VARIATION OF MIXED HODGE STRUCTURE AND ITS APPLICATIONS

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Abstract. We treat generalizations of Kollár's torsion-freeness, vanishing theorem, and so on, for projective morphisms between complex analytic spaces as an application of the theory of variations of mixed Hodge structure. The results will play a crucial role in the theory of minimal models for projective morphisms of complex analytic spaces. In this paper, we do not use Saito's theory of mixed Hodge modules.

Contents

1. Introduction

We will establish the following theorem, which is an analytic generalization of [FF1, Theorems 7.1 and 7.3]. Note that $f\colon (X, D) \to Y$ is assumed to be *algebraic* in [FF1]. Our approach in this paper is slightly different from the one in [FF1] (see Remark 1.6 below). We also note that we do not use Saito's theory of mixed Hodge modules (see [Sa1], [Sa2], [Sa3], [Sa4], [FFS], and [Sa5]) in this paper.

Theorem 1.1 (Canonical extensions of Hodge bundles, see [FF1, Theorems 7.1 and 7.3])**.** *Let (X, D) be an analytic simple normal crossing pair such that D is reduced and let $f\colon X \to Y$ be a proper surjective morphism onto a smooth complex variety Y. Assume that every stratum of (X, D) is dominant onto Y. Let Σ be a normal crossing divisor on Y such that every stratum of (X, D) is smooth over $Y^* := Y \setminus \Sigma$. We put $X^* := f^{-1}(Y^*)$, $D^* := D|_{X^*}$, and $d := \dim X - \dim Y$. If we assume that every stratum of (X, D) is a Kähler manifold in addition, then we have:*

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(i) $R^k(f|_{X^*\setminus D^*})_!\mathbb{R}_{X^*\setminus D^*}$ *underlies a graded polarizable variation of* $\mathbb{R}$*-mixed Hodge structure on* Y^* *for every* k.

We put

$$\mathcal{V}^k_{Y^*} := R^k(f|_{X^*\setminus D^*})_!\mathbb{R}_{X^*\setminus D^*}\otimes \mathcal{O}_{Y^*}$$

for every k*. The Hodge filtration and the weight filtration on* $\mathcal{V}^k_{Y^*}$ *are denoted by* F *and* L *respectively. Moreover the lower canonical extension of* $\mathcal{V}^k_{Y^*}$ *is denoted by* ${}^l\mathcal{V}^k_{Y^*}$*. The weight filtration* L *on* $\mathcal{V}^k_{Y^*}$ *is extended to* ${}^l\mathcal{V}^k_{Y^*}$ *by* $L_m({}^l\mathcal{V}^k_{Y^*}) = {}^lL_m(\mathcal{V}^k_{Y^*})$ *for every* m*. Then we have the following:*

(ii) *There exists a unique finite decreasing filtration* F *on* ${}^l\mathcal{V}^k_{Y^*}$ *such that*
- $F^p({}^l\mathcal{V}^k_{Y^*})|_{Y^*}\simeq F^p(\mathcal{V}^k_{Y^*})$*, and*
- $\mathrm{Gr}^p_F\,\mathrm{Gr}^L_m({}^l\mathcal{V}^k_{Y^*})$ *is a locally free* $\mathcal{O}_Y$*-module of finite rank*

for every k, m, p.

(iii) $R^{d-i}f_*\mathcal{O}_X(-D)$ *is isomorphic to*

$$\mathrm{Gr}^0_F({}^l\mathcal{V}^{d-i}_{Y^*}) = F^0({}^l\mathcal{V}^{d-i}_{Y^*})/F^1({}^l\mathcal{V}^{d-i}_{Y^*})$$

for every i*. In particular,* $R^{d-i}f_*\mathcal{O}_X(-D)$ *is locally free for every* i.

(iv) $R^if_*\omega_{X/Y}(D)$ *is isomorphic to*

$$\left(\mathrm{Gr}^0_F({}^l\mathcal{V}^{d-i}_{Y^*})\right)^* = \mathcal{H}om_{\mathcal{O}_Y}(\mathrm{Gr}^0_F({}^l\mathcal{V}^{d-i}_{Y^*}), \mathcal{O}_Y)$$

for every i*. In particular,* $R^if_*\omega_{X/Y}(D)$ *is locally free for every* i.

For the precise definition of upper and lower canonical extensions in Theorem 1.1, see [FF1, Remark 7.4]. In Theorem 1.1, X may be reducible, and we are mainly interested in the case where X is reducible.

Remark 1.2. We do not need the relative monodromy weight filtration for applications in the theory of minimal models (see, for example, [Fn10]). We are mainly interested in Hodge bundles and their extensions. However, we can prove the existence of the relative monodromy weight filtration in Theorem 1.1. In fact, the variations of $\mathbb{R}$-mixed Hodge structure in (i) of the theorem above are admissible. This can be checked by the same argument as in [FF1]. For the details, see Remark 4.8 below.

By Theorem 1.1, we can use the Fujita–Zucker–Kawamata semipositivity theorem in the complex analytic setting.

Theorem 1.3 (Semipositivity)**.** *In Theorem 1.1, we further assume that every local monodromy on the local system* $R^{d-i}(f|_{X^*\setminus D^*})_!\mathbb{R}_{X^*\setminus D^*}$ *around* Σ *is unipotent. Let* $\varphi\colon V\to X$ *be any morphism from a projective variety* V*. Then* $\varphi^*R^if_*\omega_{X/Y}(D)$ *is a nef locally free sheaf on* V.

In order to prove Theorem 1.1, we will establish:

Theorem 1.4 (Weight spectral sequence)**.** *Let* (X, D) *be an analytic simple normal crossing pair such that* D *is reduced and let* $f\colon X\to Y$ *be a proper morphism between complex analytic spaces. We assume that* Y *is a smooth complex variety and that there exists a normal crossing divisor* Σ *on* Y *such that every stratum of* (X, D) *is dominant onto* Y*, and smooth over* $Y\setminus\Sigma$*. If we assume that every stratum of* (X, D) *is a Kähler manifold in addition, then we have a spectral sequence:*

$$E^{p,q}_1 = \bigoplus_S R^qf_*\mathcal{O}_S \Rightarrow R^{p+q}f_*\mathcal{O}_X(-D),$$

where S runs through all $(\dim X-p)$-dimensional strata of (X,D), such that it degenerates at E_2 and its E_1-differential d_1 splits. Moreover, $R^if_*\mathcal{O}_X(-D)$ is locally free of finite rank for every i.

By combining Theorem 1.4 with Takegoshi's results (see [T]), we can prove:

Theorem 1.5 (Torsion-freeness and vanishing theorem)**.** *Let (X,D) be an analytic simple normal crossing pair such that D is reduced and let $f\colon X\to Y$ be a projective morphism between complex analytic spaces. We assume that Y is a complex variety and that every stratum of (X,D) is dominant onto Y. Then we have the following properties.*

(i) (*Torsion-freeness*). *$R^qf_*\omega_X(D)$ is a torsion-free sheaf for every q.*

(ii) (*Vanishing theorem*). *Let $\pi\colon Y\to Z$ be a projective morphism between complex analytic spaces and let $\mathcal{A}$ be a π-ample line bundle on Y. Then*

$$R^p\pi_*\left(\mathcal{A}\otimes R^qf_*\omega_X(D)\right)=0$$

holds for every $p>0$ and every q.

Of course, Theorem 1.5 is a generalization of Kollár's torsion-freeness and vanishing theorem (see [Ko1]) for reducible complex analytic spaces. We make a remark on the relationship between [FF1] and this paper.

Remark 1.6. In [FF1], we have already treated Theorems 1.1 and 1.5 when X and Y are algebraic and $f\colon X\to Y$ is projective. Roughly speaking, in [FF1, §6], we first establish Theorem 1.5 when X is quasi-projective and $f\colon X\to Y$ is algebraic. Then, by using it, we prove Theorem 1.1 under the assumption that X and Y are algebraic and $f\colon X\to Y$ is projective in [FF1, §7]. When X is quasi-projective, we can use the theory of mixed Hodge structures. Hence we can obtain desired vanishing theorems and torsion-freeness without using the theory of variations of mixed Hodge structure (for the details, see [Fn3, Chapter 5]). In this paper, we will directly prove Theorems 1.1 and 1.4 with the aid of some results established for Kähler manifolds (see [T]). Then, we will prove Theorem 1.5 as an application. Theorem 1.4 is new even when X and Y are algebraic and $f\colon X\to Y$ is projective.

By using Theorem 1.5, we have:

Theorem 1.7 (see [Fn9, Theorem 3.1])**.** *Let (X,D) be an analytic simple normal crossing pair such that D is reduced and let $f\colon X\to Y$ be a projective morphism between complex analytic spaces. Then we have the following properties.*

(i) (*Strict support condition*). *Every associated subvariety of $R^qf_*\omega_X(D)$ is the f-image of some stratum of (X,D) for every q.*

(ii) (*Vanishing theorem*). *Let $\pi\colon Y\to Z$ be a projective morphism between complex analytic spaces and let $\mathcal{A}$ be a π-ample line bundle on Y. Then*

$$R^p\pi_*\left(\mathcal{A}\otimes R^qf_*\omega_X(D)\right)=0$$

holds for every $p>0$ and every q.

(iii) (*Injectivity theorem*). *Let $\mathcal{L}$ be an f-semiample line bundle on X. Let s be a nonzero element of $H^0(X,\mathcal{L}^{\otimes k})$ for some nonnegative integer k such that the zero locus of s does not contain any strata of (X,D). Then, for every q, the map*

$$\times s\colon R^qf_*\left(\omega_X(D)\otimes\mathcal{L}^{\otimes l}\right)\to R^qf_*\left(\omega_X(D)\otimes\mathcal{L}^{\otimes k+l}\right)$$

induced by $\otimes s$ is injective for every positive integer l.

Note that Theorem 1.7 was first obtained in [Fn9, Theorem 3.1] under a weaker assumption that $f\colon X \to Y$ is Kähler by using Saito's theory of mixed Hodge modules. Theorems 1.8 and 1.9 are the main results of [Fn9]. Although they may look artificial and technical, they are very useful and indispensable for the study of varieties and pairs whose singularities are worse than kawamata log terminal (see [A], [Fn3, Chapter 6], [Fn6], [Fn7], [Fn10], [Fn11], and so on). In [Fn9], we showed that Theorems 1.8 and 1.9 follow from Theorem 1.7 (i) and (ii). Note that Theorem 1.7 (iii) is an easy consequence of Theorem 1.7 (i) and (ii). Hence this paper gives an approach to Theorems 1.8 and 1.9 without using Saito's theory of mixed Hodge modules.

Theorem 1.8 (see [Fn9, Theorem 1.1])**.** *Let (X, Δ) be an analytic simple normal crossing pair such that Δ is a boundary $\mathbb{R}$-divisor on X. Let $f\colon X \to Y$ be a projective morphism to a complex analytic space Y and let $\mathcal{L}$ be a line bundle on X. Let q be an arbitrary nonnegative integer. Then we have the following properties.*

(i) (*Strict support condition*). *If $\mathcal{L} - (\omega_X + \Delta)$ is f-semiample, then every associated subvariety of $R^q f_*\mathcal{L}$ is the f-image of some stratum of (X, Δ).*

(ii) (*Vanishing theorem*). *If $\mathcal{L} - (\omega_X + \Delta) \sim_{\mathbb{R}} f^*\mathcal{H}$ holds for some π-ample $\mathbb{R}$-line bundle $\mathcal{H}$ on Y, where $\pi\colon Y \to Z$ is a projective morphism to a complex analytic space Z, then we have $R^p\pi_* R^q f_*\mathcal{L} = 0$ for every $p > 0$.*

Theorem 1.9 (Vanishing theorem of Reid–Fukuda type, see [Fn9, Theorem 1.2])**.** *Let (X, Δ) be an analytic simple normal crossing pair such that Δ is a boundary $\mathbb{R}$-divisor on X. Let $f\colon X \to Y$ and $\pi\colon Y \to Z$ be projective morphisms between complex analytic spaces and let $\mathcal{L}$ be a line bundle on X. If $\mathcal{L} - (\omega_X + \Delta) \sim_{\mathbb{R}} f^*\mathcal{H}$ holds such that $\mathcal{H}$ is an $\mathbb{R}$-line bundle, which is nef and log big over Z with respect to $f\colon (X, \Delta) \to Y$, on Y, then $R^p\pi_* R^q f_*\mathcal{L} = 0$ holds for every $p > 0$ and every q.*

In this paper, we do not prove Theorems 1.8 and 1.9. For the details of Theorems 1.8 and 1.9, see [Fn9]. Although the motivation of the first author is obviously the minimal model theory for projective morphisms between complex analytic spaces, we do not treat the minimal model program in this paper. We recommend that the interested reader looks at [Fn8], [Fn10], [Fn11], and so on. Theorems 1.8 and 1.9 have already played a crucial role in [Fn10] and [Fn11], where we established the fundamental theorems of the theory of minimal models for projective morphisms between complex analytic spaces. Anyway, by this paper, [Fn10] and [Fn11] become free from Saito's theory of mixed Hodge modules. The relationship between [Fn9] and this paper is as follows.

Remark 1.10. In [FFS, Corollary 1 and 4.7. Remark] (see [Fn9, Theorem 2.6]), we constructed a weight spectral sequence of mixed Hodge modules. It is much more general than Theorem 1.4 in some sense. By combining it with Takegoshi's results (see [T]), we proved Theorems 1.7, 1.8, 1.9, and so on, in [Fn9]. From the Hodge theoretic viewpoint, one of the main ingredients of this paper is Steenbrink's result obtained in [St1] and [St2].

We look at the organization of this paper. In Section 2, we will briefly explain basic definitions and results necessary for this paper. In Subsection 2.1, we will explain some useful lemmas on analytic simple normal crossing pairs. In Subsection 2.2, we will briefly review Kollár's package in the complex analytic setting. In Section 3, we will present abstract arguments which help us to obtain a variation of mixed Hodge structure whose Hodge filtration can be extended to its canonical extension along a simple normal crossing divisor. Section 4 is the main part of this paper, where we will prove Theorems 1.1 and 1.4. We will also see that a generalization of the Fujita–Zucker–Kawamata semipositivity theorem holds in the complex analytic setting (see Theorem 1.3). In Section 5, we will prove Theorem 1.5. In Section 6, we will prove Theorem 1.7. Section 7 is a supplementary section, where we will explain a new construction of the rational structure for the cohomological $\mathbb{Q}$-mixed Hodge complex in [St2]. We hope that it will help the reader to understand [St1] and [St2].

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In this paper, every complex analytic space is assumed to be *Hausdorff* and *second-countable*. Note that an irreducible and reduced complex analytic space is called a *complex variety*. We will freely use the basic results on complex analytic geometry in [BS] and [Fi].

2. Preliminaries

In this section, we will collect some basic definitions. Let us start with the definition of *analytic simple normal crossing pairs*.

Definition 2.1 (Analytic simple normal crossing pairs)**.** Let X be a simple normal crossing divisor on a smooth complex analytic space M and let B be an $\mathbb{R}$-divisor on M such that the support of $B+X$ is a simple normal crossing divisor on M and that B and X have no common irreducible components. Then we put $D := B|_X$ and consider the pair (X, D). We call (X, D) an *analytic globally embedded simple normal crossing pair* and M the *ambient space* of (X, D). If the pair (X, D) is locally isomorphic to an analytic globally embedded simple normal crossing pair at any point of X and the irreducible components of X and D are all smooth, then (X, D) is called an *analytic simple normal crossing pair*.

When (X, D) is an analytic simple normal crossing pair, X has an invertible dualizing sheaf ω_X. We usually use the symbol K_X as a formal divisor class with an isomorphism $\mathcal{O}_X(K_X) \simeq \omega_X$ if there is no danger of confusion. We note that we can not always define K_X globally with $\mathcal{O}_X(K_X) \simeq \omega_X$. In general, it only exists locally on X.

The notion of *strata* plays a crucial role.

Definition 2.2 (Strata)**.** Let (X, D) be an analytic simple normal crossing pair as in Definition 2.1. Let $\nu\colon X^\nu \to X$ be the normalization. We put

$$K_{X^\nu} + \Theta = \nu^*(K_X + D).$$

This means that Θ is the union of $\nu_*^{-1}D$ and the inverse image of the singular locus of X. We note that X^ν is smooth and the support of Θ is a simple normal crossing divisor on X^ν. If W is an irreducible component of X or the ν-image of some log canonical center of (X^ν, Θ), then W is called a *stratum* of (X, D).

Remark 2.3. In this paper, D is always assumed to be reduced. Hence, Θ in Definition 2.2 is a reduced simple normal crossing divisor on X^ν. We do not need $\mathbb{Q}$-divisors nor $\mathbb{R}$-divisors in this paper.

We recall Siu's theorem on complex analytic sheaves, which is a special case of [Si, Theorem 4]. We need it for Theorem 1.7 (i) and Theorem 1.8 (i).

