$. This implies that $S^{\nu}(f_*\omega_{X/Y}^{\otimes k'})$ is ample by [Laz, Theorem 6.4.15]. Therefore, $f_*\omega_{X/Y}^{\otimes k'}$ is ample by [Hart, Proposition (2.4)]. □

Viehweg's arguments in [Vie1] and [Vie2] (see also [Fj13]) use his mysterious covering trick and fiber product trick. They are geometric. It seems to be very important to find a more direct approach to Theorem 4.10. Thus, we have:

Problem 4.11. Find an analytic (and more direct) proof of Theorem 4.10.

The example by Catanese–Dettweller (see [CD1] and [CD2]) below says that the condition $k' \geq 2$ in Theorem 4.10 is indispensable.

Example 4.12 (Catanese–Dettweller)**.** There exist a smooth projective surface X of general type and a smooth projective curve Y such that $f : X \to Y$ is semistable and that $f_*\omega_{X/Y} = A \oplus Q$, where A is an ample vector bundle of rank 2 and Q is a unitary flat vector bundle of rank 4. Moreover, Q is not semiample. In this case, $\deg\det f_*\omega_{X/Y} > 0$. By Theorem 4.10, $f_*\omega_{X/Y}^{\otimes m}$ is ample for every $m \geq 2$. However, $f_*\omega_{X/Y}$ is not ample.

Note that the construction of Example 4.12 in [CD2] depends on the theory of variation of Hodge structure. For the details, see [CD1] and [CD2].

Problem 4.13. Find similar examples to Example 4.12 without using the theory of variation of Hodge structure.

Although we do not discuss Viehweg's weak positivity in this paper, we give a remark on the weak positivity of $f_*\omega_{X/Y}^{\otimes m}$ for the interested reader.

Remark 4.14. Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Then $f_*\omega_{X/Y}^{\otimes m}$ is weakly positive for every positive integer m by Viehweg (see [Vie1]). Viehweg's original proof of his weak positivity uses Theorem 3.5. Now it is well known that we can prove the weak positivity of $f_*\omega_{X/Y}^{\otimes m}$ by

Kollár's vanishing theorem: Theorem 2.5 (ii). Moreover, Theorem 3.18 drastically simplifies the proof of Viehweg's weak positivity of $f_*\omega_{X/Y}^{\otimes m}$. For the details, see [Fj13].

Anyway, we should consider not only K_X but also mK_X with $m \geq 2$ in order to understand complex projective varieties much better.

5. On finite generation of (log) canonical rings

In this section, we quickly discuss the finite generation of (log) canonical rings due to Birkar–Cascini–Hacon–M$^{\mathrm{c}}$Kernan (see [BCHM]) and some related topics (see [FM] and [F10]). For simplicity, we only treat the absolute setting in this section although the relative setting is very important and is indispensable for some applications.

Theorem 5.1. *Let X be a smooth projective variety and let Δ be an effective $\mathbb{Q}$-divisor on X such that* $\mathrm{Supp}\Delta$ *is a simple normal crossing divisor and that the coefficients of Δ are less than one. Assume that $K_X + \Delta$ is big. Then the log canonical ring*

$$R(X, K_X + \Delta) = \bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta)\rfloor))$$

is a finitely generated $\mathbb{C}$-algebra.

Theorem 5.1 was first obtained in [BCHM]. For the proof of Theorem 5.1, see also [CL] and [Pă2]. We know that the proof of Theorem 5.1 was greatly influenced by Siu's extension argument (see [Siu1]) based on Ohsawa–Takegoshi L^2 extension theorem (see [HM], [CL], and [Pă2]) although [HM] and [CL] only use Kawamata–Viehweg vanishing theorem and do not use L^2-method. I think that Theorem 5.1 is not Hodge theoretic. It is natural to see that Theorem 5.1 is more closely related to L^2-method than to Hodge theory.

