0.1 Vanishing and injectivity theorems

sec4 [1]

The main purpose of this section is to prove Ambro's theorems (cf. [?, Theorems 3.1 and 3.2]) for embedded simple normal crossing pairs. The next lemma (cf. [?, Proposition 1.11]) is missing in the proof of [?, Theorem 3.1]. It justifies the first three lines in the proof of [?, Theorem 3.1].

re-vani-lem **Lemma 0.1** (Relative vanishing lemma)**.** *Let $f : Y \to X$ be a proper morphism from a simple normal crossing pair $(Y, T + D)$ such that $T + D$ is a boundary $\mathbb{R}$-divisor, T is reduced, and $\llcorner D \lrcorner = 0$. We assume that f is an isomorphism at the generic point of any stratum of the pair $(Y, T+D)$. Let L be a Cartier divisor on Y such that $L \sim_{\mathbb{R}} K_Y + T + D$. Then $R^q f_* \mathcal{O}_Y(L) = 0$ for $q > 0$.*

Proof. By Lemma ??, we can assume that D is a $\mathbb{Q}$-divisor and $L \sim_{\mathbb{Q}} K_Y + T + D$. We divide the proof into two steps.

1ne **Step 1.** We assume that Y is irreducible. In this case, $L - (K_Y + T + D)$ is nef and log big over X with respect to the pair $(Y, T + D)$ (see Definition ??). Therefore, $R^q f_* \mathcal{O}_Y(L) = 0$ for every $q > 0$ by the vanishing theorem (see, for example, Lemma ??).

Step 2. Let Y_1 be an irreducible component of Y and Y_2 the union of the other irreducible components of Y. Then we have a short exact sequence $0 \to i_* \mathcal{O}_{Y_1}(-Y_2|_{Y_1}) \to \mathcal{O}_Y \to \mathcal{O}_{Y_2} \to 0$, where $i : Y_1 \to Y$ is the natural closed immersion (cf. [?, Remark 2.6]). We put $L' = L|_{Y_1} - Y_2|_{Y_1}$. Then we have a short exact sequence $0 \to i_* \mathcal{O}_{Y_1}(L') \to \mathcal{O}_Y(L) \to \mathcal{O}_{Y_2}(L|_{Y_2}) \to 0$ and $L' \sim_{\mathbb{Q}} K_{Y_1} + T|_{Y_1} + D|_{Y_1}$. On the other hand, we can check that $L|_{Y_2} \sim_{\mathbb{Q}} K_{Y_2} + Y_1|_{Y_2} + T|_{Y_2} + D|_{Y_2}$. We have already known that $R^q f_* \mathcal{O}_{Y_1}(L') = 0$ for every $q > 0$ by Step 1. By the induction on the number of the irreducible components of Y, we have $R^q f_* \mathcal{O}_{Y_2}(L|_{Y_2}) = 0$ for every $q > 0$. Therefore, $R^q f_* \mathcal{O}_Y(L) = 0$ for every $q > 0$ by the exact sequence:

$$\cdots \to R^q f_* \mathcal{O}_{Y_1}(L') \to R^q f_* \mathcal{O}_Y(L) \to R^q f_* \mathcal{O}_{Y_2}(L|_{Y_2}) \to \cdots.$$

So, we finish the proof of Lemma 0.1. $\square$

The following lemma is a variant of Szabó's resolution lemma (see the resolution lemma in ??).

[1] This is a revised version of Section 2.5 of my book. 2010/3/25 version 1.01

6 **Lemma 0.2.** *Let (X, B) be an embedded simple normal crossing pair and D a permissible Cartier divisor on X. Let M be an ambient space of X. Assume that there exists an $\mathbb{R}$-divisor A on M such that* $\mathrm{Supp}(A+X)$ *is simple normal crossing on M and that $B = A|_X$. Then there exists a projective birational morphism $g : N \to M$ from a smooth variety N with the following properties. Let Y be the strict transform of X on N and $f = g|_Y : Y \to X$. Then we have*

