VANISHING AND SEMIPOSITIVITY THEOREMS FOR SEMI-LOG CANONICAL PAIRS

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Abstract. We prove an effective vanishing theorem for direct images of log pluricanonical bundles of projective semi-log canonical pairs. As an application, we obtain a semipositivity theorem for direct images of relative log pluricanonical bundles of projective semi-log canonical pairs over curves, which implies the projectivity of the moduli spaces of stable varieties. It is worth mentioning that we do not use the theory of variation of (mixed) Hodge structure.

Contents

1. Introduction

In this paper, we establish some vanishing theorems for semi-log canonical pairs and prove some semipositivity theorems for semi-log canonical pairs as applications without using the theory of graded polarizable admissible variation of mixed Hodge structure.

First we prove an effective vanishing theorem for direct images of log pluricanonical bundles of projective semi-log canonical pairs, which is a generalization of [PopS, Theorem 1.7].

Theorem 1.1 (Effective vanishing theorem)**.** *Let (X, Δ) be a projective semi-log canonical pair and let $f : X \to Y$ be a surjective morphism onto an n-dimensional projective variety Y. Let D be a Cartier divisor on X such that $D \sim_{\mathbb{R}} k(K_X + \Delta + f^*H)$ for some positive integer k, where H is an ample $\mathbb{R}$-divisor on Y. Let L be an ample Cartier divisor on Y such that $|L|$ is free. Assume that $\mathcal{O}_X(D)$ is f-generated. Then*

$$H^i(Y, f_*\mathcal{O}_X(D) \otimes \mathcal{O}_Y(lL)) = 0$$

for every $i > 0$ and every $l \geq (k-1)(n+1-t)-t+1$, where $t = \sup\{s \mid H - sL \text{ is ample}\}$. Therefore, by the Castelnuovo–Mumford regularity, $f_*\mathcal{O}_X(D) \otimes \mathcal{O}_Y(lL)$ is globally generated for every $l \geq (k-1)(n+1-t)-t+1+n$.

We note that Theorem 1.1 is a consequence of the Kollár–Ohsawa type vanishing theorem for semi-log canonical pairs (see Theorem 3.1 below). When (X, Δ) is log canonical, that is, X is normal, in Theorem 1.1, Mihnea Popa and Christian Schnell (see [PopS]) proved that Theorem 1.1 holds true without assuming that $\mathcal{O}_X(D)$ is f-generated. Therefore, Theorem

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1.1 is much weaker than [PopS, Theorem 1.7] when (X, Δ) is log canonical. However, it is sufficiently powerful.

Next we prove a semipositivity theorem for direct images of relative log pluricanonical bundles of projective semi-log canonical pairs over curves as an application of Theorem 1.1, which is a special case of [Fuj9, Theorem 1.11]. Note that the results in [Fuj9] heavily depend on the theory of graded polarizable admissible variation of mixed Hodge structure (see [FF] and [FFS]). Therefore, the reader may feel that Theorem 1.2 is more accessible than [Fuj9, Theorem 1.11]. We strongly recommend the reader to compare Theorem 1.2 with [Fuj9, Theorem 1.11].

Theorem 1.2 (Semipositivity theorem)**.** *Let (X, Δ) be a projective semi-log canonical pair and let $f : X \to Y$ be a flat morphism onto a smooth projective curve Y such that*

(i) $\operatorname{Supp}\Delta$ *avoids the generic and codimension one singular points of every fiber of f, and*

(ii) (X_y, Δ_y) *is a semi-log canonical pair for every $y \in Y$.*

Assume that $\mathcal{O}_X(k(K_X + \Delta))$ is locally free and f-generated for some positive integer k. Then $f_*\mathcal{O}_X(k(K_{X/Y} + \Delta))$ is a nef locally free sheaf.

Although Theorem 1.2 is a very special case of [Fuj9, Theorem 1.11], it seems to be sufficient for most geometric applications (see [Fuj9], [Pat2], [KovP, Lemma 7.7], [PatX, Theorem 2.13], [AT], etc.) By Kollár's projectivity criterion (see [Kol2]) and Theorem 1.2, we can easily obtain:

Theorem 1.3 ([Fuj9, Theorem 1.1])**.** *Every complete subspace of the coarse moduli space of stable varieties is* projective.

In this paper, we only sketch the proof of Theorem 1.3 for the reader's convenience. We recommend the reader to see [Kol2] and [Fuj9] for the details of Theorem 1.3 (see also [KovP]).

Finally, we give a proof of [Fuj9, Theorem 1.9], which is called the basic semipositivity theorem in [Fuj9], based on the Kollár–Ohsawa type vanishing theorem for semi-log canonical pairs (see Theorem 3.1 and Remark 4.2).

Theorem 1.4 (Basic semipositivity theorem [Fuj9, Theorem 1.9])**.** *Let (X, D) be a simple normal crossing pair such that D is reduced. Let $f : X \to C$ be a projective surjective morphism onto a smooth projective curve C. Assume that every stratum of X is dominant onto C. Then $f_*\omega_{X/C}(D)$ is nef.*

It is worth mentioning that all the semipositivity theorems in [Fuj9] follow from [Fuj9, Theorem 1.9], that is, Theorem 1.4. Therefore, by replacing the proof of [Fuj9, Theorem 1.9] with the proof of Theorem 1.4 given in this paper, the paper [Fuj9] becomes independent of the theory of graded polarizable admissible variation of mixed Hodge structure. In particular, we see that the projectivity of moduli spaces of stable varieties and pairs (see [Fuj9] and [KovP]) can be established without appealing to the theory of variation of (mixed) Hodge structure (see also Theorem 1.3). We note that the main ingredient of this paper is the vanishing theorem for simple normal crossing pairs (see [Fuj3], [Fuj6], and [Fuj7]), which comes from the theory of mixed Hodge structure on cohomology with compact support. We also note that this paper does not supersede [Fuj9] but will complement [Fuj9].