By combining Theorem 5.1 with the result in [FM], we have:

Theorem 5.2. *Let X be a smooth projective variety and let Δ be an effective $\mathbb{Q}$-divisor on X such that* $\mathrm{Supp}\Delta$ *is a simple normal crossing divisor and that the coefficients of Δ are less than one. Then the log canonical ring*

$$R(X, K_X + \Delta) = \bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta)\rfloor))$$

is a finitely generated $\mathbb{C}$-algebra.

Note that we do not assume that $K_X + \Delta$ is big in Theorem 5.2. As a corollary of Theorem 5.2, we have:

Corollary 5.3. *Let X be a smooth projective variety. Then the canonical ring*

$$R(X)=\bigoplus_{m=0}^{\infty} H^0(X,\mathcal{O}_X(mK_X))$$

is a finitely generated $\mathbb{C}$-algebra.

We note that the formulation of Theorem 5.1, which may look artificial, is indispensable for the proof of Corollary 5.3. From now on, we will see how to reduce Theorem 5.2 to Theorem 5.1. I think that this reduction step is more or less Hodge theoretic. I do not know how to prove Corollary 5.3 without using this reduction argument based on Fujino–Mori canonical bundle formula (see [FM]). We can prove the following theorem by [FM].

Theorem 5.4 (Fujino–Mori)**.** *Let X be a smooth projective variety, let Δ be an effective $\mathbb{Q}$-divisor on X such that* SuppΔ *is a simple normal crossing divisor with $\lfloor \Delta \rfloor = 0$. Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Assume that $\kappa(X_{\overline{\eta}}, K_{X_{\overline{\eta}}} + \Delta|_{X_{\overline{\eta}}}) = 0$ where $X_{\overline{\eta}}$ is the geometric generic fiber of $f : X \to Y$. Then we can construct a commutative diagram:*

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l, "p"'] \arrow[d, "f'"] \\ Y & Y' \arrow[l, "q"] \end{tikzcd}$$

with the following properties.

(i) *p and q are projective birational morphisms.*

(ii) *X' and Y' are smooth projective varieties.*

(iii) *there exists an effective $\mathbb{Q}$-divisor Δ' on X' such that* SuppΔ' *is a simple normal crossing divisor on X' with $\lfloor \Delta' \rfloor = 0$ and that*
$$H^0(X,\mathcal{O}_X(\lfloor m(K_X+\Delta)\rfloor)) \simeq H^0(X',\mathcal{O}_{X'}(\lfloor m(K_{X'}+\Delta')\rfloor))$$
for every positive integer m.

(iv) *there exist a positive integer k, a nef $\mathbb{Q}$-divisor M on Y', and an effective $\mathbb{Q}$-divisor D on Y' such that* Supp$D$ *is a simple normal crossing divisor with $\lfloor D \rfloor = 0$ and that*
$$\begin{aligned} &H^0(X',\mathcal{O}_{X'}(\lfloor mk(K_{X'}+\Delta')\rfloor)) \\ &\simeq H^0(Y',\mathcal{O}_{Y'}(\lfloor mk(K_{Y'}+M+D)\rfloor)) \end{aligned}$$
for every positive integer m.

We do not prove Theorem 5.4 here. For the details, see [FM, Theorem 4.5].

Remark 5.5. Theorem 5.4 is an application of Fujino–Mori canonical bundle formula discussed in [FM]. The $\mathbb{Q}$-divisor M is called the semistalbe part in [FM] and now is usually called the moduli part in the literature. Note that the nefness of M comes from Theorem 3.5. Therefore, Theorem 5.4 is more or less Hodge theoretic.

By Kodaira's lemma and Hironaka's resolution theorem, we can easily prove:

Proposition 5.6. *If $K_{Y'} + M + D$ is big in Theorem 5.4, then we can take a birational morphism $r : Z \to Y'$ from a smooth projective variety Z, an effective $\mathbb{Q}$-divisor Δ_Z on Z, and positive integers a and b such that* $\mathrm{Supp}\Delta_Z$ *is a simple normal crossing divisor with $\lfloor \Delta_Z \rfloor = 0$ and that*

$$\begin{aligned} &H^0(Y', \mathcal{O}_{Y'}(\lfloor ma(K_{Y'} + M + D)\rfloor)) \\ &\simeq H^0(Z, \mathcal{O}_Z(\lfloor mb(K_Z + \Delta_Z)\rfloor)) \end{aligned}$$

for every positive integer m.