(i) *$g^{-1}(D)$ is a divisor on N.* $\mathrm{Exc}(g) \cup g_*^{-1}(A+X)$ *is simple normal crossing on N, where* $\mathrm{Exc}(g)$ *is the exceptional locus of g. In particular, Y is a simple normal crossing divisor on N.*

(ii) *g and f are isomorphisms outside D, in particular, $f_*\mathcal{O}_Y \simeq \mathcal{O}_X$.*

(iii) *$f^*(D+B)$ has a simple normal crossing support on Y. More precisely, there exists an $\mathbb{R}$-divisor A' on N such that* $\mathrm{Supp}(A'+Y)$ *is simple normal crossing on N, A' and Y have no common irreducible components, and that $A'|_Y = f^*(D+B)$.*

Proof. First, we take a blow-up $M_1 \to M$ along D. Apply Hironaka's desingularization theorem to M_1 and obtain a projective birational morphism $M_2 \to M_1$ from a smooth variety M_2. Let F be the reduced divisor that coincides with the support of the inverse image of D on M_2. Apply Szabó's resolution lemma to $\mathrm{Supp}\sigma^*(A+X) \cup F$ on M_2 (see, for example, 15-resol ?? or fujino0 [?, 3.5. Resolution lemma]), where $\sigma : M_2 \to M$. Then, we obtain desired projective birational morphisms $g : N \to M$ from a smooth variety N, and $f = g|_Y : Y \to X$, where Y is the strict transform of X on N, such that Y is a simple normal crossing divisor on N, g and f are isomorphisms outside D, and $f^*(D+B)$ has a simple normal crossing support on Y. Since f is an isomorphism outside D and D is permissible on X, f is an isomorphism at the generic point of any stratum of Y. Therefore, every fiber of f is connected and then $f_*\mathcal{O}_Y \simeq \mathcal{O}_X$. □

Remark 0.3. In Lemma 6 0.2, we can directly check that $f_*\mathcal{O}_Y(K_Y) \simeq \mathcal{O}_X(K_X)$. By Lemma 5.1, $R^q f_*\mathcal{O}_Y(K_Y) = 0$ for $q > 0$. Therefore, we obtain $f_*\mathcal{O}_Y \simeq \mathcal{O}_X$ and $R^q f_*\mathcal{O}_Y = 0$ for every $q > 0$ by the Grothendieck duality.

Here, we treat the compactification problem. It is because we can use the same technique as in the proof of Lemma 6 0.2. This lemma plays important roles in this section.

comp **Lemma 0.4.** *Let $f : Z \to X$ be a proper morphism from an embedded simple normal crossing pair (Z, B). Let M be the ambient space of Z. Assume that there is an $\mathbb{R}$-divisor A on M such that* $\mathrm{Supp}(A+Z)$ *is simple normal crossing on M and that $B = A|_Z$. Let $\overline{X}$ be a projective variety such that $\overline{X}$ contains X as a Zariski open set. Then there exist a proper embedded simple normal crossing pair $(\overline{Z}, \overline{B})$ that is a compactification of (Z, B) and a proper morphism $\overline{f} : \overline{Z} \to \overline{X}$ that compactifies $f : Z \to X$. Moreover,* $\mathrm{Supp}\overline{B} \cup \mathrm{Supp}(\overline{Z} \setminus Z)$ *is a simple normal crossing divisor on $\overline{Z}$, and $\overline{Z} \setminus Z$ has no common irreducible components with $\overline{B}$. We note that $\overline{B}$ is $\mathbb{R}$-Cartier. Let $\overline{M}$, which is a compactification of M, be the ambient space of $(\overline{Z}, \overline{B})$. Then, by the construction, we can find an $\mathbb{R}$-divisor $\overline{A}$ on $\overline{M}$ such that* $\mathrm{Supp}(\overline{A}+\overline{Z})$ *is simple normal crossing on $\overline{M}$ and that $\overline{B} = \overline{A}|_{\overline{Z}}$.*

