NOTES ON THE WEAK POSITIVITY THEOREMS (PRELIMINARY VERSION)

OSAMU FUJINO

Abstract. We discuss the (twisted) weak positivity theorem. We also treat some applications.

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1. Introduction

In this paper, we discuss the (twisted) weak positivity theorem. We give a detailed proof of the following theorem, which is essentially equivalent to [Ca, Theorem 4.3] (see also [L]). The proof is based on our semipositivity theorem (see Theorem 1.5, [F1], [FF], and [FFS]). Note that Theorem 1.1 has already played important roles in [HM], [FG], and so on, when X is projective.

Theorem 1.1 (Twisted weak positivity)**.** *Let (X, Δ) be a log canonical pair such that X is in Fujiki's class $\mathcal{C}$ and let $f : X \to Y$ be a surjective*

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morphism onto a smooth projective variety Y with connected fibers. Assume that $k(K_X+\Delta)$ is Cartier. Then, for every positive integer m,

$$f_*\mathcal{O}_X(mk(K_{X/Y}+\Delta))$$

is weakly positive over some nonempty Zariski open set.

We have already discussed some generalizations of Theorem 1.1 in [F9], where Y is a curve and X is a reducible variety. They play crucial roles in order to prove the projectivity of various moduli spaces. For the details, see [F9].

In this paper, we first prove the following Hodge theoretic injectivity theorem (cf. [EV], [A], [F10], and so on).

Theorem 1.2 (Fundamental injectivity theorem)**.** *Let X be a complex manifold in Fujiki's class $\mathcal{C}$ and let Δ be a boundary $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that* SuppΔ *is a simple normal crossing divisor on X. Let $\mathcal{L}$ be a line bundle on X and let D be an effective Weil divisor on X whose support is contained in* SuppΔ*. Assume that $\mathcal{L}\sim_{\mathbb{R}} K_X+\Delta$. Then the natural homomorphism*

$$H^q(X,\mathcal{L})\to H^q(X,\mathcal{L}\otimes\mathcal{O}_X(D))$$

induced by the inclusion $\mathcal{O}_X\to\mathcal{O}_X(D)$ is injective for every q.

It is easy to see that Theorem 1.2 implies:

Theorem 1.3 (Injectivity theorem)**.** *Let X be a complex manifold in Fujiki's class $\mathcal{C}$ and let Δ be a boundary $\mathbb{R}$-divisor such that* SuppΔ *is simple normal crossing. Let $\mathcal{L}$ be a line bundle on X and let D be an effective Cartier divisor whose support contains no log canonical centers of (X,Δ). Assume the following conditions.*

(i) $\mathcal{L}\sim_{\mathbb{R}} K_X+\Delta+H$,
(ii) *H is a semi-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor, and*
(iii) *$tH\sim_{\mathbb{R}} D+D'$ for some positive real number t, where D' is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor whose support contains no log canonical centers of (X,Δ).*

Then the homomorphisms

$$H^q(X,\mathcal{L})\to H^q(X,\mathcal{L}\otimes\mathcal{O}_X(D))$$

which are induced by the natural inclusion $\mathcal{O}_X\to\mathcal{O}_X(D)$ are injective for all q.

As an application of Theorem 1.3, we obtain:

Theorem 1.4 (Torsion-freeness and vanishing theorem)**.** *Let Y be a complex manifold in Fujiki's class $\mathcal{C}$ and let Δ be a boundary $\mathbb{R}$-divisor*

such that $\mathrm{Supp}\Delta$ *is simple normal crossing. Let* $f : Y \to X$ *be a surjective morphism onto a projective variety* X *and let* $\mathcal{L}$ *be a line bundle on* Y *such that* $\mathcal{L} - (K_Y + \Delta)$ *is* f*-semi-ample.*

(i) *Let* q *be an arbitrary nonnegative integer. Every associated prime of* $R^q f_*\mathcal{L}$ *is the generic point of the* f*-image of some stratum of* (Y, Δ)*.*

(ii) *Assume that* $\mathcal{L} - (K_Y + \Delta) \sim_{\mathbb{R}} f^*H$ *for some ample* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor* H *on* X*. Then* $H^p(X, R^q f_*\mathcal{L}) = 0$ *for every* $p > 0$ *and* $q \geq 0$*.*

When X and Y are projective, Theorem 1.3 and Theorem 1.4 are well known and play crucial roles in [F6].

By using Theorem 1.4, we can establish:

Theorem 1.5 (Semipositivity theorem)**.** *Let* X *be a compact Kähler manifold and let* Y *be a smooth projective variety, and let* $f : X \to Y$ *be a surjective morphism. Let* D *be a simple normal crossing divisor on* X *such that every stratum of* D *is dominant onto* Y*. Let* Σ *be a simple normal crossing divisor on* Y*. We put* $Y_0 = Y \setminus \Sigma$*. If* f *is smooth and* D *is relatively normal crossing over* Y_0*, then* $R^i f_*\omega_{X/Y}(D)$ *is the upper canonical extension of the bottom Hodge filtration. In particular, it is locally free.*

We further assume that all the local monodromies on the local system $R^{d+i} f_{0*}\mathbb{C}_{X_0 - D_0}$ *around* Σ *are unipotent, then* $R^i f_*\omega_{X/Y}(D)$ *is semi-positive, where* $d = \dim X - \dim Y$*,* $X_0 = f^{-1}(Y_0)$*, and* $D_0 = D|_{X_0}$*.*

Theorem 1.5 is the main ingredient of Theorem 1.1. In this paper, we do not use [Kw1, Theorem 32] for the proof of Theorem 1.1 (see Remark 5.4). Note that Theorem 6.7 and Corollary 6.9, which directly follow from Theorem 1.5, seem to be new.

Let us discuss some applications of Theorem 1.1. The following conjecture is a natural formulation of Iitaka's conjecture for the minimal model program.

Conjecture 1.6 (Log Iitaka conjecture)**.** *Let* (X, Δ) *be a projective log canonical pair and let* $f : X \to Y$ *be a surjective morphism onto a normal projective variety* Y *with connected fibers. Then*

$$\kappa(X, K_X + \Delta) \geq \kappa(X_y, K_{X_y} + \Delta|_{X_y}) + \kappa(Y)$$

where X_y *is a sufficiently general fiber of* $f : X \to Y$*. Note that* $\kappa(Y)$ *denotes the Kodaira dimension of* Y*, that is,* $\kappa(Y) = \kappa(\widetilde{Y}, K_{\widetilde{Y}})$*, where* $\widetilde{Y} \to Y$ *is a resolution of singularities.*

When $\dim X = n$ and $\dim Y = m$ in Conjecture 1.6, we sometimes call it Conjecture $C^{\log}_{n,m}$. If X and Y are smooth and $\Delta = 0$, then

Conjecture 1.6 is nothing but Iitaka's original conjecture (see [I1]). We can easily check that Conjecture 1.6 holds true when Y is of general type. Note that Theorem 1.8 is contained in [Ca] (see also [N2, Chapter V. 4.1. Theorem (2)]). Moreover, Campana raised the orbifold version of Iitaka conjecture. For the details, see [Ca, Section 4] (see also [L]).

Remark 1.7. By Nakayama (see [N2, Chapter V. 4.4. Theorem (1)]), we have

$$\kappa_\sigma(X, K_X + \Delta) \geq \kappa_\sigma(X_y, K_{X_y} + \Delta|_{X_y}) + \kappa_\sigma(\widetilde{Y}, K_{\widetilde{Y}}),$$

where κ_σ denotes Nakayama's numerical dimension. In general, it is conjectured that $\kappa_\sigma(X, K_X + \Delta) = \kappa(X, K_X + \Delta)$, which is sometimes called *the generalized abundance conjecture*. If $\kappa_\sigma(X, K_X + \Delta) = \kappa(X, K_X + \Delta)$, then we have

$$\begin{aligned}\kappa(X, K_X + \Delta) &\geq \kappa_\sigma(X_y, K_{X_y} + \Delta|_{X_y}) + \kappa_\sigma(\widetilde{Y}, K_{\widetilde{Y}}) \\ &\geq \kappa(X_y, K_{X_y} + \Delta|_{X_y}) + \kappa(\widetilde{Y}, K_{\widetilde{Y}}).\end{aligned}$$

Therefore, Conjecture 1.6 follows from the generalized abundance conjecture.

Theorem 1.8 (Addition formula)**.** *Let (X, Δ) be a projective log canonical pair and let $f : X \to Y$ be a surjective morphism onto a normal projective variety Y with connected fibers. Assume that $\kappa(Y) = \dim Y$. Then*

$$\begin{aligned}\kappa(X, K_X + \Delta) &= \kappa(X_y, K_{X_y} + \Delta|_{X_y}) + \kappa(Y) \\ &= \kappa(X_y, K_{X_y} + \Delta|_{X_y}) + \dim Y\end{aligned}$$

where X_y is a sufficiently general fiber of $f : X \to Y$.

Theorem 1.9 is due to Maehara (see [Ma, Corollary 2]). We recover it as an application of Theorem 1.1.

Theorem 1.9 (Addition formula for logarithmic Kodaira dimensions)**.** *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Let D_X (resp. D_Y) be a simple normal crossing divisor on X (resp. Y). Assume that $\mathrm{Supp} f^*D_Y \subset \mathrm{Supp} D_X$. We further assume that $\kappa(Y, K_Y + D_Y) = \dim Y$. Then we have*

$$\begin{aligned}\kappa(X, K_X + D_X) &= \kappa(F, K_F + D_X|_F) + \kappa(Y, K_Y + D_Y) \\ &= \kappa(F, K_F + D_X|_F) + \dim Y,\end{aligned}$$

where F is a sufficiently general fiber of $f : X \to Y$.

We put $X^0 = X \setminus D_X$, $Y^0 = Y \setminus D_Y$, *and* $F^0 = F|_{X^0}$. *Then the above equality is nothing but*

$$\begin{aligned}\overline{\kappa}(X^0) &= \overline{\kappa}(F^0) + \overline{\kappa}(Y^0)\\ &= \overline{\kappa}(F^0) + \dim Y^0.\end{aligned}$$

Note that $\overline{\kappa}$ *denotes Iitaka's logarithmic Kodaira dimension* (*see* [I2]).

We quickly prove Theorem 1.8 and Theorem 1.9 in Section 7 and Section 8 by using Theorem 1.1 and [AK].

We summarize the contents of this paper. Section 2 collects some basic results and definitions. In Section 3, we prove the fundamental injectivity theorem: Theorem 1.2. The proof of Theorem 1.2 uses the theory of mixed Hodge structures. Subsection 3.1 is devoted to the theory of mixed Hodge structures for cohomology with compact support. In Section 4, we prove Theorem 1.3 and Theorem 1.4. These are direct consequences of Theorem 1.2. In Section 5, we explain the semipositivity theorem: Theorem 1.5. Section 6 is the main part of this paper. It is devoted to the proof of the twisted weak positivity theorem: Theorem 1.1. We prove Theorem 1.8 (resp. Theorem 1.9) in Section 7 (resp. Section 8) as an application of Theorem 1.1.

In this paper, we discuss neither Nakayama's sophisticated treatment of weak positivity in [N2, Chapter V. §3] nor Schnell's results on weak positivity coming from Saito's theory of mixed Hodge modules. We naively discuss some generalizations of Viehweg's weak positivity following [V2], [Ca], and so on.

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We will use the standard notation of the minimal model program as in [F6]. In this paper, we always assume that complex varieties are Hausdorff and countable at infinity. For the basic theory of complex varieties, see, for example, [BS], [Fi], and [N2]. The style of this paper is the same as that of [F6] (cf. [F3], [F4], [FF], and so on). Our results depend on the theory of variations of mixed Hodge structures (cf. [F1], [FF], [FFS], and so on).

2. Preliminaries

Let us start with some remarks on *canonical divisors*.

2.1 (Canonical divisors)**.** We consider complex variety X, which is not necessarily algebraic.

Remark 2.2. (i) Let $\omega_X^\bullet$ be the *dualizing complex* of a complex variety X (see, for example, [RR], [RRV], [BS], and so on). We put $\omega_X = \mathcal{H}^{-d}(\omega_X^\bullet)$, where $d = \dim X$, and call it the *canonical sheaf* of X. When X is a compact complex manifold, it is well known that $\omega_X \simeq \Omega_X^d$. For the details of $\omega_X^\bullet$, see, for example, [BS, Chapter VII §2].

(iii) Some complex variety X does not admit any nonzero meromorphic section of ω_X. However, if there is no risk of confusion, we use the symbol K_X as a formal divisor class with an isomorphism $\mathcal{O}_X(K_X) \simeq \omega_X$ and call it the *canonical divisor* of X. See [N2, Chapter II. §4].

Remark 2.3. Let D be a Cartier divisor and let $\mathcal{L}$ be a line bundle on a complex variety X. If there is no risk of confusion, we sometimes write

$$\mathcal{O}_X(K_X + D + \mathcal{L})$$

in order to express

$$\omega_X \otimes \mathcal{O}_X(D) \otimes \mathcal{L}.$$

For simplicity, we sometimes use $\mathcal{L}^N$ to denote $\mathcal{L}^{\otimes N}$ if there is no risk of confusion.

