# ON EXTREMAL CONTRACTIONS OF LOG CANONICAL PAIRS

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Throughout this note, we will work over $\mathbb{C}$. We will freely use the basic definitions and results of the minimal model theory for log canonical pairs in [F1] and [F2]. The following theorems generalize Kawamata's famous result in [K].

**Theorem 1** ([F5, Theorem 1.13]). *Let $(X, \Delta)$ be a log canonical pair and let $\pi\colon X \to S$ be a projective morphism onto a variety $S$. Let $R$ be a $(K_X + \Delta)$-negative extremal ray of $\overline{\mathrm{NE}}(X/S)$ and let $\varphi_R\colon X \to W$ be the contraction morphism over $S$ associated to $R$. We put*

$$d := \min_E \dim E,$$

*where $E$ runs over all positive-dimensional irreducible components of $\varphi_R^{-1}(P)$ for all $P \in W$. Then $R$ is spanned by a possibly singular rational curve $\ell$ with*

$$0 < -(K_X + \Delta) \cdot \ell \leq 2d$$

More generally, we have:

**Theorem 2** ([F5, Theorem 1.12]). *Let $(X, \Delta)$ be a log canonical pair and let $\varphi\colon X \to W$ be a projective morphism between varieties such that $-(K_X + \Delta)$ is $\varphi$-ample. Let $P$ be an arbitrary closed point of $W$. Let $E$ be any positive-dimensional irreducible component of $\varphi^{-1}(P)$. Then $E$ is covered by (possibly singular) rational curves $\ell$ with*

$$0 < -(K_X + \Delta) \cdot \ell \leq 2 \dim E.$$

*In particular, $E$ is uniruled.*

Our approach in [F5] is different from and is independent of Kawamata's in [K]. We think that the argument in [K] does not work for pairs whose singularities are worse than kawamata log terminal. Lemma 3 is a key lemma for the proof of Theorems 1 and 2.

**Lemma 3** ([F5, Lemma 12.1]). *Let $(X, \Delta)$ be a log canonical pair and let $\varphi\colon X \to W$ be a projective morphism. We take an arbitrary closed point $P$ of $W$. Let $E$ be any positive-dimensional irreducible component of $\varphi^{-1}(P)$. Let $\nu\colon \overline{E} \to E$ be tne normalization. Then, for every ample $\mathbb{R}$-divisor $H$ on $\overline{E}$, there exists an effective $\mathbb{R}$-divisor $\Delta_{\overline{E},H}$ on $\overline{E}$ such that*

$$\nu^*(K_X + \Delta) + H \sim_{\mathbb{R}} K_{\overline{E}} + \Delta_{\overline{E},H}.$$

We note that Theorems 1, 2, and Lemma 3 are formulated for quasi-log schemes in [F5]. Hence the results in [F5] are much more general.

Let us recall the definition of quasi-log schemes. For the details, see [F2, Chapter 6], [F4], and so on.

**Definition 4** (Quasi-log schemes). A *quasi-log scheme* is a scheme $X$ endowed with an $\mathbb{R}$-line bundle $\omega$ on $X$, a closed subscheme $X_{-\infty} \subsetneq X$, and a finite collection $\{C\}$ of reduced and irreducible subschemes of $X$ such that there is a proper morphism $f\colon (Y, B_Y) \to X$ from a globally embedded simple normal crossing pair satisfying the following properties:

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(1) $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$.
(2) The natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil)$ induces an isomorphism

$$\mathcal{I}_{X_{-\infty}} \xrightarrow{\simeq} f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil - \lfloor B_Y^{>1}\rfloor),$$

where $\mathcal{I}_{X_{-\infty}}$ is the defining ideal sheaf of $X_{-\infty}$.
(3) The collection of reduced and irreducible subschemes $\{C\}$ coincides with the images of the strata of $(Y, B_Y)$ that are not included in $X_{-\infty}$.

We note that $Y$ may be reducible in Definition 4 and that any log canonical pair has a natural quasi-log scheme structure. The notion of a basic slc-trivial fibration was first introduced in [F3] (see also [FH]).

**Definition 5** (Basic slc-trivial fibrations)**.** A *basic slc-trivial fibration* $f\colon (X, B) \to Y$ consists of a projective surjective morphism $f\colon X \to Y$ and a simple normal crossing pair $(X, B)$ satisfying the following properties:

(1) $Y$ is a normal variety,
(2) every stratum of $X$ is dominant onto $Y$ and $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$,
(3) $B$ is a $\mathbb{Q}$-divisor such that $B = B^{\leq 1}$ holds over the generic point of $Y$,
(4) there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D$ on $Y$ such that $K_X + B \sim_{\mathbb{Q}} f^*D$, and
(5) $\mathrm{rank} f_*\mathcal{O}_X(\lceil -(B^{<1})\rceil) = 1$.

We also note that $X$ in Definition 5 is not necessarily irreducible. The author thinks that the theory of quasi-log schemes is a powerful framework to use mixed Hodge structures on cohomology with compact support for the study of higher-dimensional algebraic varieties. He constructed the theory of basic slc-trivial fibrations in order to make the theory of variations of mixed Hodge structure on cohomology with compact support applicable for various geometric problems.

Anyway, in [F5], we use quasi-log schemes and basic slc-trivial fibrations to establish Theorems 1, 2, and Lemma 3.

## References


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