Proof. Let $\overline{Z}, \overline{A} \subset \overline{M}$ be any compactification. By blowing up $\overline{M}$ inside $\overline{Z} \setminus Z$, we can assume that $f : Z \to X$ extends to $\overline{f} : \overline{Z} \to \overline{X}$. By Hironaka's desingularization and the resolution lemma, we can assume that $\overline{M}$ is smooth and $\mathrm{Supp}(\overline{Z} + \overline{A}) \cup \mathrm{Supp}(\overline{M} \setminus M)$ is a simple normal crossing divisor on $\overline{M}$. It is not difficult to see that the above compactification has the desired properties. □

rem-2 **Remark 0.5.** There exists a big trouble to compactify normal crossing varieties. When we treat normal crossing varieties, we can not directly compactify them. For the details, see fujino0 [?, 3.6, Whitney umbrella], especially, Corollary 3.6.10 and Remark 3.6.11 in fujino0 [?]. Therefore, the first two lines in the proof of ambro [?, Theorem 3.2] is nonsense.

It is the time to state the main injectivity theorem (cf. ambro [?, Theorem 3.1]) for embedded *simple* normal crossing pairs. For applications, this formulation seems to be sufficient. We note that we will recover ambro [?, Theorem 3.1] in full generality in Section sec6 ?? (see Theorem 61 ??).

5.1 **Theorem 0.6** (cf. ambro [?, Theorem 3.1])**.** *Let $(X, S+B)$ be an embedded simple normal crossing pair such that X is proper, $S+B$ is a boundary $\mathbb{R}$-divisor, S is reduced, and $\llcorner B \lrcorner = 0$. Let L be a Cartier divisor on X and D an effective Cartier divisor that is permissible with respect to $(X, S+B)$. Assume the following conditions.*

(i) *$L \sim_{\mathbb{R}} K_X + S + B + H$,*

(ii) *H is a semi-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor, and*

(iii) $tH \sim_{\mathbb{R}} D+D'$ *for some positive real number* t*, where* D' *is an effective* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor that is permissible with respect to* $(X, S+B)$*.*

Then the homomorphisms

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L+D)),$$

which are induced by the natural inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$*, are injective for all* q*.*

Proof. First, we use Lemma ??. Thus, we can assume that there exists a divisor A on M, where M is the ambient space of X, such that $\mathrm{Supp}(A+X)$ is simple normal crossing on M and that $A|_X = S$. Apply Lemma 0.2 to an embedded simple normal crossing pair (X, S) and a divisor $\mathrm{Supp}(D+D'+B)$ on X. Then we obtain a projective birational morphism $f: Y \to X$ from an embedded simple normal crossing variety Y such that f is an isomorphism outside $\mathrm{Supp}(D+D'+B)$, and that the union of the support of $f^*(S+B+D+D')$ and the exceptional locus of f has a simple normal crossing support on Y. Let B' be the strict transform of B on Y. We can assume that $\mathrm{Supp}B'$ is disjoint from any strata of Y that are not irreducible components of Y by taking blow-ups. We write $K_Y + S' + B' = f^*(K_X + S + B) + E$, where S' is the strict transform of S, and E is f-exceptional. By the construction of $f: Y \to X$, S' is Cartier and B' is $\mathbb{R}$-Cartier. Therefore, E is also $\mathbb{R}$-Cartier. It is easy to see that $E_+ = \ulcorner E \urcorner \geq 0$. We put $L' = f^*L + E_+$ and $E_- = E_+ - E \geq 0$. We note that E_+ is Cartier and E_- is $\mathbb{R}$-Cartier because $\mathrm{Supp}E$ is simple normal crossing on Y. Since f^*H is an $\mathbb{R}_{>0}$-linear combination of semi-ample Cartier divisors, we can write $f^*H \sim_{\mathbb{R}} \sum_i a_i H_i$, where $0 < a_i < 1$ and H_i is a general Cartier divisor on Y for every i. We put $B'' = B' + E_- + \frac{\varepsilon}{t} f^*(D+D') + (1-\varepsilon)\sum_i a_i H_i$ for some $0 < \varepsilon \ll 1$. Then $L' \sim_{\mathbb{R}} K_Y + S' + B''$. By the construction, $\llcorner B'' \lrcorner = 0$, the support of $S' + B''$ is simple normal crossing on Y, and $\mathrm{Supp}B'' \supset \mathrm{Supp}f^*D$. So, Proposition ?? implies that the homomorphisms $H^q(Y, \mathcal{O}_Y(L')) \to H^q(Y, \mathcal{O}_Y(L' + f^*D))$ are injective for all q. By Lemma 0.1, $R^q f_* \mathcal{O}_Y(L') = 0$ for any $q > 0$ and it is easy to see that $f_* \mathcal{O}_Y(L') \simeq \mathcal{O}_X(L)$. By the Leray spectral sequence, the homomorphisms $H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L+D))$ are injective for all q. □