In this paper, all complex varieties are *algebraic* or *compact*. Therefore, there are no subtle problems in the following definitions.

2.4 (Singularities of pairs)**.** Let us recall the definition of singularities of pairs.

Let X be a normal variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $f : Y \to X$ be a resolution such that $\mathrm{Exc}(f) \cup f_*^{-1}\Delta$ has a simple normal crossing support, where $f_*^{-1}\Delta$ is the strict transform of Δ on Y. We write

$$K_Y = f^*(K_X + \Delta) + \sum_i a_i E_i$$

and $a(E_i, X, \Delta) = a_i$. We say that (X, Δ) is *lc* if and only if $a_i \geq -1$ for every i. Note that the *discrepancy* $a(E, X, \Delta) \in \mathbb{R}$ can be defined for every prime divisor E *over* X. Let (X, Δ) be an lc pair. If there is a resolution $f : Y \to X$ such that $\mathrm{Exc}(f)$ is a divisor, $\mathrm{Exc}(f) \cup f_*^{-1}\Delta$ has a simple normal crossing support, and $a(E, X, \Delta) > -1$ for every f-exceptional divisor E, then (X, Δ) is called *dlt*. Here, lc (resp. dlt) is an abbreviation of *log canonical* (resp. *divisorial log terminal*).

For the details and various examples of singularities of pairs, see, for example, [F2].

Remark 2.5 (Szabó's resolution lemma)**.** We note that Szabó's resolution lemma (see, for example, [F2, 3.5 Resolution lemma]) now holds

for compact complex varieties. For the details, see, for example, [Ko, Theorem 10.45, Proposition 10.49, and the proof of (10.45)]. We will use Szabó's resolution lemma repeatedly in this paper.

Let us recall the definition of *log canonical centers*.

Definition 2.6 (Log canonical center)**.** Let (X, Δ) be a log canonical pair. If there is a resolution $f : Y \to X$ and a prime divisor E on Y such that $a(E, X, \Delta) = -1$, then $f(E)$ is called a *log canonical center* of (X, Δ).

Definition 2.7 is useful for torsion-free theorem.

Definition 2.7 (Stratum)**.** Let (X, Δ) be a log canonical pair. A *stratum* of (X, Δ) is X itself or a log canonical center of (X, Δ).

2.8 (Divisors)**.** Let us recall some basic operations for $\mathbb{Q}$-divisors and $\mathbb{R}$-divisors.

For an $\mathbb{R}$-divisor $D = \sum_{i=1}^{r} d_i D_i$ such that D_i is a prime divisor for every i and $D_i \neq D_j$ for $i \neq j$, we define the *round-down* $\lfloor D \rfloor = \sum_{i=1}^{r} \lfloor d_i \rfloor D_i$ (resp. the *round-up* $\lceil D \rceil = \sum_{i=1}^{r} \lceil d_i \rceil D_i$), where for every real number x, $\lfloor x \rfloor$ (resp. $\lceil x \rceil$) is the integer defined by $x - 1 < \lfloor x \rfloor \leq x$ (resp. $x \leq \lceil x \rceil < x + 1$). The *fractional part* $\{D\}$ of D denotes $D - \lfloor D \rfloor$. We also define $D^{=1} = \sum_{d_i=1} D_i$. We call D a *boundary* $\mathbb{R}$-divisor if $0 \leq d_i \leq 1$ for every i.

Remark 2.9. Let X be a compact complex manifold and let D_1, D_2, $\cdots$, D_k be Cartier divisors on X. We consider the linear map

$$\varphi : \mathbb{R}^k \longrightarrow \mathrm{Pic}(X) \otimes \mathbb{R}$$

defined by $\varphi(r_1, r_2, \cdots, r_k) = r_1 D_1 + r_2 D_2 + \cdots + r_k D_k$, which is defined over $\mathbb{Q}$. Let $\mathcal{L}$ be a line bundle on X. Then $\mathcal{L} \sim_{\mathbb{R}} \sum_{i=1}^{k} r_i D_i$ means $\mathcal{L} = \varphi(r_1, r_2, \cdots, r_k)$ in $\mathrm{Pic}(X) \otimes \mathbb{R}$. Note that $\varphi^{-1}(\mathcal{L})$ is an affine subspace of $\mathbb{R}^k$ defined over $\mathbb{Q}$. Therefore, we can find $(r'_1, r'_2, \cdots, r'_k) \in \mathbb{Q}^k$ such that $\mathcal{L} \sim_{\mathbb{Q}} \sum_{i=1}^{k} r'_i D_i$, that is, $\mathcal{L} = \varphi(r'_1, r'_2, \cdots, r'_k)$ in $\mathrm{Pic}(X) \otimes \mathbb{Q}$ if $\varphi^{-1}(\mathcal{L})$ is not empty.

2.10 (Fujiki's class $\mathcal{C}$)**.** In this paper, we use the notion of complex varieties in Fujiki's class $\mathcal{C}$.

Definition 2.11 (Fujiki's class $\mathcal{C}$)**.** Let X be a compact reduced complex analytic space. Then X is *in Fujiki's class* $\mathcal{C}$ if and only if there is a surjective morphism $f : Y \to X$ with Y a compact Kähler manifold. It is well known that X is in Fujiki's class $\mathcal{C}$ if and only if there is a bimeromorphic morphism $g : V \to X$ from a compact Kähler manifold V (see, for example, [Va, Théorème 3]).

It is well known that some basic results on the minimal model program can be generalized for varieties in Fujiki's class $\mathcal{C}$. See, for example, [N1], [F5, Section 4], and so on.

Remark 2.12. For the details of complex varieties in Fujiki's class $\mathcal{C}$, (locally) Kähler morphisms, and so on, see [Fk1], [Fk2], and [Va].

2.13 (Simple normal crossing varieties)**.** In Subsection 3.1, we will use the *Mayer–Vietoris simplicial resolution* of a simple normal crossing variety X in order to discuss various mixed Hodge structures.

Definition 2.14 (Mayer–Vietoris simplicial resolution)**.** Let X be a simple normal crossing variety with the irreducible decomposition $X = \bigcup_{i\in I} X_i$. Let I_n be the set of strictly increasing sequences $(i_0, \cdots, i_n)$ in I and $X^n = \coprod_{I_n} X_{i_0}\cap\cdots\cap X_{i_n}$ the disjoint union of the intersections of X_i. Let $\varepsilon_n : X^n \to X$ be the disjoint union of the natural inclusions. Then $\{X^n, \varepsilon_n\}_n$ has a natural semi-simplicial structure. The face operator is induced by $\lambda_{j,n}$, where

$$\lambda_{j,n} : X_{i_0} \cap \cdots \cap X_{i_n} \to X_{i_0} \cap \cdots \cap X_{i_{j-1}} \cap X_{i_{j+1}} \cap \cdots \cap X_{i_n}$$

is the natural closed embedding for $j \leq n$ (cf. [E2, 3.5.5]). We denote it by $\varepsilon : X^\bullet \to X$ and call it the *Mayer–Vietoris simplicial resolution* of X. The complex

$$0 \to \varepsilon_{0*}\mathcal{O}_{X^0} \to \varepsilon_{1*}\mathcal{O}_{X^1} \to \cdots \to \varepsilon_{k*}\mathcal{O}_{X^k} \to \cdots,$$

where the differential $d_k : \varepsilon_{k*}\mathcal{O}_{X^k} \to \varepsilon_{k+1*}\mathcal{O}_{X^{k+1}}$ is $\sum_{j=0}^{k+1}(-1)^j\lambda_{j,k+1}^*$ for every $k \geq 0$, is denoted by $\mathcal{O}_{X^\bullet}$. It is easy to see that $\mathcal{O}_{X^\bullet}$ is quasi-isomorphic to $\mathcal{O}_X$. By tensoring $\mathcal{L}$, any line bundle on X, to $\mathcal{O}_{X^\bullet}$, we obtain a complex

$$0 \to \varepsilon_{0*}\mathcal{L}^0 \to \varepsilon_{1*}\mathcal{L}^1 \to \cdots \to \varepsilon_{k*}\mathcal{L}^k \to \cdots,$$

where $\mathcal{L}^n = \varepsilon_n^*\mathcal{L}$. It is denoted by $\mathcal{L}^\bullet$. Of course, $\mathcal{L}^\bullet$ is quasi-isomorphic to $\mathcal{L}$. We note that $\mathbb{H}^q(X, \mathcal{L}^\bullet)$ is obviously isomorphic to $H^q(X, \mathcal{L})$ for every $q \geq 0$ because $\mathcal{L}^\bullet$ is quasi-isomorphic to $\mathcal{L}$.

We note that a *stratum* of X means an irreducible component of $X_{i_0} \cap \cdots \cap X_{i_k}$ for some $\{i_0, \cdots, i_k\} \subset I$.

2.15 (Flat base change theorem)**.** In the proof of Theorem 1.9, we will use the flat base change theorem for relative dualizing sheaves (see [V2, §3] and [Mo, Sectino 4]). We need the following statement.

Theorem 2.16. *Let $f : V \to W$ be a surjective morphism from a Cohen–Macaulay projective variety V to a smooth projective variety*

W. Let $g : W' \to W$ be a finite flat morphism from a smooth projective variety W'. We consider the following diagram:

$$\begin{array}{ccc} W' \times_W V & \xrightarrow{h} & V \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ W' & \xrightarrow[g]{} & W \end{array}$$

Then we have

$$h^*\omega_{V/W} = \omega_{W' \times_W V/W'}.$$

Note that

$$\omega_{V/W} = \omega_V \otimes f^*\omega_W^{\otimes -1} \quad \text{and} \quad \omega_{W' \times_W V/W'} = \omega_{W' \times_W V} \otimes f'^*\omega_{W'}^{\otimes -1}.$$

Theorem 2.16 is a very special case of the flat base change theorem (see [Vd, Theorem 2]). See also [H], [Co], and so on. The author does not know if the flat base change theorem ([Vd, Theorem 2]) is true or not in the analytic category (cf. [RR] and [RRV]). Therefore, we do not use the flat base change theorem in the proof of Theorem 1.1 (see [V2, Lemma 3.2.] and [Mo, (4.10) Base change theorem]). Note that X in Theorem 1.1 is not necessarily *algebraic*.

2.17 (Relative vanishing theorems)**.** The following theorem is a relative version of the Kawamata–Viehweg vanishing theorem for generically finite morphisms.

Theorem 2.18 (cf. [N1, Theorem 3.6])**.** *Let $f : X \to Y$ be a proper generically finite morphism from a compact complex manifold X onto a complex variety Y and let Δ be a $\mathbb{Q}$-divisor on X such that* SuppΔ *is a simple normal crossing divisor and $\lfloor \Delta \rfloor = 0$. Let $\mathcal{L}$ be a line bundle on X. Assume that $\mathcal{L} - (K_X + \Delta)$ is f-nef. Then $R^i f_* \mathcal{L} = 0$ for $i > 0$.*

Theorem 2.18 is a special case of [F7, Corollary 1.3]. For the details, see [N1], [F7], and so on. Lemma 2.19, which is an easy consequence of Theorem 2.18, is very useful and indispensable.

Lemma 2.19 (Reid–Fukuda type)**.** *Let X be a compact complex manifold and let Δ be a boundary $\mathbb{Q}$-divisor on X such that* SuppΔ *is a simple normal crossing divisor on X. Let $f : X \to Y$ be a bimeromorphic morphism onto a compact complex variety Y. Assume that f is an isomorphism at the generic point of any log canonical center of (X, Δ) and that $\mathcal{L}$ is a line bundle on X such that $\mathcal{L} - (K_X + \Delta)$ is f-nef. Then $R^i f_* \mathcal{L} = 0$ for every $i > 0$.*

Proof. By using induction on the number of irreducible components of $\lfloor \Delta \rfloor$ and on the dimension of X, we can quickly prove Lemma 2.19

by Theorem 2.18. For the details, see, for example, the proof of [F6, Lemma 6.2]. □

We close this section with a remark on the relative Kawamata–Viehweg vanishing theorem. Anyway, the proof of Theorem 2.18 when Y is not algebraic is much harder than the case when Y is algebraic.

Remark 2.20 (Projective versus Kähler)**.** We are mainly interested in *projective* varieties. This is because the minimal model program works well only for *projective* varieties. However, in this paper, we treat Kähler manifolds and complex varieties in Fujiki's class $\mathcal{C}$ in order to cover Campana's result (see [Ca, Theorem 4.13], which is essentially equivalent to Theorem 1.1). If the reader is only interested in *projective* varieties, then we recommend the reader to read this paper assuming that all the varieties are *projective*.

Let $f : X \to Y$ be a projective bimeromorphic morphism from a compact complex manifold X to a compact Kähler manifold Y. Let D be an f-nef Cartier divisor on X such that the support of the fractional part $\{D\}$ of D is a simple normal crossing divisor on X. Then $R^i f_*\mathcal{O}_X(K_X + \lceil D \rceil) = 0$ for every $i > 0$ by Theorem 2.18.