Proof. This is an easy consequence of Kodaira's lemma on big $\mathbb{Q}$-divisors and Hironaka's resolution of singularities. We note that we can choose Δ_Z with $\lfloor \Delta_Z \rfloor = 0$ since M is nef and $\lfloor D \rfloor = 0$. More precisely, by Kodaira, we have $K_{Y'} + M + D \sim_{\mathbb{Q}} A + E$, where A is an ample $\mathbb{Q}$-divisor and E is an effective $\mathbb{Q}$-divisor. Thus we have $(1+\varepsilon)(K_{Y'} + M + D) \sim_{\mathbb{Q}} K_{Y'} + (M + \varepsilon A) + D + \varepsilon E$ for every positive rational number ε. If ε is sufficiently small, then $(Y', D + \varepsilon E)$ is kawamata log terminal. Since $M + \varepsilon A$ is ample, we can take an effective $\mathbb{Q}$-divisor B such that $B \sim_{\mathbb{Q}} M + \varepsilon A$ and that $(Y', \Delta_{Y'})$ is still kawamata log terminal, where $\Delta_{Y'} = B + D + \varepsilon E$. Therefore, we can find positive integers a and b such that $a(K_{Y'} + M + D) \sim b(K_{Y'} + \Delta_{Y'})$. By Hironaka's resolution theorem, we can take $r : Z \to Y'$ and Δ_Z such that $\mathrm{Supp}\Delta_Z$ is a simple normal crossing divisor with $\lfloor \Delta_Z \rfloor = 0$ and that $H^0(Z, \mathcal{O}_Z(\lfloor m(K_Z + \Delta_Z)\rfloor)) \simeq H^0(Y', \mathcal{O}_{Y'}(\lfloor m(K_{Y'} + \Delta_{Y'})\rfloor))$ for every positive integer m. Thus we have the desired properties. $\square$

Let us see how to prove Theorem 5.2 by using Theorem 5.1, Theorem 5.4, and Proposition 5.6.

Proof of Theorem 5.2. Let X and Δ be as in Theorem 5.2. Assume that $\kappa(X, K_X + \Delta) \geq 0$ and that $K_X + \Delta$ is not big. We consider the Iitaka fibration $\Phi_{|m(K_X+\Delta)|} : X \dashrightarrow Y$ for some large and divisible positive integer m. By taking a suitable birational modification, we may assume that $f : X \to Y$ is a surjective morphism between smooth projective varieties with connected fibers. Then we apply Theorem 5.4

and Proposition 5.6 to $f : X \to Y$. Thus, we see that $R(X, K_X + \Delta)$ is a finitely generated $\mathbb{C}$-algebra if and only if $R(Z, K_Z + \Delta_Z)$ is a finitely generated $\mathbb{C}$-algebra. By Theorem 5.1, we know that $R(Z, K_Z + \Delta_Z)$ is a finitely generated $\mathbb{C}$-algebra. Therefore, we obtain Theorem 5.2. $\square$

As I explained in Remark 5.5, Theorem 5.4 is more or less Hodge theoretic. So, we have:

Problem 5.7. Prove Corollary 5.3 without using Hodge theory.

Remark 5.8. Theorem 3.5 holds true under the assumption that X is only a compact Kähler manifold. This is because we can use the theory of variation of Hodge structure even for compact Kähler manifolds. Therefore, Theorem 5.4 holds true under the assumption that X is a compact complex manifold in Fujiki's class $\mathcal{C}$. As a consequence, we see that Theorem 5.2 holds for compact complex manifolds in Fujiki's class $\mathcal{C}$. For the details, see [F10, Section 5]. As a special case, we have the corollary below.