The following theorem is another main theorem of this section. It is essentially the same as [?, Theorem 3.2]. We note that we assume that

$(Y, S+B)$ is a *simple* normal crossing pair. It is a small but technically important difference. For the full statement, see Theorem ?? below.

8 **Theorem 0.7** (cf. [?, Theorem 3.2]). *Let $(Y, S+B)$ be an embedded simple normal crossing pair such that $S+B$ is a boundary $\mathbb{R}$-divisor, S is reduced, and $\llcorner B \lrcorner = 0$. Let $f : Y \to X$ be a proper morphism and L a Cartier divisor on Y such that $H \sim_{\mathbb{R}} L - (K_Y + S + B)$ is f-semi-ample.*

(i) *every non-zero local section of $R^q f_* \mathcal{O}_Y(L)$ contains in its support the f-image of some stratum of $(Y, S+B)$.*

(ii) *let $\pi : X \to V$ be a projective morphism and assume that $H \sim_{\mathbb{R}} f^* H'$ for some π-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor H' on X. Then $R^q f_* \mathcal{O}_Y(L)$ is π_*-acyclic, that is, $R^p \pi_* R^q f_* \mathcal{O}_Y(L) = 0$ for every $p > 0$ and $q \geq 0$.*

Remark 0.8. It is obvious that the statement of Theorem 0.7 (i) is equivalent to the following one.

(i′) every associated prime of $R^q f_* \mathcal{O}_Y(L)$ is the generic point of the f-image of some stratum of $(Y, S+B)$.

Proof. Let M be the ambient space of Y. Then, by Lemma ??, we can assume that there exists an $\mathbb{R}$-divisor D on M such that $\mathrm{Supp}(D+Y)$ is simple normal crossing on M and that $D|_Y = S+B$. Therefore, we can use Lemma 0.4 in Step 2 of (i) and Step 3 of (ii) below.

(i) We have already proved a very spacial case in Lemma 0.1.

Step 1. First, we assume that X is projective. We can assume that H is semi-ample by replacing L (resp. H) with $L + f^*A'$ (resp. $H + f^*A'$), where A' is a very ample Cartier divisor. Assume that $R^q f_* \mathcal{O}_Y(L)$ has a local section whose support does not contain the f-images of any strata of $(Y, S+B)$. More precisely, let U be a non-empty Zariski open set and $s \in \Gamma(U, R^q f_* \mathcal{O}_Y(L))$ a non-zero section of $R^q f_* \mathcal{O}_Y(L)$ on U whose support $V \subset U$ does not contain the f-images of any strata of $(Y, S+B)$. Let $\overline{V}$ be the closure of V in X. We note that $\overline{V} \setminus V$ may contain the f-image of some stratum of $(Y, S+B)$. Let Y_1 be the union of the irreducible components of Y that are mapped into $\overline{V} \setminus V$ and let Y_2 be the union of the other irreducible components of Y. We put