1.5 (Historical comments)**.** In 2012, I wrote and submitted [Fuj9]. Unfortunately, some referee kept it for a very long time without making decisions. In 2017, the editor changed the referee. Then the process became a usual one. By the way, I obtained the results of this paper in 2015. I think that they make [Fuj9] more accessible. However, since the referee was keeping [Fuj9], I could not revise [Fuj9]. So I wrote this paper separately. Anyway, I recommend the reader to read [Fuj9] too.

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We will work over $\mathbb{C}$, the complex number field, throughout this paper. Note that, by the Lefschetz principle, all the results in this paper hold over any algebraically closed field k of characteristic zero. For the standard notations and conventions of the log minimal model program, see [Fuj1] and [Fuj7]. In this paper, a *variety* means a separated reduced scheme of finite type over $\mathbb{C}$.

2. Preliminaries

In this section, we collect some basic definitions and results. Note that we are mainly interested in non-normal reducible equidimensional varieties.

We need the notion of *simple normal crossing pairs* for various vanishing theorems (see, for example, [Fuj3], [Fuj6], [Fuj7], and Theorem 3.1 below). We note that a simple normal crossing pair is sometimes called a *semi-snc* pair in the literature (see [BieVP, Definition 1.1] and [Kol3, Definition 1.10]).

Definition 2.1 (Simple normal crossing pairs)**.** We say that the pair (X, D) is *simple normal crossing* at a point $a \in X$ if X has a Zariski open neighborhood U of a that can be embedded in a smooth variety Y, where Y has regular system of parameters $(x_1, \cdots, x_p, y_1, \cdots, y_r)$ at $a = 0$ in which U is defined by a monomial equation

$$x_1 \cdots x_p = 0$$

and

$$D = \sum_{i=1}^{r} \alpha_i (y_i = 0)|_U, \quad \alpha_i \in \mathbb{R}.$$

We say that (X, D) is a *simple normal crossing pair* if it is simple normal crossing at every point of X. We sometimes say that D is a *simple normal crossing divisor* on X if (X, D) is a simple normal crossing pair and D is reduced.

2.2 ($\mathbb{Q}$-divisors and $\mathbb{R}$-divisors)**.** Let D be an $\mathbb{R}$-divisor (resp. a $\mathbb{Q}$-divisor) on an equidimensional variety X, that is, D is a finite formal $\mathbb{R}$-linear (resp. $\mathbb{Q}$-linear) combination

$$D = \sum_i d_i D_i$$

of irreducible reduced subschemes D_i of codimension one such that $D_i \neq D_j$ for $i \neq j$. We define the *round-up* $\lceil D \rceil = \sum_i \lceil d_i \rceil D_i$, where for every real number x, $\lceil x \rceil$ is the integer defined by $x \leq \lceil x \rceil < x + 1$. We set

$$D^{<1} = \sum_{d_i < 1} d_i D_i.$$

We say that D is a *boundary* (resp. *subboundary*) $\mathbb{R}$-divisor if $0 \leq d_i \leq 1$ (resp. $d_i \leq 1$) for every i.

2.3 ($\mathbb{R}$-linear equivalence)**.** Let B_1 and B_2 be two $\mathbb{R}$-Cartier divisors on a variety X. Then $B_1 \sim_{\mathbb{R}} B_2$ means that B_1 is $\mathbb{R}$-*linearly equivalent* to B_2.

Let us recall the definition of *strata* of simple normal crossing pairs.

Definition 2.4 (Stratum)**.** Let (X, D) be a simple normal crossing pair such that D is a boundary $\mathbb{R}$-divisor. Let $\nu : X^\nu \to X$ be the normalization. We put $K_{X^\nu}+\Theta = \nu^*(K_X+D)$, that is, Θ is the sum of the inverse images of D and the singular locus of X. A *stratum* of (X, D) is an irreducible component of X or the ν-image of a log canonical center of (X^ν, Θ). We note that (X^ν, Θ) is log canonical since (X, D) is a simple normal crossing pair and D is a boundary $\mathbb{R}$-divisor.

For the reader's convenience, we recall the definition of *semi-log canonical pairs*.

Definition 2.5 (Semi-log canonical pairs)**.** Let X be an equidimensional variety which satisfies Serre's S_2 condition and is normal crossing in codimension one. Let Δ be an effective $\mathbb{R}$-divisor on X such that no irreducible component of $\operatorname{Supp}\Delta$ is contained in the singular locus of X. The pair (X, Δ) is called a *semi-log canonical pair* (an *slc pair*, for short) if

(1) $K_X + \Delta$ is $\mathbb{R}$-Cartier, and
(2) (X^ν, Θ) is log canonical, where $\nu : X^\nu \to X$ is the normalization and $K_{X^\nu} + \Theta = \nu^*(K_X + \Delta)$, that is, Θ is the sum of the inverse images of Δ and the conductor of X.

If $(X, 0)$ is a semi-log canonical pair, then we simply say that X is a *semi-log canonical variety* or X has only *semi-log canonical singularities*.

For the details of semi-log canonical pairs and the basic notations, see [Fuj2] and [Kol3].

In the recent literature, an equidimensional variety which is normal crossing in codimension one and satisfies Serre's S_2 condition is sometimes said to be *demi-normal*.

Definition 2.6 (Kollár)**.** An equidimensional variety X is said to be *demi-normal* if X satisfies Serre's S_2 condition and is normal crossing in codimension one.

By definition, if (X, Δ) is a semi-log canonical pair, then X is demi-normal. For the details of divisors and divisorial sheaves on demi-normal varieties and semi-log canonical pairs, see [Kol3, Section 5.1].