If Y is *projective*, then the above vanishing easily follows from the usual Kawamata–Viehweg vanishing theorem for *projective* varieties (cf. [KM, Proposition 2.69] and the proof of Proposition 6.11). This means that the relative vanishing theorem follows from the vanishing theorem for projective varieties. On the other hand, if Y is Kähler but is not projective, then the above vanishing theorem is much harder to prove.

3. Proof of Theorem 1.2: Fundamental injectivity theorem

In this section, we prove Theorem 1.2. Theorem 1.2 is a direct consequence of the E_1-degeneration of Hodge to de Rham spectral sequence associated to the mixed Hodge structure for cohomology with compact support. We discuss the E_1-degeneration in Subsection 3.1.

Proof of Theorem 1.2. Without loss of generality, we may assume that X is connected. We put $S = \lfloor \Delta \rfloor$ and $B = \{\Delta\}$. By perturbing B, we may assume that B is a $\mathbb{Q}$-divisor (see Remark 2.9). We put $\mathcal{M} = \mathcal{O}_X(\mathcal{L} - K_X - S)$. Let N be the smallest positive integer such that $N\mathcal{L} \sim N(K_X + S + B)$. In particular, NB is an integral Weil

divisor. We take the N-fold cyclic cover

$$\pi' : Y' = \mathrm{Specan} \bigoplus_{i=0}^{N-1} \mathcal{M}^{-i} \to X$$

associated to the section $NB \in |\mathcal{M}^N|$. More precisely, let $s \in H^0(X, \mathcal{M}^N)$ be a section whose zero divisor is NB. Then the dual of $s : \mathcal{O}_X \to \mathcal{M}^N$ defines an $\mathcal{O}_X$-algebra structure on $\bigoplus_{i=0}^{N-1} \mathcal{M}^{-i}$. Let $Y \to Y'$ be the normalization and let $\pi : Y \to X$ be the composition morphism. It is well known that

$$Y = \mathrm{Specan} \bigoplus_{i=0}^{N-1} \mathcal{M}^{-i}(\lfloor iB \rfloor).$$

For the details, see [EV, 3.5. Cyclic covers]. Note that Y has only quotient singularities. We put $T = \pi^* S$. We note that T is Cartier. Hence the locally free sheaf $\mathcal{O}_Y(-T)$ is the defining ideal sheaf of T on Y. The E_1-degeneration of

$$(\clubsuit) \qquad E_1^{p,q} = \mathbb{H}^q(Y, \widetilde{\Omega}_Y^p(\log T)(-T)) \Rightarrow H^{p+q}(Y, j_! \mathbb{C}_{Y-T})$$

implies that the homomorphism

$$H^q(Y, j_! \mathbb{C}_{Y-T}) \to H^q(Y, \mathcal{O}_Y(-T))$$

induced by the natural inclusion

$$j_! \mathbb{C}_{Y-T} \subset \mathcal{O}_Y(-T)$$

is surjective for every q. We will discuss the E_1-degeneration of $(\clubsuit)$ in Subsection 3.1 below. By taking a suitable direct summand

$$\mathcal{C} \subset \mathcal{M}^{-1}(-S)$$

of

$$\pi_*(j_! \mathbb{C}_{Y-T}) \subset \pi_* \mathcal{O}_Y(-T),$$

we obtain a surjection

$$H^q(X, \mathcal{C}) \to H^q(X, \mathcal{M}^{-1}(-S))$$

induced by the natural inclusion $\mathcal{C} \subset \mathcal{M}^{-1}(-S)$ for every q. We can check the following simple property by examining the monodromy action of the Galois group $\mathbb{Z}/N\mathbb{Z}$ of $\pi : Y \to X$ on $\mathcal{C}$ around $\mathrm{Supp} B$.

Lemma 3.1 (cf. [KM, Corollary 2.54])**.** *Let $U \subset X$ be a connected open subset such that $U \cap \mathrm{Supp}\Delta \neq \emptyset$. Then $H^0(U, \mathcal{C}|_U) = 0$.*

Proof. If $U \cap \mathrm{Supp} B \neq \emptyset$, then $H^0(U, \mathcal{C}|_U) = 0$ since the monodromy action on $\mathcal{C}$ around $\mathrm{Supp} B$ is nontrivial. If $U \cap \mathrm{Supp} S \neq \emptyset$, then $H^0(U, \mathcal{C}|_U) = 0$ since $\mathcal{C}$ is a direct summand of $\pi_*(j_! \mathbb{C}_{Y-T})$ and $T = \pi^* S$. □

Lemma 3.1 is useful by the following fact. The proof of Lemma 3.2 is obvious.

Lemma 3.2 (cf. [KM, Lemma 2.55])**.** *Let F be a sheaf of Abelian groups on a topological space X and let F_1 and F_2 be subsheaves of F. Let $Z \subset X$ be a closed subset. Assume that*

(1) $F_2|_{X-Z} = F|_{X-Z}$, *and*
(2) *if $U \subset X$ is a connected open subset with $U \cap Z \neq \emptyset$, then $H^0(U, F_1|U) = 0$.*

Then F_1 is a subsheaf of F_2.

Therefore, we obtain:

Corollary 3.3 (cf. [KM, Corollary 2.56])**.** *Let $M \subset \mathcal{M}^{-1}(-S)$ be a subsheaf such that $M|_{X-\mathrm{Supp}\Delta} = \mathcal{M}^{-1}(-S)|_{X-\mathrm{Supp}\Delta}$. Then the injection*

$$\mathcal{C} \to \mathcal{M}^{-1}(-S)$$

factors as

$$\mathcal{C} \to M \to \mathcal{M}^{-1}(-S).$$

Therefore,

$$H^q(X, M) \to H^q(X, \mathcal{M}^{-1}(-S))$$

is surjective for every q.

Proof. The first part is clear from Lemma 3.1 and Lemma 3.2. This implies that we have maps

$$H^q(X, \mathcal{C}) \to H^q(X, M) \to H^q(X, \mathcal{M}^{-1}(-S)).$$

As we saw above, the composition is surjective. Hence so is the map on the right. □

Therefore, $H^q(X, \mathcal{M}^{-1}(-S-D)) \to H^q(X, \mathcal{M}^{-1}(-S))$ is surjective for every q. By Serre duality, we obtain that

$$H^q(X, \mathcal{O}_X(K_X) \otimes \mathcal{M}(S)) \to H^q(X, \mathcal{O}_X(K_X) \otimes \mathcal{M}(S+D))$$

is injective for every q. This means that

$$H^q(X, \mathcal{L}) \to H^q(X, \mathcal{L} \otimes \mathcal{O}_X(D))$$

is injective for every q. □

3.1. **MHS for cohomology with compact support.** In this subsection, we prove the E_1-degeneration of (♣) in the proof of Theorem 1.2 for the reader's convenience. It is more or less well known to the experts.

From 3.1.1 to 3.1.3, we recall some well-known results on mixed Hodge structures. We use the notations in [D] freely. The basic references on this topic are [D, Section 8], [E1, Part II], [E2, Chapitres 2 and 3], and the book [PS].

First, we start with the pure Hodge structures on complex manifolds in Fujiki's class $\mathcal{C}$.

3.1.1. Let X be a complex manifold in Fujiki's class $\mathcal{C}$. Then the triple $(\mathbb{Z}_X, (\Omega_X^\bullet, F), \alpha)$, where $\Omega_X^\bullet$ is the holomorphic de Rham complex with the filtration bête F and $\alpha : \mathbb{C}_X \to \Omega_X^\bullet$ is the inclusion, is a cohomological Hodge complex (CHC, for short) of weight zero.

If we define weight filtrations as follows:

$$W_m\mathbb{Q}_X = \begin{cases} 0 & \text{if} \quad m < 0 \\ \mathbb{Q}_X & \text{if} \quad m \geq 0 \end{cases}$$

and

$$W_m\Omega_X^\bullet = \begin{cases} 0 & \text{if} \quad m < 0 \\ \Omega_X^\bullet & \text{if} \quad m \geq 0, \end{cases}$$

then we can see that

$$(\mathbb{Z}_X, (\mathbb{Q}_X, W), (\Omega_X^\bullet, F, W))$$

is a cohomological mixed Hodge complex (CMHC, for short). We need these weight filtrations in the following arguments.

The next one is also a fundamental example. For the details, see [E1, I.1.] or [E2, 3.5].

3.1.2. Let D be a simple normal crossing variety in Fujiki's class $\mathcal{C}$. Let $\varepsilon : D^\bullet \to D$ be the Mayer–Vietoris simplicial resolution (see Definition 2.14). We use similar notations to those in Definition 2.14. The following complex of sheaves, denoted by $\mathbb{Q}_{D^\bullet}$,

$$\varepsilon_{0*}\mathbb{Q}_{D^0} \to \varepsilon_{1*}\mathbb{Q}_{D^1} \to \cdots \to \varepsilon_{k*}\mathbb{Q}_{D^k} \to \cdots,$$

is a resolution of $\mathbb{Q}_D$. More explicitly, the differential $d_k : \varepsilon_{k*}\mathbb{Q}_{D^k} \to \varepsilon_{k+1*}\mathbb{Q}_{D^{k+1}}$ is $\sum_{j=0}^{k+1}(-1)^j\lambda_{j,k+1}^*$ for every $k \geq 0$. The weight filtration W on $\mathbb{Q}_{D^\bullet}$ is defined by

$$W_{-q}(\mathbb{Q}_{D^\bullet}) = (0 \to \cdots \to 0 \to \varepsilon_{q*}\mathbb{Q}_{D^q} \to \varepsilon_{q+1*}\mathbb{Q}_{D^{q+1}} \to \cdots).$$

We obtain the resolution $\Omega^\bullet_{D^\bullet}$ of $\mathbb{C}_D$ as follows:

$$\varepsilon_{0*}\Omega^\bullet_{D^0} \to \varepsilon_{1*}\Omega^\bullet_{D^1} \to \cdots \to \varepsilon_{k*}\Omega^\bullet_{D^k} \to \cdots .$$

Of course, $d_k : \varepsilon_{k*}\Omega^\bullet_{D^k} \to \varepsilon_{k+1*}\Omega^\bullet_{D^{k+1}}$ is $\sum_{j=0}^{k+1}(-1)^j\lambda^*_{j,k+1}$. Let $s(\Omega^\bullet_{D^\bullet})$ be the single complex associated to the double complex $\Omega^\bullet_{D^\bullet}$. The Hodge filtration F on $s(\Omega^\bullet_{D^\bullet})$ is defined by

$$F^p = s(0 \to \cdots \to 0 \to \varepsilon_*\Omega^p_{D^\bullet} \to \varepsilon_*\Omega^{p+1}_{D^\bullet} \to \cdots).$$

We note that

$$\varepsilon_*\Omega^p_{D^\bullet} = (\varepsilon_{0*}\Omega^p_{D^0} \to \varepsilon_{1*}\Omega^p_{D^1} \to \cdots \to \varepsilon_{k*}\Omega^p_{D^k} \to \cdots)$$

for every p. The weight filtration W on $s(\Omega^\bullet_{D^\bullet})$ is defined by

$$W_{-q}(s(\Omega^\bullet_{D^\bullet})) = s(0 \to \cdots \to 0 \to \varepsilon_{q*}\Omega^\bullet_{D^q} \to \varepsilon_{q+1*}\Omega^\bullet_{D^{q+1}} \to \cdots).$$

We note that

$$\mathrm{Gr}^W_{-q}\mathbb{Q}_{D^\bullet} \simeq \varepsilon_{q*}\mathbb{Q}_{D^q}[-q]$$

and

$$\mathrm{Gr}^W_{-q}(s(\Omega^\bullet_{D^\bullet})) \simeq \varepsilon_{q*}\Omega^\bullet_{D^q}[-q].$$

Then

$$(\mathbb{Z}_D, (\mathbb{Q}_{D^\bullet}, W), (s(\Omega^\bullet_{D^\bullet}), W, F))$$

is a CMHC. Here, we omitted the quasi-isomorphisms $\alpha : \mathbb{Z}_D \otimes \mathbb{Q} \to \mathbb{Q}_{D^\bullet}$ and $\beta : (\mathbb{Q}_{D^\bullet}, W) \otimes \mathbb{C} \to (s(\Omega^\bullet_{D^\bullet}), W)$ since there is no risk of confusion. This CMHC induces a natural mixed Hodge structure on $H^\bullet(D, \mathbb{Z})$. We note that the spectral sequence with respect to W on $\mathbb{Q}_{D^\bullet}$ is

$$\begin{aligned} {}_WE_1^{p,q} = H^{p+q}(D, \mathrm{Gr}^W_{-p}\mathbb{Q}_{D^\bullet}) &= H^{p+q}(D, \varepsilon_{p*}\mathbb{Q}_{D^p}[-p]) \\ &= H^q(D^p, \mathbb{Q}) \\ &\Rightarrow H^{p+q}(D, \mathbb{Q}) \end{aligned}$$

such that the differential $d_1^{p,q} : {}_WE_1^{p,q} \to {}_WE_1^{p+1,q}$ is given by

$$d_1^{p,q} = \sum_{j=0}^{p+1}(-1)^j\lambda^*_{j,p+1} : H^q(D^p, \mathbb{Q}) \to H^q(D^{p+1}, \mathbb{Q})$$

and it degenerates at E_2. The spectral sequence with respect to F is

$${}_FE_1^{p,q} = \mathbb{H}^{p+q}(D, \mathrm{Gr}^p_F(s(\Omega^\bullet_{D^\bullet}))) \Rightarrow H^{p+q}(D, \mathbb{C})$$

and it degenerates at E_1.