$$K_{Y_1} + S_1 + B_1 = (K_Y + S + B)|_{Y_1}$$

such that S_1 is reduced and that $\llcorner B_1 \lrcorner = 0$. By replacing Y, S, B, L, and H with Y_1, S_1, B_1, $L|_{Y_1}$, and $H|_{Y_1}$, we can assume that no irreducible components of Y are mapped into $\overline{V} \setminus V$. Let C be a stratum of $(Y, S+B)$ that is mapped into $\overline{V} \setminus V$. Let $\sigma : M' \to M$ be the blow-up along C and $Y' = \sigma^{-1}(Y)$ as in Lemma useful-lem2 ??. We can write $Y' = Y_1' \cup Y_2'$ where $Y_2' = \sigma^{-1}(C)$. We put

$$K_{Y'} + S' + B' = \sigma^*(K_Y + S + B)$$

such that S' is reduced and $\llcorner B' \lrcorner = 0$. We define

$$K_{Y_1'} + S_1' + B_1' = (K_{Y'} + S' + B')|_{Y_1'}$$

such that S_1' is reduced and $\llcorner B_1' \lrcorner = 0$. Thus,

$$\sigma^* H \sim_{\mathbb{R}} \sigma^* L - (K_{Y'} + S' + B')$$

and

$$\sigma^* H|_{Y_1'} \sim_{\mathbb{R}} \sigma^* L|_{Y_1'} - Y_2'|_{Y_1'} - (K_{Y_1'} + (S_1' - Y_2'|_{Y_1'}) + B_1').$$

We note that $S_1' - Y_2'|_{Y_1'}$ is effective. We replace Y, H, L, S, and B with Y_1', $\sigma^* H|_{Y_1'}$, $\sigma^* L|_{Y_1'}$, $S_1' - Y_2'|_{Y_1'}$, and B_1'. By repeating this process finitely many times, we can assume that $\overline{V}$ does not contain f-images of any strata of $(Y, S+B)$. Then we can find a very ample Cartier divisor A with the following properties.

(a) f^*A is permissible with respect to $(Y, S+B)$, and

(b) $R^q f_* \mathcal{O}_Y(L) \to R^q f_* \mathcal{O}_Y(L) \otimes \mathcal{O}_X(A)$ is not injective.

We can assume that $H - f^*A$ is semi-ample by replacing L (resp. H) with $L + f^*A$ (resp. $H + f^*A$). If necessary, we replace L (resp. H) with $L + f^*A''$ (resp. $H + f^*A''$), where A'' is a very ample Cartier divisor. Then, we have $H^0(X, R^q f_* \mathcal{O}_Y(L)) \simeq H^q(Y, \mathcal{O}_Y(L))$ and $H^0(X, R^q f_* \mathcal{O}_Y(L) \otimes \mathcal{O}_X(A)) \simeq H^q(Y, \mathcal{O}_Y(L + f^*A))$. We obtain that

$$H^0(X, R^q f_* \mathcal{O}_Y(L)) \to H^0(X, R^q f_* \mathcal{O}_Y(L) \otimes \mathcal{O}_X(A))$$

is not injective by (b) if A'' is sufficiently ample. So, $H^q(Y, \mathcal{O}_Y(L)) \to H^q(Y, \mathcal{O}_Y(L + f^*A))$ is not injective. It contradicts Theorem 5.1 0.6. We finish the proof when X is projective.