2.7 (cf. [Pat1])**.** For a complex $\mathcal{C}^\bullet$ of sheaves, $h^i(\mathcal{C}^\bullet)$ is the i-th cohomology sheaf of $\mathcal{C}^\bullet$. For a morphism $f : X \to Y$ between separated schemes of finite type over $\mathbb{C}$, we put $\omega^\bullet_{X/Y} = f^!\mathcal{O}_Y$, where $f^!$ is the functor obtained in [Har1, Chapter VII, Corollary 3.4 (a)] (see also [Con]). If f has equidimensional fibers of dimension n, then we put $\omega_{X/Y} = h^{-n}(\omega^\bullet_{X/Y})$ and call it the *relative canonical sheaf* of $f : X \to Y$. We recommend the reader to see [Pat1, Section 3] for some basic properties of (relative) canonical sheaves and base change properties.

Although the following definition is slightly different from [KovP, Definition 3.3], there are no problems since almost all varieties we treat in this paper satisfy Serre's S_2 condition.

Definition 2.8. Let Z be an equidimensional variety. A *big open subset* U of Z is a Zariski open subset $U \subset Z$ such that $\operatorname{codim}_Z(Z \setminus U) \geq 2$.

We discuss the *divisorial pull-backs* of $\mathbb{Q}$-divisors and $\mathbb{R}$-divisors under some suitable assumptions (cf. [KovP, Notation 3.7]).

2.9. Let $f : X \to Y$ be a flat projective morphism from a demi-normal variety X to a smooth curve Y. We further assume that every fiber X_y of f is demi-normal. Let D be a $\mathbb{Q}$-divisor on X that avoids the generic and codimension one singular points of every fiber of f. We will denote by D_y the *divisorial pull-back* of D to X_y, which is defined as follows: As D avoids the singular codimension one points of X_y, we can take a big open subset $U \subset X$ such that $D|_U$ is $\mathbb{Q}$-Cartier and that $U_y = U|_{X_y}$ is also a big open subset of X_y. We define

D_y to be the unique $\mathbb{Q}$-divisor on X_y whose restriction to U_y is $(D|_U)|_{U_y}$. By adjunction, we have $(K_X+X_y)|_{X_y}=K_{X_y}$. Note that X_y is a Cartier divisor on X since Y is a smooth curve and that X_y is Gorenstein in codimension one since X_y is demi-normal. Therefore, in Theorem 1.2, we have $(K_{X/Y}+\Delta)|_{X_y}=K_{X_y}+\Delta_y$. Moreover, if $m(K_X+\Delta)$ is Cartier for some positive integer m in Theorem 1.2, then $\mathcal{O}_X(m(K_{X/Y}+\Delta))|_{X_y}\simeq \mathcal{O}_{X_y}(m(K_{X_y}+\Delta_y))$.

2.10. Let $f: X\to Y$ be a flat morphism between demi-normal varieties. Let D be an $\mathbb{R}$-divisor on Y that avoids the codimension one singular points of Y. Then we will denote by $f^{-1}D$ the *divisorial pull-back* of D to X, which is defined as follows: As D avoids the singular codimension one points of Y, there is a big open subset $U\subset Y$ such that $D|_U$ is $\mathbb{R}$-Cartier. We define $f^{-1}D$ to be the unique $\mathbb{R}$-divisor on X whose restriction to $f^{-1}(U)$ is $f^*(D|_U)$.

The following two lemmas are well-known and easy to check. So we leave the proof as exercises for the reader. We will use these lemmas in the proof of Lemma 2.13.

Lemma 2.11. *Let (X,Δ) be a semi-log canonical pair. Let* $\operatorname{Supp}\Delta=\sum_i B_i$ *be the irreducible decomposition. We define a finite-dimensional $\mathbb{R}$-vector space $V=\bigoplus_i \mathbb{R}B_i$. Then we see that*

$$\mathcal{L}=\{D\in V\,|\,(X,D) \text{ is semi-log canonical}\}$$

is a rational polytope in V. Therefore, we can write

$$K_X+\Delta=\sum_{i=1}^{k} r_i(K_X+D_i),$$

where

(i) *$D_i\in\mathcal{L}$ for every i;*
(ii) *D_i is a $\mathbb{Q}$-divisor for every i;*
(iii) *$0<r_i<1$, $r_i\in\mathbb{R}$ for every i, and $\sum_{i=1}^{k} r_i=1$.*

Lemma 2.12. *Let (V_i,D_i) be a log canonical pair such that $K_{V_i}+D_i$ is $\mathbb{Q}$-Cartier for $i=1,2$. We put $V=V_1\times V_2$ and $D=p_1^{-1}D_1+p_2^{-1}D_2$, where $p_i: V\to V_i$ is the i-th projection for $i=1,2$. Then (V,D) is log canonical.*

We will use the following lemma, which seems to be well-known to the experts, in the proof of Theorem 1.2.

Lemma 2.13. *Let (X_i,Δ_i) be a semi-log canonical pair for $i=1,2$. We put $X=X_1\times X_2$ and $\Delta=p_1^{-1}\Delta_1+p_2^{-1}\Delta_2$, where $p_i: X\to X_i$ is the i-th projection for $i=1,2$. Then (X,Δ) is a semi-log canonical pair as well.*

Proof. By Lemma 2.11, we may assume that Δ_1 and Δ_2 are $\mathbb{Q}$-divisors on X_1 and X_2 respectively. We see that $X=X_1\times X_2$ satisfies Serre's S_2 condition and is normal crossing in codimension one. We take a positive integer m such that $m(K_{X_1}+\Delta_1)$ and $m(K_{X_2}+\Delta_2)$ are Cartier. Then we have

$$\mathcal{O}_X(m(K_X+\Delta))\simeq p_1^*\mathcal{O}_{X_1}(m(K_{X_1}+\Delta_1))\otimes p_2^*\mathcal{O}_{X_2}(m(K_{X_2}+\Delta_2)). \tag{2.1}$$