For the precise definitions of CHC and CMHC (CHMC, in French), see [D, Section 8] or [E2, Chapitre 3]. See also [PS, 2.3.3 and 3.3].

The third example is not so standard but is indispensable for our injectivity theorems.

3.1.3. Let X be a complex manifold in Fujiki's class $\mathcal{C}$ and let D be a simple normal crossing divisor on X. We consider the mixed cones of $\phi : \mathbb{Q}_X \to \mathbb{Q}_{D^\bullet}$ and $\psi : \Omega_X^\bullet \to \Omega_{D^\bullet}^\bullet$ with suitable shifts of complexes and weight filtrations (for the details, see, for example, [E1, I.3.], [E2, 3.7.14] or [PS, Theorem 3.22]), where ϕ and ψ are induced by the natural restriction map. More precisely, we define a complex

$$\mathbb{Q}_{X-D^\bullet} = \mathrm{Cone}^\bullet(\phi)[-1].$$

Then we have

$$(\mathbb{Q}_{X-D^\bullet})^p = (\mathbb{Q}_X)^p \oplus (\mathbb{Q}_{D^\bullet})^{p-1}.$$

The weight filtration on $\mathbb{Q}_{X-D^\bullet}$ is defined as follows:

$$(W_m\mathbb{Q}_{X-D^\bullet})^p = (W_m\mathbb{Q}_X)^p \oplus (W_{m+1}(\mathbb{Q}_{D^\bullet}))^{p-1}.$$

We note that $\mathbb{Q}_{X-D^\bullet}$ is quasi-isomorphic to $j_!\mathbb{Q}_{X-D}$, where $j : X-D \to X$ is the natural open immersion. We put

$$\Omega_{X-D^\bullet}^\bullet = \mathrm{Cone}^\bullet(\psi)[-1].$$

We note that

$$\Omega_{X-D^\bullet}^p = \Omega_X^p \oplus (s\Omega_{D^\bullet}^\bullet)^{p-1}.$$

We define filtrations on $\Omega_{X-D^\bullet}^\bullet$ as follows:

$$(W_m\Omega_{X-D^\bullet}^\bullet)^p = (W_m\Omega_X^\bullet)^p \oplus (W_{m+1}(s\Omega_{D^\bullet}^\bullet))^{p-1}$$

and

$$(F^r\Omega_{X-D^\bullet}^\bullet)^p = (F^r\Omega_X^\bullet)^p \oplus (F^r(s\Omega_{D^\bullet}^\bullet))^{p-1}.$$

Then we obtain that the triple

$$(j_!\mathbb{Z}_{X-D}, (\mathbb{Q}_{X-D^\bullet}, W), (\Omega_{X-D^\bullet}^\bullet, W, F))$$

is a CMHC. It defines a natural mixed Hodge structure on $H_c^\bullet(X - D, \mathbb{Z})$. We note that

$$\mathrm{Gr}_0^W\mathbb{Q}_{X-D^\bullet} = \mathbb{Q}_X$$

and

$$\mathrm{Gr}_{-p}^W\mathbb{Q}_{X-D^\bullet} = \mathrm{Gr}_{1-p}^W\mathbb{Q}_{D^\bullet}[-1] = \varepsilon_{p-1*}\mathbb{Q}_{D^{p-1}}[-p]$$

for $p \geq 1$. The spectral sequence with respect to W

$${}_WE_1^{p,q} = H^{p+q}(X, \mathrm{Gr}_{-p}^W\mathbb{Q}_{X-D^\bullet}) \Rightarrow H_c^{p+q}(X - D, \mathbb{Q})$$

degenerates at E_2, where

$${}_WE_1^{0,q} = H^q(X, \mathbb{Q})$$

and

$${}_WE_1^{p,q} = H^q(D^{p-1}, \mathbb{Q})$$

for every $p \geq 1$. We put

$$\Omega_X^\bullet(\log D)(-D) = \Omega_X^\bullet(\log D) \otimes \mathcal{O}_X(-D).$$

Since we can check that the complex

$$0 \to \Omega_X^\bullet(\log D)(-D) \to \Omega_X^\bullet \to \varepsilon_{0*}\Omega_{D^0}^\bullet \\ \to \varepsilon_{1*}\Omega_{D^1}^\bullet \to \cdots \to \varepsilon_{k*}\Omega_{D^k}^\bullet \to \cdots$$

is exact by direct local calculations, we see that $(\Omega_{X-D^\bullet}^\bullet, F)$ is quasi-isomorphic to $(\Omega_X^\bullet(\log D)(-D), F)$ in $D^+F(X, \mathbb{C})$, where

$$F^p\Omega_X^\bullet(\log D)(-D) \\ = (0 \to \cdots \to 0 \to \Omega_X^p(\log D)(-D) \to \Omega_X^{p+1}(\log D)(-D) \to \cdots).$$

Therefore, the spectral sequence with respect to F

$$E_1^{p,q} = H^q(X, \Omega_X^p(\log D)(-D)) \Rightarrow \mathbb{H}^{p+q}(X, \Omega_X^\bullet(\log D)(-D))$$

degenerates at E_1. Note that the right hand side is isomorphic to $H_c^{p+q}(X - D, \mathbb{C})$. We also note that

$$\mathrm{Gr}_F^0\Omega_X^\bullet(\log D)(-D) \simeq \mathcal{O}_X(-D).$$

Let us recall the notion of *V-manifolds*. We need it for the proof of Theorem 1.2.

Definition 3.1.4 (V-manifold)**.** A *V-manifold* of dimension d is a complex analytic space that admits an open covering $\{U_i\}$ such that each U_i is analytically isomorphic to V_i/G_i, where $V_i \subset \mathbb{C}^d$ is an open ball and G_i is a finite subgroup of $\mathrm{GL}(d, \mathbb{C})$. In this paper, G_i is always an abelian group for every i.

Let X be a V-manifold and let Σ be its singular locus. Then we define

$$\widetilde{\Omega}_X^\bullet = j_*\Omega_{X-\Sigma}^\bullet,$$

where $j : X - \Sigma \to X$ is the natural open immersion. A divisor D on X is called a *divisor with V-normal crossings* if locally on X we have $(X, D) \simeq (V, E)/G$ with $V \subset \mathbb{C}^d$ an open domain, $G \subset \mathrm{GL}(d, \mathbb{C})$ a small subgroup acting on V, and $E \subset V$ a G-invariant divisor with only normal crossing singularities. We define

$$\widetilde{\Omega}_X^\bullet(\log D) = j_*\Omega_{X-\Sigma}^\bullet(\log D).$$

Furthermore, if D is Cartier, then we put

$$\widetilde{\Omega}_X^\bullet(\log D)(-D) = \widetilde{\Omega}_X^\bullet(\log D) \otimes \mathcal{O}_X(-D).$$

Let us go back to the proof of the E_1-degeneration of (♣) in the proof of Theorem 1.2.

Proof of the E_1-degeneration of (♠) *in the proof of Theorem 1.2.* Here, we use the notation in the proof of Theorem 1.2. In this case, Y has

only quotient singularities, that is, Y is a V-manifold. Then we obtain that

$$(\mathbb{Z}_Y, (\widetilde{\Omega}_Y^\bullet, F), \alpha)$$

is a CHC, where F is the filtration bête and $\alpha : \mathbb{C}_Y \to \widetilde{\Omega}_Y^\bullet$ is the inclusion. For the details, see [St, (1.6)]. It is easy to see that T is a divisor with V-normal crossings on Y (see Definition 3.1.4 and [St, (1.16) Definition]). We can easily check that Y is singular only over the singular locus of Supp$B$. Let $\varepsilon : T^\bullet \to T$ be the Mayer–Vietoris simplicial resolution. Although T has singularities, Definition 2.14 makes sense without any modifications. We note that T^n has only quotient singularities for every $n \geq 0$ by the construction of $\pi : Y \to X$. We can also check that the same construction in 3.1.2 works with only minor modifications. Hence we have a CMHC

$$(\mathbb{Z}_T, (\mathbb{Q}_{T^\bullet}, W), (s(\widetilde{\Omega}_{T^\bullet}^\bullet), W, F))$$

that induces a natural mixed Hodge structure on $H^\bullet(T, \mathbb{Z})$. By the same arguments as in 3.1.3, we can construct a triple

$$(j_!\mathbb{Z}_{Y-T}, (\mathbb{Q}_{Y-T^\bullet}, W), (K_\mathbb{C}, W, F)),$$

where $j : Y - T \to Y$ is the natural open immersion. It is a CMHC that induces a natural mixed Hodge structure on $H_c^\bullet(Y - T, \mathbb{Z})$ and $(K_\mathbb{C}, F)$ is quasi-isomorphic to $(\widetilde{\Omega}_Y^\bullet(\log T)(-T), F)$ in $D^+F(Y, \mathbb{C})$, where

$$\begin{aligned} &F^p\widetilde{\Omega}_Y^\bullet(\log T)(-T) \\ &\quad = (0 \to \cdots \to 0 \to \widetilde{\Omega}_Y^p(\log T)(-T) \to \widetilde{\Omega}_Y^{p+1}(\log T)(-T) \to \cdots). \end{aligned}$$

Therefore, the spectral sequence with respect to F

$$E_1^{p,q} = H^q(Y, \widetilde{\Omega}_Y^p(\log T)(-T)) \Rightarrow \mathbb{H}^{p+q}(Y, \widetilde{\Omega}_Y^\bullet(\log T)(-T))$$

degenerates at E_1. Note that the right hand side is isomorphic to $H_c^{p+q}(Y - T, \mathbb{C}) = H^{p+q}(Y, j_!\mathbb{C}_{Y-T})$. We also note that

$$\mathrm{Gr}_F^0\widetilde{\Omega}_Y^\bullet(\log T)(-T) \simeq \mathcal{O}_Y(-T).$$

□

4. Proof of Theorems 1.3 and 1.4: Injectivity, torsion-free, and vanishing theorems

Once we establish Theorem 1.2, we can easily prove Theorem 1.3. Moreover, Theorem 1.4 is an easy consequence of Theorem 1.3.

Proof of Theorem 1.3. We can obtain Theorem 1.3 as an application of Theorem 1.2. More precisely, by Theorem 1.2, the proof of [F6, Theorem 6.1] works with some suitable modifications. Note that the

vanishing theorem of Reid–Fukuda type for birational morphisms (see [F6, Lemma 6.2]) holds for bimeromorphic morphisms between complex varieties by Lemma 2.19. The desingularization used in the proof of [F6, Theorem 6.1] holds also for compact complex varieties (see Remark 2.5). We leave the details as exercises for the reader. □

Proof of Theorem 1.4. Theorem 1.4 follows from Theorem 1.3 by some standard arguments. For (i), Step 1 in the proof of [F6, Theorem 6.3 (i)] is sufficient by Theorem 1.3 since X is projective. Step 1 in the proof of [F6, Theorem 6.3 (ii)] works without any modifications by Theorem 1.3. Therefore, we obtain the statement (ii). For the details, see [F6]. □

5. Proof of Theorem 1.5: Semipositivity

The following result is a special case of Theorem 1.4 (ii).

Corollary 5.1. *Let X be a compact Kähler manifold and let $f : X \to Y$ be a surjective morphism onto a projective variety Y. Let D be a simple normal crossing divisor on X such that every stratum of D is dominant onto Y. Then $R^i f_*\mathcal{O}_X(K_X + D)$ is torsion-free for every i.*

Once we establish Corollary 5.1, it is easy to prove Theorem 1.5.

Proof of Theorem 1.5. By Corollary 5.1, the arguments in [F1, Section 3] work without any modifications. Therefore, we obtain Theorem 1.5. □

Remark 5.2. Theorem 1.5 is contained in [FFS]. See [FFS, 4.7. Remark]. Note that the argument in [FFS] heavily depends on Saito's theory of mixed Hodge modules.

Remark 5.3. The semipositivity of $R^i f_*\mathcal{O}_X(K_{X/Y} + D)$ in Theorem 1.5 follows from [FF, Theorem 1.3] or [FFS, Theorem 3]. We do not use [Kw2, Theorem 2], which depends on [Kw1]. For some comments on the semipositivity theorem in [Kw1], see [FFS, 4.6. Remark].