Theorem 2.4. *Let $\mathcal{F}$ be a coherent sheaf on a complex analytic space X. Then there exists a locally finite family $\{Y_i\}_{i\in I}$ of complex analytic subvarieties of X such that*

$$\mathrm{Ass}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) = \{\mathfrak{p}_{x,1}, \ldots, \mathfrak{p}_{x,r(x)}\}$$

holds for every point $x \in X$, where $\mathfrak{p}_{x,1}, \ldots, \mathfrak{p}_{x,r(x)}$ are the prime ideals of $\mathcal{O}_{X,x}$ associated to the irreducible components of the germs $Y_{i,x}$ of Y_i at x with $x \in Y_i$. We note that each Y_i is called an associated subvariety *of $\mathcal{F}$.*

Definition 2.5 (Relatively nef, ample, and big line bundles)**.** Let $f\colon X \to Y$ be a projective morphism of complex analytic spaces and let $\mathcal{L}$ be a line bundle on X. Then we say that

- $\mathcal{L}$ is f*-nef* if $\mathcal{L} \cdot C \geq 0$ holds for every curve C on X such that $f(C)$ is a point, and
- $\mathcal{L}$ is f*-ample* if $\mathcal{L}|_{f^{-1}(y)}$ is ample in the usual sense for every $y \in Y$.

We further assume that $f\colon X \to Y$ is a projective surjective morphism of complex varieties. Then we say that

- $\mathcal{L}$ is f*-big* if there exists some positive real number c such that $\operatorname{rank} f_*\mathcal{L}^{\otimes m} > c \cdot m^d$ holds for $m \gg 0$, where $d = \dim X - \dim Y$.

We need the notion of *nef locally free sheaves* in Theorem 1.3.

Definition 2.6 (Nef locally free sheaves)**.** Let $\mathcal{E}$ be a locally free sheaf of finite rank on a projective variety V. If $\mathcal{O}_{\mathbb{P}_V(\mathcal{E})}(1)$ is nef, that is, $\mathcal{O}_{\mathbb{P}_V(\mathcal{E})}(1) \cdot C \geq 0$ holds for every curve C on $\mathbb{P}_V(\mathcal{E})$, then $\mathcal{E}$ is called a *nef* locally free sheaf on V.

A nef locally free sheaf is sometimes called a *semipositive vector bundle* or a *semipositive locally free sheaf* in the literature.

2.1. **Lemmas on analytic simple normal crossing pairs.** In this subsection, we will collect some useful lemmas on analytic simple normal crossing pairs. We will repeatedly use these lemmas in subsequent sections.

Lemma 2.7 (see [Fn9, Lemmas 2.13 and 2.15])**.** *Let (X, D) and (X', D') be simple normal crossing pairs such that D and D' are reduced. Let $g\colon X' \to X$ be a projective bimeromorphic morphism. Assume that there exists a Zariski open subset U of X such that $g\colon U' := g^{-1}(U) \to U$ is an isomorphism and that U (resp. U') intersects every stratum of (X, D) (resp. (X', D')). Then $R^ig_*\mathcal{O}_{X'} = 0$ and $R^ig_*\mathcal{O}_{X'}(K_{X'} + D') = 0$ for every $i > 0$, and $g_*\mathcal{O}_{X'} \simeq \mathcal{O}_X$ and $g_*\mathcal{O}_{X'}(K_{X'} + D') \simeq \mathcal{O}_X(K_X + D)$ hold.*

Proof. By [Fn9, Lemma 2.15], we have $R^ig_*\mathcal{O}_{X'} = 0$ for every $i > 0$ and $g_*\mathcal{O}_{X'} \simeq \mathcal{O}_X$. Since D and D' are reduced, we can easily check that

$$K_{X'} + D' = g^*(K_X + D) + E \tag{2.1}$$

holds for some effective g-exceptional Cartier divisor E on X' and that $D' = g_*^{-1}D$ holds. By (2.1), we have $g_*\mathcal{O}_{X'}(K_{X'} + D') \simeq \mathcal{O}_X(K_X + D)$. By [Fn9, Lemma 2.13], we obtain $R^ig_*\mathcal{O}_{X'}(K_{X'} + D') = 0$ for every $i > 0$. We finish the proof. □

Lemma 2.8 (see [Fn9, Lemma 5.1])**.** *Let (X, D) be an analytic simple normal crossing pair such that D is reduced and let $f\colon X \to Y$ be a projective morphism between complex analytic spaces. Let L be a Cartier divisor on X. We take an arbitrary point $P \in Y$.*

Then, after shrinking Y around P suitably, we can construct the following commutative diagram:

$$\begin{array}{ccc} Z & \overset{\iota}{\hookrightarrow} & M \\ {\scriptstyle p}\downarrow & & \Big\downarrow \\ X & & {\scriptstyle q} \\ {\scriptstyle f}\downarrow & & \Big\downarrow \\ Y & \underset{\iota_Y}{\hookrightarrow} & \Delta^m \end{array}$$

such that

(i) $\iota_Y\colon Y \hookrightarrow \Delta^m$ *is a closed embedding into a polydisc* Δ^m *with* $\iota_Y(P) = 0 \in \Delta^m$,
(ii) (Z, D_Z) *is an analytic globally embedded simple normal crossing pair such that* D_Z *is reduced,*
(iii) M *is the ambient space of* (Z, D_Z) *and is projective over* Δ^m,
(iv) *there exists a Cartier divisor* L_Z *on* Z *satisfying*

$$L_Z - (K_Z + D_Z) = p^*(L - (K_X + D)),$$

$p_*\mathcal{O}_Z(L_Z) \simeq \mathcal{O}_X(L)$, *and* $R^ip_*\mathcal{O}_Z(L_Z) = 0$ *for every* $i > 0$,
(v) $p(W)$ *is a stratum of* (X, D) *for every stratum* W *of* (Z, D_Z),
(vi) *there exists a Zariski open subset* U *of* X, *which intersects every stratum of* X, *such that* p *is an isomorphism over* U,
(vii) p *maps every stratum of* Z *bimeromorphically onto some stratum of* X, *and*
(viii) *for any stratum* S *of* (X, D), *there exists a stratum* W *of* (Z, D_Z) *such that* $S = p(W)$.

In particular, we have:

(ix) $p_*\mathcal{O}_Z(K_Z + D_Z) \simeq \mathcal{O}_X(K_X + D)$ *and* $R^ip_*\mathcal{O}_Z(K_Z + D_Z) = 0$ *holds for every* $i > 0$.

Proof. The proof of [Fn9, Lemma 5.1], where we allow D to be a boundary $\mathbb{R}$-divisor, works without any modifications. Thus we have the desired commutative diagram satisfying (i)–(viii). When $L = K_X + D$, we have $L_Z = K_Z + D_Z$ by (iv). Hence we obtain (ix). ☐

Lemma 2.9 (see [Fn9, Lemma 2.17])**.** *Let* (X, D) *be an analytic globally embedded simple normal crossing pair such that* D *is reduced and let* M *be the ambient space of* (X, D). *Let* C *be a stratum of* (X, D), *which is not an irreducible component of* X. *Let* $\sigma\colon M' \to M$ *be the blow-up along* C *and let* X' *denote the reduced structure of the total transform of* X *on* M'. *We put*

$$K_{X'} + D' := g^*(K_X + D),$$

where $g := \sigma|_{X'}$. *Then we have the following properties:*

(i) (X', D') *is an analytic globally embedded simple normal crossing pair such that* D' *is reduced,*
(ii) M' *is the ambient space of* (X', D'),
(iii) $g_*\mathcal{O}_{X'} \simeq \mathcal{O}_X$ *holds and* $R^ig_*\mathcal{O}_{X'} = 0$ *for every* $i > 0$,
(iv) *the strata of* (X, D) *are exactly the images of the strata of* (X', D'), *and*
(v) $\sigma^{-1}(C)$ *is a maximal* (*with respect to the inclusion*) *stratum of* (X', D'), *that is,* $\sigma^{-1}(C)$ *is an irreducible component of* X'.

Proof. The proof of [Fn9, Lemma 2.17], where we allow D to be a boundary $\mathbb{R}$-divisor, works without any modifications. ☐

2.2. **Complex analytic generalization of Kollár's package.** Here, let us briefly review Kollár's package (see [Ko1] and [Ko2]) in the complex analytic setting. We recommend that the interested reader looks at [Fn2], [N3, Chapter V. 3.7. Theorem], and [T].

Theorem 2.10 is a variant of Takegoshi's vanishing theorem (see [T, Theorem IV Relative vanishing Theorem] and [Fn2, Corollary 1.5]). We note that it is well known when $f\colon X \to Y$ and $\pi\colon Y \to Z$ are projective morphisms of algebraic varieties.

Theorem 2.10 (Vanishing theorem)**.** *Let $f\colon X \to Y$ and $\pi\colon Y \to Z$ be projective surjective morphisms between complex varieties such that X is smooth. Let $\mathcal{M}$ be a line bundle on Y. Assume that $\mathcal{M}$ is π-nef and π-big over Z. Then*

$$R^p\pi_*\left(\mathcal{M}\otimes R^qf_*\omega_X\right)=0 \tag{2.2}$$

holds for every $p>0$ and every q. In particular, if further π is bimeromorphic, then

$$R^p\pi_*R^qf_*\omega_X=0 \tag{2.3}$$

holds for every $p>0$ and every q.

Proof. The vanishing theorem (2.2) is more or less well known to the experts. For the details, see, for example, [Fn2, Corollary 1.5]. Note that (2.3) is a special case of (2.2). This is because the trivial line bundle on Y is π-nef and π-big when π is bimeromorphic. □

Lemma 2.11 is an easy consequence of Theorem 2.10.

Lemma 2.11. *Let $f_i\colon X_i \to Y$ be a projective surjective morphism of complex varieties such that X_i is smooth for every $1\le i\le k$. Let $\pi\colon Y\to Z$ be a projective bimeromorphic morphism between complex varieties. We put*

$$\mathcal{F}:=\bigoplus_{i=1}^{k}R^{q_i}f_{i*}\omega_{X_i},$$

where q_i is some nonnegative integer for every i. Let $\mathcal{G}$ be a coherent sheaf on Y. Assume that $\mathcal{G}$ is a direct summand of $\mathcal{F}$. Then we have $R^p\pi_*\mathcal{F}=0$ and $R^p\pi_*\mathcal{G}=0$ for every $p>0$. In particular, $\pi_*\mathcal{G}$ is a direct summand of

$$\bigoplus_{i=1}^{k}R^{q_i}(\pi\circ f_i)_*\omega_{X_i}.$$

Proof. By Lemma 2.10, $R^p\pi_*R^{q_i}f_{i*}\omega_{X_i}=0$ holds for every $p>0$. Hence we have $R^p\pi_*\mathcal{F}=0$ for every $p>0$. Since $\mathcal{G}$ is a direct summand of $\mathcal{F}$, we obtain $R^p\pi_*\mathcal{G}=0$ for every $p>0$. It is obvious that $\pi_*\mathcal{G}$ is a direct summand of $\pi_*\mathcal{F}$. Since

$$\pi_*\mathcal{F}=\bigoplus_{i=1}^{k}\pi_*R^{q_i}f_{i*}\omega_{X_i}\simeq\bigoplus_{i=1}^{k}R^{q_i}(\pi\circ f_i)_*\omega_{X_i},$$

$\pi_*\mathcal{G}$ is a direct summand of $\bigoplus_{i=1}^{k}R^{q_i}(\pi\circ f_i)_*\omega_{X_i}$. We finish the proof. □

Theorem 2.12 below is a special case of Takegoshi's torsion-freeness (see [T, Theorem II Torsion freeness Theorem] and [Fn2, Corollary 1.2]). When $f\colon X\to Y$ is a projective surjective morphism between projective varieties, it is nothing but Kollár's famous torsion-freeness (see [Ko1, Theorem 2.1 (i)]).

Theorem 2.12 (Torsion-freeness)**.** *Let $f\colon X\to Y$ be a projective surjective morphism of complex varieties such that X is smooth. Then $R^qf_*\omega_X$ is torsion-free for every q.*

When $f\colon X\to Y$ is algebraic, Theorem 2.13 below was first obtained independently by Kollár (see [Ko2, Theorem 2.6]) and Nakayama (see [N2, Theorem 1]). When $f\colon X\to Y$ is a projective morphism of smooth complex varieties, it was obtained by Moriwaki (see [Mo, Theorem (2.4)]).

Theorem 2.13 (Hodge filtration, see [T, Theorem V Local freeness Theorem (ii)] and [N3, Chapter V, 3.7. Theorem (4)])**.** *Let $f\colon X\to Y$ be a proper surjective morphism between smooth complex varieties and let Σ be a normal crossing divisor on Y such that f is smooth over $Y^*:=Y\setminus\Sigma$. We assume that X is a Kähler manifold. Then $R^qf_*\omega_{X/Y}$ is locally free and is characterized as the upper canonical extension of the corresponding bottom Hodge filtration on Y^* for every q.*

We make a remark on the proof of Theorem 2.13.

Remark 2.14. One of the main ingredients of [N2] is Steenbrink's result established in [St1] and [St2] (see [N2, Theorem 3]). Although it was explicitly stated only for projective morphisms, it also holds for proper morphisms from Kähler manifolds (see Remark 4.4 below). Hence the argument in [N2] works for Kähler manifolds with the aid of [T]. We recommend that the interested reader looks at [N1, Conjectures 7.2 and 7.3] and [N2].

3. On variations of mixed Hodge structure

In this section, we present abstract arguments which help us to obtain a variation of mixed Hodge structure whose Hodge filtration can be extended to its canonical extension along a simple normal crossing divisor. Although the arguments look rather technical, they give us an appropriate viewpoint for the proofs of Theorems 1.1 and 1.4.

3.1. Let X be a complex manifold and

$$K=((K_{\mathbb{R}},W),(K_{\mathcal{O}},W,F),\alpha)$$

be a triple consisting of

- a bounded complex of $\mathbb{R}$-sheaves $K_{\mathbb{R}}$ on X equipped with a finite increasing filtration W,
- a bounded complex of $\mathcal{O}_Y$-modules $K_{\mathcal{O}}$ on X equipped with a finite increasing filtration W and a finite decreasing filtration F, and
- a morphism of complexes of $\mathbb{R}$-sheaves $\alpha\colon K_{\mathbb{R}}\to K_{\mathcal{O}}$ preserving the filtration W.

Such a triple K yields a pair of spectral sequences

$$E_r^{p,q}(K,W)=(E_r^{p,q}(K_{\mathbb{R}},W),(E_r^{p,q}(K_{\mathcal{O}},W),F),E_r^{p,q}(\alpha))$$

where F on $E_r^{p,q}(K_{\mathcal{O}},W)$ stands for the first direct filtration (cf. [D2, (1.3.8)]), that is,

$$F^aE_r^{p,q}(K_{\mathcal{O}},W)=\mathrm{Image}(E_r^{p,q}(F^aK_{\mathcal{O}},W)\to E_r^{p,q}(K_{\mathcal{O}},W))$$

for $r=0,1,2,\dots,\infty$ and for $a,p,q\in\mathbb{Z}$. Here we remark that F on $E_r^{p,q}(K_{\mathcal{O}},W)$ above is not necessarily a filtration by *subbundles*. The morphism of E_r-terms is denoted by $d_r^{p,q}$ for every p,q,r.

3.2. For a triple K as in 3.1, we consider the following conditions:

(3.2.1) The triple

$$(H^n(\mathrm{Gr}_m^W K_{\mathbb{R}}), (H^n(\mathrm{Gr}_m^W K_{\mathcal{O}}), F), H^n(\mathrm{Gr}_m^W \alpha))$$

is a variation of $\mathbb{R}$-Hodge structure of weight $n+m$ on X for every m, n.

(3.2.2) The spectral sequence associated to $(\mathrm{Gr}_m^W K_{\mathcal{O}}, F)$ degenerates at E_1-terms for every m.

The following lemma is a counterpart of [D3, Scholie (8.1.9)] for variations of (mixed) Hodge structure.

Lemma 3.3. *If K satisfies* (3.2.1) *and* (3.2.2)*, then the following holds:*

(3.3.1) *$E_r^{p,q}(K, W)$ is a variation of $\mathbb{R}$-Hodge structure of weight q on X for every $r = 1, 2, \ldots, \infty$ and for every p, q.*

(3.3.2) *The triple*

$$(\mathrm{Gr}_m^W H^n(K_{\mathbb{R}}), (\mathrm{Gr}_m^W H^n(K_{\mathcal{O}}), F), \mathrm{Gr}_m^W H^n(\alpha))$$

is a variation of $\mathbb{R}$-Hodge structure of weight $n+m$ on X for every $m, n \in \mathbb{Z}$.

(3.3.3) *The spectral sequence associated to $(K_{\mathcal{O}}, W)$ degenerates at E_2-terms.*

(3.3.4) *The spectral sequence associated to $(K_{\mathcal{O}}, F)$ degenerates at E_1-terms.*

(3.3.5) *There exists an isomorphism*

$$E_r^{p,q}(\mathrm{Gr}_F^a K_{\mathcal{O}}, W) \simeq \mathrm{Gr}_F^a E_r^{p,q}(K_{\mathcal{O}}, W),$$

under which the morphism of the E_r-terms of the left hand side coincides with $\mathrm{Gr}_F^a d_r^{p,q}$ on the right hand side for every a, p, q, r.