Corollary 5.9. *Let X be a compact Kähler manifold. Then the canonical ring*

$$R(X) = \bigoplus_{m=0}^{\infty} H^0(X, \omega_X^{\otimes m})$$

is a finitely generated $\mathbb{C}$-algebra.

Remark 5.10. There exists a compact complex non-Kähler manifold whose canonical ring is not a finitely generated $\mathbb{C}$-algebra (see [F10, Example 6.4], which is essentially due to Wilson). This means that Corollary 5.9 does not hold for compact complex non-Kähler manifolds. The reader can find a smooth morphism $f : X \to Y$ from a compact complex non-Kähler manifold X onto $Y = \mathbb{P}^1$ with connected fibers such that $f_*\omega_{X/Y} \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$ (see [F10, Example 6.1]). This example is due to Atiyah. Therefore, Theorem 3.5 does not hold for compact non-Kähler manifolds. Of course, by this example, we see that Theorem 5.4 does not hold true without assuming that X is a compact complex manifold in Fujiki's class $\mathcal{C}$. For the details, see [F10].

Related to Theorem 5.2, we have:

Conjecture 5.11. *Let X be a smooth projective variety and let Δ be an effective $\mathbb{Q}$-divisor on X such that* SuppΔ *is a simple normal crossing divisor and that the coefficients of Δ are less than or equal to one. Then the log canonical ring*

$$R(X, K_X + \Delta) = \bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor))$$

is a finitely generated $\mathbb{C}$-algebra.

Remark 5.12. Of course, we think that Conjecture 5.11 also holds for compact complex manifolds in Fujiki's class $\mathcal{C}$. However, I could not reduce Conjecture 5.11 for compact complex manifolds in Fujiki's class $\mathcal{C}$ to Conjecture 5.11 for projective varieties. Therefore, Conjecture 5.11 for compact complex manifolds in Fujiki's class $\mathcal{C}$ may be much harder than for projective varieties. For the details, see [F10].

I have already discussed Conjecture 5.11 in details in a joint paper with Yoshinori Gongyo (see [FG]). In [FG], we clarified the relationship among various conjectures in the minimal model program related to Conjecture 5.11. So we do not repeat the discussions about Conjecture 5.11 here. We strongly recommend the interested reader to see [FG].

In order to prove Conjecture 5.11, the following famous conjecture seems to be unavoidable. I think that it is a very difficult open problem for higher-dimensional algebraic varieties.

Conjecture 5.13 (Nonvanishing conjecture)**.** *Let X be a smooth projective variety such that the canonical divisor K_X is pseudoeffective. Then we have $H^0(X, \mathcal{O}_X(mK_X)) \neq 0$ for some positive integer m, equivalently, $\kappa(X) \geq 0$.*

For the reader's convenience, let us recall the definition of pseudoeffective divisors.

Definition 5.14. Let X be a smooth projective variety and let D be a Cartier divisor on X. Then D is pseudoeffective if $D + A$ is big for every ample $\mathbb{Q}$-divisor A on X.

The characterization of pseudoeffective divisors via singular hermitian metrics may be helpful.

Remark 5.15. Let X be a smooth projective variety and let D be a Cartier divisor on X. Then D is pseudoeffective if and only if $\mathcal{O}_X(D)$ has a singular hermitian metric h with $\sqrt{-1}\Theta_h \geq 0$ in the sense of currents.

The characterization of uniruled varieties due to Boucksom–Demailly–Păun–Peternell (see [BDPP]) is important and helps us understand Conjecture 5.13.

Theorem 5.16. *Let X be a smooth projective variety. Then X is uniruled if and only if K_X is not pseudoeffective.*

For the reader's convenience, we recall the definition of uniruled varieties.