8-2 **Step 2.** Next, we assume that X is not projective. Note that the problem is local. So, we can shrink X and assume that X is affine. By the argument similar to the one in Step 1 in the proof of (ii) below, we can assume that H is a semi-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. We compactify X and apply Lemma comp 0.4. Then we obtain a compactification $\overline{f}:\overline{Y}\to\overline{X}$ of $f:Y\to X$. Let $\overline{H}$ be the closure of H on $\overline{Y}$. If $\overline{H}$ is not a semi-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, then we take blowing-ups of $\overline{Y}$ inside $\overline{Y}\setminus Y$ and obtain a semi-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $\overline{H}$ on $\overline{Y}$ such that $\overline{H}|_Y=H$. Let $\overline{L}$ (resp. $\overline{B}$, $\overline{S}$) be the closure of L (resp. B, S) on $\overline{Y}$. We note that $\overline{H}\sim_{\mathbb{R}}\overline{L}-(K_{\overline{Y}}+\overline{S}+\overline{B})$ does not necessarily hold. We can write $H+\sum_i a_i(f_i)=L-(K_Y+S+B)$, where a_i is a real number and $f_i\in\Gamma(Y,\mathcal{K}_Y^*)$ for every i. We put $E=\overline{H}+\sum_i a_i(f_i)-(\overline{L}-(K_{\overline{Y}}+\overline{S}+\overline{B}))$. We replace $\overline{L}$ (resp. $\overline{B}$) with $\overline{L}+\ulcorner E\urcorner$ (resp. $\overline{B}+\{-E\}$). Then we obtain the desired property of $R^q\overline{f}_*\mathcal{O}_{\overline{Y}}(\overline{L})$ since $\overline{X}$ is projective. We note that Supp$E$ is in $\overline{Y}\setminus Y$. So, we finish the whole proof.

(ii) We divide the proof into three steps.

Step 1. We assume that $\dim V=0$. The following arguments are well known and standard. We describe them for the reader's convenience. In this case, we can write $H'\sim_{\mathbb{R}}H'_1+H'_2$, where H'_1 (resp. H'_2) is a π-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor (resp. π-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor) on X. So, we can write $H'_2\sim_{\mathbb{R}}\sum_i a_iH_i$, where $0<a_i<1$ and H_i is a general very ample Cartier divisor on X for every i. Replacing B (resp. H') with $B+\sum_i a_if^*H_i$ (resp. H'_1), we can assume that H' is a π-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. We take a general member $A\in|mH'|$, where m is a sufficiently large and divisible integer, such that $A'=f^*A$ and $R^qf_*\mathcal{O}_Y(L+A')$ is π_*-acyclic for all q. By (i), we have the following short exact sequences,

$$0\to R^qf_*\mathcal{O}_Y(L)\to R^qf_*\mathcal{O}_Y(L+A')\to R^qf_*\mathcal{O}_{A'}(L+A')\to 0.$$

for every q. Note that $R^qf_*\mathcal{O}_{A'}(L+A')$ is π_*-acyclic by induction on $\dim X$ and $R^qf_*\mathcal{O}_Y(L+A')$ is also π_*-acyclic by the above assumption. Thus, $E_2^{pq}=0$ for $p\geq 2$ in the following commutative diagram of spectral sequences.

$$\begin{array}{ccc} E_2^{pq}=R^p\pi_*R^qf_*\mathcal{O}_Y(L) & \Longrightarrow & R^{p+q}(\pi\circ f)_*\mathcal{O}_Y(L) \\ \downarrow{\scriptstyle \varphi^{pq}} & & \downarrow{\scriptstyle \varphi^{p+q}} \\ \overline{E}_2^{pq}=R^p\pi_*R^qf_*\mathcal{O}_Y(L+A') & \Longrightarrow & R^{p+q}(\pi\circ f)_*\mathcal{O}_Y(L+A') \end{array}$$

We note that φ^{1+q} is injective by Theorem 5.1 0.6. We have that $E_2^{1q}\to R^{1+q}(\pi\circ f)_*\mathcal{O}_Y(L)$ is injective by the fact that $E_2^{pq}=0$ for $p\geq 2$. We also have that $\overline{E}_2^{1q}=0$ by the above assumption. Therefore, we obtain $E_2^{1q}=0$ since the injection $E_2^{1q} \to R^{1+q}(\pi \circ f)_* \mathcal{O}_Y(L + A')$ factors through $\overline{E}_2^{1q} = 0$. This implies that $R^p\pi_* R^q f_* \mathcal{O}_Y(L) = 0$ for every $p > 0$.