(3.3.6) *The spectral sequence associated to $(\mathrm{Gr}_F^a K_{\mathcal{O}}, W)$ degenerates at E_2-terms for every $a \in \mathbb{Z}$.*

Proof. From the assumption (3.2.2), the differential of $\mathrm{Gr}_m^W K_{\mathcal{O}}$ is strictly compatible with F for every m. Then we obtain the conclusions from the assumption (3.2.1) by a similar argument to [D3, Proposition (7.2.8) and Scholie (8.1.9)] and by Lemma 3.4 below. □

Lemma 3.4. *The category of the variations of $\mathbb{R}$-Hodge structure of a fixed weight on a complex manifold is an abelian category.*

Proof. By Lemma 3.14 (i) of [FF1] (replaced $\mathbb{Q}$ by $\mathbb{R}$), it suffices to prove that the kernel and the cokernel of a morphism of variations of $\mathbb{R}$-Hodge structure satisfy the Griffiths transversality. Let $\varphi\colon \mathbb{V}_1 \to \mathbb{V}_2$ be a morphism of variations of $\mathbb{R}$-Hodge structure of the same weight on a complex manifold X. Then, from the commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathcal{O}_X \otimes \mathrm{Ker}(\varphi) & \longrightarrow & \mathcal{O}_X \otimes \mathbb{V}_1 & \longrightarrow & \mathcal{O}_X \otimes \mathbb{V}_2 & \longrightarrow & \mathcal{O}_X \otimes \mathrm{Coker}(\varphi) & \longrightarrow 0 \\ & \downarrow{\scriptstyle d\otimes\mathrm{id}} & & \downarrow{\scriptstyle d\otimes\mathrm{id}} & & \downarrow{\scriptstyle d\otimes\mathrm{id}} & & \downarrow{\scriptstyle d\otimes\mathrm{id}} & \\ 0 \longrightarrow & \Omega_X^1 \otimes \mathrm{Ker}(\varphi) & \longrightarrow & \Omega_X^1 \otimes \mathbb{V}_1 & \longrightarrow & \Omega_X^1 \otimes \mathbb{V}_2 & \longrightarrow & \Omega_X^1 \otimes \mathrm{Coker}(\varphi) & \longrightarrow 0, \end{array}$$

we can easily check that the first and the last vertical arrows satisfy the Griffiths transversality. Here we note that the Hodge filtrations on $\mathcal{O}_X \otimes \mathrm{Ker}(\varphi)$ and $\mathcal{O}_X \otimes \mathrm{Coker}(\varphi)$ are induced from the Hodge filtrations on $\mathcal{O}_X \otimes \mathbb{V}_1$ and $\mathcal{O}_X \otimes \mathbb{V}_2$ respectively. □

3.5. Now we study the case of a pair (X, Σ), where X is a complex manifold and Σ a reduced simple normal crossing divisor on X having finitely many irreducible components. We set $X^* = X \setminus \Sigma$.

The following elementary lemma will be used several times in the present and next sections.

Lemma 3.6. *Let $\mathcal{F}$ and $\mathcal{G}$ be locally free $\mathcal{O}_X$-modules of finite rank on X and $\varphi, \psi\colon \mathcal{F} \to \mathcal{G}$ morphisms of $\mathcal{O}_X$-modules. If $\varphi|_{X^*} = \psi|_{X^*}$, then $\varphi = \psi$. In particular, if $\varphi|_{X^*} = 0$ then $\varphi = 0$.*

Proof. It is obvious. □

3.7. For the case of (X, Σ), we consider a triple

$$K = ((K_{\mathbb{R}}, W), (K_{\mathcal{O}}, W, F), \alpha)$$

consisting of

- a bounded complex of $\mathbb{R}$-sheaves $K_{\mathbb{R}}$ on X^* equipped with a finite increasing filtration W,
- a bounded complex of $\mathcal{O}_X$-modules $K_{\mathcal{O}}$ on X equipped with a finite increasing filtration W and a finite decreasing filtration F, and
- a morphism of complexes of $\mathbb{R}$-sheaves $\alpha\colon K_{\mathbb{R}} \to K_{\mathcal{O}}|_{X^*}$ preserving the filtration W.

Note that only $(K_{\mathcal{O}}, W, F)$ is defined over the whole X. We set

$$K|_{X^*} = ((K_{\mathbb{R}}, W), (K_{\mathcal{O}}, W, F)|_{X^*}, \alpha),$$

which is a triple on X^* considered in 3.1, and consider the following conditions:

(3.7.1) $K|_{X^*}$ satisfies (3.2.1) and (3.2.2) on X^*.

(3.7.2) For every m, n, the $\mathbb{R}$-local system $H^n(\mathrm{Gr}_m^W K_{\mathbb{R}})$ is of quasi-unipotent local monodromies around all the irreducible components of Σ.

(3.7.3) For every m, n, the variation of $\mathbb{R}$-Hodge structure on X^* in (3.2.1) for $K|_{X^*}$ is polarizable.

If K satisfies these conditions, then the lower canonical extension of $\mathcal{O}_{X^*} \otimes H^n(\mathrm{Gr}_m^W K_{\mathbb{R}}) \simeq H^n(\mathrm{Gr}_m^W K_{\mathcal{O}})|_{X^*}$ is denoted by ${}^{\ell}H^n(\mathrm{Gr}_m^W K_{\mathcal{O}})|_{X^*}$ for every m, n. By Schmid's nilpotent orbit theorem [Sc, (4.12)], the Hodge filtration F on $H^n(\mathrm{Gr}_m^W K_{\mathcal{O}})|_{X^*}$ extends to a filtration (by subbundles) on ${}^{\ell}H^n(\mathrm{Gr}_m^W K_{\mathcal{O}})|_{X^*}$, which is denoted by the same letter F. Moreover, the condition (3.2.2) for $K|_{X^*}$ implies that there exists a natural isomorphism

$$H^n(\mathrm{Gr}_F^a \mathrm{Gr}_m^W K_{\mathcal{O}})|_{X^*} \simeq \mathrm{Gr}_F^a H^n(\mathrm{Gr}_m^W K_{\mathcal{O}})|_{X^*} \tag{3.1}$$

for every a, m, n. Here we remark that this morphism induces the isomorphism in (3.3.5) for $r = 1$.

On the other hand, we conclude (3.3.1)–(3.3.6) for $K|_{X^*}$ by Lemma 3.3. In addition, we can easily obtain the following:

(3.7.4) $H^n(K_{\mathbb{R}})$ is an $\mathbb{R}$-local system on X^* of quasi-unipotent local monodromy along Σ for every n.

(3.7.5) $W_m H^n(K_{\mathcal{O}})|_{X^*} \simeq \mathcal{O}_{X^*} \otimes W_m H^n(K_{\mathbb{R}})$ for every m, n. In particular, we have $H^n(K_{\mathcal{O}})|_{X^*} \simeq \mathcal{O}_{X^*} \otimes H^n(K_{\mathbb{R}})$ for every n.

(3.7.6) The variation of $\mathbb{R}$-Hodge structure $E_r^{p,q}(K|_{X^*}, W)$ is polarizable for every p, q, r. In particular, the variation of $\mathbb{R}$-Hodge structure

$$(\mathrm{Gr}_m^W H^n(K_{\mathbb{R}}), (\mathrm{Gr}_m^W H^n(K_{\mathcal{O}}), F)|_{X^*}, \mathrm{Gr}_m^W H^n(\alpha))$$

is polarizable.

The following lemma and theorem play essential roles in the proofs of Theorems 1.1 and 1.4.

Lemma 3.8. *Let* (X, Σ) *be as in 3.5 and* $K = ((K_{\mathbb{R}}, W), (K_{\mathcal{O}}, W, F), \alpha)$ *as in 3.7 satisfying the conditions* (3.7.1)–(3.7.3). *For* $a \in \mathbb{Z}$, *if there exists an isomorphism*

$$H^n(\mathrm{Gr}_F^a \mathrm{Gr}_m^W K_{\mathcal{O}}) \xrightarrow{\simeq} \mathrm{Gr}_F^a({}^{\ell}H^n(\mathrm{Gr}_m^W K_{\mathcal{O}})|_{X^*})$$

whose restriction to X^* *coincides with the isomorphism* (3.1) *for every* m, n, *then the spectral sequence associated to* $(\mathrm{Gr}_F^a K_{\mathcal{O}}, W)$ *degenerates at* E_2*-terms and the morphism of* E_1*-terms splits. Moreover* $\mathrm{Gr}_m^W H^n(\mathrm{Gr}_F^a K_{\mathcal{O}})$ *is locally free of finite rank for every* m, n. *In particular, so is* $H^n(\mathrm{Gr}_F^a K_{\mathcal{O}})$ *for every* n.

Proof. Because the variation of $\mathbb{R}$-Hodge structure $E_r^{p,q}(K|_{X^*}, W)$ is of quasi-unipotent local monodromy and polarizable as mentioned in (3.7.5), we obtain a filtered $\mathcal{O}_X$-module $({}^{\ell}E_r^{p,q}(K_{\mathcal{O}}, W)|_{X^*}, F)$ for every p, q, r by using Schmid's nilpotent orbit theorem as in 3.7.

Because $d_1^{p,q}|_{X^*}$ is a morphism of variations of $\mathbb{R}$-Hodge structure of weight q, we obtain a polarizable variation of $\mathbb{R}$-Hodge structure $I^{p,q}$ of weight q on X^* by

$$\begin{aligned} I^{p,q} &= (I_{\mathbb{R}}^{p,q}, (I_{\mathcal{O}}^{p,q}, F)) \\ &= \mathrm{Image}(d_1^{p-1,q}|_{X^*} \colon E_1^{p-1,q}(K, W)|_{X^*} \to E_1^{p,q}(K, W)|_{X^*}) \end{aligned}$$

for every p, q. It is clear that $I_{\mathbb{R}}^{p,q}$ is of quasi-unipotent local monodromy along Σ. Thus we obtain a filtered $\mathcal{O}_X$-module $({}^{\ell}I_{\mathcal{O}}^{p,q}, F)$ as in 3.7 again.

By the semisimplicity of the polarizable variations of $\mathbb{R}$-Hodge structure, we have the direct sum decomposition

$$E_1^{p,q}(K|_{X^*}, W) \simeq E_2^{p,q}(K|_{X^*}, W) \oplus I^{p,q} \oplus I^{p+1,q}$$

as variations of $\mathbb{R}$-Hodge structure on X^*. Therefore we obtain the direct sum decomposition

$$({}^{\ell}E_1^{p,q}(K_{\mathcal{O}}, W)|_{X^*}, F) \simeq ({}^{\ell}E_2^{p,q}(K_{\mathcal{O}}, W)|_{X^*}, F) \oplus ({}^{\ell}I_{\mathcal{O}}^{p,q}, F) \oplus ({}^{\ell}I_{\mathcal{O}}^{p+1,q}, F) \tag{3.2}$$

as filtered $\mathcal{O}_X$-modules because the extension of the filtration is unique by [FF1, Corollary 5.2]. It is clear that the morphism ${}^{\ell}d_1^{p,q}|_{X^*}$ is identified with the composite of the projection

$${}^{\ell}E_2^{p,q}(K_{\mathcal{O}}, W)|_{X^*} \oplus {}^{\ell}I_{\mathcal{O}}^{p,q} \oplus {}^{\ell}I_{\mathcal{O}}^{p+1,q} \to {}^{\ell}I_{\mathcal{O}}^{p+1,q}$$

and the inclusion

$${}^{\ell}I_{\mathcal{O}}^{p+1,q} \hookrightarrow {}^{\ell}E_2^{p+1,q}(K_{\mathcal{O}}, W)|_{X^*}$$

under the isomorphism (3.2).

We fix $a \in \mathbb{Z}$ satisfying the assumption, and consider the spectral sequence

$$E_r^{p,q}(\mathrm{Gr}_F^a K_{\mathcal{O}}, W) \Rightarrow H^{p+q}(\mathrm{Gr}_F^a K_{\mathcal{O}})$$

with the morphism of E_r-terms $d_r^{p,q}(\mathrm{Gr}_F^a K_{\mathcal{O}}, W)$. By the assumption, there exists an isomorphism

$$E_1^{p,q}(\mathrm{Gr}_F^a K_{\mathcal{O}}, W) \simeq \mathrm{Gr}_F^a({}^{\ell}E_1^{p,q}(K_{\mathcal{O}}, W)|_{X^*}) \tag{3.3}$$

for every p, q, because we have

$$\begin{aligned} E_1^{p,q}(\mathrm{Gr}_F^a K_{\mathcal{O}}, W) &\simeq H^{p+q}(\mathrm{Gr}_{-p}^W \mathrm{Gr}_F^a K_{\mathcal{O}}) \\ &\simeq H^{p+q}(\mathrm{Gr}_F^a \mathrm{Gr}_{-p}^W K_{\mathcal{O}}) \\ &\simeq \mathrm{Gr}_F^a({}^{\ell}H^{p+q}(\mathrm{Gr}_{-p}^W K_{\mathcal{O}})|_{X^*}) \\ &\simeq \mathrm{Gr}_F^a({}^{\ell}E_1^{p,q}(K_{\mathcal{O}}, W)|_{X^*}). \end{aligned}$$

In particular, $E_1^{p,q}(\operatorname{Gr}_F^a K_{\mathcal{O}}, W)$ is locally free of finite rank for every p, q. Moreover, the restriction of the isomorphism (3.3) to X^* identifies $d_1^{p,q}(\operatorname{Gr}_F^a K_{\mathcal{O}}, W)|_{X^*}$ and $\operatorname{Gr}_F^a d_1^{p,q}|_{X^*}$ by the assumption and by (3.3.5) for $K|_{X^*}$. Then

$$d_1^{p,q}(\operatorname{Gr}_F^a K_{\mathcal{O}}, W) = \operatorname{Gr}_F^a({}^{\ell}d_1^{p,q}|_{X^*})$$

under the identification (3.3) by Lemma 3.6. Therefore there exists an isomorphism

$$E_2^{p,q}(\operatorname{Gr}_F^a K_{\mathcal{O}}, W) \simeq \operatorname{Gr}_F^a({}^{\ell}E_2^{p,q}(K_{\mathcal{O}}, W)|_{X^*})$$

and $E_2^{p,q}(\operatorname{Gr}_F^a K_{\mathcal{O}}, W)$ is isomorphic to a direct factor of $E_1^{p,q}(\operatorname{Gr}_F^a K_{\mathcal{O}}, W)$ by the direct sum decomposition (3.2). In particular, $E_2^{p,q}(\operatorname{Gr}_F^a K_{\mathcal{O}}, W)$ is locally free of finite rank. Then $d_2^{p,q}(\operatorname{Gr}_F^a K_{\mathcal{O}}, W) = 0$ because $d_2^{p,q}(\operatorname{Gr}_F^a K_{\mathcal{O}}, W)|_{X^*} = 0$ by (3.3.6) of Lemma 3.3. Inductively, $E_r^{p,q}(\operatorname{Gr}_F^a K_{\mathcal{O}}, W)$ is locally free of finite rank, and $d_r^{p,q}(\operatorname{Gr}_F^a K_{\mathcal{O}}, W) = 0$ for $r \geq 2$ by (3.3.6) of Lemma 3.3 again. Thus the spectral sequence (3.3) degenerates at E_2-terms. Moreover,

$$\operatorname{Gr}_m^W H^n(\operatorname{Gr}_F^a K_{\mathcal{O}}) \simeq E_2^{-m,n+m}(\operatorname{Gr}_F^a K_{\mathcal{O}}, W) \simeq \operatorname{Gr}_F^a({}^{\ell}E_2^{-m,n+m}(K_{\mathcal{O}}, W)|_{X^*})$$

is locally free of finite rank for every m, n. □

Theorem 3.9. *Let (X, Σ) be as in 3.5 and $K = ((K_{\mathbb{R}}, W), (K_{\mathcal{O}}, W, F), \alpha)$ as in 3.7 satisfying the conditions* (3.7.1)–(3.7.3)*. If there exists an isomorphism*

$$H^n(\operatorname{Gr}_m^W K_{\mathcal{O}}) \simeq {}^{\ell}H^n(\operatorname{Gr}_m^W K_{\mathcal{O}})|_{X^*} \tag{3.4}$$

whose restriction to X^* is the identity for every m, n, then we have the following:

(3.9.1) *There exist isomorphisms*

$$H^n(K_{\mathcal{O}}) \simeq {}^{\ell}H^n(K_{\mathcal{O}})|_{X^*},$$
$$W_m H^n(K_{\mathcal{O}}) \simeq {}^{\ell}W_m H^n(K_{\mathcal{O}})|_{X^*},$$

whose restriction to X^* coincide with the identities, for every $m, n \in \mathbb{Z}$. In particular, $\operatorname{Gr}_m^W H^n(K_{\mathcal{O}})$ is locally free of finite rank on X for every m, n.