Thus $K_X+\Delta$ is $\mathbb{Q}$-Cartier. Let $\nu_i: X_i^\nu\to X_i$ be the normalization. We put $K_{X_i^\nu}+\Theta_i=\nu_i^*(K_{X_i}+\Delta_i)$ as in Definition 2.5 for $i=1,2$. By definition, (X_i^ν,Θ_i) is log canonical for $i=1,2$. We note that $\nu=\nu_1\times\nu_2: X^\nu=X_1^\nu\times X_2^\nu\to X=X_1\times X_2$ is the normalization and $K_{X^\nu}+\Theta=\nu^*(K_X+\Delta)$, where $\Theta=q_1^{-1}\Theta_1+q_2^{-1}\Theta_2$ such that $q_i: X^\nu\to X_i^\nu$ is the i-th projection for $i=1,2$. By Lemma 2.12, (X^ν,Θ) is log canonical. Therefore, (X,Δ) is semi-log canonical. □

We will need the following definition in Section 4.

Definition 2.14. Let $\mathcal{F}$ be a coherent sheaf on a projective variety X. If the natural map

$$H^0(X, \mathcal{F}) \otimes \mathcal{O}_X \to \mathcal{F}$$

is generically surjective, then $\mathcal{F}$ is said to be *generically globally generated*.

We close this section with the definition of *nef locally free sheaves*.

Definition 2.15 (Nef locally free sheaves)**.** A locally free sheaf $\mathcal{E}$ of finite rank on a complete variety X is *nef* if the following equivalent conditions are satisfied:

(i) $\mathcal{E} = 0$ or $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1)$ is nef on $\mathbb{P}_X(\mathcal{E})$.
(ii) For every map from a smooth projective curve $f : C \to X$, every quotient line bundle of $f^*\mathcal{E}$ has non-negative degree.

A nef locally free sheaf was originally called a (*numerically*) *semipositive* sheaf in the literature.

3. Vanishing and semipositivity theorems

Let us start the following vanishing theorem for semi-log canonical pairs.

Theorem 3.1 (Vanishing theorem)**.** *Let (X, Δ) be a projective semi-log canonical pair and let $f : X \to Y$ be a surjective morphism onto a projective variety Y. Let D be a Cartier divisor on X such that $D - (K_X + \Delta) \sim_{\mathbb{R}} f^*H$ for some ample $\mathbb{R}$-divisor H on Y. Then $H^i(Y, f_*\mathcal{O}_X(D)) = 0$ for every $i > 0$.*

We can see Theorem 3.1 as a generalization of the Kollár–Ohsawa vanishing theorem for projective semi-log canonical pairs (see [Ohs, Theorem 3.1] and [Kol1, Theorem 2.1]). We note that both the vanishing theorems ([Ohs, Theorem 3.1] and [Kol1, Theorem 2.1]) are now special cases of [Mat, Theorem 1.3]. Theorem 3.1 is a key ingredient of the proof of Theorem 1.1. It follows from the theory of mixed Hodge structure on cohomology with compact support (see [Fuj3] and [Fuj7]).

Proof. Let $\pi : \widetilde{X} \to X$ be a natural double cover due to Kollár (see [Kol3, 5.23 (A natural double cover)]). Then $\mathcal{O}_X(D)$ is a direct summand of $\pi_*\mathcal{O}_{\widetilde{X}}(\pi^*D)$. Therefore, by replacing X and D with $\widetilde{X}$ and π^*D, respectively, we may assume that X is simple normal crossing in codimension one. Let U be the largest Zariski open subset of X where $(U, \Delta|_U)$ is a simple normal crossing pair. Then $\operatorname{codim}_X(X \setminus U) \geq 2$. By the theorem of Bierstone–Vera Pacheco (see [BieVP, Theorem 1.4]), there exists a birational morphism $g : Z \to X$ from a projective simple normal crossing variety Z such that g is an isomorphism over U, g maps $\operatorname{Sing} Z$ birationally onto the closure of $\operatorname{Sing} U$ in X, and that $K_Z + \Delta_Z = g^*(K_X + \Delta)$, where Δ_Z is a subboundary $\mathbb{R}$-divisor such that $\operatorname{Supp} \Delta_Z$ is a simple normal crossing divisor on Z. We put $E = \lceil -\Delta_Z^{<1} \rceil$. Then E is an effective g-exceptional Cartier divisor. By the definition of E, $\Delta_Z + E$ is a boundary $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on Z such that $\operatorname{Supp}(\Delta_Z + E)$ is a simple normal crossing divisor on Z. In particular, $(Z, \Delta_Z + E)$ is a simple normal crossing pair. Since we have

$$g^*D + E - (K_Z + \Delta_Z + E) \sim_{\mathbb{R}} g^*f^*H,$$

we obtain

$$H^i(Y, (f \circ g)_*\mathcal{O}_Z(g^*D + E)) = 0$$

for every $i > 0$ (see [Fuj3, Theorem 1.1] and [Fuj7]). This implies that $H^i(Y, f_*\mathcal{O}_X(D)) = 0$ for every $i > 0$ because $g_*\mathcal{O}_Z(g^*D + E) \simeq \mathcal{O}_X(D)$. ∎

By the Castelnuovo–Mumford regularity and Theorem 3.1, we have:

Corollary 3.2. *Let (X, Δ) be a projective semi-log canonical pair such that $K_X + \Delta$ is Cartier and let $f : X \to Y$ be a surjective morphism onto a projective variety Y with* $\dim Y = n$. *Let L be an ample Cartier divisor on Y such that $|L|$ is free. Then $f_*\mathcal{O}_X(K_X + \Delta) \otimes \mathcal{O}_Y(lL)$ is globally generated for every $l \geq n + 1$.*

Proof. We put $D = K_X + \Delta + f^*(lL)$ with $l \geq n+1$. Then we have that $H^i(Y, f_*\mathcal{O}_X(D) \otimes \mathcal{O}_Y(-iL)) = 0$ for every $i > 0$ by Theorem 3.1. Therefore, by the Castelnuovo–Mumford regularity, we obtain that $f_*\mathcal{O}_X(K_X + \Delta) \otimes \mathcal{O}_Y(lL)$ is globally generated for every $l \geq n + 1$. $\square$

We note that we will use Corollary 3.2 in the proof of Theorem 1.4 in Section 4.