o2 **Step 2.** We assume that V is projective. By replacing H' (resp. L) with $H' + \pi^* G$ (resp. $L + (\pi \circ f)^* G$), where G is a very ample Cartier divisor on V, we can assume that H' is an ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. By the same argument as in Step 1, we can assume that H' is ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor and $H \sim_{\mathbb{Q}} f^* H'$. If G is a sufficiently ample Cartier divisor on V, $H^k(V, R^p\pi_* R^q f_* \mathcal{O}_Y(L) \otimes G) = 0$ for every $k \geq 1$,

$$\begin{aligned} H^0(V, R^p\pi_* R^q f_* \mathcal{O}_Y(L) \otimes \mathcal{O}_V(G)) &\simeq H^p(X, R^q f_* \mathcal{O}_Y(L) \otimes \mathcal{O}_X(\pi^* G)) \\ &\simeq H^p(X, R^q f_* \mathcal{O}_Y(L + f^*\pi^* G)), \end{aligned}$$

and $R^p\pi_* R^q f_* \mathcal{O}_Y(L) \otimes \mathcal{O}_V(G)$ is generated by its global sections. Since

$$\begin{aligned} &H + f^*\pi^* G \sim_{\mathbb{R}} L + f^*\pi^* G - (K_Y + S + B), \\ &H + f^*\pi^* G \sim_{\mathbb{Q}} f^*(H' + \pi^* G), \end{aligned}$$

and $H' + \pi^* G$ is ample, we can apply Step 1 and obtain $H^p(X, R^q f_* \mathcal{O}_Y(L + f^*\pi^* G)) = 0$ for every $p > 0$. Thus, $R^p\pi_* R^q f_* \mathcal{O}_Y(L) = 0$ for every $p > 0$ by the above arguments.

o3 **Step 3.** When V is not projective, we shrink V and assume that V is affine. By the same argument as in Step 1 above, we can assume that H' is $\mathbb{Q}$-Cartier. We compactify V and X, and can assume that V and X are projective. By Lemma 0.4, we can reduce it to the case when V is projective. This step is essentially the same as Step 2 in the proof of (i). So, we omit the details here.

We finish the whole proof of (ii). □

smooth-ne **Remark 0.9.** In Theorem 0.6, if X is smooth, then Proposition ?? is enough for the proof of Theorem 0.6. In the proof of Theorem 0.7, if Y is smooth, then Theorem 0.6 for a smooth X is sufficient. Lemmas 0.1, 0.2, and 0.4 are easy and well known for smooth varieties. Therefore, the reader can find that our proof of Theorem 0.7 becomes much easier if we assume that Y is smooth. Ambro's original proof of [?, Theorem 3.2 (ii)] used embedded simple normal crossing pairs even when Y is smooth (see (b) in the proof of [?, Theorem 3.2 (ii)]). It may be a technically important difference. I could not follow Ambro's original argument in (a) in the proof of [?, Theorem 3.2 (ii)].

9-1 **Remark 0.10.** It is easy to see that Theorem 0.6 is a generalization of Kollár's injectivity theorem. Theorem 0.7 (i) (resp. (ii)) is a generalization of Kollár's torsion-free (resp. vanishing) theorem.

We treat an easy vanishing theorem for lc pairs as an application of Theorem 0.7 (ii). It seems to be buried in [?]. We note that we do not need the notion of embedded simple normal crossing pairs to prove Theorem 0.11. See Remark 0.9.

lc **Theorem 0.11** (Kodaira vanishing theorem for lc pairs)**.** *Let (X, B) be an lc pair such that B is a boundary $\mathbb{R}$-divisor. Let L be a $\mathbb{Q}$-Cartier Weil divisor on X such that $L-(K_X+B)$ is π-ample, where $\pi : X \to V$ is a projective morphism. Then $R^q\pi_*\mathcal{O}_X(L)=0$ for every $q>0$.*