(3.9.2) *The spectral sequence associated to $(K_{\mathcal{O}}, W)$ degenerates at E_2-terms on X.*

If we further assume that $H^n(\operatorname{Gr}_F^a \operatorname{Gr}_m^W K_{\mathcal{O}})$ is locally free of finite rank for every a, m, n and that $K_{\mathcal{O}}$ satisfies (3.2.2) *on the whole X, then we have the following:*

(3.9.3) *The spectral sequence associated to $(K_{\mathcal{O}}, F)$ degenerates at E_1-terms.*

(3.9.4) $\operatorname{Gr}_F^a \operatorname{Gr}_m^W H^n(K_{\mathcal{O}})$ *is a locally free $\mathcal{O}_X$-module of finite rank for every a, m, n.*

(3.9.5) *The spectral sequence associated to $(\operatorname{Gr}_F^a K_{\mathcal{O}}, W)$ degenerates at E_2-terms for every a.*

Proof. We use the same notation as in the proof of Lemma 3.8.

From (3.4), we have

$$E_1^{p,q}(K_{\mathcal{O}}, W) \simeq {}^{\ell}E_1^{p,q}(K_{\mathcal{O}}, W)|_{X^*} \tag{3.5}$$

for every p, q. In particular, $E_1^{p,q}(K_{\mathcal{O}}, W)$ is locally free of finite rank. Then, by Lemma 3.6, we have $d_1^{p,q} = {}^{\ell}d_1^{p,q}|_{X^*}$ under the isomorphism (3.5) because $({}^{\ell}d_1^{p,q}|_{X^*})|_{X^*} = d_1^{p,q}|_{X^*}$. Therefore

$$E_2^{p,q}(K_{\mathcal{O}}, W) \simeq {}^{\ell}E_2^{p,q}(K_{\mathcal{O}}, W)|_{X^*} \tag{3.6}$$

by (3.2). In particular, $E_2^{p,q}(K_{\mathcal{O}}, W)$ is locally free of finite rank for every p, q. Lemma 3.6 implies $d_2^{p,q} = 0$ because $d_2^{p,q}|_{X^*} = 0$ by (3.3.3) of Lemma 3.3. Inductively, $E_r^{p,q}(K_{\mathcal{O}}, W)$ is locally free of finite rank and $d_r^{p,q} = 0$ for $r \geq 2$. Thus we obtain (3.9.2). We have

$$\operatorname{Gr}_m^W H^n(K_{\mathcal{O}}) \simeq E_2^{-m,n+m}(K_{\mathcal{O}}, W) \simeq {}^{\ell}E_2^{-m,n+m}(K_{\mathcal{O}}, W)|_{X^*} \simeq {}^{\ell}(\operatorname{Gr}_m^W H^n(K_{\mathcal{O}})|_{X^*}),$$

from which we obtain (3.9.1).

Next, we will prove the latter half of the theorem. By the assumption (3.2.2) on the whole X, the sequence of the canonical morphisms

$$0 \to H^n(F^{a+1} \operatorname{Gr}_m^W K_{\mathcal{O}}) \to H^n(F^a \operatorname{Gr}_m^W K_{\mathcal{O}}) \to H^n(\operatorname{Gr}_F^a \operatorname{Gr}_m^W K_{\mathcal{O}}) \to 0$$

is exact for every a, m, n. Then we have

$$\operatorname{Gr}_F^a H^n(\operatorname{Gr}_m^W K_{\mathcal{O}}) \simeq H^n(\operatorname{Gr}_F^a \operatorname{Gr}_m^W K_{\mathcal{O}})$$

for every a, m, n, from which $\operatorname{Gr}_F^a H^n(\operatorname{Gr}_m^W K_{\mathcal{O}})$ is locally free of finite rank by the assumption. Thus $F^a H^n(\operatorname{Gr}_m^W K_{\mathcal{O}})$ turns out to be a subbundle for every a. Since the extension of the filtration F on $E_1^{p,q}(K_{\mathcal{O}}, W)|_{X^*}$ is unique by [FF1, Corollary 5.2], the isomorphism (3.4) induces an isomorphism

$$(H^n(\operatorname{Gr}_m^W K_{\mathcal{O}}), F) \simeq ({}^{\ell}H^n(\operatorname{Gr}_m^W K_{\mathcal{O}})|_{X^*}, F) \tag{3.7}$$

of filtered $\mathcal{O}_X$-modules. In particular, we have isomorphisms

$$H^n(\operatorname{Gr}_F^a \operatorname{Gr}_m^W K_{\mathcal{O}}) \simeq \operatorname{Gr}_F^a H^n(\operatorname{Gr}_m^W K_{\mathcal{O}}) \simeq \operatorname{Gr}_F^a({}^{\ell}H^n(\operatorname{Gr}_m^W K_{\mathcal{O}})|_{X^*})$$

whose restriction to X^* coincides with the natural isomorphism (3.1). Thus we obtain (3.9.5) by Lemma 3.8. From (3.7), it turns out that (3.5) is, in fact, an isomorphism of filtered $\mathcal{O}_X$-modules

$$(E_1^{p,q}(K_{\mathcal{O}}, W), F) \simeq ({}^{\ell}E_1^{p,q}(K_{\mathcal{O}}, W)|_{X^*}, F), \tag{3.8}$$

under which $d_1^{p,q} = {}^{\ell}d_1^{p,q}|_{X^*}$. Therefore $d_1^{p,q}$ is strictly compatible with F by the direct sum decomposition (3.2). The assumption (3.2.2) implies that

$$d_0^{p,q} \colon E_0^{p,q}(K_{\mathcal{O}}, W) \to E_0^{p,q+1}(K_{\mathcal{O}}, W)$$

is strictly compatible with the filtration F. By (3.9.2), we already have $d_r^{p,q} = 0$ for all $r \geq 2$. Thus the morphism $d_r^{p,q}$ is strictly compatible with the filtration F for every p, q, r. Hence we conclude (3.9.3) by the lemma on two filtrations (see e.g. [D3, Proposition (7.2.8)], [PS, Theorem 3.12, 3)]).

From (3.8) and the direct sum decomposition (3.2), the isomorphism (3.6) induces an isomorphism of filtered $\mathcal{O}_X$-modules

$$(E_2^{p,q}(K_{\mathcal{O}}, W), F_{\mathrm{rec}}) \simeq ({}^{\ell}E_2^{p,q}(K_{\mathcal{O}}, W)|_{X^*}, F)$$

where F_{rec} denotes the inductive filtration (la filtration récurrente in [D2, (1.3.11)]). On the other hand, $F = F_{\mathrm{rec}}$ on $E_2^{p,q}(K_{\mathcal{O}}, W)$ by the lemma on two filtrations again (see e.g. [D3, Proposition (7.2.5)], [PS, Theorem 3.12, 1)]. Thus we have

$$\operatorname{Gr}_F^a E_2^{p,q}(K_{\mathcal{O}}, W) \simeq \operatorname{Gr}_{F_{\mathrm{rec}}}^a E_2^{p,q}(K_{\mathcal{O}}, W) \simeq \operatorname{Gr}_F^a({}^{\ell}E_2^{p,q}(K_{\mathcal{O}}, W)|_{X^*})$$

for every a, p, q. In particular

$$\operatorname{Gr}_F^a E_2^{p,q}(K_{\mathcal{O}}, W)$$

is locally free of finite rank for every a, p, q. Moreover, the filtration F on $E_2^{p,q}(K_{\mathcal{O}}, W)$ coincides with the filtration F on $\operatorname{Gr}_{-p}^W H^{p+q}(K_{\mathcal{O}})$ under the isomorphism

$$E_2^{p,q}(K_{\mathcal{O}}, W) \simeq \operatorname{Gr}_{-p}^W H^{p+q}(K_{\mathcal{O}})$$

by the lemma on two filtrations (see e.g. [PS, Theorem 3.12, 2)]). Thus we obtain (3.9.4) from

$$\operatorname{Gr}_F^a E_2^{-m,n+m}(K_{\mathcal{O}}) \simeq \operatorname{Gr}_F^a \operatorname{Gr}_m^W H^n(K_{\mathcal{O}})$$

for every a, m, n. □

4. Proofs of Theorems 1.1, 1.3, and 1.4

In this section, we will prove Theorems 1.1, 1.3, and 1.4. Our approach to Theorem 1.1 (ii)–(iv) here is different from [FF1] (see also [Fn5, Section 13]) because we do not assume that (X, D) is projective over Y in this section. We use the terminologies in [FF1, Section 4].

4.1. First, we briefly recall several constructions and results in [FF1, Section 4], which are necessary for the proofs of Theorems 1.1 and 1.4.

Let $f\colon (X, D) \to Y$ be as in Theorems 1.1 and 1.4. Let

$$X = \bigcup_{i\in I} X_i \quad \text{and} \quad D = \bigcup_{\lambda\in\Lambda} D_\lambda$$

be the irreducible decompositions of X and D, respectively. Fixing orders $<$ on Λ and I, we put

$$D_k \cap X_l = \coprod_{\substack{\lambda_0<\lambda_1<\cdots<\lambda_k \\ i_0<i_1<\cdots<i_l}} D_{\lambda_0} \cap D_{\lambda_1} \cap \cdots \cap D_{\lambda_k} \cap X_{i_0} \cap X_{i_1} \cap \cdots \cap X_{i_l}$$

for $k, l \geq 0$ (see [FF1, 4.14]). Here we use the convention

$$D_k = D_k \cap X_{-1} = \coprod_{\lambda_0<\lambda_1<\cdots<\lambda_k} D_{\lambda_0} \cap D_{\lambda_1} \cap \cdots \cap D_{\lambda_k}$$

$$X_l = D_{-1} \cap X_l = \coprod_{i_0<i_1<\cdots<i_l} X_{i_0} \cap X_{i_1} \cap \cdots \cap X_{i_l}$$

for $k, l \geq 0$. By setting

$$(X, D)_n := (D \cap X)_n \setminus D_n = \coprod_{\substack{k+l+1=n \\ l\geq 0}} D_k \cap X_l,$$

we obtain an augmented semisimplicial variety $\varepsilon\colon (X, D)_\bullet \to X$. Note that $(X, D)_n$ is the disjoint union of all the strata of (X, D) of dimension $\dim X - n$ for all $n \in \mathbb{Z}_{\geq 0}$. We set $f_n := f\varepsilon_n\colon (X, D)_n \to Y$ for every n. Then f_n is smooth over $Y^* = Y \setminus \Sigma$. Then the complex $\varepsilon_*\mathbb{R}_{(X,D)_\bullet}$ is given by

$$(\varepsilon_*\mathbb{R}_{(X,D)_\bullet})^n = (\varepsilon_n)_*\mathbb{R}_{(X,D)_n} = \bigoplus_{l\geq 0} \mathbb{R}_{D_{n-l-1}\cap X_l}$$

with the Čech type morphism δ as the differential. Note that this complex is the single complex associated to the double complex obtained by deleting the first vertical column of the double complex in [FF1, p.626, 4.14], and by replacing $\mathbb{Q}$ with $\mathbb{R}$. Then we have quasi-isomorphisms

$$i_!\mathbb{R}_{X\setminus D} \xrightarrow{\simeq} (\mathbb{R}_X \to \mathbb{R}_{D_0} \xrightarrow{\delta} \mathbb{R}_{D_1} \xrightarrow{\delta} \cdots) \xrightarrow{\simeq} \varepsilon_*\mathbb{R}_{(X,D)_\bullet}$$

from the double complex in [FF1] mentioned above, where i denotes the open immersion $X \setminus D \hookrightarrow X$. By setting

$$L_m(\varepsilon_*\mathbb{R}_{(X,D)_\bullet})^n = \begin{cases} 0 & n < -m \\ (\varepsilon_n)_*\mathbb{R}_{(X,D)_n} & n \geq -m \end{cases}$$

a finite increasing filtration L is defined on $\varepsilon_*\mathbb{R}_{(X,D)_\bullet}$. We have the relative de Rham complex $\Omega_{(X,D)_\bullet/Y}$ for the morphism $f\varepsilon\colon (X,D)_\bullet \to Y$. Then the complex $\varepsilon_*\Omega_{(X,D)_\bullet/Y}$ is given by

$$(\varepsilon_*\Omega_{(X,D)_\bullet/Y})^n = \bigoplus_{k\geq 0}(\varepsilon_k)_*\Omega^{n-k}_{(X,D)_k/Y}$$

with the differential $\delta+(-1)^k d$ on $(\varepsilon_k)_*\Omega^{n-k}_{(X,D)_k/Y}$, where δ denotes the Čech type morphism for $(X,D)_\bullet$ and d denotes the differential of the relative de Rham complex $\Omega_{(X,D)_n/Y}$. By setting

$$L_m(\varepsilon_*\Omega_{(X,D)_\bullet/Y})^n = \bigoplus_{k\geq -m}(\varepsilon_k)_*\Omega^{n-k}_{(X,D)_k/Y}$$
$$F^p(\varepsilon_*\Omega_{(X,D)_\bullet/Y})^n = \bigoplus_{0\leq k\leq n-p}(\varepsilon_k)_*\Omega^{n-k}_{(X,D)_k/Y},$$

a finite increasing filtration L and a finite decreasing filtration F on $\varepsilon_*\Omega_{(X,D)_\bullet/Y}$ are defined. The canonical morphism $\mathbb{R}_{(X,D)_n} \to \mathcal{O}_{(X,D)_n}$ induces a morphism of complexes $\iota\colon \varepsilon_*\mathbb{R}_{(X,D)_\bullet} \to \varepsilon_*\Omega_{(X,D)_\bullet/Y}$.

By setting

$$\begin{aligned} K &= ((K_\mathbb{R}, L), (K_\mathcal{O}, L, F), \alpha) \\ &= ((Rf_*\varepsilon_*\mathbb{R}_{(X,D)_\bullet}, L)|_{Y^*}, (Rf_*\varepsilon_*\Omega_{(X,D)_\bullet/Y}, L, F), Rf_*\iota|_{Y^*}) \end{aligned} \tag{4.1}$$

(see [FF1, 4.1]), we obtain a triple K as in 3.7 satisfying the following:

(4.1.1) There exists a quasi-isomorphism $R(f|_{X^*\setminus D^*})_!\mathbb{R}_{X^*\setminus D^*} \simeq K_\mathbb{R}$.

(4.1.2) There exists a quasi-isomorphism $\mathrm{Gr}^p_F K_\mathcal{O} \simeq Rf_*\varepsilon_*\Omega^p_{(X,D)_\bullet/Y}[-p]$. for every p. In particular, $Rf_*\mathcal{O}_X(-D) \simeq \mathrm{Gr}^0_F K_\mathcal{O}$.

(4.1.3) For every $m \in \mathbb{Z}$,

$$\begin{aligned} \mathrm{Gr}^L_m K &= (\mathrm{Gr}^L_m K_\mathbb{R}, (\mathrm{Gr}^L_m K_\mathcal{O}, F), \mathrm{Gr}^L_m \alpha) \\ &\simeq \bigoplus_S (R(f_{S^*})_*\mathbb{R}_{S^*}[m], (R(f_S)_*\Omega_{S/Y}[m], F), R(f_{S^*})_*\iota_{S^*}[m]), \end{aligned}$$

where S runs through all $(\dim X + m)$-dimensional strata of (X,D), $f_S = f|_S$, $S^* = f_S^{-1}(Y^*)$, $f_{S^*} = (f_S)|_{S^*}$, and ι_{S^*} is the composite $\mathbb{R}_{S^*} \hookrightarrow \mathbb{C}_{S^*} \to \Omega_{S^*/Y^*}$.

First, we will prove Theorem 1.4.

Proof of Theorem 1.4. We use the notations and terminologies in 4.1. We will prove that the spectral sequence

$$E_r^{p,q}(\mathrm{Gr}^0_F K_\mathcal{O}, L) \Rightarrow H^{p+q}(\mathrm{Gr}^0_F K_\mathcal{O})$$

associated to the filtered complex $(\mathrm{Gr}^0_F K_\mathcal{O}, L)$ satisfies the desired properties.

By (4.1.2), we have an isomorphism

$$H^{p+q}(\mathrm{Gr}^0_F K_\mathcal{O}) \simeq R^{p+q}f_*\mathcal{O}_X(-D)$$

as desired.

By (4.1.3), K satisfies (3.7.1)–(3.7.3) because $f_S\colon S \to Y$ is smooth over Y^* for every stratum S of (X, D). Moreover, we have

$$H^n(\mathrm{Gr}_F^0 \mathrm{Gr}_m^L K_{\mathcal{O}}) \simeq \bigoplus_S R^{n+m}(f_S)_*\mathcal{O}_S$$

where S runs through all $(\dim X + m)$-dimensional strata of (X, D). Therefore K satisfies the assumption of Lemma 3.8 by the dual of Theorem 2.13. Thus we obtain the conclusion by Lemma 3.8. □

Next, we will prove Theorem 1.1 (i).