Let us start the proof of Theorem 1.1, which is a clever application of Theorem 3.1.

Proof of Theorem 1.1. We closely follow the proof of [PopS, Theorem 1.7]. Since L is ample, there exists the smallest integer $m \geq 0$ such that $f_*\mathcal{O}_X(D) \otimes \mathcal{O}_Y(mL)$ is globally generated. Since $f^*f_*\mathcal{O}_X(D) \to \mathcal{O}_X(D)$ is surjective by assumption, $\mathcal{O}_X(D) \otimes f^*\mathcal{O}_Y(mL)$ is globally generated as well. Let B be a general member of the free linear system $|\mathcal{O}_X(D) \otimes f^*\mathcal{O}_Y(mL)|$. Then

$$k(K_X + \Delta + f^*H) + mf^*L \sim_{\mathbb{R}} B.$$

Therefore, we have

$$(k-1)(K_X + \Delta + f^*H) \sim_{\mathbb{R}} \frac{k-1}{k}B - \frac{k-1}{k}mf^*L.$$

For any integer l, we can write

$$D + lf^*L \sim_{\mathbb{R}} K_X + \Delta + \frac{k-1}{k}B + f^*\left(H + \left(l - \frac{k-1}{k}m\right)L\right).$$

We note that the $\mathbb{R}$-divisor

$$H + \left(l - \frac{k-1}{k}m\right)L$$

on Y is ample if $l + t - \frac{k-1}{k}m > 0$. We also note that $(X, \Delta + \frac{k-1}{k}B)$ is semi-log canonical since B is a general member of the free linear system $|\mathcal{O}_X(D) \otimes f^*\mathcal{O}_Y(mL)|$. Thus we obtain that

$$H^i(Y, f_*\mathcal{O}_X(D) \otimes \mathcal{O}_Y(lL)) = 0$$

for all $i > 0$ and $l > \frac{k-1}{k}m - t$ by Theorem 3.1. Therefore, for every $l > \frac{k-1}{k}m - t + n$, we have

$$H^i(Y, f_*\mathcal{O}_X(D) \otimes \mathcal{O}_X(lL) \otimes \mathcal{O}_Y(-iL)) = 0$$

for every $i > 0$. Hence, $f_*\mathcal{O}_X(D) \otimes \mathcal{O}_Y(lL)$ is globally generated for $l > \frac{k-1}{k}m - t + n$ by the Castelnuovo–Mumford regularity. Given our minimal choice of m, we conclude that for the smallest integer l_0 which is greater than $\frac{k-1}{k}m - t$ we have $m \leq l_0 + n$. This implies that

$$m \leq l_0 + n \leq \frac{k-1}{k}m + n + 1 - t,$$

which is equivalent to $m \leq k(n + 1 - t)$. Thus,

$$H^i(Y, f_*\mathcal{O}_X(D) \otimes \mathcal{O}_Y(lL)) = 0$$

for every $i > 0$ and $l \geq (k-1)(n + 1 - t) - t + 1$. $\square$

As a direct consequence of Theorem 1.1, we have:

Corollary 3.3. *Let (X, Δ) be a projective semi-log canonical pair and let $f : X \to Y$ be a surjective morphism onto an n-dimensional projective variety Y. Assume that $\mathcal{O}_X(k(K_X + \Delta))$ is locally free and f-generated for some positive integer k. Let L be an ample Cartier divisor on Y such that $|L|$ is free. Then*

$$H^i(Y, f_*\mathcal{O}_X(k(K_X + \Delta)) \otimes \mathcal{O}_Y(lL)) = 0$$

for every $i > 0$ and every $l \geq k(n + 1) - n$. Therefore, by the Castelnuovo–Mumford regularity, $f_*\mathcal{O}_X(k(K_X + \Delta)) \otimes \mathcal{O}_Y(lL)$ is globally generated for every $l \geq k(n + 1)$.

Proof. We put $D = k(K_X + \Delta + f^*L)$ and apply Theorem 1.1. Then we obtain

$$H^i(Y, f_*\mathcal{O}_X(D) \otimes \mathcal{O}_Y((l - k)L)) = 0$$

for every $i > 0$ and every $l - k \geq (k - 1)n$, equivalently, $l \geq k(n + 1) - n$. □

Corollary 3.3 will play a crucial role in the proof of Theorem 1.2. Let us prove Theorem 1.2. The idea of the proof of Theorem 1.2 is the same as [Fuj4, Section 5] (see also [Fuj5] and [Fuj8, Subsection 3.1]).