Remark 5.4. In Theorem 1.5, the case when $i = 0$ is similar to [Kw1, Theorem 32]. Unfortunately, our results and arguments do not recover Kawamata's original statement (see [Kw1, Theorem 32]). Anyway, our formulation of Theorem 1.5 is natural and the statement of Theorem 1.5 is sufficient for various applications. We do not use [Kw1, Theorem 32] in this paper. Therefore, we do not touch [Kw1, Theorem 32] here anymore. We note that [Kw1] was written before [SZ] and [Ks], where the theory of (admissible) variations of mixed Hodge structure was investigated.

Anyway, we have established Theorem 1.5, which is the main ingredient of the twisted weak positivity: Theorem 1.1.

6. Proof of Theorem 1.1: Twisted weak positivity

In this section, we discuss *weakly positive sheaves* introduced by Viehweg (see [V1] and [V2]). For the basic properties of weakly positive sheaves and related results, see, for example, [V3, Section 2]. Here we closely follow [V2], [V3, Section 2], [Ca, Section 4.4], and [Mo].

Definition 6.1 (Weak positivity)**.** Let W be a smooth projective variety and let $\mathcal{F}$ be a torsion-free coherent sheaf on W. Let U be a Zariski open set of W. We call $\mathcal{F}$ *weakly positive over* U, if for every ample line bundle $\mathcal{H}$ on W and every positive integer α there exists some positive integer β such that $\widehat{S}^{\alpha\beta}(\mathcal{F})\otimes\mathcal{H}^{\otimes\beta}$ is generated by global sections over U. This means that the natural map

$$H^0(W,\widehat{S}^{\alpha\beta}(\mathcal{F})\otimes\mathcal{H}^{\otimes\beta})\otimes\mathcal{O}_W\to\widehat{S}^{\alpha\beta}(\mathcal{F})\otimes\mathcal{H}^{\otimes\beta}$$

is surjective over U. We call $\mathcal{F}$ *weakly positive*, if there exists some nonempty Zariski open set U such that $\mathcal{F}$ is weakly positive over U.

Remark 6.2. In Definition 6.1, let $\widehat{W}$ be the largest Zariski open subset of W such that $\mathcal{F}|_{\widehat{W}}$ is locally free. Then we put

$$\widehat{S}^k(\mathcal{F})=i_*S^k(i^*\mathcal{F})$$

where $i:\widehat{W}\to W$ is the natural open immersion and S^k denotes the k-th symmetric product. Note that $\operatorname{codim}_W(W\setminus\widehat{W})\geq 2$ since $\mathcal{F}$ is torsion-free.

Remark 6.3. Let W and $\mathcal{F}$ be as in Definition 6.1. By Definition 6.1, $\mathcal{F}$ is weakly positive if and only if so is $\widehat{S}^1(\mathcal{F})=\mathcal{F}^{**}$, where $\mathcal{F}^{**}$ is the double-dual of $\mathcal{F}$.

Remark 6.4. Let W be a smooth projective variety and let $\mathcal{F}$ be a locally free sheaf of finite rank on W. Then $\mathcal{F}$ is semipositive, that is, $\mathcal{O}_{\mathbb{P}_W(\mathcal{F})}(1)$ is a nef line bundle on $\mathbb{P}_W(\mathcal{F})$, if and only if $\mathcal{F}$ is weakly positive over W.

Remark 6.5. Let $\mathcal{F}$ be a line bundle on a smooth projective variety W. Then $\mathcal{F}$ is weakly positive over a nonempty Zariski open set of W if and only if $\mathcal{F}$ is psuedo-effective.

Although Lemma 6.6 is obvious by the definition of weakly positive sheaves, it is very useful. We will repeatedly use Lemma 6.6 in this section.

Lemma 6.6 (cf. [V2, Lemma 1.4. 1)])**.** *Let W be a smooth projective variety and let $\mathcal{F}$ and $\mathcal{G}$ be torsion-free coherent sheaves on W. If $\mathcal{F} \to \mathcal{G}$ is a morphism which is surjective over U and if $\mathcal{F}$ is weakly positive over U, then $\mathcal{G}$ is also weakly positive over U.*

Let us prove the following generalization of Viehweg's theorem (cf. [V2, Theorem 4.1]), which follows from Theorem 1.5. Viehweg only considered the case when $i = 0$.

Theorem 6.7 (Fundamental weak positivity theorem)**.** *Let $f : V \to W$ be a surjective morphism from a compact Kähler manifold V to a smooth projective variety W. Let D be a simple normal crossing divisor on V such that every irreducible component of D is dominant onto W. Let Σ be a simple normal crossing divisor on W such that f is smooth over $W_0 = W \setminus \Sigma$ and that D is relatively normal crossing over W_0, and $\mathrm{Supp}(D + f^*\Sigma)$ is a simple normal crossing divisor on V. Then the locally free sheaf $R^i f_*\mathcal{O}_V(K_{V/W} + D)$ is weakly positive over W_0 for every i.*

Proof. We put $V_0 = f^{-1}(W_0)$, $f_0 = f|_{V_0}$, $D_0 = D|_{V_0}$, and $d = \dim V - \dim W$. We take a finite morphism $g : W' \to W$ from a smooth projective variety, which induces a unipotent reduction for the local system $R^{d+i}f_{0*}\mathbb{C}_{V_0 - D_0}$, such that $\mathrm{Supp}(g^*\Sigma)$ is a simple normal crossing divisor on W' (see, for example, [KM, Proposition 2.67]). We can construct a commutative diagram:

$$\begin{array}{ccc} V' & \longrightarrow & V \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ W' & \xrightarrow[g]{} & W \end{array}$$

with the following properties:

(i) V' is a compact Kähler manifold which is a resolution of $V \times_W W'$,
(ii) f' is smooth over $W'_0 = W' \setminus \Sigma'$, where $\Sigma' = \mathrm{Supp}(g^*\Sigma)$,
(iii) D' is a simple normal crossing divisor on V' such that D' and $f'^*\Sigma'$ have no common irreducible components and that $\mathrm{Supp}(D' + f'^*\Sigma')$ is a simple normal crossing divisor on V', and
(iv) $f' : (V', D') \to W'$ is nothing but the base change of $f : (V, D) \to W$ over W'_0.

Then we obtain a natural inclusion of locally free sheaves

$$\varphi^i : R^i f'_*\mathcal{O}_{V'}(K_{V'/W'} + D') \subset g^*R^i f_*\mathcal{O}_V(K_{V/W} + D).$$

such that φ^i is the identity over W'_0. Note that $R^i f_*\mathcal{O}_V(K_{V/W} + D)$ is the upper canonial extension of the bottom Hodge filtration (see

Theorem 1.5). By Theorem 1.5, $R^if'_*\mathcal{O}_{V'}(K_{V'/W'}+D')$ is semipositive. In particular, $R^if'_*\mathcal{O}_{V'}(K_{V'/W'}+D')$ is weakly positive over W'_0. Thus, $R^if_*\mathcal{O}_V(K_{V/W}+D)$ is weakly positive over W_0 by [V2, Lemma 1.4. 5)]. □

Remark 6.8. In general, the locally free sheaf $R^if_*\mathcal{O}_V(K_{V/W}+D)$ is not necessarily semipositive. See [FF, Section 8] for some examples.

By using the basic properties of weakly positive sheaves, we can easily obtain the following corollary of Theorem 6.7, which seems to be new when $i>0$.

Corollary 6.9. *Let $f : V \to W$ be a surjective morphism from a compact Kähler manifold V to a smooth projective varieity W. Let D be a simple normal crossing divisor on V. Then the torsion-free part of $R^if_*\mathcal{O}_V(K_{V/W}+D)$, that is,*

$$R^if_*\mathcal{O}_V(K_{V/W}+D)/\text{torsion},$$

is weakly positive over some nonempty Zariski open set of W for every i.

Proof. By replacing D with its horizontal part, we may assume that every irreducible component of D is dominant onto W (see Lemma 6.6). If there is a log canonical center C of (V,D) such that $f(C)\subsetneq W$, then we take the blow-up $h : V' \to V$ along C. We put

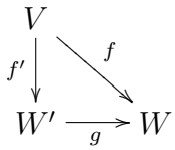
$$K_{V'}+D' = h^*(K_V+D).$$

Then D' is a simple normal crossing divisor on V' and

$$R^if_*\mathcal{O}_V(K_{V/W}+D) \simeq R^i(f\circ h)_*\mathcal{O}_{V'}(K_{V'/W}+D')$$

for every i. Therefore, we can replace (V,D) with (V',D'). Then we replace D with its horizontal part (see Lemma 6.6). By repeating this process finitely many times, we may assume that every stratum of D is dominant onto W. Now we take a closed subset Σ of W such that f is smooth over $W\setminus\Sigma$ and that D is relatively normal crossing over $W\setminus\Sigma$. Let $g : W' \to W$ be a birational morphism from a smooth projective variety W' such that $\Sigma' = g^{-1}(\Sigma)$ is a simple normal crossing divisor. By taking some suitable blow-ups of V in $f^{-1}(\Sigma)$ and replacing D with its strict transform, we may further assume the following conditions:

(i) $f' = g^{-1}\circ f : V \to W'$ is a morphism,
(ii) f' is smooth over $W'\setminus\Sigma'$ and D is relatively normal crossing over $W'\setminus\Sigma'$, and
(iii) every irreducible component of D is dominant onto W and $\mathrm{Supp}(f'^*\Sigma'+D)$ is a simple normal crossing divisor on V.



Here we used Szabó's resolution lemma (see Remark 2.5) and Lemma 2.19. Then, by Theorem 6.7, $R^if'_*\mathcal{O}_V(K_{V/W'}+D)$ is weakly positive over a nonempty Zariski open set of W'. Note that

$$R^if'_*\mathcal{O}_V(K_{V/W}+D)\simeq R^if'_*\mathcal{O}_V(K_{V/W'}+D)\otimes\mathcal{O}_{W'}(E)$$

where E is a g-exceptional effective divisor such that $K_{W'}=g^*K_W+E$. Thus $R^if'_*\mathcal{O}_V(K_{V/W}+D)$ is weakly positive over a nonempty Zariski open set of W'. We note that

$$g_*R^if'_*\mathcal{O}_V(K_{V/W}+D)\simeq R^if_*\mathcal{O}_V(K_{V/W}+D).$$

Here we used the fact that

$$R^pg_*R^qf'_*\mathcal{O}_V(K_{V/W}+D)=0$$

for every $p>0$ and $q\geq 0$ by Proposition 6.11 below. Therefore, $R^if_*\mathcal{O}_V(K_{V/W}+D)$ is weakly positive over a nonempty Zariski open set of W (see, for example, [V2, Lemma 1.4. 4)]). □

Before we give a proof of Proposition 6.11, we prepare an easy lemma.

Lemma 6.10. *Theorem 1.4 (ii) holds true even when Y is a complex manifold in Fujiki's class $\mathcal{C}$ and Δ is an effective $\mathbb{R}$-divisor on Y such that (Y,Δ) is dlt.*

Proof. Let $h:Y'\to Y$ be a resolution such that h is an isomorphism over the generic point of any log canonical center of (Y,Δ) (see Remark 2.5). We can write

$$K_{Y'}+\Delta_{Y'}=h^*(K_Y+\Delta)+E$$

where $\Delta_{Y'}$ and E are effective, E is Cartier and h-exceptional, $\Delta_{Y'}$ is a boundary $\mathbb{R}$-divisor, and $\mathrm{Supp}(\Delta_{Y'}+E)$ is a simple normal crossing divisor on Y'. Then

$$\begin{aligned}h^*\mathcal{L}\otimes\mathcal{O}_{Y'}(E)-(K_{Y'}+\Delta_{Y'})&=h^*(\mathcal{L}-(K_Y+\Delta))\\&\sim_{\mathbb{R}}h^*f^*H.\end{aligned}$$

By Theorem 1.4 (ii), we obtain that

$$H^p(X,R^q(f\circ h)_*(h^*\mathcal{L}\otimes\mathcal{O}_{Y'}(E)))=0$$

for every $p>0$ and $q\geq 0$. Note that $R^ih_*(h^*\mathcal{L}\otimes\mathcal{O}_{Y'}(E))=0$ for every $i>0$ by Lemma 2.19. Thus we obtain

$$H^p(X,R^qf_*\mathcal{L})\simeq H^p(X,R^q(f\circ h)_*(h^*\mathcal{L}\otimes\mathcal{O}_{Y'}(E)))=0$$

for every $p > 0$ and $q \geq 0$. Note that $h_*(h^*\mathcal{L} \otimes \mathcal{O}_{Y'}(E)) \simeq \mathcal{L}$. □

Proposition 6.11. *Let $f : X \to Y$ be a surjective morphism such that X is a complex manifold in Fujiki's class $\mathcal{C}$ and Y is a projective variety. Let D be a simple normal crossing divisor on X such that every stratum of D is dominant onto Y. Let $g : Y \to Z$ be a birational morphism between projective varieties. Then*

$$R^p g_* R^q f_* \mathcal{O}_X(K_X + D) = 0$$

for every $p > 0$ and $q \geq 0$.