Proof of Theorem 1.1 (i). As already mentioned above, K satisfies the conditions (3.7.1)–(3.7.3). By applying Lemma 3.3 together with (3.7.4)–(3.7.6), the triple

$$(\mathrm{Gr}_m^L H^k(K_{\mathbb{R}}), (\mathrm{Gr}_m^L H^k(K_{\mathcal{O}}), F)|_{Y^*}, \mathrm{Gr}_m^L H^k(\alpha))$$

is a polarizable variation of $\mathbb{R}$-Hodge structure of weight $k+m$ for every k, m. By (4.1.1), we have $R^k(f|_{X^*\setminus D^*})_!\mathbb{R}_{X^*\setminus D^*} \simeq H^k(K_{\mathbb{R}})$, which implies $\mathcal{V}_{Y^*}^k \simeq H^k(K_{\mathcal{O}})|_{Y^*}$ as in (3.7.5) for all k. By using these isomorphism, we introduce filtrations L on $R^k(f|_{X^*\setminus D^*})_!\mathbb{R}_{X^*\setminus D^*}$ and $\mathcal{V}_{Y^*}^k$, F on $\mathcal{V}_{Y^*}^k$, and then obtain a polarizable variation of $\mathbb{R}$-Hodge structure

$$(\mathrm{Gr}_m^{L[k]} R^k(f|_{X^*\setminus D^*})_!\mathbb{R}_{X^*\setminus D^*}, (\mathrm{Gr}_m^{L[k]} \mathcal{V}_{Y^*}^k, F), \mathrm{Gr}_m^{L[k]} \beta)$$

of weight m on Y^* for every k, m, where the natural morphism $R^k(f|_{X^*\setminus D^*})_!\mathbb{R}_{X^*\setminus D^*} \to \mathcal{V}_{Y^*}^k = R^k(f|_{X^*\setminus D^*})_!\mathbb{R}_{X^*\setminus D^*} \otimes \mathcal{O}_{Y^*}$ is denoted by β.

On the other hand, the Griffiths transversality for

$$(\mathcal{V}_{Y^*}^k, F) \simeq (H^k(K_{\mathcal{O}}), F)|_{Y^*} \simeq (R^k f_* \varepsilon_* \Omega_{(X,D)_\bullet/Y}, F)|_{Y^*}$$

can be easily seen by the same way as in the proof of Lemma 4.5 of [FF1]. Therefore

$$((R^k(f|_{X^*\setminus D^*})_!\mathbb{R}_{X^*\setminus D^*}, L[k]), (\mathcal{V}_{Y^*}^k, L[k], F), \beta)$$

is a graded polarizable variation of $\mathbb{R}$-mixed Hodge structure on Y^* as desired. □

In order to prove Theorem 1.1 (ii)–(iv), we recall results in [St1] and [St2] in a slightly generalized form.

Definition 4.2. Let $f\colon X \to Y$ be a surjective morphism of smooth complex varieties and Σ a simple normal crossing divisor on Y. We assume that $E = (f^*\Sigma)_{\mathrm{red}}$ is a simple normal crossing divisor on X. For such f, we set

$$\Omega_{X/Y}^1(\log E) = \mathrm{Coker}(f^*\Omega_Y^1(\log \Sigma) \to \Omega_X^1(\log E))$$

and

$$\Omega_{X/Y}^p(\log E) = \bigwedge^p \Omega_{X/Y}^1(\log E)$$

for every p. An $f^{-1}\mathcal{O}_Y$-differential $d\colon \Omega_{X/Y}^p(\log E) \to \Omega_{X/Y}^{p+1}(\log E)$ can be uniquely defined by the commutative diagram

$$\begin{array}{ccc} \Omega_X^p(\log E) & \longrightarrow & \Omega_{X/Y}^p(\log E) \\ \downarrow d & & \downarrow d \\ \Omega_X^{p+1}(\log E) & \longrightarrow & \Omega_{X/Y}^{p+1}(\log E), \end{array}$$

where the horizontal arrows are the canonical surjections induced from the surjection $\Omega^1_X(\log E) \to \Omega^1_{X/Y}(\log E)$. Thus we obtain a complex of $f^{-1}\mathcal{O}_Y$-modules $\Omega_{X/Y}(\log E)$, which is called the relative log de Rham complex of f.

Lemma 4.3. *Let $f\colon X \to Y$ be a proper surjective morphism from a Kähler manifold X to a smooth complex variety Y. Assume that there exists a smooth divisor Σ on Y such that*

(4.3.1) *f is smooth over $Y^* = Y \setminus \Sigma$,*

(4.3.2) *$E = (f^*\Sigma)_{\mathrm{red}}$ is a simple normal crossing divisor on X having finitely many irreducible components, and*

(4.3.3) *$\Omega^1_{X/Y}(\log E)$ is a locally free $\mathcal{O}_X$-module of finite rank.*

Then we have

$$R^if_*\Omega_{X/Y}(\log E) \simeq {}^l(R^if_*\Omega_{X/Y}(\log E)|_{Y^*}) \simeq {}^l(\mathcal{O}_{Y^*} \otimes (R^if_*\mathbb{C}_X)|_{Y^*})$$

for all i, where ${}^l(\cdot)$ stands for the lower canonical extension as before. In particular, $R^if_*\Omega_{X/Y}(\log E)$ is a locally free $\mathcal{O}_Y$-module of finite rank for all i. Moreover, $R^if_*\Omega^p_{X/Y}(\log E)$ is also a locally free $\mathcal{O}_Y$-module of finite rank, and the stupid filtration (filtration bête in [D2, (1.4.7)]*) F on $\Omega_{X/Y}(\log E)$ induces the natural exact sequence*

$$0 \to R^if_*F^{p+1}\Omega_{X/Y}(\log E) \to R^if_*F^p\Omega_{X/Y}(\log E) \to R^if_*\Omega^p_{X/Y}(\log E) \to 0 \tag{4.2}$$

for all i, p.

Proof. We may assume $Y = \Delta^k$ with the coordinates $t_1, \dots, t_k$ and $\Sigma = \{t_1 = 0\}$. For any $x \in E$, we can take local coordinates $x_1, \dots, x_n$ centered at x on X with

$$f^*t_1 = x_1^{a_1} \cdots x_l^{a_l}$$

for some $a_1, \dots, a_l \in \mathbb{Z}_{>0}$ by (4.3.2). We set $f_i = f^*t_i$ for $i = 2, \dots, k$. On the other hand, we have the canonical exact sequence

$$0 \to f^*\Omega^1_Y(\log \Sigma)_x \otimes \mathbb{C}(x) \to \Omega^1_X(\log E)_x \otimes \mathbb{C}(x) \to \Omega^1_{X/Y}(\log E)_x \otimes \mathbb{C}(x) \to 0, \tag{4.3}$$

where $\mathbb{C}(x)$ denotes the residue field at x, because $\Omega^1_{X/Y}(\log E)$ is a locally free $\mathcal{O}_X$-module of rank $\dim X - \dim Y$ by (4.3.3). Under the isomorphisms

$$\Omega^1_Y(\log \Sigma) \simeq \mathcal{O}_Y \frac{dt_1}{t_1} \oplus (\bigoplus_{i=2}^k \mathcal{O}_Y dt_i),$$

$$\Omega^1_X(\log E) \simeq (\bigoplus_{i=1}^l \mathcal{O}_X \frac{dx_i}{x_i}) \oplus (\bigoplus_{i=l+1}^n \mathcal{O}_X dx_i)$$

the morphism $f^*\Omega^1_Y(\log \Sigma)_x \otimes \mathbb{C}(x) \to \Omega^1_X(\log E)_x \otimes \mathbb{C}(x)$ is represented by the matrix

$$\left(\begin{array}{ccc|ccc} a_1 & \dots & a_l & 0 & \dots & 0 \\ \hline 0 & \dots & 0 & & & \\ \vdots & \ddots & \vdots & & \dfrac{\partial f_i}{\partial x_j}(0) & \\ 0 & \dots & 0 & & & \end{array}\right) \tag{4.4}$$

where i and j run through $2, \dots, k$ and $l+1, \dots, n$ respectively. The exactness of (4.3) implies that the matrix (4.4) is of rank k, and then we may assume

$$\operatorname{rank}\left(\frac{\partial f_i}{\partial x_j}(0)\right)_{2 \le i \le k, l+1 \le j \le l+k-1} = k-1$$

by changing the order of $x_{l+1}, \dots, x_n$. Replacing $x_{l+1}, \dots, x_{l+k-1}$ by $f_2, \dots, f_k$, we obtain a new local coordinates $(x_1, \dots, x_n)$ at x, under which the morphism f is given in the form

$$t_1 = x_1^{a_1} \cdots x_l^{a_l}, t_2 = x_{l+1}, \dots, t_k = x_{l+k-1} \tag{4.5}$$

around x. We set $f_s \colon X_s \to \Delta = \Delta \times \{s\}$ by the Cartesian square

$$\begin{array}{ccc} X_s & \longrightarrow & X \\ {\scriptstyle f_s}\downarrow & & \downarrow \\ \Delta & \longrightarrow & Y \end{array}$$

for any $s = (t_2, \dots, t_k) \in \Delta^{k-1}$. Then X_s is smooth, f_s is smooth over $\Delta^* = \Delta \setminus \{0\}$ and $\operatorname{Supp} f_s^{-1}(0)$ is a simple normal crossing divisor on X_s by the local description (4.5). Hence $R^i(f_s)_*\Omega_{X_s/\Delta}(\log(E \cap X_s))$ and $R^i(f_s)_*\Omega^p_{X_s/\Delta}(\log(E \cap X_s))$ are locally free of finite rank for every i, p by [St1, (2.18) Theorem] and by [St2, (2.11) Theorem]. Therefore $R^i f_*\Omega_{X/Y}(\log E)$ and $R^i f_*\Omega^p_{X/Y}(\log E)$ are locally free $\mathcal{O}_Y$-modules of finite rank for all i, p by the base change theorem. Once we know that $R^i f_*\Omega_{X/Y}(\log E)$ is locally free, it is the lower canonical extension of its restriction to $Y^* = Y \setminus \Sigma$ by [St1, (2.20) Proposition].

Next, we consider the spectral sequence

$$E_r^{p,q}(Rf_*\Omega_{X/Y}(\log E), F) \Rightarrow E^{p+q}(Rf_*\Omega_{X/Y}(\log E)) = R^{p+q} f_*\Omega_{X/Y}(\log E) \tag{4.6}$$

and denote the morphism of E_r-terms by

$$d_r^{p,q} \colon E_r^{p,q}(Rf_*\Omega_{X/Y}(\log E), F) \to E_r^{p+r,q-r+1}(Rf_*\Omega_{X/Y}(\log E), F)$$

for a while. Then $d_r^{p,q}|_{Y^*} = 0$ for all p, q and $r \ge 1$ because the restriction of this spectral sequence to Y^* degenerates at E_1-terms. Since

$$E_1^{p,q}(Rf_*\Omega_{X/Y}(\log E), F) \simeq R^q f_*\Omega^p_{X/Y}(\log E)$$

is a locally free $\mathcal{O}_Y$-module of finite rank for all p, q, we have $d_1^{p,q} = 0$ for all p, q by Lemma 3.6. This implies that

$$E_2^{p,q}(Rf_*\Omega_{X/Y}(\log E), F) \simeq E_1^{p,q}(Rf_*\Omega_{X/Y}(\log E), F)$$

is locally free for all p, q and that $d_2^{p,q} = 0$ for all p, q by Lemma 3.6 again. Inductively, we obtain $d_r^{p,q} = 0$ for all p, q and $r \ge 1$. Thus the spectral sequence (4.6) degenerates at E_1-terms, or equivalently, (4.2) is exact. □

Remark 4.4. In [St2], f_s is assumed to be a projective morphism. However, we can check that the proof of (2.11) Theorem in [St2] is also valid to a proper morphism from a Kähler manifold by using results in [PS, I.2.5 Almost Kähler V-manifolds]. See also Theorem 7.9 below.

Corollary 4.5. *In the situation of Lemma 4.3, we have the canonical isomorphisms*

$$R^i f_* F^p\Omega_{X/Y}(\log E) \simeq F^p R^i f_*\Omega_{X/Y}(\log E),$$
$$R^i f_*\Omega^p_{X/Y}(\log E) \simeq \operatorname{Gr}_F^p R^i f_*\Omega_{X/Y}(\log E)$$

for all i,p. In particular, $F^pR^if_*\Omega_{X/Y}(\log E)$ is a subbundle of $R^if_*\Omega_{X/Y}(\log E)$.

Lemma 4.6. *Let $f\colon X\to Y$ be a proper surjective morphism between smooth complex varieties. Assume that there exists a smooth divisor Σ such that*

- *f is smooth over $Y^*=Y\setminus\Sigma$, and*
- *$E=(f^*\Sigma)_{\mathrm{red}}$ is a simple normal crossing divisor on X having finitely many irreducible components.*

Then there exists a closed analytic subset $\Sigma_0\subset\Sigma$ with $\dim\Sigma_0\le\dim Y-2$, such that $\Omega^1_{X/Y}(\log E)$ is locally free on $f^{-1}(Y\setminus\Sigma_0)$.

Proof. We may assume that Σ is irreducible. Let $E=\sum_{i=1}^N E_i$ be the irreducible decomposition of E. For a nonempty subset $I\subset\{1,\dots,N\}$, we set $E_I=\bigcap_{i\in I}E_i$, which is a smooth closed subvariety of X. If $f(E_I)\neq\Sigma$, we set $\Sigma_I=f(E_I)$, which is a closed analytic subset of Σ. If $f(E_I)=\Sigma$, then there exists a closed analytic subset $\Sigma_I\subsetneq\Sigma$ such that $f|_{E_I}\colon E_I\to\Sigma$ is smooth over $\Sigma\setminus\Sigma_I$. We are going to check that the closed analytic subset

$$\Sigma_0:=\bigcup_{\emptyset\neq I\subset\{1,\dots,N\}}\Sigma_I$$

satisfies the desired property. We have $\Sigma_0\neq\Sigma$, by definition. Therefore $\dim\Sigma_0\le\dim Y-2$ because Σ is irreducible. Then, it suffices to prove that $\Omega^1_{X/Y}(\log E)$ is locally free on $f^{-1}(Y\setminus\Sigma_0)$. A point $x\in E\cap f^{-1}(Y\setminus\Sigma_0)$ defines a nonempty subset $I\subset\{1,\dots,N\}$ by $I=\{i\mid x\in E_i\}$. Then $x\in E_I$ and $f(E_I)=\Sigma$. Take local coordinates $x_1,\dots,x_n$ and $t_1,\dots,t_k$ centered at x and $f(x)$ on X and Y respectively, satisfying the following conditions:

- $\Sigma=\{t_1=0\}$ on Y, and
- $f^*t_1=x_1^{a_1}\cdots x_l^{a_l}$ for some $a_1,\dots,a_l\in\mathbb{Z}_{>0}$.

We set $f_i=f^*t_i$ for $i=2,\dots,k$. Then $E_I=\{x_1=\cdots=x_l=0\}$ and the morphism $(f|_{E_I})^*\Omega^1_\Sigma\to\Omega^1_{E_I}$ is represented by the matrix

$$\left(\frac{\partial f_i}{\partial x_j}(0,\dots,0,x_{l+1},\dots,x_n)\right)_{2\le i\le k,\,l+1\le j\le n}$$

via the isomorphisms $(f|_{E_I})^*\Omega^1_\Sigma\simeq\bigoplus_{j=2}^k\mathcal{O}_{E_I}f^*dt_j$ and $\Omega^1_{E_I}\simeq\bigoplus_{i=l+1}^n\mathcal{O}_{E_I}dx_i$. Since $x\in f^{-1}(\Sigma\setminus\Sigma_I)$, the morphism $f|_{E_I}$ is smooth at x. Then

$$\operatorname{rank}\left(\frac{\partial f_i}{\partial x_j}(0)\right)_{2\le i\le k,\,l+1\le j\le n}=k-1,$$

which implies that the matrix (4.4) in the proof of Lemma 4.3 is of rank k. Therefore the canonical morphism $f^*\Omega^1_Y(\log\Sigma)_x\otimes\mathbb{C}(x)\to\Omega^1_X(\log E)_x\otimes\mathbb{C}(x)$ is injective, by which we conclude that $\Omega^1_{X/Y}(\log E)$ is locally free around x. □

4.7. We return to the situation in 4.1. For the moment, we assume that there exist another semisimplicial variety $Z_\bullet$ and a morphism of semisimplicial varieties $\sigma\colon Z_\bullet\to(X,D)_\bullet$ satisfying the conditions

- Z_n is smooth and Kähler,
- $\sigma_n\colon Z_n\to(X,D)_n$ is a projective surjective morphism,
- for $g_n:=f_n\sigma_n=f\varepsilon_n\sigma_n\colon Z_n\to Y$, the divisor $E_n:=(g_n^*\Sigma)_{\mathrm{red}}$ is a simple normal crossing divisor on Z_n having finitely many irreducible components, and
- $\sigma_n\colon Z_n\to(X,D)_n$ is isomorphic over Y^*

for every $n \in \mathbb{Z}_{\geq 0}$. We obtain an augmentation $\eta\colon Z_\bullet \to X$ by setting $\eta = \varepsilon\sigma$. The relative log de Rham complex of Z_n over Y is denoted by $\Omega_{Z_n/Y}(\log E_n)$. Then $\{\Omega_{Z_n/Y}(\log E_n)\}_{n\in\mathbb{Z}_{\geq 0}}$ forms a complex on the semisimplicial variety $Z_\bullet$.