Proof of Theorem 1.2. Let s be an arbitrary positive integer. Let

$$X^{(s)} = \underbrace{X \times_Y X \times_Y \cdots \times_Y X}_{s}$$

be the s-fold fiber product of X over Y and let $f^{(s)} : X^{(s)} \to Y$ be the induced natural map. Let p_i be the i-th projection $X^{(s)} \to X$ for $1 \leq i \leq s$. We put $\Delta_{X^{(s)}} = \sum_{i=1}^{s} p_i^{-1}\Delta$. Then we have:

$$\mathcal{O}_{X^{(s)}}(k(K_{X^{(s)}/Y} + \Delta_{X^{(s)}})) \simeq \bigotimes_{i=1}^{s} p_i^*\mathcal{O}_X(k(K_{X/Y} + \Delta)), \tag{3.1}$$

and

$$(X^{(s)}, \Delta_{X^{(s)}}) \text{ is semi-log canonical.} \tag{3.2}$$

We will check the isomorphism (3.1) and the statement (3.2). We use induction on s. If $s = 1$, then the isomorphism (3.1) and the statement (3.2) are obvious. By the induction hypothesis, we have

$$\mathcal{O}_{X^{(s-1)}}(k(K_{X^{(s-1)}/Y} + \Delta_{X^{(s-1)}})) \simeq \bigotimes_{i=1}^{s-1} p_i^*\mathcal{O}_X(k(K_{X/Y} + \Delta)). \tag{3.3}$$

Therefore, it is sufficient to prove that

$$\begin{aligned} &\mathcal{O}_{X^{(s)}}(k(K_{X^{(s)}/Y} + \Delta_{X^{(s)}})) \\ &\simeq p_s^*\mathcal{O}_X(k(K_{X/Y} + \Delta)) \otimes q_s^*\mathcal{O}_{X^{(s-1)}}(k(K_{X^{(s-1)}/Y} + \Delta_{X^{(s-1)}})), \end{aligned} \tag{3.4}$$

where $q_s = (p_1, \cdots, p_{s-1}) : X^{(s)} \to X^{(s-1)}$. The following commutative diagram

$$\begin{array}{ccc} X^{(s)} & \xrightarrow{q_s} & X^{(s-1)} \\ {\scriptstyle p_s}\downarrow & \searrow^{f^{(s)}} & \downarrow{\scriptstyle f^{(s-1)}} \\ X & \xrightarrow[f]{} & Y \end{array} \tag{3.5}$$

may be helpful. By the commutative diagram (3.5) and the induction hypothesis, we see that $X^{(s)}$ is demi-normal (see, for example, Step 1 in the proof of [Pat2, Lemma 2.12]). By throwing out codimension at most two closed subsets, we may find a Zariski open subset $V \subset X^{(s)}$ such that $X|_{p_s(V)}$ and $X^{(s-1)}|_{q_s(V)}$ are Gorenstein and $k\Delta|_{p_i(V)}$ is Cartier for every i. By the flat base change theorem [Har1, Chapter VII, Corollary 3.4] (see also [Con]), the isomorphism (3.4) holds over V. On the other hand, in (3.4), the right hand side is locally free and the left hand side is reflexive. Therefore, we obtain the desired isomorphism (3.4) over $X^{(s)}$. This implies the isomorphism (3.1). By (3.1), we see that $\mathcal{O}_{X^{(s)}}(k(K_{X^{(s)}}+\Delta_{X^{(s)}}))$ is locally free and $f^{(s)}$-generated. By a special case of [Pat2, Lemma 2.12] and Lemma 2.13, we see that $(X^{(s)}, \Delta_{X^{(s)}})$ is semi-log canonical. This is essentially the inversion of adjunction on semi-log canonicity (see [Pat2, Lemma 2.10 and Corollary 2.11], which is based on Kawakita's inversion of adjunction on log canonicity [Kaw]). Moreover, we have:

$$f^{(s)}_*\mathcal{O}_{X^{(s)}}(k(K_{X^{(s)}/Y}+\Delta_{X^{(s)}}))\simeq \bigotimes^{s} f_*\mathcal{O}_X(k(K_{X/Y}+\Delta)). \tag{3.6}$$

We will check the isomorphism (3.6). We use induction on s. If $s=1$, then it is obvious. By (3.4), we have

$$\begin{aligned}
&f^{(s)}_*\mathcal{O}_{X^{(s)}}(k(K_{X^{(s)}/Y}+\Delta_{X^{(s)}}))\\
&\simeq f_*p_{s*}(p_s^*\mathcal{O}_X(k(K_{X/Y}+\Delta))\otimes q_s^*\mathcal{O}_{X^{(s-1)}}(k(K_{X^{(s-1)}/Y}+\Delta_{X^{(s-1)}})))\\
&\simeq f_*(\mathcal{O}_X(k(K_{X/Y}+\Delta))\otimes p_{s*}q_s^*\mathcal{O}_{X^{(s-1)}}(k(K_{X^{(s-1)}/Y}+\Delta_{X^{(s-1)}})))\\
&\simeq f_*(\mathcal{O}_X(k(K_{X/Y}+\Delta))\otimes f^*f^{(s-1)}_*\mathcal{O}_{X^{(s-1)}}(k(K_{X^{(s-1)}/Y}+\Delta_{X^{(s-1)}})))\\
&\simeq f_*\mathcal{O}_X(k(K_{X/Y}+\Delta))\otimes f^{(s-1)}_*\mathcal{O}_{X^{(s-1)}}(k(K_{X^{(s-1)}/Y}+\Delta_{X^{(s-1)}}))\\
&\simeq \bigotimes^{s} f_*\mathcal{O}_X(k(K_{X/Y}+\Delta))
\end{aligned}$$

by the projection formula and the flat base change theorem (see [Har2, Chapter III, Proposition 9.3]). This is the desired isomorphism (3.6). Note that $f_*\mathcal{O}_X(k(K_{X/Y}+\Delta))$ is locally free since Y is a smooth projective curve.

Let L be an ample Cartier divisor on Y such that $|L|$ is free. We put $M=kK_Y+2kL$. Then

$$f^{(s)}_*\mathcal{O}_{X^{(s)}}(k(K_{X^{(s)}/Y}+\Delta_{X^{(s)}}))\otimes\mathcal{O}_Y(M)\simeq\left(\bigotimes^{s} f_*\mathcal{O}_X(k(K_{X/Y}+\Delta))\right)\otimes\mathcal{O}_Y(M)$$

is globally generated by Corollary 3.3. Note that M is independent of s. This implies that $f_*\mathcal{O}_X(k(K_{X/Y}+\Delta))$ is a nef locally free sheaf by Lemma 3.4 below. □

The following well-known lemma was already used in the proof of Theorem 1.2. We include it here for the reader's convenience.