Proof. Let $\mathcal{A}$ be a sufficiently ample line bundle on Z such that

$$H^r(Z, R^p g_* R^q f_* \mathcal{O}_X(K_X + D) \otimes \mathcal{A}) = 0$$

for $p > 0, q \geq 0$, and $r > 0$ and that

$$R^p g_* R^q f_* \mathcal{O}_X(K_X + D) \otimes \mathcal{A}$$

is generated by global sections for $p > 0$ and $q \geq 0$. By Leray, we have

$$\begin{aligned} &H^0(Z, R^p g_* R^q f_* \mathcal{O}_X(K_X + D) \otimes \mathcal{A}) \\ &\simeq H^p(Y, R^q f_* \mathcal{O}_X(K_X + D + f^*g^*\mathcal{A})). \end{aligned}$$

Therefore, it is sufficient to prove that

$$H^p(Y, R^q f_* \mathcal{O}_X(K_X + D + f^*g^*\mathcal{A})) = 0$$

for $p > 0$ and $q \geq 0$. By Kodaira, we can write

$$g^*\mathcal{A} \sim_{\mathbb{Q}} H + E$$

where H is an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Y and E is an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Y. Let ε be a sufficiently small positive number. Then $(X, D + \varepsilon f^*E)$ is dlt and

$$\mathcal{O}_X(K_X + D + f^*g^*\mathcal{A}) - (K_X + D + \varepsilon f^*E) \sim_{\mathbb{Q}} f^*((1-\varepsilon)g^*\mathcal{A} + \varepsilon H).$$

Therefore, by Lemma 6.10, we obtain that

$$H^p(Y, R^q f_* \mathcal{O}_X(K_X + D + f^*g^*\mathcal{A})) = 0$$

for $p > 0$ and $q \geq 0$. □

Before we start the proof of Theorem 1.1, we prepare a very important lemma.

Lemma 6.12 (cf. [Ca, Lemma 4.19])**.** *Let $f : X \to Y$ be a surjective morphism from a compact Kähler manifold X to a smooth projective variety Y with connected fibers. Let Δ be a boundary $\mathbb{Q}$-divisor on X such that* SuppΔ *is a simple normal crossing divisor and that every stratum of* SuppΔ *is dominant onto Y. Let l be a positive integer such*

that $l(K_{X/Y}+\Delta)$ is Cartier. Let A' be an ample Cartier divisor on Y. We put $A=f^*A'$. Assume that

$$\widehat{S}^N(f_*\mathcal{O}_X(l(K_{X/Y}+\Delta)+lA))$$

is generated by global sections on some nonempty Zariski open set of Y. Then

$$f_*\mathcal{O}_X(l(K_{X/Y}+\Delta)+(l-1)A)$$

is weakly positive over a nonempty Zariski open set of Y.

Proof. We consider

$$\mathcal{M}:=\mathrm{Im}(f^*f_*\mathcal{O}_X(l(K_{X/Y}+\Delta)+lA)\to\mathcal{O}_X(l(K_{X/Y}+\Delta)+lA)).$$

We may assume that the relative base locus of $l(K_{X/Y}+\Delta)+lA$ does not contain any component of $l\Delta$ by decreasing the relevant coefficients of Δ. Furthermore, if necessary, by taking blow-ups of X and decreasing the relevant coefficients of Δ, we may assume that $\mathcal{M}$ is a line bundle, $l(K_{X/Y}+\Delta)+lA=\mathcal{M}+E$, where E is an effective divisor on X, E and $l\Delta$ have no common components, and $\mathrm{Supp}(E+\Delta)$ is a simple normal crossing divisor on X. By assumption, there is an effective divisor S on X such that $\mathrm{codim}_Y f(S)\geq 2$ and that $\mathcal{M}^N\otimes\mathcal{O}_X(NlS)$ is generated by global sections over $f^{-1}(U)\subset X$, where U is a nonempty Zariski open set of Y (cf. the proof of [N2, Chapter III. 5.10. Lemma (1)]). We put

$$L=K_{X/Y}+\Delta^{=1}+l\{\Delta\}+A+S$$

and

$$L^{(l-1)}=(l-1)L-\lfloor\frac{l-1}{l}(E+(l-1)l\{\Delta\})\rfloor.$$

We note that

$$\Delta^{=1}=\lfloor\Delta\rfloor\quad\text{and}\quad\Delta=\Delta^{=1}+\{\Delta\}$$

because Δ is a boundary $\mathbb{Q}$-divisor. We also note that

$$\begin{aligned}lL&=lK_{X/Y}+l\Delta+lA+lS+(l-1)l\{\Delta\}\\&=\mathcal{M}+lS+E+(l-1)l\{\Delta\}.\end{aligned}$$

By the usual covering argument, we obtain that

$$f_*\mathcal{O}_X(K_{X/Y}+\Delta^{=1}+L^{(l-1)})$$

is weakly positive over a nonempty Zariski open set by Lemma 6.13 below. We can easily see that

$$\begin{aligned}&K_{X/Y}+\Delta^{=1}+(l-1)L-\lfloor(l-1)^2\{\Delta\}\rfloor\\&=l(K_{X/Y}+\Delta^{=1})+l(l-1)\{\Delta\}-\lfloor(l^2-2l+1)\{\Delta\}\rfloor+(l-1)(A+S)\\&=l(K_{X/Y}+\Delta)+(l-1)(A+S).\end{aligned}$$

Therefore, we can also check that

$$\begin{aligned}&f_*\mathcal{O}_X(K_{X/Y}+\Delta^{=1}+L^{(l-1)})\\&\subset f_*\mathcal{O}_X(K_{X/Y}+\Delta^{=1}+(l-1)L-\lfloor(l-1)^2\{\Delta\}\rfloor)\end{aligned}$$

and that they coincide over the generic point of Y because E is the relative base locus of $l(K_{X/Y}+\Delta)+lA$, $A=f^*A'$, $f(S)\subsetneq Y$, and

$$L^{(l-1)}=(l-1)L-\lfloor(l-1)^2\{\Delta\}\rfloor-\lfloor\frac{l-1}{l}E\rfloor.$$

Hence we obtain that

$$\begin{aligned}&f_*\mathcal{O}_X(l(K_{X/Y}+\Delta)+(l-1)(A+S))\\&=f_*\mathcal{O}_X(K_{X/Y}+\Delta^{=1}+(l-1)L-\lfloor(l-1)^2\{\Delta\}\rfloor)\end{aligned}$$

is weakly positive over a nonempty Zariski open set by Lemma 6.6. Therefore,

$$(f_*\mathcal{O}_X(l(K_{X/Y}+\Delta)+(l-1)A))^{**}$$

is weakly positive because $\operatorname{codim}_Y f(S)\geq 2$. This means that

$$f_*\mathcal{O}_X(l(K_{X/Y}+\Delta)+(l-1)A)$$

is weakly positive (see Remark 6.3). □

Lemma 6.13 is a slight generalization of [V2, Lemma 5.1]. It follows from Corollary 6.9 by the usual covering trick.

Lemma 6.13 (cf. [V2, Lemma 5.1])**.** *Let $f:X\to Y$ be a surjective morphism from a compact Kähler manifold X to a smooth projective variety Y. Let D be a simple normal crossing divisor on X such that every stratum of D is dominant onto Y. Let $\mathcal{L}$ and $\mathcal{N}$ be line bundles on X and let E be an effective divisor on X such that $\mathcal{L}^N=\mathcal{N}+E$ for some positive integer N, D and E have no common components, and $\operatorname{Supp}(D+E)$ is a simple normal crossing divisor on X. Assume that there is a nonempty Zariski open set U of Y such that some power of $\mathcal{N}$ is generated over $f^{-1}(U)$ by global sections. Then the sheaf*

$$f_*\mathcal{O}_X(K_{X/Y}+D+\mathcal{L}^{(i)})$$

is weakly positive for $0\leq i\leq N-1$, where

$$\mathcal{L}^{(i)}=\mathcal{L}^i\otimes\mathcal{O}_X\left(-\lfloor\frac{iE}{N}\rfloor\right).$$

Proof. Since the statement is compatible with replacing N by NN', E by $N'E$, and $\mathcal{N}$ by $\mathcal{N}^{N'}$ for some positive integer N', we may assume that $\mathcal{N}$ itself is generated by global sections over $f^{-1}(U)$. Let $B+F$ be the zero-set of a general section of $\mathcal{N}$ such that every irreducible

component of B is dominant onto Y and that $\mathrm{Supp} F \subset X \setminus f^{-1}(U)$. By Bertini, B is smooth and $\mathrm{Supp}(B+D+E)$ is a simple normal crossing divisor on $f^{-1}(U)$. We note that $\mathcal{N} = \mathcal{O}_X(B+F)$. By taking a suitable bimeromorphic modification outside $f^{-1}(U)$, we may assume that B is smooth and that $\mathrm{Supp}(B+D+E+F)$ is a simple normal crossing divisor. In fact, if $h : \widetilde{X} \to X$ is a bimeromorphic modification which is an isomorphism over $f^{-1}(U)$ and if $\widetilde{\mathcal{L}} = h^*\mathcal{L}$, $\widetilde{\mathcal{N}} = h^*\mathcal{N}$, $\widetilde{E} = h^*E$, and $\widetilde{D}$ is the strict transform of D, then we can easily check that $h_*\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}/Y} + \widetilde{D} + \widetilde{\mathcal{L}}^{(i)})$ is contained in $\mathcal{O}_X(K_{X/Y} + D + \mathcal{L}^{(i)})$. By construction, $h_*\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}/Y} + \widetilde{D} + \widetilde{\mathcal{L}}^{(i)})$ coincides with $\mathcal{O}_X(K_{X/Y} + D + \mathcal{L}^{(i)})$ over $f^{-1}(U)$. By replacing E with $E+F$, we may assume that $F = 0$ (see Lemma 6.6). Note that every irreducible component of F is vertical with respect to $f : X \to Y$. By taking a cyclic cover $p : Z' \to X$ associated to $\mathcal{L}^N = B + E$, that is, Z' is the normalization of $\mathrm{Spec}_{\mathrm{an}} \bigoplus_{i=1}^{N-1} \mathcal{L}^{-i}$. Let Z be a resolution of the cyclic cover and let $g : Z \to Y$ be the corresponding morphism.

$$\begin{array}{ccccc} Z & \xrightarrow{q} & Z' & \xrightarrow{p} & X \\ & \searrow^{g} & & & \downarrow{\scriptstyle f} \\ & & & & Y \end{array}$$

It is well known that Z' has only quotient singularities and

$$p_*\mathcal{O}_{Z'}(K_{Z'}) \simeq \bigoplus_{i=0}^{N-1} \mathcal{O}_X(K_X + \mathcal{L}^{(i)}).$$

Let $D^\dagger$ be the union of the strict transform of p^*D and the exceptional divisor of $q : Z \to Z'$. Then

$$q_*\mathcal{O}_Z(K_Z + D^\dagger) \simeq \mathcal{O}_{Z'}(K_{Z'} + p^*D).$$

Note that (Z', p^*D) is log canonical. Then we obtain

$$g_*\mathcal{O}_Z(K_{Z/Y} + D^\dagger) \simeq \bigoplus_{i=0}^{N-1} f_*\mathcal{O}_X(K_{X/Y} + D + \mathcal{L}^{(i)}).$$

Therefore, by Corollary 6.9, $f_*\mathcal{O}_X(K_{X/Y} + D + \mathcal{L}^{(i)})$ is weakly positive for every i. □

Let us start the proof of Theorem 1.1.

Proof of Theorem 1.1. We divide the proof into several steps.

Step 1. Let $\widetilde{X} \to X$ be a resolution such that $\widetilde{X}$ is a compact Kähler manifold with

$$K_{\widetilde{X}} + \widetilde{\Delta} = \pi^*(K_X + \Delta) + E,$$

where E and $\widetilde{\Delta}$ are effective and have no common components. By replacing (X, Δ) with $(\widetilde{X}, \widetilde{\Delta})$, we may assume that X is a compact Kähler manifold and that $\mathrm{Supp}\Delta$ is a simple normal crossing divisor. By taking more blow-ups and using Lemma 6.6, we may further assume that every stratum of $\mathrm{Supp}\Delta$ is dominant onto Y. By replacing mk with k, we may assume that $m = 1$.