For an augmentation of a semisimplicial variety, we can define the direct image functor as in [FF1, 4.1, 4.2] (for the detail, see e.g. [D3, 5.1, 5.2], [PS, 5.1.2]). The complex $R\varepsilon_*\Omega_{(X,D)_\bullet}$ is isomorphic to $\varepsilon_*\Omega_{(X,D)_\bullet}$ defined in the proof of Theorem 1.1 (i) in the derived category because $\varepsilon_n\colon (X,D)_n \to X$ is a finite morphism for all n. On the other hand, we obtain a complex $R\eta_*\Omega_{Z_\bullet/Y}(\log E_\bullet)$ on X. Here, we briefly recall the definitions of this complex, of the finite increasing filtration L, and of the finite decreasing filtration F on it. First, the complex $R\eta_*\Omega_{Z_\bullet/Y}(\log E_\bullet)$ is given as the total single complex associated to the double complex

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & (R(\eta_p)_*\Omega_{Z_p/Y}(\log E_p))^q & \xrightarrow{\delta} & (R(\eta_{p+1})_*\Omega_{Z_{p+1}/Y}(\log E_{p+1}))^q & \longrightarrow & \cdots \\
 & & {\scriptstyle (-1)^p d}\downarrow & & \downarrow{\scriptstyle (-1)^{p+1} d} & & \\
\cdots & \longrightarrow & (R(\eta_p)_*\Omega_{Z_p/Y}(\log E_p))^{q+1} & \xrightarrow{\delta} & (R(\eta_{p+1})_*\Omega_{Z_{p+1}/Y}(\log E_{p+1}))^{q+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & &
\end{array}
$$

that is,

$$(R\eta_*\Omega_{Z_\bullet/Y}(\log E_\bullet))^n = \bigoplus_p (R(\eta_p)_*\Omega_{Z_p/Y}(\log E_p))^{n-p},$$

where $R(\eta_p)_*\Omega_{Z_p/Y}(\log E_p)$ is regarded as *a genuine complex* on X by using the Godement resolutions (cf. [FF1, 4.1]). The filtrations L and F are defined by

$$L_m(R\eta_*\Omega_{Z_\bullet/Y}(\log E_\bullet))^n = \bigoplus_{p\geq -m} (R(\eta_p)_*\Omega_{Z_p/Y}(\log E_p))^{n-p},$$
$$F^r(R\eta_*\Omega_{Z_\bullet/Y}(\log E_\bullet))^n = \bigoplus_p F^r(R(\eta_p)_*\Omega_{Z_p/Y}(\log E_p))^{n-p}$$

for all m, n, r. Therefore we have

$$(\mathrm{Gr}_m^L R\eta_*\Omega_{Z_\bullet/Y}(\log E_\bullet), F) \simeq (R(\eta_{-m})_*\Omega_{Z_{-m}/Y}(\log E_{-m})[m], F) \tag{4.7}$$

in the derived category. From the morphism $\sigma\colon Z_\bullet \to (X,D)_\bullet$, we obtain a morphism of bifiltered complexes

$$(\varepsilon_*\Omega_{(X,D)_\bullet/Y}, L, F) \to (R\eta_*\Omega_{Z_\bullet/Y}(\log E_\bullet), L, F), \tag{4.8}$$

which induces a morphism

$$\begin{aligned}\mathrm{Gr}_m^L \mathrm{Gr}_F^0 \varepsilon_*\Omega_{(X,D)_\bullet/Y} &\simeq (\varepsilon_{-m})_*\mathcal{O}_{(X,D)_{-m}} \\ &\to R(\eta_{-m})_*\mathcal{O}_{Z_{-m}} \simeq \mathrm{Gr}_m^L \mathrm{Gr}_F^0 R\eta_*\Omega_{Z_\bullet/Y}(\log E_\bullet)\end{aligned} \tag{4.9}$$

for all m. Because σ_n induces the isomorphism $\mathcal{O}_{(X,D)_n} \xrightarrow{\simeq} R(\sigma_n)_*\mathcal{O}_{Z_n}$ for all n, we have the isomorphisms

$$(\varepsilon_{-m})_*\mathcal{O}_{(X,D)_{-m}} \simeq R(\varepsilon_{-m})_*\mathcal{O}_{(X,D)_{-m}} \simeq R(\varepsilon_{-m})_*R(\sigma_{-m})_*\mathcal{O}_{Z_{-m}} \simeq R(\eta_{-m})_*\mathcal{O}_{Z_{-m}}$$

for all m. Therefore the morphism (4.9) is an isomorphism for all m in the derived category, which implies

$$(\mathrm{Gr}_F^0 \varepsilon_* \Omega_{(X,D)_\bullet/Y}, L) \simeq (\mathrm{Gr}_F^0 R\eta_* \Omega_{Z_\bullet/Y}(\log E_\bullet), L) \tag{4.10}$$

in the filtered derived category.

Now, we complete the proof of Theorem 1.1.

Proof of Theorem 1.1 (ii)–(iv). First, we prove (ii). The uniqueness of the filtration F on ${}^l\mathcal{V}^k_{Y^*}$ follows from [FF1, Corollary 5.2]. Therefore we may work locally on Y. Then after shrinking Y to a relatively compact open subset, we can take $Z_\bullet$ and $\sigma_\bullet \colon Z_\bullet \to (X,D)_\bullet$ in 4.7 by the theorem of resolution of singularities (see [BM, Section 13]). By Lemma 4.6, there exists a closed analytic subset $\Sigma_0 \subset \Sigma$ with $\dim \Sigma_0 \le \dim Y - 2$ such that $\Sigma \setminus \Sigma_0$ is a smooth divisor in $Y \setminus \Sigma_0$, and that $\Omega^1_{Z_n/Y}(\log E_n)$ is locally free over $g_n^{-1}(Y \setminus \Sigma_0)$ for all $n \in \mathbb{Z}_{\ge 0}$. By setting $Y_0 := Y \setminus \Sigma_0$, we trivially have $Y^* \subset Y_0 \subset Y$.

Now we set

$$K(\log)_{\mathcal{O}} = Rf_* R\eta_* \Omega_{Z_\bullet/Y}(\log E_\bullet)$$

equipped with the induced filtrations L and F. Because σ is isomorphic over Y^*, the morphism (4.8) induces an isomorphism

$$(K_{\mathcal{O}}, L, F)|_{Y^*} \xrightarrow{\simeq} (K(\log)_{\mathcal{O}}, L, F)|_{Y^*}, \tag{4.11}$$

and then a morphism of complexes $\beta \colon K_{\mathbb{R}} \to K(\log)_{\mathcal{O}}|_{Y^*}$ is defined by

$$\beta \colon K_{\mathbb{R}} \xrightarrow{\alpha} K_{\mathcal{O}}|_{Y^*} \xrightarrow{\simeq} K(\log)_{\mathcal{O}}|_{Y^*},$$

where $K_{\mathbb{R}}$ and α are given in (4.1). A triple

$$K(\log) = ((K_{\mathbb{R}}, L), (K(\log)_{\mathcal{O}}, L, F), \beta)$$

satisfies the conditions (3.7.1)–(3.7.3) because $K|_{Y^*} \simeq K(\log)|_{Y^*}$ as above. Then the triple on Y_0

$$K(\log)|_{Y_0} = ((K_{\mathbb{R}}, L), (K(\log)_{\mathcal{O}}, L, F)|_{Y_0}, \beta)$$

satisfies all the assumptions in Theorem 3.9 by (4.7) and Lemma 4.3. Applying Theorem 3.9 to $K(\log)|_{Y_0}$, we conclude

$$H^n(K(\log)_{\mathcal{O}})|_{Y_0} \simeq ({}^\ell H^n(K(\log)_{\mathcal{O}})|_{Y^*})|_{Y_0} \simeq ({}^\ell H^n(K_{\mathcal{O}})|_{Y^*})|_{Y_0}$$
$$L_m H^n(K(\log)_{\mathcal{O}})|_{Y_0} \simeq ({}^\ell L_m H^n(K(\log)_{\mathcal{O}})|_{Y^*})|_{Y_0} \simeq ({}^\ell L_m H^n(K_{\mathcal{O}})|_{Y^*})|_{Y_0}$$

from (4.11), and that $\mathrm{Gr}_F^a \mathrm{Gr}_m^L H^n(K(\log)_{\mathcal{O}})|_{Y_0}$ is locally free of finite rank for every a, m, n. By the isomorphism $\mathcal{V}^k_{Y^*} \simeq H^k(K_{\mathcal{O}})|_{Y^*}$ as in the proof of Theorem 1.1 (i) above, we obtain a filtration F on $({}^\ell\mathcal{V}^k_{Y^*})|_{Y_0}$ satisfying the two conditions in Theorem 1.1 (ii) on Y_0. Then Lemma 1.11.2 in [Ka] together with Schmid's nilpotent orbit theorem (see [Sc, (4.12)]) for each $\mathrm{Gr}_m^L \mathcal{V}^k_{Y^*}$ implies the conclusion of Theorem 1.1 (ii) on the whole Y.

Next, we will prove (iii). By (3.9.3) of Theorem 3.9 for $K(\log)|_{Y_0}$, we have

$$H^k(\mathrm{Gr}_F^a K(\log)_{\mathcal{O}})|_{Y_0} \simeq \mathrm{Gr}_F^a H^k(K(\log)_{\mathcal{O}})|_{Y_0} \simeq \mathrm{Gr}_F^a({}^\ell\mathcal{V}^k_{Y^*})|_{Y_0}$$

for every a, k. On the other hand,

$$\begin{aligned} \mathrm{Gr}_F^0 K(\log)_{\mathcal{O}} &= \mathrm{Gr}_F^0 Rf_* R\eta_* \Omega_{Z_\bullet/Y}(\log E_\bullet) \\ &\simeq \mathrm{Gr}_F^0 Rf_* \varepsilon_* \Omega_{(X,D)_\bullet/Y} = \mathrm{Gr}_F^0 K_{\mathcal{O}} \simeq Rf_* \mathcal{O}_X(-D) \end{aligned}$$

by (4.10) and (4.1.2). Thus we obtain an isomorphism

$$R^{d-i}f_*\mathcal{O}_X(-D)|_{Y_0} \simeq H^{d-i}(\mathrm{Gr}_F^0 K(\log)_{\mathcal{O}})|_{Y_0} \simeq \mathrm{Gr}_F^0({}^{\ell}\mathcal{V}_{Y^*}^{d-i})|_{Y_0}$$

for every i. Because $R^{d-i}f_*\mathcal{O}_X(-D)$ is locally free of finite rank by Theorem 1.4, and because $\Sigma_0 = Y \setminus Y_0$ has the codimension at least two, the isomorphism above extends uniquely to the whole Y.

By Grothendieck duality (see [RRV]), we obtain (iv) from (iii). □

Remark 4.8. As already mentioned in Remark 1.2, the local system $R^k(f|_{X^*\setminus D^*})_!\mathbb{R}_{X^*\setminus D^*}$ underlies *an admissible* graded polarizable variation of $\mathbb{R}$-mixed Hodge structure on Y^*. In order to check the admissibility, we may assume that $(Y, \Sigma) = (\Delta, \{0\})$ from the beginning. We may further assume that all the monodromy automorphisms of $R^i(f_S)_*\mathbb{R}_S|_{\Delta^*}$ around the origin are unipotent for all strata S and for all $i \in \mathbb{Z}$ by [Ka, Lemma 1.9.1]. Then the extension of the Hodge filtration has been already given by Theorem 1.1 (ii). On the other hand, the existence of the relative monodromy weight filtration can be proved by the same way as the proof of Lemma 4.10 of [FF1]. Here we remark that the coincidence of the monodromy weight filtration and the weight filtration on the limit mixed Hodge structure in [St1, (5.9) Theorem] holds true for a proper surjective morphism $f\colon X \to \Delta$ from a Kähler manifold X to the unit disc Δ such that f is smooth over Δ^*, that the local system $R^if_*\mathbb{R}|_{\Delta^*}$ is of unipotent monodromy for every i, and that $f^{-1}(0)_{\mathrm{red}}$ is a simple normal crossing divisor (cf. [Sa1, 4.2.5 Remarque], [GN, (5.2) Théorème]).

The following theorem is an easy consequence of the proof of Theorem 1.4. We will use it in the proof of Theorem 1.5.

Theorem 4.9. *In Theorem 1.1, for every i, there exists a finite filtration of locally free sheaves*

$$0 = \mathcal{E}_0^i \subset \mathcal{E}_1^i \subset \cdots \subset \mathcal{E}_{l_i}^i = R^if_*\omega_{X/Y}(D)$$

such that

$$\mathcal{E}_{j+1}^i/\mathcal{E}_j^i$$

is isomorphic to a direct summand of

$$\bigoplus_{\text{finite}} R^\alpha f_*\omega_{S_\beta/Y},$$

where α is a nonnegative integer and S_β is a stratum of (X, D), for every j.

Moreover, if $\pi\colon Y \to Z$ is a projective bimeromorphic morphism of complex varieties, then

$$R^p\pi_*R^if_*\omega_X(D) = 0$$

holds for every $p > 0$. In particular, we have

$$\pi_*R^if_*\omega_X(D) \simeq R^i(\pi\circ f)_*\omega_X(D).$$

Proof. By Theorem 1.4, there exists a finite filtration of locally free sheaves

$$0 = \mathcal{F}_0^{d-i} \subset \mathcal{F}_1^{d-i} \subset \cdots \subset \mathcal{F}_{l_i}^{d-i} = R^{d-i}f_*\mathcal{O}_X(-D)$$

such that

$$\mathcal{F}_{j+1}^{d-i}/\mathcal{F}_j^{d-i}$$

is isomorphic to a direct summand of

$$\bigoplus_{\text{finite}} R^rf_*\mathcal{O}_{S_\beta},$$

where S_β is a stratum of (X, D) and r is a nonnegative integer, for every j. We put

$$\mathcal{E}^i_j := \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}^{d-i}_{l_i}/\mathcal{F}^{d-i}_{l_i-j}, \mathcal{O}_Y)$$

for every j. Then, by Grothendieck duality (see [RRV]), we obtain a desired filtration of $R^if_*\omega_{X/Y}(D)$. By Theorem 2.10, we have $R^p\pi_*R^\alpha f_*\omega_{S_\beta} = 0$ for every $p > 0$. This implies that $R^p\pi_*\left(\mathcal{E}^i_j \otimes \omega_Y\right) = 0$ holds for every $p > 0$ and every j. Thus we obtain $R^p\pi_*R^if_*\omega_X(D) = 0$ for every $p > 0$. Hence we have $\pi_*R^if_*\omega_X(D) \simeq R^i(\pi\circ f)_*\omega_X(D)$. We finish the proof. □

We close this section with the proof of Theorem 1.3.

Proof of Theorem 1.3. This theorem is obvious by Theorem 1.1 (iv) and the Fujita–Zucker–Kawamata semipositivity theorem. For the details of the Fujita–Zucker–Kawamata semipositivity theorem, see, for example, [FF1, Section 5], [FFS, Corollary 2], [FF2], and so on. □

We note that Theorems 1.1 and 1.3 have already played a crucial role when $f\colon (X, D) \to Y$ is algebraic. We recommend that the interested reader looks at [Fn4], [Fn5], [Fn6], [Fn7], [FFL], [FH], and so on.

5. Proof of Theorem 1.5

In this section, we will prove Theorem 1.5 by using Theorem 4.9. In Section 6, we will see that Theorem 1.7 follows from Theorem 1.5.

Proof of Theorem 1.5. In Step 1 and Step 2, we will prove (i) and (ii), respectively.

Step 1. In this step, we will prove (i).

We take an arbitrary point $P \in Y$. It is sufficient to prove (i) around P. By Lemma 2.8, we may assume that (X, D) is an analytic globally embedded simple normal crossing pair and that there exists the following commutative diagram:

$$\begin{array}{ccc} X & \hookrightarrow & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q_M} \\ Y & \underset{\iota_Y}{\hookrightarrow} & \Delta^m, \end{array}$$

where M is the ambient space of (X, D), such that q_M is projective and $\iota_Y(P) = 0 \in \Delta^m$. By taking a suitable resolution of singularities of Y (see [BM, Sections 12 and 13]), there exist a projective bimeromorphic morphism $\psi\colon Y' \to Y$ from a smooth complex variety Y' and a simple normal crossing divisor Σ' on Y' such that every stratum of (X, D) is smooth over $Y \setminus \psi(\Sigma')$. Then, by taking a suitable resolution of singularities of M (see [BM, Sections 12 and 13]) and applying Lemma 2.7, we may assume that

$$f'\colon X \xrightarrow{f} Y \overset{\psi^{-1}}{\dashrightarrow} Y'$$

is a projective morphism. Hence we have the following commutative diagram:

$$\begin{array}{ccc} X & = & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \underset{\psi}{\longrightarrow} & Y \end{array}$$

such that every stratum of (X, D) is smooth over $Y' \setminus \Sigma'$. By Theorem 4.9, $R^q f'_* \omega_{X/Y'}(D)$ is locally free and has a finite filtration as in Theorem 4.9. Since ψ is projective bimeromorphic and $R^q f_* \omega_X(D) \simeq \psi_* R^q f'_* \omega_X(D)$ by Theorem 4.9, $R^q f_* \omega_X(D)$ is torsion-free. This is what we wanted.