Lemma 3.4. *Let $\mathcal{E}$ be a non-zero locally free sheaf of finite rank on a smooth projective variety V. Assume that there exists a line bundle $\mathcal{M}$ such that $\mathcal{E}^{\otimes s}\otimes\mathcal{M}$ is globally generated for every positive integer s. Then $\mathcal{E}$ is nef.*

Proof. We put $\pi: W=\mathbb{P}_V(\mathcal{E})\to V$ and $\mathcal{O}_W(1)=\mathcal{O}_{\mathbb{P}_V(\mathcal{E})}(1)$. Since $\mathcal{E}^{\otimes s}\otimes\mathcal{M}$ is globally generated, $\mathrm{Sym}^s\mathcal{E}\otimes\mathcal{M}$ is also globally generated for every positive integer s. This implies that $\mathcal{O}_W(s)\otimes\pi^*\mathcal{M}$ is globally generated for every positive integer s. Thus, we obtain that $\mathcal{O}_W(1)$ is nef, equivalently, $\mathcal{E}$ is nef. □

We sketch the proof of Theorem 1.3 for the reader's convenience. For the details, see [Kol2] and [Fuj9].

Sketch of Proof of Theorem 1.3. As in [Kol2, Section 2], we may assume that we start with a bounded moduli functor $\mathcal{M}$. We consider $(f: X\to C)\in\mathcal{M}(C)$, where C is a smooth projective curve. Then, by [Pat2, Lemma 2.12], X is a semi-log canonical variety. Since $\mathcal{M}$ is bounded, we can take a large and divisible positive integer k, which is independent of C, such that $\mathcal{O}_X(kK_{X/C})$ is locally free and f-generated. By Theorem 1.2, $f_*\mathcal{O}_X(klK_{X/C})$ is nef for every positive integer l. By Kollár's projectivity criterion, this implies that every complete subspace of the moduli space of stable varieties is projective. For the details, see [Kol2, Sections 2 and 3]. ☐

We close this section with a remark on *slc morphisms*.

Remark 3.5. In Theorem 1.2, (i) and (ii) are equivalent to the condition that $f : (X, \Delta) \to Y$ is an slc morphism (see [Fuj2, Definition 6.11]) since Y is a smooth curve.

4. Proof of the basic semipositivity theorem

In this section, we will give a proof of Theorem 1.4, which was first proved in [Fuj9], without using the theory of graded polarizable admissible variation of mixed Hodge structure (see [FF] and [FFS]). Our approach to the basic semipositivity theorem (see Theorem 1.4 and [Fuj9, Theorem 1.9]) is arguably simpler than the arguments in [Fuj9].

Let us start with the following easy lemma, which is a variant of Lemma 3.4.

Lemma 4.1. *Let $\mathcal{E}$ be a non-zero locally free sheaf of finite rank on a smooth projective curve C. Let $\mathcal{M}$ be a fixed line bundle on C. Assume that $\mathcal{E}^{\otimes s} \otimes \mathcal{M}$ is generically globally generated for every positive integer s. Then $\mathcal{E}$ is nef.*

Proof. Let $p : C' \to C$ be a finite morphism from a smooth projective curve C'. Let $\mathcal{L}$ be a quotient line bundle of $p^*\mathcal{E}$. Then we have a surjective morphism

$$p^*(\mathcal{E}^{\otimes s} \otimes \mathcal{M}) \to \mathcal{L}^{\otimes s} \otimes p^*\mathcal{M}.$$

By assumption, $p^*(\mathcal{E}^{\otimes s} \otimes \mathcal{M})$ is generically globally generated for every positive integer s. Therefore, $\deg(\mathcal{L}^{\otimes s} \otimes p^*\mathcal{M}) \geq 0$ for every positive integer s. This means that $\deg \mathcal{L} \geq 0$. Therefore, $\mathcal{E}$ is nef. ☐

We will prove Theorem 1.4 by using the vanishing theorem for semi-log canonical pairs: Theorem 3.1.

Proof of Theorem 1.4. We will modify the proof of Theorem 1.2. Let s be an arbitrary positive integer. Let

$$X^{(s)} = \underbrace{X \times_C X \times_C \cdots \times_C X}_{s}$$

be the s-fold fiber product of X over C and let $f^{(s)} : X^{(s)} \to C$ be the induced natural map. Let p_i be the i-th projection $X^{(s)} \to X$ for $1 \leq i \leq s$. We put $D^{(s)} = \sum_{i=1}^{s} p_i^* D$. Then we have

$$\mathcal{O}_{X^{(s)}}(K_{X^{(s)}/C} + D^{(s)}) \simeq \bigotimes_{i=1}^{s} p_i^* \mathcal{O}_X(K_{X/C} + D) \tag{4.1}$$

by the flat base change theorem. The isomorphism (4.1) can be checked similarly to the isomorphism (3.1) in the proof of Theorem 1.2. We note that the isomorphism (4.1) is equivalent to

$$\mathcal{O}_{X^{(s)}}(K_{X^{(s)}/C}) \simeq \bigotimes_{i=1}^{s} p_i^* \mathcal{O}_X(K_{X/C}) \tag{4.2}$$

by the definition of $D^{(s)} = \sum_{i=1}^{s} p_i^* D$. Moreover, we have

$$f_*^{(s)} \mathcal{O}_{X^{(s)}}(K_{X^{(s)}/C} + D^{(s)}) \simeq \bigotimes^{s} f_* \mathcal{O}_X(K_{X/C} + D). \tag{4.3}$$

Note that we can prove the isomorphism (4.3) similarly to the isomorphism (3.6) in the proof of Theorem 1.2. The following commutative diagram

$$\begin{array}{ccc} X^{(s)} & \xrightarrow{q_s} & X^{(s-1)} \\ {\scriptstyle p_s}\downarrow & \searrow^{f^{(s)}} & \downarrow{\scriptstyle f^{(s-1)}} \\ X & \xrightarrow[f]{} & C, \end{array}$$

where $q_s = (p_1, \cdots, p_{s-1}) : X^{(s)} \to X^{(s-1)}$, and the isomorphism

$$\mathcal{O}_{X^{(s)}}(K_{X^{(s)}/C} + D^{(s)}) \simeq p_s^*\mathcal{O}_X(K_{X/C} + D) \otimes q_s^*\mathcal{O}_{X^{(s-1)}}(K_{X^{(s-1)}/C} + D^{(s-1)}), \tag{4.4}$$

may be helpful.