Step 2. Let H be an ample Cartier divisor on Y. We put

$$r = \min\{s > 0\,;\, f_*\mathcal{O}_X(k(K_{X/Y}+\Delta))\otimes\mathcal{O}_Y((sk-1)H) \text{ is weakly positive}\}.$$

By definition, we can find $\nu > 0$ such that

$$\widehat{S}^{\nu}(f_*\mathcal{O}_X(k(K_{X/Y} + \Delta))) \otimes \mathcal{O}_Y((rk\nu - \nu)H) \otimes \mathcal{O}_Y(\nu H)$$

is generated by global sections over a nonempty Zariski open set. By Lemma 6.12,

$$f_*\mathcal{O}_X(k(K_{X/Y} + \Delta) \otimes \mathcal{O}_Y((rk - r)H)$$

is weakly positive. The choice of r allows this only if $(r-1)k-1 < rk-r$, equivalently, $r \leq k$. Hence we obtained the weak positivity of

$$f_*\mathcal{O}_Y(k(K_{X/Y} + \Delta)) \otimes \mathcal{O}_Y((k^2 - k)H).$$

Step 3. By Lemma 6.14, we can take a finite flat morphism $g : Y' \to Y$ from a smooth projective variety Y' such that $g^*H \sim dH'$ for $d \gg 0$, $X' = X \times_Y Y'$ is a compact Kähler manifold. We put $\tau : X' \to X$ and $\Delta' = \tau^*\Delta$. Thus, by Lemma 6.14, we have

$$f'_*\mathcal{O}_{X'}(k(K_{X'/Y'} + \Delta')) \simeq g^*f_*\mathcal{O}_X(k(K_{X/Y} + \Delta)),$$

where $f' : X' \to Y'$. By the same argument as above (see Step 1), we assume that $\mathrm{Supp}\Delta'$ is a simple normal crossing divisor, every stratum of $\mathrm{Supp}\Delta'$ is dominant onto Y', and there is a natural inclusion

$$f'_*\mathcal{O}_{X'}(k(K_{X'/Y'} + \Delta')) \subset g^*f_*\mathcal{O}_X(k(K_{X/Y} + \Delta))$$

which is an isomorphism over some nonempty Zariski open set. Then we obtain that

$$g^*f_*\mathcal{O}_X(k(K_{X/Y} + \Delta)) \otimes \mathcal{O}_{Y'}((k^2 - k)H')$$

is weakly positive by Lemma 6.6. This is because

$$f'_*\mathcal{O}_{X'}(k(K_{X'/Y'} + \Delta')) \otimes \mathcal{O}_{Y'}((k^2 - k)H')$$

is weakly positive by applying the above result (see Step 2) to $f' : (X', \Delta') \to Y'$. If α is a positive integer, then we choose $d = 2\alpha(k^2 - k) + 1$. Let β be a sufficiently large positive integer, then we have that

$$\begin{aligned}&\widehat{S}^{2\alpha\beta}(g^*f_*\mathcal{O}_X(k(K_{X/Y} + \Delta)) \otimes \mathcal{O}_{Y'}((k^2 - k)H')) \otimes \mathcal{O}_{Y'}(\beta H')\\ &= g^*\widehat{S}^{2\alpha\beta}(f_*\mathcal{O}_X(k(K_{X/Y} + \Delta))) \otimes g^*\mathcal{O}_Y(\beta H)\end{aligned}$$

is generated by global sections over a nonempty Zariski open set. Over the Zariski open set $\widehat{Y}$ of Y where

$$\widehat{S}^{2\alpha\beta}(f_*\mathcal{O}_X(k(K_{X/Y} + \Delta)))$$

is locally free, we have a surjection

$$\begin{aligned}&g_*g^*\widehat{S}^{2\alpha\beta}(f_*\mathcal{O}_X(k(K_{X/Y} + \Delta))) \otimes \mathcal{O}_Y(\beta H)\\ &\to \widehat{S}^{2\alpha\beta}(f_*\mathcal{O}_X(k(K_{X/Y} + \Delta))) \otimes \mathcal{O}_Y(\beta H).\end{aligned}$$

Therefore, we have a homomorphism

$$\bigoplus_{\text{finite}}(\mathcal{O}_Y(\beta H) \otimes g_*\mathcal{O}_{Y'}) \to \widehat{S}^{2\alpha\beta}(f_*\mathcal{O}_X(k(K_{X/Y} + \Delta))) \otimes \mathcal{O}_Y(2\beta H)$$

which is surjective over a nonempty Zariski open set. If β is sufficiently large, then $g_*\mathcal{O}_{Y'} \otimes \mathcal{O}_Y(\beta H)$ is generated by global sections. Therefore,

$$\widehat{S}^{2\alpha\beta}(f_*\mathcal{O}_X(k(K_{X/Y} + \Delta))) \otimes \mathcal{O}_Y(2\beta H)$$

is generated by global sections over a nonempty Zariski open set.

This means that $f_*\mathcal{O}_X(k(K_{X/Y} + \Delta))$ is weakly positive. □

In the above proof of Theorem 1.1, we have already used the following lemma. It is a variant of Kawamata's cover (see [Kw1, Theorem 17]). The description of Kawamata's covering trick in [EV, 3.19. Lemma] is very useful for our purpose. See also [AK, 5.3. Kawamata's covering] and [V3, Lemma 2.5].

Lemma 6.14. *Let $f : X \to Y$ be a surjective morphism from a compact Kähler manifold X to a smooth projective variety Y and let H be a Cartier divisor on Y. Let d be an arbitrary positive integer. Then we can take a finite flat morphism $g : Y' \to Y$ from a smooth projective variety Y' and a Cartier divisor H' on Y' such that $g^*H \sim dH'$, $X' = X \times_Y Y'$ is a compact Kähler manifold with $\omega_{X'/Y'} = \tau^*\omega_{X/Y}$, where $\tau : X' \to X$.*

Furthermore, let D be a Cartier divisor on X. We put $f' : X' \to Y'$. Then there is a natural isomorphism

$$f'_*\mathcal{O}_{X'}(nK_{X'/Y'} + \tau^*D) \simeq g^*f_*\mathcal{O}_X(nK_{X/Y} + D)$$

for every integer n.

Proof. We take general very ample Cartier divisors D_1 and D_2 with the following properties.

(i) $H \sim D_1 - D_2$,
(ii) D_1, D_2, f^*D_1, and f^*D_2 are smooth,
(iii) D_1 and D_2 have no common components, and
(iv) $\mathrm{Supp}(D_1+D_2)$ and $\mathrm{Supp}(f^*D_1+f^*D_2)$ are simple normal crossing divisors.

We take a finite flat cover due to Kawamata with respect to Y and $D_1 + D_2$ (see [Kw1, Theorem 17]), we obtain $g : Y' \to Y$ and H' such that $g^*H \sim dH'$. By the construction of the above Kawamata cover $g : Y' \to Y$, we can assume that the ramification locus Σ of g in Y is a general simple normal crossing divisor. This means that f^*P is a smooth divisor for any irreducible component P of Σ and that $f^*\Sigma$ is a simple normal crossing divisor on X. In this situation, we can easily check that $X' = X \times_Y Y'$ is a compact Kähler manifold.

$$\begin{array}{ccc} X' & \xrightarrow{\tau} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

By the construction, we can also easily check that $\omega_{X'/Y'} = \tau^*\omega_{X/Y}$. Therefore, we have

$$\mathcal{O}_{X'}(nK_{X'/Y'} + \tau^*D) \simeq \tau^*\mathcal{O}_X(nK_{X/Y} + D)$$

for every integer n. Thus, we obtain

$$\begin{aligned} f'_*\mathcal{O}_{X'}(nK_{X'/Y'} + \tau^*D) &\simeq f'_*\tau^*\mathcal{O}_X(nK_{X/Y} + D) \\ &\simeq g^*f_*\mathcal{O}_X(nK_{X/Y} + D) \end{aligned}$$

for every integer n. □

As a special case of Theorem 1.1, we have:

Corollary 6.15 (cf. [V2, Theorem III])**.** *Let $f : X \to Y$ be a surjective morphism with connected fibers such that X is a complex manifold in Fujiki's class $\mathcal{C}$ and Y is a smooth projective variety. Then $f_*\mathcal{O}_X(kK_{X/Y})$ is weakly positive for every $k > 0$.*

As is well known, Corollary 6.15 is a very famous fundamental result by Viehweg when X is projective.

By Theorem 1.1, we can easily recover Campana's twisted weak positivity (see [Ca, Theorem 4.13]).

Corollary 6.16. *Let $f : X \to Y$ be a surjective morphism from a complex manifold X in Fujiki's class $\mathcal{C}$ to a smooth projective variety Y. Let D be a divisor on X. We put $D = D^h + D^v$ where D^h (resp. D^v) is the horizontal (resp. vertical) part of D with respect to $f : X \to Y$. Assume that $\mathrm{Supp} D^h$ is a simple normal crossing divisor and the coefficients of D^h is less than or equal to m, where m is a positive integer. Then $f_*\mathcal{O}_X(mK_{X/Y} + D)$ is weakly positive.*

Proof. We put $\Delta = \frac{1}{m}D$. Then (X, Δ) is log canonical over the generic point of Y. We take a resolution $g : X' \to X$. Then

$$K_{X'} + \Delta' = g^*(K_X + \Delta) + E$$

where Δ' and E are effective and have no common components such that $\mathrm{Supp}(\Delta' + E)$ is a simple normal crossing divisor on X'. Let $\widetilde{\Delta}$ be the horizontal part of Δ'. Then $(X', \widetilde{\Delta})$ is log canonical. By Lemma 6.6, we can replace (X, Δ) with $(X', \widetilde{\Delta})$. By Theorem 1.1, we obtain that $f_*\mathcal{O}_X(m(K_{X/Y}+\Delta)) = f_*\mathcal{O}_X(mK_{X/Y}+D)$ is weakly positive. □

7. Proof of Theorem 1.8: Addition formula

Theorem 1.8 is an easy application of Theorem 1.1. It is contained in [Ca]. See also [L] and [N2].

Proposition 7.1 is a slight reformulation of [V2, Corollary 7.1].

Proposition 7.1 (cf. [V2, Corollary 7.1])**.** *Let $f : V \to W$ be an equidimensional surjective morphism from a normal projective variety V to a smooth projective variety W with connected fibers. Let (V, Δ) be a log canonical pair. Let H be an ample Cartier divisor on W. Then there are some positive integers a and l such that $a(K_X + \Delta)$ is Cartier and the linear system Λ associated to*

$$H^0(V, \mathcal{O}_V(al(K_{V/W} + \Delta)) \otimes f^*\mathcal{O}_W(lH))$$

defines a rational map $\Phi : V \dashrightarrow X$ with

$$\dim X = \kappa(V_w, K_{V_w} + \Delta|_{V_w}) + \dim W,$$

where V_w is a sufficiently general fiber of f Moreover, there is a rational map $\pi : X \dashrightarrow W$ such that $f = \pi \circ \Phi$.

$$\begin{array}{ccc} V & \overset{\Phi}{\dashrightarrow} & X \\ {\scriptstyle f}\downarrow & \swarrow_{\pi} & \\ W & & \end{array}$$

Proof. We take a positive integer a such that $a(K_X + \Delta)$ is Cartier and $f_*\mathcal{O}_V(a(K_{V/W} + \Delta))$ is nontrivial. By the twisted weak positivity theorem: Theorem 1.1, there is some $b > 0$ such that

$$\widehat{S}^{2b}(f_*\mathcal{O}_V(a(K_{V/W} + \Delta))) \otimes \mathcal{O}_W(bH)$$

is generated by global sections on some nonempty Zariski open set. Since the natural map

$$\begin{aligned}&\widehat{S}^{2b}(f_*\mathcal{O}_V(a(K_{V/W} + \Delta))) \otimes \mathcal{O}_W(bH)\\ &\to f_*\mathcal{O}_V(2ba(K_{V/W} + \Delta)) \otimes \mathcal{O}_W(bH)\end{aligned}$$

is nontrivial, we obtain an inclusion $f^*\mathcal{O}_W(bH)$ into $\mathcal{O}_V(2ba(K_{V/W} + \Delta)) \otimes f^*\mathcal{O}_W(2bH)$. Note that $f_*\mathcal{O}_V(2ba(K_{V/W}+\Delta))$ is reflexive since f is equi-dimensional. Without loss of generality, we may assume that bH is very ample by replacing b with $b'b$ with $b' \gg 0$. We put $l = 2b$. If b is sufficiently large, then the sections of $\mathcal{O}_V(2ba(K_{V/W}+\Delta)) \otimes f^*\mathcal{O}_W(2bH)$ define a rational map $\Phi : V \dashrightarrow X$ such that $\mathbb{C}(X)$ is algebraically closed in $\mathbb{C}(V)$. Since bH is very ample and there is an inclusion $f^*\mathcal{O}_W(bH)$ into $\mathcal{O}_V(2ba(K_{V/W}+\Delta)) \otimes f^*\mathcal{O}_W(2bH)$, we obtain a rational map $\pi : X \dashrightarrow W$ such that $f = \pi \circ \Phi$. The easy addition formula gives

$$\begin{aligned}&\kappa(V_w, K_{V_w} + \Delta|_{V_w})\\ &\leq \dim \Phi(V_w) + \kappa(F, (\mathcal{O}_V(2ba(K_{V/W} + \Delta)) \otimes f^*\mathcal{O}_W(2bH))|_F)\\ &= \dim \Phi(V_w)\end{aligned}$$

where F is a sufficiently general fiber of $\Phi : V \dashrightarrow X$ (if necessary, we take an elimination of points of indeterminacy of Φ). On the other hand, the restriction of the linear system Λ to V_w is a subsystem of $H^0(V_w, \mathcal{O}_{V_w}(2ba(K_{V_w} + \Delta|_{V_w}))$. Therefore, $\dim \Phi(V_w) \leq \kappa(V_w, K_{V_w} + \Delta|_{V_w})$. Hence, we obtain

$$\kappa(V_w, K_{V_w} + \Delta|_{V_w}) = \dim \Phi(V_w) = \dim X - \dim W.$$

□

Remark 7.2. In Theorem 7.1, it is sufficient to assume that a is a positive integer such that $a(K_V+\Delta)$ is Cartier and that $f_*\mathcal{O}_V(a(K_{V/W}+\Delta))$ is nontrivial.