Definition 5.17. Let X be a smooth projective variety with $\dim X = n$. Then X is uniruled if there exist a smooth projective variety Y with $\dim Y = n-1$ and a dominant rational map $Y \times \mathbb{P}^1 \dashrightarrow X$.

Therefore, Conjecture 5.13 says that the Kodaira dimension of X is nonnegative if X is not covered by rational curves. Thus, Conjecture 5.13 looks very reasonable. However, I do not know how to attack Conjecture 5.13.

6. Appendix

In this appendix, we collect some definitions, which may help us understand this paper, for the reader's convenience. For the details, see [Fj3] and [Fj11]. Note that a scheme means a separated scheme of finite type over $\mathbb{C}$ in this paper.

6.1 (Iitaka dimension and Kodaira dimension)**.** Let X be a normal projective variety and let D be a $\mathbb{Q}$-Cartier divisor on X. Then $\kappa(X, D)$ denotes the Iitaka dimension of D. Let X be a smooth projective variety. Then we put $\kappa(X) = \kappa(X, K_X)$ and call it the Kodaira dimension of X.

6.2 ($\mathbb{Q}$-factorial)**.** Let X be a normal variety. Then X is called $\mathbb{Q}$-factorial if every prime divisor D on X is $\mathbb{Q}$-Cartier.

Note that $\mathbb{Q}$-factoriality sometimes plays crucial roles in the minimal model program. The notion of terminal singularities and canonical singularities is indispensable in the minimal model program.

6.3 (Terminal singularities and Canonical singularities)**.** Let X be a normal variety such that K_X is $\mathbb{Q}$-Cartier. If there exists a projective birational morphism $f : Y \to X$ from a smooth variety Y such that the exceptional locus $\mathrm{Exc}(f) = \sum_{i \in I} E_i$ is a simple normal crossing divisor on Y. In this situation, we can write

$$K_Y = f^*K_X + \sum_{i \in I} a_i E_i.$$

If $a_i > 0$ for every $i \in I$, then we say that X has only terminal singularities. If $a_i \geq 0$ for every $i \in I$, then we say that X has only canonical singularities. It is well known that X has only rational singularities when X has only canonical singularities.

6.4 (Round-down of $\mathbb{Q}$-divisors)**.** Let $D = \sum a_i D_i$ be a $\mathbb{Q}$-divisor on a normal variety X. Note that D_i is a prime divisor for every i and that $D_i \neq D_j$ for $i \neq j$. Of course, $a_i \in \mathbb{Q}$ for every i. We put $\lfloor D \rfloor = \sum \lfloor a_i \rfloor D_i$ and call it the round-down of D. Note that, for every rational number x, $\lfloor x \rfloor$ is the integer defined by $x - 1 < \lfloor x \rfloor \leq x$.

In the minimal model program, we usually use the notion of pairs.

6.5 (Singularities of pairs)**.** A pair (X, Δ) consists of a normal variety X and an effective $\mathbb{R}$-divisor Δ on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. A pair (X, Δ) is called kawamata log terminal (resp. log canonical) if for any projective birational morphism $f : Y \to X$ from a normal variety Y, $a(E, X, \Delta) > -1$ (resp. ≥ -1) for every E, where

$$K_Y = f^*(K_X + \Delta) + \sum_E a(E, X, \Delta)E.$$

Let (X, Δ) be a log canonical pair and let W be a closed subset of X. Then W is called a log canonical center of (X, Δ) if there are a projective birational morphism $f : Y \to X$ from a normal variety Y and a prime divisor E on Y such that $a(E, X, \Delta) = -1$ and that $f(E) = W$.

In order to understand Theorem 2.12, Theorem 2.13, and Theorem 3.6, we quickly explain the notion of simple normal crossing pairs and some related topics.