Step 2. In this step, we will prove (ii).

We take an arbitrary point $P \in Z$. It is sufficient to prove (ii) around P. As in Step 1, after shrinking Z suitably, by Lemma 2.8, a suitable resolution of singularities (see [BM, Sections 12 and 13]), and Lemma 2.7, we may assume that there exists the following commutative diagram:

$$\begin{array}{ccccc} X & = & X & \hookrightarrow & M \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \\ Y' & \xrightarrow{\psi} & Y & & \downarrow{\scriptstyle q_M} \\ & & \downarrow{\scriptstyle \pi} & & \\ & & Z & \overset{\iota_Z}{\hookrightarrow} & \Delta^m \end{array}$$

such that $\iota_Z(P) = 0 \in \Delta^m$. By Theorem 4.9, we have

$$R^p\pi_* \left(\mathcal{A} \otimes R^q f_* \omega_X(D)\right) \simeq R^p\pi_* \left(\mathcal{A} \otimes \psi_* R^q f'_* \omega_X(D)\right) \simeq R^p(\pi \circ \psi)_* \left(\psi^* \mathcal{A} \otimes R^q f'_* \omega_X(D)\right).$$

By Theorem 4.9 again, $R^q f'_* \omega_{X/Y'}(D)$ has a finite filtration as in Theorem 4.9. Thus we can reduce the problem to the case where X is smooth and $D = 0$. Since $\psi^*\mathcal{A}$ is $(\pi \circ \psi)$-nef and $(\pi \circ \psi)$-big over Z, we get the desired vanishing theorem by Theorem 2.10. We finish the proof of (ii).

We finish the proof of Theorem 1.5. □

Remark 5.1. By the above proof, we see that Theorem 1.5 (ii) holds under a weaker assumption that $\mathcal{A}$ is π-nef and π-big over Z (see Theorem 2.10).

6. Proof of Theorem 1.7

In this section, we will prove Theorem 1.7 by using Theorem 1.5. As we mentioned before, Theorem 1.7 (iii) is an easy consequence of Theorem 1.7 (i) and (ii).

Proof of Theorem 1.7. In Step 1, we will prove Theorem 1.7 (i). Then, in Steps 2 and 3, we will prove Theorem 1.7 (ii) and (iii), respectively.

Step 1. In this step, we will prove Theorem 1.7 (i).

By replacing Y with $f(X)$, we may assume that $f(X) = Y$. Let $P \in Y$ be an arbitrary point. It is sufficient to prove the statement after shrinking Y around P suitably. By Lemma 2.8, we may assume that (X, D) is an analytic globally embedded simple normal crossing pair and that there exists the following commutative diagram:

$$\begin{array}{ccc} X & \hookrightarrow & M \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle q_M} \\ Y & \overset{\iota_Y}{\hookrightarrow} & \Delta^m, \end{array}$$

where M is the ambient space of (X, D), such that q_M is projective and $\iota_Y(P) = 0 \in \Delta^m$. By using Lemma 2.9 finitely many times, we can decompose $X = X' + X''$ as follows: X'

is the union of all strata of (X, D) that are not mapped onto irreducible components of $Y = f(X)$ and $X'' = X - X'$. We put

$$K_{X'} + D_{X'} := (K_X + D)|_{X'}$$

and

$$K_{X''} + D_{X''} := (K_X + D)|_{X''} - X'|_{X''}.$$

We note that $(X'', D_{X''})$ is an analytic globally embedded simple normal crossing pair such that $D_{X''}$ is reduced and that every stratum of $(X'', D_{X''})$ is mapped onto some irreducible component of Y. We consider the following short exact sequence:

$$0 \to \mathcal{O}_{X''}(K_{X''} + D_{X''}) \to \mathcal{O}_X(K_X + D) \to \mathcal{O}_{X'}(K_{X'} + D_{X'}) \to 0.$$

By Theorem 1.5 (i), every associated subvariety of $R^q f_* \mathcal{O}_{X''}(K_{X''} + D_{X''})$ is an irreducible component of Y for every q. Note that every associated subvariety of $R^q f_* \mathcal{O}_{X'}(K_{X'} + D_{X'})$ is contained in $f(X')$ for every q. Thus, the connecting homomorphisms

$$\delta \colon R^q f_* \mathcal{O}_{X'}(K_{X'} + D_{X'}) \to R^{q+1} f_* \mathcal{O}_{X''}(K_{X''} + D_{X''})$$

are zero for all q. Hence we obtain the following short exact sequence

$$0 \to R^q f_* \mathcal{O}_{X''}(K_{X''} + D_{X''}) \to R^q f_* \mathcal{O}_X(K_X + D) \to R^q f_* \mathcal{O}_{X'}(K_{X'} + D_{X'}) \to 0 \tag{6.1}$$

for every q. By induction on $\dim f(X)$, every associated subvariety of $R^q f_* \mathcal{O}_{X'}(K_{X'} + D_{X'})$ is the f-image of some stratum of $(X', D_{X'})$ for every q. Therefore, every associated subvariety of $R^q f_* \mathcal{O}_X(K_X + D)$ is the f-image of some stratum of (X, D) for every q by (6.1).

Step 2. In this step, we will prove Theorem 1.7 (ii).

We may assume that $f(X) = Y$ and $\pi \circ f(X) = Z$. Let $P \in Z$ be an arbitrary point. It is sufficient to prove the desired vanishing theorem after shrinking Z around P suitably. As in Step 1, by Lemma 2.8, we have the following commutative diagram:

$$\begin{array}{ccc} X & \hookrightarrow & M \\ {\scriptstyle \pi \circ f} \downarrow & & \downarrow {\scriptstyle q_M} \\ Z & \underset{\iota_Z}{\hookrightarrow} & \Delta^m, \end{array}$$

where M is the ambient space of (X, D), such that q_M is projective and $\iota_Z(P) = 0 \in \Delta^m$. By the same argument as in Step 1, we obtain

$$0 \to R^q f_* \mathcal{O}_{X''}(K_{X''} + D_{X''}) \to R^q f_* \mathcal{O}_X(K_X + D) \to R^q f_* \mathcal{O}_{X'}(K_{X'} + D_{X'}) \to 0$$

for every q. By applying Theorem 1.5 (ii) to every connected component of X'', we see that

$$R^p \pi_* \left(\mathcal{A} \otimes R^q f_* \mathcal{O}_{X''}(K_{X''} + D_{X''}) \right) = 0$$

holds for every $p > 0$. By induction on $\dim f(X)$, we obtain

$$R^p \pi_* \left(\mathcal{A} \otimes R^q f_* \mathcal{O}_{X'}(K_{X'} + D_{X'}) \right) = 0$$

for every $p > 0$. This implies

$$R^p \pi_* \left(\mathcal{A} \otimes R^q f_* \mathcal{O}_X(K_X + D) \right) = 0$$

for every $p > 0$. This is what we wanted.

Step 3. In this step, we will prove Theorem 1.7 (iii).

Since we have already proved the strict support condition (see (i)) and the vanishing theorem (see (ii)) in Steps 1 and 2, respectively, the proof of [Fn9, Theorem 3.1 (iii)] works. Hence we obtain the desired injectivity in (iii).

We finish the proof of Theorem 1.7. □

Remark 6.1. Theorem 1.7 (ii) holds under a weaker assumption that $\mathcal{A}$ is nef and log big over Z with respect to $f\colon (X, D) \to Y$. We can easily check it by the above proof of Theorem 1.7 (ii) and Remark 5.1. We do not discuss the details here because we have already known a more general statement, that is, the vanishing theorem of Reid–Fukuda type (see Theorem 1.9).

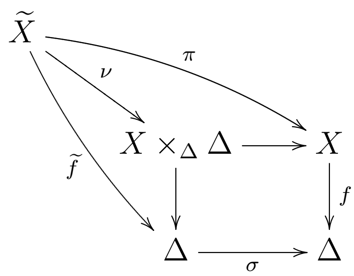
7. Supplement to [St2]

In this section, we give a remark on the construction of the cohomological $\mathbb{Q}$-mixed Hodge complex $((A_{\mathbb{Q}}, W), (A_{\mathbb{C}}, W, F))$ in [St2, p.536]. More precisely, we will present a new construction of $(A_{\mathbb{Q}}, W)$ here. In the context of log geometry, such a construction is originated in [St3] and used in other articles (e.g. [FN], [Fs2] and so on). For the case of a semistable reduction, a new construction of $(A_{\mathbb{Q}}, W)$, which is similar to [St3], is given in [PS, 11.2.6 The Rational Structure]. (For the case of a semistable morphism over the polydisc, see e.g. [Fs1].) Here we will see that the construction in [Fs2] works in the situation of [St2].

7.1. Let $f\colon X \to \Delta$ be a proper surjective morphism from a smooth complex variety X to the unit disc Δ satisfying the conditions

- f is smooth over $\Delta^* = \Delta \setminus \{0\}$, and
- $\operatorname{Supp} f^{-1}(0)$ is a simple normal crossing divisor on X

as in [St2, (2.1) Notations]. Note that $f^{-1}(0)$ is *not* assumed to be reduced. We fix $N \in \mathbb{Z}_{>0}$, which is a multiple of all the multiplicities of the irreducible components of $\operatorname{Supp} f^{-1}(0)$, and consider the morphism $\sigma\colon \Delta \to \Delta$ given by $\sigma(t) = t^N$. We define $\widetilde{X}, \pi$ and $\widetilde{f}$ by the commutative diagram



where ν is the normalization. We set $E = \operatorname{Supp} \widetilde{f}^{-1}(0)$, which is an effective Cartier divisor on $\widetilde{X}$. The irreducible decomposition of E is written in $E = \bigcup_{i=1}^{l} E_i$. The closed immersion $E_i \hookrightarrow \widetilde{X}$ is denoted by a_i.

7.2. We recall the local description of $\widetilde{X}$ and $\widetilde{f}$ given in the proof of [St2, (2.2) Lemma]. For any point of $\widetilde{X}$, there exist an open neighborhood $\widetilde{U}$ in $\widetilde{X}$, $d_1, \ldots, d_k \in \mathbb{Z}_{>0}$ with $\gcd(d_1, \ldots, d_k) = 1$, and $e \in \mathbb{Z}_{>0} \cap (\bigcap_{i=1}^{k} d_i\mathbb{Z})$ with $N \in e\mathbb{Z}$ such that $\widetilde{U}$ and $\widetilde{f}|_{\widetilde{U}}$ are described by using $d_1, \ldots, d_k, e$ as follows. By setting $c_i := e/d_i \in \mathbb{Z}_{>0}$ and $G := \bigoplus_{i=1}^{k} \mathbb{Z}/c_i\mathbb{Z}$, the kernel of the morphism

$$G=\bigoplus_{i=1}^{k}\mathbb{Z}/c_i\mathbb{Z}\ni(b_1,\dots,b_k)\mapsto\sum_{i=1}^{k}d_ib_i\in\mathbb{Z}/e\mathbb{Z}$$

is denoted by H. The finite abelian group G acts on the polydisc Δ^n by

$$(b_1,\dots,b_k)\cdot y_i=\begin{cases}\exp(2\pi\sqrt{-1}b_i/c_i)y_i & \text{for } 1\le i\le k\\ y_i & \text{for } k+1\le i\le n,\end{cases}$$

where $(y_1,\dots,y_n)$ is the coordinate of Δ^n. Then $\widetilde{U}\simeq\Delta^n/H$ and $\widetilde{f}^*t=y_1\cdots y_k$, where t is the coordinate of Δ. Note that $y_1\cdots y_k$ is H-invariant. Moreover, $U=\pi(\widetilde{U})$ is an open subset of X, and we also have $U\simeq\Delta^n/G$ and $f^*t=(y_1\cdots y_k)^N$. Here we note that $(y_1\cdots y_k)^N$ is G-invariant because $N\in e\mathbb{Z}$. The G-invariant functions $y_1^{c_1},\dots,y_k^{c_k},y_{k+1},\dots,y_n$ give us a coordinate on U.

From the local description above, $\widetilde{X}$ is trivially a V-manifold. We can easily see that E_i is a reduced Cartier divisor on $X\setminus\bigcup_{j\neq i}E_j$. Moreover, E_i is locally irreducible at any point because $\pi(E_i)$ is an irreducible component of $\operatorname{Supp}f^{-1}(0)$ and because $\operatorname{Supp}f^{-1}(0)$ is a simple normal crossing divisor on X.

7.3. In the situation 7.1, the log structure on $\widetilde{X}$ associated to the effective divisor E is denoted by $\mathcal{M}$, that is,

$$\mathcal{M}:=\mathcal{O}_{\widetilde{X}}\cap j_*\mathcal{O}^*_{\widetilde{X}\setminus E}$$

in $j_*\mathcal{O}_{\widetilde{X}\setminus E}$, where j denotes the open immersion $\widetilde{X}\setminus E\hookrightarrow\widetilde{X}$. The abelian sheaf associated to the monoid sheaf $\mathcal{M}$ is denoted by $\mathcal{M}^{\mathrm{gp}}$. By using the fact that E_i is locally irreducible, a morphism of monoid sheaves $\mathcal{M}\to(a_i)_*\mathbb{N}_{E_i}$ can be defined by

$$\mathcal{M}=\mathcal{O}_{\widetilde{X}}\cap j_*\mathcal{O}^*_{\widetilde{X}\setminus E}\ni a\mapsto\operatorname{ord}_{E_i}(a)\in(a_i)_*\mathbb{N}_{E_i}\tag{7.1}$$

for any i, where ord_{E_i} denotes the vanishing order of a holomorphic function on $\widetilde{X}$ along the divisor E_i. The direct sum of the morphisms (7.1) for all i induces a morphism

$$\mathcal{M}^{\mathrm{gp}}\to\bigoplus_{i=1}^{l}(a_i)_*\mathbb{Z}_{E_i},\tag{7.2}$$

which fits in an exact sequence

$$0\to\mathcal{O}^*_{\widetilde{X}}\to\mathcal{M}^{\mathrm{gp}}\to\bigoplus_{i=1}^{l}(a_i)_*\mathbb{Z}_{E_i}\tag{7.3}$$

by definition.

The following is a key lemma for the construction of $(A_{\mathbb{Q}},W)$.

Lemma 7.4. *We obtain the exact sequence*

$$0\to\mathcal{O}^*_{\widetilde{X}}\otimes_{\mathbb{Z}}\mathbb{Q}\to\mathcal{M}^{\mathrm{gp}}\otimes_{\mathbb{Z}}\mathbb{Q}\to\bigoplus_{i=1}^{l}(a_i)_*\mathbb{Q}_{E_i}\to 0$$

by tensoring $\mathbb{Q}$ *to* (7.3).

Proof. We may work in the local situation described in 7.2. Since $y_i^{c_i}$ is H-invariant, it gives us a holomorphic function on $\widetilde{U}$ for $i = 1, \dots, k$. We may assume that $E_i = \mathrm{Supp}\{y^{c_i} = 0\}$ for $1 \le i \le k$ and $E_i \cap \widetilde{U} = \emptyset$ for $k+1 \le i \le l$ by changing the indices. Because E_i is the zero set of $\widetilde{f}^* t = y_1 \cdots y_k$ on $\widetilde{U} \setminus \bigcup_{j \neq i}(E_j \cap \widetilde{U})$, the image of $y_i^{c_i} \in \mathcal{M} \subset \mathcal{M}^{\mathrm{gp}}$ by the morphism (7.2) is $(0, \dots, 0, c_i, 0 \dots, 0) \in \bigoplus_{j=1}^{l} (a_j)_* \mathbb{Z}_{E_j}$, where c_i is on the i-th entry. Thus we obtain the conclusion. □

7.5. We briefly recall the constructions of the Koszul complexes and related objects. For the detail, see [Fs2, Sections 1 and 2] and [PS, §4.4 and §11.2.6] (cf. [I], [St3] and so on).