Let U be a non-empty Zariski open subset of C such that every stratum of $(X, D)|_{f^{-1}(U)}$ is smooth over U. Then we can directly see that $(X^{(s)}, D^{(s)})|_{f^{(s)-1}(U)}$ is semi-log canonical and that every irreducible component of $f^{(s)-1}(U)$ is smooth. Let V be the largest Zariski open subset of $X^{(s)}$ such that $(V, D^{(s)}|_V)$ is a simple normal crossing pair. By construction, we see that $\operatorname{codim}_{f^{(s)-1}(U)}(f^{(s)-1}(U) \setminus V) \geq 2$. By the theorem of Bierstone–Vera Pacheco (see [BieVP, Theorem 1.4]), we have a projective surjective birational morphism $g : Z \to X^{(s)}$, which is given by a composite of blow-ups, an isomorphism over V, and maps $\operatorname{Sing} Z$ birationally onto the closure of $\operatorname{Sing} V$ in $X^{(s)}$, such that $\operatorname{Exc}(g) \cup \operatorname{Supp} g^*D^{(s)}$ is a simple normal crossing divisor on Z. Since g is birational, we have a generically isomorphic injection $g_*\omega_Z \subset \omega_{X^{(s)}}$. Note that Z and $X^{(s)}$ are both Gorenstein. Therefore, we have a generically isomorphic injection

$$g_*\omega_Z(g^*D^{(s)}) \subset \omega_{X^{(s)}}(D^{(s)}). \tag{4.5}$$

We can take a reduced Weil divisor Δ_Z on Z such that $\operatorname{Supp} \Delta_Z \subset \operatorname{Exc}(g) \cup \operatorname{Supp} g^*D^{(s)}$ and that

$$g_*\omega_Z(\Delta_Z) \simeq \omega_{X^{(s)}}(D^{(s)}) \tag{4.6}$$

holds on $f^{(s)-1}(U)$. We note that $(X^{(s)}, D^{(s)})|_{f^{(s)-1}(U)}$ is semi-log canonical. More precisely, we put

$$\Delta_Z = \sum_i E_i + \sum_j F_j \tag{4.7}$$

where E_i (resp. F_j) runs over the irreducible components of $\operatorname{Exc}(g)$ (resp. $g_*^{-1}D^{(s)}$) such that $f^{(s)}(E_i) \cap U \neq \emptyset$ (resp. $f^{(s)}(F_j) \cap U \neq \emptyset$). Then $g_*\omega_Z(\Delta_Z) \simeq \omega_{X^{(s)}}(D^{(s)})$ holds on $f^{(s)-1}(U)$. By the definition of Δ_Z, we have

$$0 \leq \Delta_Z \leq g^*D^{(s)} + \sum_i E_i. \tag{4.8}$$

Therefore, we have natural inclusions

$$g_*\omega_Z(\Delta_Z) \subset g_*\omega_Z\left(g^*D^{(s)} + \sum_i E_i\right) \subset \omega_{X^{(s)}}(D^{(s)}). \tag{4.9}$$

Note that $\operatorname{codim}_{X^{(s)}}(g(\sum_i E_i)) \geq 2$ and that $\omega_{X^{(s)}}(D^{(s)})$ is locally free. By taking some suitable blow-ups of Z outside $(f^{(s)} \circ g)^{-1}(U)$, if necessary, we may further assume that Δ_Z is a simple normal crossing divisor on Z, that is, Δ_Z is Cartier (see, for example, [Fuj9, Lemma 2.11]). Anyway, we have a natural inclusion

$$(f^{(s)} \circ g)_*\omega_{Z/C}(\Delta_Z) \subset f_*^{(s)}\omega_{X^{(s)}/C}(D^{(s)}) \simeq \bigotimes^s f_*\omega_{X/C}(D) \tag{4.10}$$

by the inclusions (4.9) and the isomorphism (4.3), which is an isomorphism on U. We put $\mathcal{M} = \omega_C \otimes \mathcal{L}^{\otimes 2}$, where $\mathcal{L}$ is an ample line bundle on C such that $|\mathcal{L}|$ is free. Then $(f^{(s)} \circ g)_* \omega_{Z/C}(\Delta_Z) \otimes \mathcal{M}$ is globally generated by Corollary 3.2. Therefore, we obtain that

$$\left(\bigotimes^{s} f_* \omega_{X/C}(D) \right) \otimes \mathcal{M} \tag{4.11}$$

is generically globally generated for every positive integer s. By Lemma 4.1, we obtain that $f_* \omega_{X/C}(D)$ is nef. Anyway, we obtain that $f_* \omega_{X/C}(D)$ is nef without using the theory of variation of (mixed) Hodge structure. □

Remark 4.2. In the above proof of Theorem 1.4, we did not use the assumption that every stratum of X is dominant onto C. It is sufficient to assume that $f : X \to C$ is flat for Theorem 1.4 (see also [Fuj9, Theorem 1.10], which is more general than Theorem 1.4).

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