6.6 (Simple normal crossing pairs)**.** Let Z be a simple normal crossing divisor on a smooth variety M and let B be an $\mathbb{R}$-divisor on M such that $\mathrm{Supp}(B + Z)$ is a simple normal crossing divisor and that B and Z have no common irreducible components. We put $\Delta_Z = B|_Z$ and consider the pair (Z, Δ_Z). We call (Z, Δ_Z) a globally embedded simple normal crossing pair. A pair (X, Δ) is called a simple normal crossing pair if it is Zariski locally isomorphic to a globally embedded simple normal crossing pair. If (X, Δ) is a simple normal crossing pair and X is a divisor on a smooth variety M, then (X, Δ) is called an embedded simple normal crossing pair. Of course, a globally embedded simple normal crossing pair is automatically an embedded simple normal crossing pair.

6.7 (Strata and permissibility)**.** Let (X, Δ) be a simple normal crossing pair. Assume that the coefficients of Δ are in $[0, 1]$. Let $\nu : X^\nu \to X$ be the normalization. We put

$$K_{X^\nu} + \Theta = \nu^*(K_X + \Delta).$$

Then we see that (X^ν, Θ) is log canonical. Let W be a closed subset of X. Then W is called a stratum of (X, Δ) if W is an irreducible component of X or the ν-image of some log canonical center of (X^ν, Θ). A Cartier divisor D on X is permissible with respect to (X, Δ) if D contains no strata of (X, Δ) in its support. A finite $\mathbb{R}$-linear combination of permissible Cartier divisors with respect to (X, Δ) is called a permissible $\mathbb{R}$-divisor with respect to (X, Δ).

We need the notion of nef and log big divisors for Theorem 2.13.

6.8 (Nef and log big divisors)**.** Let $f : (Y, \Delta) \to X$ be a proper morphism from an embedded simple normal crossing pair (Y, Δ) to a scheme X. Assume that the coefficients of Δ are in $[0, 1]$. Let $\pi : X \to V$ be a proper morphism between schemes. Let H be an $\mathbb{R}$-Cartier divisor on X. Then H is nef and log big over V with respect to $f : (Y, \Delta) \to X$ if H is nef over V and $H|_{f(W)}$ is big over $\pi \circ f(W)$ for every stratum W of (Y, Δ). Note that if H is ample over V then H is nef and log big over V with respect to $f : (Y, \Delta) \to X$.

For Theorem 2.12 and Theorem 2.13, let us recall the definition of $\mathbb{R}$-linear equivalence.

6.9 ($\mathbb{R}$-divisors)**.** Let B_1 and B_2 be two $\mathbb{R}$-Cartier divisors on a scheme X. Then B_1 is linearly (resp. $\mathbb{Q}$-linearly, or $\mathbb{R}$-linearly) equivalent to B_2, denoted by $B_1 \sim B_2$ (resp. $B_1 \sim_{\mathbb{Q}} B_2$, or $B_1 \sim_{\mathbb{R}} B_2$) if

$$B_1 = B_2 + \sum_{i=1}^{k} r_i(f_i)$$

such that $f_i \in \Gamma(X, \mathcal{K}_X^*)$ and $r_i \in \mathbb{Z}$ (resp. $r_i \in \mathbb{Q}$, or $r_i \in \mathbb{R}$) for every i. Here, $\mathcal{K}_X$ is the sheaf of total quotient rings of $\mathcal{O}_X$ and $\mathcal{K}_X^*$ is the sheaf of invertible elements in the sheaf of rings $\mathcal{K}_X$. We note that (f_i) is a principal Cartier divisor associated to f_i, that is, the image of f_i by $\Gamma(X, \mathcal{K}_X^*) \to \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$, where $\mathcal{O}_X^*$ is the sheaf of invertible elements in $\mathcal{O}_X$.

Let $f : X \to Y$ be a morphism between schemes. If there is an $\mathbb{R}$-Cartier divisor B on Y such that

$$B_1 \sim_{\mathbb{R}} B_2 + f^*B,$$

then B_1 is said to be relatively $\mathbb{R}$-linearly equivalent to B_2. It is denoted by $B_1 \sim_{\mathbb{R}, f} B_2$ or $B_1 \sim_{\mathbb{R}, Y} B_2$.

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