A morphism of abelian sheaves $\mathbf{e} \colon \mathcal{O}_{\widetilde{X}} \to \mathcal{M}^{\mathrm{gp}}$ is defined as the composite of the exponential map

$$\mathcal{O}_{\widetilde{X}} \ni a \mapsto e^{2\pi\sqrt{-1}a} \in \mathcal{O}^*_{\widetilde{X}}$$

and the inclusion $\mathcal{O}^*_{\widetilde{X}} \hookrightarrow \mathcal{M}^{\mathrm{gp}}$. Then the morphism $\mathbf{e} \otimes \mathrm{id} \colon \mathcal{O}_{\widetilde{X}} \simeq \mathcal{O}_{\widetilde{X}} \otimes \mathbb{Q} \to \mathcal{M}^{\mathrm{gp}} \otimes \mathbb{Q}$ is obtained. Note that $1 \in \Gamma(X, \mathcal{O}_{\widetilde{X}})$ is contained in the kernel of $\mathbf{e} \otimes \mathrm{id}$. We set $\mathcal{M}^{\mathrm{gp}}_{\mathbb{Q}} = \mathcal{M}^{\mathrm{gp}} \otimes \mathbb{Q}$ for short. For $p \in \mathbb{Z}$, a $\mathbb{Q}$-sheaf $\mathrm{Kos}(\mathcal{M})^p$ on $\widetilde{X}$ is defined by

$$\mathrm{Kos}(\mathcal{M})^p := \varinjlim_n \mathrm{Sym}^{n-p}_{\mathbb{Q}}(\mathcal{O}_{\widetilde{X}}) \otimes_{\mathbb{Q}} \bigwedge^p \mathcal{M}^{\mathrm{gp}}_{\mathbb{Q}},$$

where $\mathrm{Sym}^{n-p}_{\mathbb{Q}}(\mathcal{O}_{\widetilde{X}})$ denotes the symmetric tensor product of degree $n-p$ of $\mathcal{O}_{\widetilde{X}}$ over $\mathbb{Q}$, and where the inductive limit is taken over the inductive system defined by the morphisms

$$\mathrm{Sym}^{n-p}_{\mathbb{Q}}(\mathcal{O}_{\widetilde{X}}) \otimes_{\mathbb{Q}} \bigwedge^p \mathcal{M}^{\mathrm{gp}}_{\mathbb{Q}} \ni a \otimes b \mapsto (1 \cdot a) \otimes b \in \mathrm{Sym}^{n+1-p}_{\mathbb{Q}}(\mathcal{O}_{\widetilde{X}}) \otimes_{\mathbb{Q}} \bigwedge^p \mathcal{M}^{\mathrm{gp}}_{\mathbb{Q}} \tag{7.4}$$

for all $n \ge p$. A morphism of $\mathbb{Q}$-sheaves

$$\mathrm{Sym}^{n-p}_{\mathbb{Q}}(\mathcal{O}_{\widetilde{X}}) \otimes_{\mathbb{Q}} \bigwedge^p \mathcal{M}^{\mathrm{gp}}_{\mathbb{Q}} \to \mathrm{Sym}^{n-p-1}_{\mathbb{Q}}(\mathcal{O}_{\widetilde{X}}) \otimes_{\mathbb{Q}} \bigwedge^{p+1} \mathcal{M}^{\mathrm{gp}}_{\mathbb{Q}}$$

is defined by

$$a_1^{n_1} \cdots a_k^{n_k} \otimes b \mapsto \sum_{j=1}^{k} n_j a_1^{n_1} \cdots a_j^{n_j - 1} \cdots a_k^{n_k} \otimes (\mathbf{e} \otimes \mathrm{id})(a_j) \wedge b \tag{7.5}$$

where $n_1, \dots, n_k$ are positive integers with $n_1 + \cdots + n_k = n - p$. These morphisms form a morphism of inductive systems defined by (7.4) for p and $p+1$ because $1 \in \Gamma(X, \mathcal{O}_{\widetilde{X}})$ is contained in the kernel of $\mathbf{e} \otimes \mathrm{id}$. Then a morphism of $\mathbb{Q}$-sheaves

$$d \colon \mathrm{Kos}(\mathcal{M})^p \to \mathrm{Kos}(\mathcal{M})^{p+1}$$

is induced. We can easily see the equality $d^2 = 0$. Thus we obtain a complex of $\mathbb{Q}$-sheaves $\mathrm{Kos}(\mathcal{M})$ on $\widetilde{X}$. Replacing $\mathcal{M}^{\mathrm{gp}}$ by $\mathcal{O}^*_{\widetilde{X}}$, we obtain a complex of $\mathbb{Q}$-sheaves $\mathrm{Kos}(\mathcal{O}^*_{\widetilde{X}})$.

We set

$$W_m(\mathrm{Sym}^{n-p}_{\mathbb{Q}}(\mathcal{O}_{\widetilde{X}}) \otimes_{\mathbb{Q}} \bigwedge^p \mathcal{M}^{\mathrm{gp}}_{\mathbb{Q}}) = \mathrm{Sym}^{n-p}_{\mathbb{Q}}(\mathcal{O}_{\widetilde{X}}) \otimes_{\mathbb{Q}} \bigwedge^{p-m} (\mathcal{O}^*_{\widetilde{X}} \otimes \mathbb{Q}) \otimes_{\mathbb{Q}} \bigwedge^m \mathcal{M}^{\mathrm{gp}}_{\mathbb{Q}},$$

which is a $\mathbb{Q}$-subsheaf of $\mathrm{Sym}^{n-p}_{\mathbb{Q}}(\mathcal{O}_{\widetilde{X}}) \otimes_{\mathbb{Q}} \bigwedge^p \mathcal{M}^{\mathrm{gp}}_{\mathbb{Q}}$ for every $m \in \mathbb{Z}$. Since the morphism (7.4) trivially preserves W_m on the both sides, a subsheaf $W_m \mathrm{Kos}(\mathcal{M})^p$ of $\mathrm{Kos}(\mathcal{M})^p$ is obtained by

$$W_m \mathrm{Kos}(\mathcal{M})^p = \varinjlim_n W_m(\mathrm{Sym}^{n-p}_{\mathbb{Q}}(\mathcal{O}_{\widetilde{X}}) \otimes_{\mathbb{Q}} \bigwedge^p \mathcal{M}^{\mathrm{gp}}_{\mathbb{Q}})$$

for every m. It can be easily checked that they define a finite increasing filtration W on the complex $\mathrm{Kos}(\mathcal{M})$.

The singular locus of $\widetilde{X}$ is denoted by $\mathrm{Sing}(\widetilde{X})$ and the smooth locus $\widetilde{X}_{\mathrm{sm}}$ is defined by $\widetilde{X}_{\mathrm{sm}} = \widetilde{X} \setminus \mathrm{Sing}(\widetilde{X})$. Note that the restriction $E \cap \widetilde{X}_{\mathrm{sm}}$ of E to $\widetilde{X}_{\mathrm{sm}}$ is a simple normal crossing divisor on $\widetilde{X}_{\mathrm{sm}}$. Then we have a morphism of monoid sheaves

$$\mathrm{dlog}\colon \mathcal{M}|_{\widetilde{X}_{\mathrm{sm}}} \to \Omega^1_{\widetilde{X}_{\mathrm{sm}}}(\log E \cap \widetilde{X}_{\mathrm{sm}})$$

defined by $\mathrm{dlog}(a) := a^{-1}da$ for $a \in \mathcal{M}|_{\widetilde{X}_{\mathrm{sm}}} \subset \mathcal{O}_{\widetilde{X}_{\mathrm{sm}}}$. Thus we obtain a morphism

$$\mathcal{M} \to \widetilde{\Omega}^1_{\widetilde{X}}(\log E)$$

denoted by the same letter dlog as above. This morphism induces a morphism of $\mathbb{Q}$-sheaves

$$\bigwedge^p \mathcal{M}^{\mathrm{gp}}_{\mathbb{Q}} \to \widetilde{\Omega}^p_{\widetilde{X}}(\log E)$$

denoted by $\bigwedge^p \mathrm{dlog}$ for every p. A morphism of $\mathbb{Q}$-sheaves

$$\mathrm{Sym}^{n-p}_{\mathbb{Q}}(\mathcal{O}_{\widetilde{X}}) \otimes_{\mathbb{Q}} \bigwedge^p \mathcal{M}^{\mathrm{gp}}_{\mathbb{Q}} \to \widetilde{\Omega}^p_{\widetilde{X}}(\log E),$$

defined by

$$a_1^{n_1} \cdots a_k^{n_k} \otimes b \mapsto (2\pi\sqrt{-1})^{-p} a_1^{n_1} \cdots a_k^{n_k} \bigwedge^p \mathrm{dlog}(b)$$

for positive integers $n_1, \ldots, n_k$ with $n_1 + \cdots + n_k = n - p$, is compatible with the morphisms (7.4) and (7.5). Therefore we have a morphism of complexes of $\mathbb{Q}$-sheaves

$$\mathrm{Kos}(\mathcal{M}) \to \widetilde{\Omega}_{\widetilde{X}}(\log E),$$

which is denoted by ψ as in [Fs2, (2.4)]. It can be easily seen that the morphism ψ preserves the filtration W on the both sides.

The global section $\widetilde{f}^*t \in \Gamma(\widetilde{X}, \mathcal{M})$ defines a morphism of complexes

$$(\widetilde{f}^*t)\wedge\colon \mathrm{Kos}(\mathcal{M}) \to \mathrm{Kos}(\mathcal{M})[1],$$

which sends $W_m \mathrm{Kos}(\mathcal{M})^n$ to $W_{m+1} \mathrm{Kos}(\mathcal{M})^{n+1}$ as in [Fs2, (1.11) and (1.12)]. It can be easily checked that the diagram

$$\begin{array}{ccc} \mathrm{Kos}(\mathcal{M}) & \xrightarrow{\psi} & \widetilde{\Omega}_{\widetilde{X}}(\log E) \\ {\scriptstyle (\widetilde{f}^*t)\wedge}\downarrow & & \downarrow{\scriptstyle \theta\wedge} \\ \mathrm{Kos}(\mathcal{M})[1] & \xrightarrow[(2\pi\sqrt{-1})\psi]{} & \widetilde{\Omega}_{\widetilde{X}}(\log E)[1] \end{array} \tag{7.6}$$

is commutative, where $\theta = \widetilde{f}^*(dt/t) \in \widetilde{\Omega}^1_{\widetilde{X}}(\log E)$.

For $\mathrm{Kos}(\mathcal{M})$ and ψ above, we have the following lemmas.

Lemma 7.6. *In the situation above, we set*

$$E^{(m)} = \coprod_{1 \le i_1 < \cdots < i_m \le l} E_{i_1} \cap \cdots \cap E_{i_m}$$

for $m \in \mathbb{Z}_{>0}$. Moreover, we set $E^{(0)} = \widetilde{X}$. The natural morphism $E^{(m)} \to \widetilde{X}$ is denoted by a_m for $m \in \mathbb{Z}_{\geq 0}$. Then there exists a quasi-isomorphism

$$(a_m)_* \mathbb{Q}_{E^{(m)}}[-m] \to \mathrm{Gr}_m^W \mathrm{Kos}(\mathcal{M})$$

for all $m \in \mathbb{Z}$.

Proof. We have an isomorphism

$$\bigwedge^m (\mathcal{M}^{\mathrm{gp}} \otimes \mathbb{Q}/\mathcal{O}^*_{\widetilde{X}} \otimes \mathbb{Q}) \otimes \mathrm{Kos}(\mathcal{O}^*_{\widetilde{X}})[-m] \simeq \mathrm{Gr}_m^W \mathrm{Kos}(\mathcal{M})$$

by [Fs2, Proposition 1.10], and a quasi-isomorphism $\mathbb{Q}_{\widetilde{X}} \to \mathrm{Kos}(\mathcal{O}^*_{\widetilde{X}})$ by [Fs2, Corollary 1.15]. Therefore we obtain the conclusion by Lemma 7.4. □

Lemma 7.7. *In the situation above, we have the commutative diagram*

$$\begin{CD} (a_m)_* \mathbb{Q}_{E^{(m)}}[-m] @>{(2\pi\sqrt{-1})^{-m}\iota[-m]}>> (a_m)_* \widetilde{\Omega}_{E^{(m)}}[-m] \\ @VVV @VV{\simeq}V \\ \mathrm{Gr}_m^W \mathrm{Kos}(\mathcal{M}) @>>{\mathrm{Gr}_m^W \psi}> \mathrm{Gr}_m^W \widetilde{\Omega}_{\widetilde{X}}(\log E) \end{CD} \tag{7.7}$$

where ι is the natural morphism induced from the inclusion $\mathbb{Q} \to \mathcal{O}_{E^{(m)}}$, the left vertical arrow is the quasi-isomorphism in Lemma 7.6, and the right vertical arrow is the inverse of the residue isomorphism in [St2, (1.18) Definition and (1.19) Lemma] (*see also* [D1, 3.5]). *In particular, the morphism*

$$\mathrm{Kos}(\mathcal{M}) \otimes \mathbb{C} \to \widetilde{\Omega}_{\widetilde{X}}(\log E)$$

induced by ψ is a filtered quasi-isomorphism with respect to W on the both sides.

Proof. The commutativity of the diagram (7.7) can be checked by the direct computation from the definition of ψ (cf. [Fs2, (2.4)]). Then the latter conclusion follows from [St2, (1.9) Corollary]. □

Once we obtain these two lemmas, it is more or less clear that the construction, parallel to $A_{\mathbb{C}}$ in [St1, (4.14) and (4.17)] and [St2, (2.8)], works for $A_{\mathbb{Q}}$.

Definition 7.8. In the situation 7.1, a bifiltered complex of $\mathbb{C}$-sheaves $(A_{\mathbb{C}}, W, F)$ on $\widetilde{X}$ is defined by

$$A^n_{\mathbb{C}} := \bigoplus_{q \geq 0} \widetilde{\Omega}^{n+1}_{\widetilde{X}}(\log E)/W_q \widetilde{\Omega}^{n+1}_{\widetilde{X}}(\log E),$$

$$W_m A^n_{\mathbb{C}} := \bigoplus_{q \geq 0} W_{m+2q+1} \widetilde{\Omega}^{n+1}_{\widetilde{X}}(\log E)/W_q \widetilde{\Omega}^{n+1}_{\widetilde{X}}(\log E),$$

$$F^p A^n_{\mathbb{C}} := \bigoplus_{0 \leq q \leq n-p} \widetilde{\Omega}^{n+1}_{\widetilde{X}}(\log E)/W_q \widetilde{\Omega}^{n+1}_{\widetilde{X}}(\log E)$$

with the differential $-d - \theta\wedge$, where d denotes the differential of the complex $\widetilde{\Omega}_{\widetilde{X}}(\log E)$ as in [St1, (4.17)]. Similarly, a filtered complex of $\mathbb{Q}$-sheaves $(A_{\mathbb{Q}}, W)$ on $\widetilde{X}$ is defined by

$$A^n_{\mathbb{Q}} := \bigoplus_{q \geq 0} \mathrm{Kos}(\mathcal{M})^{n+1}/W_q \mathrm{Kos}(\mathcal{M})^{n+1}$$

$$W_m A^n_{\mathbb{Q}} := \bigoplus_{q \geq 0} W_{m+2q+1} \mathrm{Kos}(\mathcal{M})^{n+1}/W_q \mathrm{Kos}(\mathcal{M})^{n+1}$$

with the differential $-d-(\widetilde{f}^*t)\wedge$, where d denotes the differential of the complex $\mathrm{Kos}(\mathcal{M})$. The direct sum of the morphisms of $\mathbb{Q}$-sheaves

$$(2\pi\sqrt{-1})^{q+1}\psi\colon\ \mathrm{Kos}(\mathcal{M})^{n+1}/W_q\,\mathrm{Kos}(\mathcal{M})^{n+1} \to \widetilde{\Omega}^{n+1}_{\widetilde{X}}(\log E)/W_q\widetilde{\Omega}^{n+1}_{\widetilde{X}}(\log E)$$

gives us a morphism of $\mathbb{Q}$-sheaves

$$A^n_{\mathbb{Q}} = \bigoplus_{q\geq 0}\mathrm{Kos}(\mathcal{M})^{n+1}/W_q\,\mathrm{Kos}(\mathcal{M})^{n+1} \to \bigoplus_{q\geq 0}\widetilde{\Omega}^{n+1}_{\widetilde{X}}(\log E)/W_q\widetilde{\Omega}^{n+1}_{\widetilde{X}}(\log E) = A^n_{\mathbb{C}}$$

which is compatible with the differentials by the commutativity of the diagram (7.6). Thus we obtain a morphism of filtered complexes of $\mathbb{Q}$-sheaves $\alpha\colon (A_{\mathbb{Q}}, W) \to (A_{\mathbb{C}}, W)$. Note that the supports of $A^n_{\mathbb{C}}$ and $A^n_{\mathbb{Q}}$ are contained in E for every n. Therefore they are the (bi)filtered complexes on E.

Theorem 7.9 (cf. [St2, (2.8)])**.** *Let $f\colon X \to \Delta$ be as in 7.1. If we assume that X is Kähler, then $((A_{\mathbb{Q}}, W), (A_{\mathbb{C}}, W, F), \alpha)$ is a cohomological $\mathbb{Q}$-mixed Hodge complex on E.*

Proof. By Lemmas 7.6 and 7.7, $(\mathrm{Gr}^W_m A_{\mathbb{Q}}, (\mathrm{Gr}^W_m A_{\mathbb{C}}, F), \mathrm{Gr}^W_m \alpha)$ is identified with the direct sum of the direct images of

$$(\mathbb{Q}(-m-q)[-m-2q], (\widetilde{\Omega}_{E^{(m+2q+1)}}[-m-2q], F[-m-q]))$$

by the finite morphism a_{m+2q+1} for all $q \geq \max(0, -m)$. Since $\widetilde{X}$ is an almost Kähler V-manifold as in [PS, I.2.5] by the assumption for X being Kähler, we obtain the conclusion by Theorem 2.43 of [PS]. □

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