Let us start the proof of Theorem 1.8.

Proof of Theorem 1.8. By [AK], we can construct a commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

with the following properties:

(i) $f' : X' \to Y'$ is an equi-dimensional surjective morphism from a normal projective variety X' to a smooth projective variety Y',
(ii) h and g are birational, and
(iii) X' has only quotient singularities and $(U_{X'} \subset X')$ is toroidal for some nonempty Zariski open set $U_{X'}$.

By [AK], we may assume that $\mathrm{Exc}(h) \cup \mathrm{Supp} h_*^{-1}\Delta$ is contained in $X' \setminus U_{X'}$. We put

$$K_{X'} + \Delta' = h^*(K_X + \Delta) + E$$

such that (X', Δ') is log canonical and E is effective and h-exceptional. Let H be an ample Cartier divisor on Y'. Since $K_{Y'}$ is a big divisor, by Kodaira's lemma, $aK_{Y'} \sim H + F$ for some effective divisor F on Y' and a sufficiently divisible positive integer a. By Theorem 7.1 (see also Remark 7.2), we have

$$\begin{aligned} \kappa(X, K_X + \Delta) &= \kappa(X', K_{X'} + \Delta') \\ &\geq \kappa(X', a(K_{X'} + \Delta') - af'^*K_{Y'} + f'^*H) \\ &\geq \kappa(X'_y, K_{X'_y} + \Delta'|_{X'_y}) + \dim Y' \\ &= \kappa(X_y, K_{X_y} + \Delta|_{X_y}) + \dim Y. \end{aligned}$$

Note that

$$\kappa(X, K_X + \Delta) \leq \kappa(X_y, K_{X_y} + \Delta|_{X_y}) + \dim Y$$

always holds by the easy addition formula. Therefore, we obtain

$$\kappa(X, K_X + \Delta) = \kappa(X_y, K_{X_y} + \Delta|_{X_y}) + \dim Y.$$

It is the desired equality. □

8. Proof of Theorem 1.9: Addition for log Kodaira dimensions

We prove Theorem 1.9, which is due to Maehara (see [Ma]), as an application of Theorem 1.1: Twisted weak positivity. Before we start

the proof of Theorem 1.9, we give some comments on Maehara's works (see [Ma]).

Remark 8.1. In Theorem 1.9,

$$f_*\mathcal{O}_X(k(K_X+D_X))\otimes\mathcal{O}_Y(-k(K_Y+D_Y))\otimes\mathcal{O}_Y(D_Y)$$

is weakly 1-positive in the sense of Maehara for any $k>0$. It is the main theorem of [Ma]. Maehara obtained Theorem 1.9 as a corollary of the above weak 1-positivity (see [Ma, Corollary 2]). In this section, we do not use Maehara's results and prove Theorem 1.9 as an application of Theorem 1.1 by using the weak semistable reduction theorem (see [AK]).

Let us start the proof of Theorem 1.9.

Proof of Theorem 1.9. By [AK], we may assume that

(i) $f:(U_X\subset X)\to(U_Y\subset Y)$ is toroidal and is equi-dimensional,
(ii) D_Y is contained in $Y\setminus U_Y$ and D_X is contained in $X\setminus U_X$,
(iii) X has only quotient singularities, Y is smooth, and
(iv) f is smooth over U_Y.

Moreover, there is a Kawamata cover $g:Y'\to Y$ such that the normalization X' of $\widetilde{X}=X\times_Y Y'$ is a weak semistable reduction over Y'.

Lemma 8.2 (cf. [Ma, Main Theorem])**.** *Under the above assumptions, there is an effective Cartier divisor B on Y such that*

$$f_*\mathcal{O}_X(k(K_X+D_X))\otimes\mathcal{O}_Y(-k(K_Y+D_Y))\otimes\mathcal{O}_Y(B)$$

is weakly positive for some nonempty Zariski open set of Y for every divisible positive integer k.

Once we establish Lemma 8.2, we can easily check that

$$\kappa(X,K_X+D_X)\geq\kappa(F,K_F+D_X|_F)+\dim Y$$

when $\kappa(Y,K_Y+D_Y)=\dim Y$. The proof is essentially the same as the proof of Theorem 1.8 in Section 7. We leave the details as exercises for the reader (see Remark 8.4 below). Therefore, it is sufficient to prove Lemma 8.2. For the proof of Lemma 8.2, we can replace D_Y with $\Delta_Y=Y\setminus U_Y$ and D_X with $\mathrm{Supp}(D_X+f^*\Delta_Y)$. Moreover, by adding some divisors to D_Y, we may further assume that g is étale over U_Y. We put $f':X'\to Y'$, $p:X'\to\widetilde{X}$, $q:\widetilde{X}\to X$, and

$K_{X'}+D_{X'}=\tau^*(K_X+D_X)$ where $\tau=q\circ p : X'\to X$.

$$\begin{array}{ccccc} X' & \xrightarrow{p} & \widetilde{X} & \xrightarrow{q} & X \\ & \searrow^{f'} & \downarrow{\scriptstyle \widetilde{f}} & & \downarrow{\scriptstyle f} \\ & & Y' & \xrightarrow[g]{} & Y \end{array}$$

We also put $K_{Y'}+D_{Y'}=g^*(K_Y+D_Y)$. Note that $D_{X'}$ and $D_{Y'}$ are reduced. Let Σ be the singular locus of X. We put $\widetilde{\Sigma}=q^{-1}(\Sigma)$. Then $\operatorname{codim}_{\widetilde{X}}\widetilde{\Sigma}\geq 2$ and $\omega_{\widetilde{X}}|_{X^\dagger}$ is locally free by the flat base change theorem (cf. Theorem 2.16) with $X^\dagger=\widetilde{X}\setminus\widetilde{\Sigma}$. We put $\omega_{\widetilde{X}}^{[k]}=\iota_*((\omega_{\widetilde{X}}|_{X^\dagger})^{\otimes k})$ where $\iota : X^\dagger\to\widetilde{X}$. We note that $\widetilde{X}$ is Cohen–Macaulay by the local description of Kawamata's cover. We also note that $\omega_{\widetilde{X}}^{[k]}$ is invertible if so is $\mathcal{O}_X(kK_X)$. We can check:

Lemma 8.3 (cf. [Ma, Lemma C])**.** *We have the following inclusion*

$$\begin{aligned}(\spadesuit)\qquad & p_*\mathcal{O}_{X'}(k(K_{X'/Y'}+D_{X'}-f'^*D_{Y'}))\otimes\mathcal{O}_{\widetilde{X}}(\widetilde{f}^*D_{Y'})\\ & \subset\omega_{\widetilde{X}/Y'}^{[k]}\otimes\mathcal{O}_{\widetilde{X}}(k(q^*(D_X-f^*D_Y)))\otimes\mathcal{O}_{\widetilde{X}}(q^*f^*D_Y)\end{aligned}$$

for every divisible positive integer k.

Proof of Lemma 8.3. Note that the both hand sides are S_2 since k is divisible. Therefore, by enlarging Σ, we may assume that $X\setminus\Sigma$ is smooth, $X^\dagger$ is Gorenstein, and $p^{-1}(X^\dagger)$ is also smooth. By replacing $\widetilde{X}$ (resp. X') with $X^\dagger$ (resp. $p^{-1}(X^\dagger)$), we may assume that $\widetilde{X}$ (resp. X') is Gorenstein (resp. smooth). We have

$$p_*\mathcal{O}_{X'}(K_{X'})\subset\omega_{\widetilde{X}}$$

because p is the normalization. Since $D_{X'}-\tau^*D_X\leq 0$, we obtain

$$p_*\mathcal{O}_{X'}(K_{X'/Y'}+D_{X'}-\tau^*D_X)\subset\omega_{\widetilde{X}/Y'}.$$

This is equivalent to

$$\begin{aligned}(\heartsuit)\qquad & p_*\mathcal{O}_{X'}(K_{X'/Y'}+D_{X'}-f'^*D_{Y'})\otimes\mathcal{O}_{\widetilde{X}}(\widetilde{f}^*D_{Y'})\\ & \subset\omega_{\widetilde{X}/Y'}\otimes\mathcal{O}_{\widetilde{X}}(q^*(D_X-f^*D_Y))\otimes\mathcal{O}_{\widetilde{X}}(q^*f^*D_Y).\end{aligned}$$

Therefore, the desired inclusion holds for $k=1$. Note that

$$q^*\mathcal{O}_X(K_X+D_X-f^*(K_Y+D_Y))\simeq\omega_{\widetilde{X}/Y'}\otimes\mathcal{O}_{\widetilde{X}}(q^*(D_X-f^*D_Y))$$

by the flat base change theorem (cf. Theorem 2.16) and

$$\begin{aligned}&p^*q^*\mathcal{O}_X(K_X+D_X-f^*(K_Y+D_Y))\\ &\simeq\mathcal{O}_{X'}(K_{X'}+D_{X'}-f'^*(K_{Y'}+D_{Y'})).\end{aligned}$$

By taking

$$\otimes\omega_{\widetilde{X}/Y'}^{\otimes k-1}\otimes\mathcal{O}_{\widetilde{X}}((k-1)q^*(D_X-f^*D_Y))$$

with (♡), we obtain the desired inclusion by the projection formula. □

Let us go back to the proof of Lemma 8.2. Note that

$$\begin{aligned}&\omega_{\widetilde{X}/Y'}^{[k]}\otimes\mathcal{O}_{\widetilde{X}}(k(q^*(D_X-f^*D_Y)))\\&\simeq q^*(\mathcal{O}_X(k(K_X+D_X))\otimes f^*\mathcal{O}_Y(-k(K_Y+D_Y)))\end{aligned}$$

when k is a divisible positive integer. By applying $\widetilde{f}_*$, (♠) implies

$$\begin{aligned}&f'_*\mathcal{O}_{X'}(k(K_{X'}+D_{X'}))\otimes\mathcal{O}_{Y'}(-k(K_{Y'}+D_{Y'}))\\&\subset g^*(f_*\mathcal{O}_X(k(K_X+D_X))\otimes\mathcal{O}_Y(-k(K_Y+D_Y))\otimes\mathcal{O}_Y(D_Y))\end{aligned}$$

because g is flat and $\widetilde{f}^*D_{Y'}\geq 0$. Since

$$\begin{aligned}&f'_*\mathcal{O}_{X'}(k(K_{X'}+D_{X'}))\otimes\mathcal{O}_{Y'}(-k(K_{Y'}+D_{Y'}))\\&=f'_*\mathcal{O}_{X'}(k(K_{X'/Y'}+D_{X'}-f'^*D_{Y'}))\end{aligned}$$

is weakly positive by Theorem 1.1,

$$f_*\mathcal{O}_X(k(K_X+D_X))\otimes\mathcal{O}_Y(-k(K_Y+D_Y))\otimes\mathcal{O}_Y(D_Y)$$

is also weakly positive (see, for example, [V2, Lemma 1.4. 5)]). Note that $D_{X'}-f'^*D_{Y'}$ is effective since f' is weakly semistable. Thus we obtain Lemma 8.2. We also note that the original D_Y was replaced by a larger effective Cartier divisor in the proof of Lemma 8.2.

As we mentioned above, it implies the desired inequality (see Remark 8.4 below). By the easy addition formula,

$$\kappa(X,K_X+D_X)\leq\kappa(F,K_F+D_X|_F)+\dim Y$$

always holds. Therefore, we obtain

$$\kappa(X,K_X+D_X)=\kappa(F,K_F+D_X|_F)+\dim Y.$$

□

Remark 8.4. If a is a sufficiently large and divisible positive integer, then $a(K_X+D_X)$ is Cartier,

$$f_*\mathcal{O}_X(a(K_X+D_X))\otimes\mathcal{O}_Y(-a(K_Y+D_Y))\otimes\mathcal{O}_Y(B)$$

is weakly positive by Lemma 8.2, and

$$a(K_Y+D_Y)-B\sim H+F$$

for an ample Cartier divisor H and some effective divisor F on Y by Kodaira. Therefore, we can use the same arguments as in Section 7 and obtain

$$\kappa(X,K_X+D_X)\geq\kappa(F,K_F+D_X|_F)+\dim Y.$$

We leave the details as exercises for the reader.

Note that Theorem 1.9 contains a generalization of [Kw1, Theorem 30], which plays important role for Kawamata's birational characterization of *quasi-Abelian* varieties (see [Kw1, Corollary 29]).

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Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, Japan

E-mail address: fujino@math.kyoto-u.ac.jp