Proof. Let $f : Y \to X$ be a log resolution of (X, B) such that $K_Y = f^*(K_X+B)+\sum_i a_iE_i$ with $a_i \geq -1$ for every i. We can assume that $\sum_i E_i \cup \mathrm{Supp} f^*L$ is a simple normal crossing divisor on Y. We put $E=\sum_i a_iE_i$ and $F=\sum_{a_j=-1}(1-b_j)E_j$, where $b_j = \mathrm{mult}_{E_j}\{f^*L\}$. We note that $A = L-(K_X+B)$ is π-ample by the assumption. So, we have $f^*A = f^*L - f^*(K_X+B) = \ulcorner f^*L+E+F\urcorner - (K_Y+F+\{-(f^*L+E+F)\})$. We can easily check that $f_*\mathcal{O}_Y(\ulcorner f^*L+E+F\urcorner) \simeq \mathcal{O}_X(L)$ and that $F+\{-(f^*L+E+F)\}$ has a simple normal crossing support and is a boundary $\mathbb{R}$-divisor on Y. By Theorem 0.7 (ii), we obtain that $\mathcal{O}_X(L)$ is π_*-acyclic. Thus, we have $R^q\pi_*\mathcal{O}_X(L)=0$ for every $q>0$. □

We note that Theorem 0.11 contains a complete form of [?, Theorem 0.3] as a corollary. For the related topics, see [?, Corollary 1.3].

lcvar **Corollary 0.12** (Kodaira vanishing theorem for lc varieties)**.** *Let X be a projective lc variety and L an ample Cartier divisor on X. Then*

$$H^q(X, \mathcal{O}_X(K_X+L))=0$$

for every $q>0$. Furthermore, if we assume that X is Cohen–Macaulay, then $H^q(X, \mathcal{O}_X(-L))=0$ for every $q<\dim X$.

Remark 0.13. We can see that Corollary 0.12 is contained in [?, Theorem 2.6], which is a very special case of Theorem 0.7 (ii). I forgot to state Corollary 0.12 explicitly in [?]. There, we do not need embedded simple normal crossing pairs. We note that there are typos in the proof of [?, Theorem 2.6]. In the commutative diagram, $R^if_*\omega_X(D)$'s should be replaced by $R^jf_*\omega_X(D)$'s.

We close this section with an easy example.

Example 0.14. Let X be a projective lc threefold which has the following properties: (i) there exists a projective birational morphism $f : Y \to X$ from a smooth projective threefold, and (ii) the exceptional locus E of f is an Abelian surface with $K_Y = f^*K_X - E$. For example, X is a cone over a normally projective Abelian surface in $\mathbb{P}^N$ and $f : Y \to X$ is the blow-up at the vertex of X. Let L be an ample Cartier divisor on X. By the Leray spectral sequence, we have

$$\begin{aligned} 0 &\to H^1(X, f_*f^*\mathcal{O}_X(-L)) \to H^1(Y, f^*\mathcal{O}_X(-L)) \to H^0(X, R^1f_*f^*\mathcal{O}_X(-L)) \\ &\to H^2(X, f_*f^*\mathcal{O}_X(-L)) \to H^2(Y, f^*\mathcal{O}_X(-L)) \to \cdots . \end{aligned}$$

Therefore, we obtain

$$H^2(X, \mathcal{O}_X(-L)) \simeq H^0(X, \mathcal{O}_X(-L) \otimes R^1f_*\mathcal{O}_Y),$$

because $H^1(Y, f^*\mathcal{O}_X(-L)) = H^2(Y, f^*\mathcal{O}_X(-L)) = 0$ by the Kawamata–Viehweg vanishing theorem. On the other hand, we have $R^qf_*\mathcal{O}_Y \simeq H^q(E, \mathcal{O}_E)$ for every $q > 0$ since $R^qf_*\mathcal{O}_Y(-E) = 0$ for every $q > 0$. Thus, $H^2(X, \mathcal{O}_X(-L)) \simeq \mathbb{C}^2$. In particular, $H^2(X, \mathcal{O}_X(-L)) \neq 0$. We note that X is not Cohen–Macaulay. In the above example, if we assume that E is a $K3$-surface, then $H^q(X, \mathcal{O}_X(-L)) = 0$ for $q < 3$ and X is Cohen–Macaulay. For the details, see the subsection ??, especially, Lemma ??.