AN AFFINE LOCAL CRITERION FOR TORIC PROJECTIVE SPACE BUNDLES

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ABSTRACT. We study when an equidimensional toric morphism is forced to be a projective-space bundle. Our main result is an affine rigidity theorem: if the base space is affine, the toric relative canonical divisor is $\mathbb{Q}$-Cartier, and its negative has degree greater than the relative dimension on every complete curve, then the morphism is equivariantly a trivial projective-space bundle. As an application, we derive a projective-space-bundle theorem for equidimensional toric contractions associated to long extremal rays, without assuming $\mathbb{Q}$-factoriality.

1. INTRODUCTION

Projective-space-bundle structures arise naturally in the study of toric extremal contractions associated to long extremal rays; see [FS20a], [FS20b], [FS24], and [FS25]. The authors proved that if X is a $\mathbb{Q}$-factorial projective toric variety and

$$\varphi_R \colon X \longrightarrow W$$

is a Fano contraction associated to a K_X-negative extremal ray R, then the inequality

$$l(R) > \dim X - \dim W$$

implies that φ_R is a projective-space bundle; see [FS20b, Corollary 3.3]. The purpose of this paper is to establish an affine local version which does not require the total space to be $\mathbb{Q}$-factorial.

Let

$$\varphi \colon X \longrightarrow Y$$

be a proper surjective equidimensional toric morphism with connected fibers. We write

$$d := \dim X - \dim Y.$$

For a torus-invariant Weil divisor D on Y, we denote by

$$\varphi^{[*]}D$$

the torus-invariant Weil divisor on X obtained by pulling back D over the smooth locus of Y and then taking its closure in X. In particular, we define the toric relative canonical divisor by

$$K_{X/Y} := K_X - \varphi^{[*]}K_Y.$$

If K_Y is $\mathbb{Q}$-Cartier, then $\varphi^{[*]}K_Y = \varphi^*K_Y$, and this agrees with the usual relative canonical divisor.

Our main result is an affine criterion for φ to be a trivial projective-space bundle. More precisely, Theorem 4.1 states that if Y is affine, $K_{X/Y}$ is $\mathbb{Q}$-Cartier, and

$$-K_{X/Y} \cdot C > d$$

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for every complete integral curve $C \subset X$, then there is an equivariant isomorphism over Y

$$X \simeq \mathbb{P}^d \times Y.$$

Thus, although neither X nor Y is assumed to be $\mathbb{Q}$-factorial, the relative numerical condition forces the morphism to have the simplest possible toric structure.

We briefly explain the main steps of the proof. First, the numerical assumption implies that the general fiber of φ is $\mathbb{P}^d$. Hence the fan in the kernel of the corresponding lattice homomorphism is the standard fan of $\mathbb{P}^d$.

Second, we study the inverse image of the toric chart associated to a ray of the fan of Y. After separating a torus factor, the resulting morphism is a toric morphism to $\mathbb{A}^1$. A relative version of the long-ray argument of the authors shows that each ray of the fan of Y has exactly one lift to the fan of X, and that a primitive generator is mapped to the primitive generator of the corresponding ray of the base.

Third, equidimensionality and properness force these lifted rays to form a horizontal subfan. Moreover, at the level of real cones, the fan of X is the sum of this horizontal subfan and the standard fan of $\mathbb{P}^d$. In other words, the fan is weakly split.

It remains to rule out a finite lattice-index defect. If the horizontal lattice were not saturated, then one of the standard walls of the fiber fan would determine a complete torus-invariant curve $C \subset X$ such that

$$-K_{X/Y} \cdot C = \frac{d+1}{q} \leq d$$

for some integer $q \geq 2$. This contradicts the numerical assumption. The fan is therefore split in the integral sense, and the standard split-fan criterion gives

$$X \simeq \mathbb{P}^d \times Y$$

over the affine toric variety Y.

As an application of the affine theorem, we obtain a global projective-space-bundle statement for equidimensional toric contractions satisfying the corresponding relative length condition. The relative Picard number is not needed in the affine theorem itself. In the global extremal setting, it is used to ensure that every complete curve contained in the inverse image of an affine toric chart belongs to the same contracted extremal ray and hence satisfies the required numerical inequality.

Finally, we give an example showing that equidimensionality is essential. Without this assumption, a torus-invariant divisor may be contained in a special fiber, and the dimension of that fiber may be larger than the dimension of the general fiber, even when the relevant relative length is larger than the relative dimension.

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2. Preliminaries

We work over an algebraically closed field. We use the standard notation and terminology of toric geometry and toric Mori theory. For details, see, for example, [Oda88], [Ful93], [CLS11] and [Mat02, Chapter 14]. Throughout the paper, a curve means an integral curve.

We first recall some known results. Lemma 2.1 and Lemma 2.2 extract the necessary parts from [Fuj03, Theorem 0.1] and [FS20a, Theorem 3.2.1], respectively.

Lemma 2.1 ([Fuj03, Theorem 0.1]). *Let X be a $\mathbb{Q}$-Gorenstein projective toric variety of dimension n, and let $R \subset \mathrm{NE}(X)$ be a K_X-negative extremal ray. Then*

$$l(R) \le n$$

unless $X \simeq \mathbb{P}^n$, where

$$l(R) := \min\{-K_X \cdot C \mid C \text{ is a complete integral curve and } [C] \in R\}$$

is the length of R.

Lemma 2.2 ([FS20a, Theorem 3.2.1]). *Let $f\colon X \to Y$ be a projective toric morphism with $\dim X = n$. Assume that K_X is $\mathbb{Q}$-Cartier. Let R be a K_X-negative extremal ray of $\mathrm{NE}(X/Y)$ and let $\varphi_R\colon X \to W$ be the contraction morphism associated to R. Assume that φ_R is birational. Then we obtain*

$$l(R) < e + 1,$$

where

$$e = \max_{w \in W} \dim \varphi_R^{-1}(w) \le n - 1.$$

When $e = n - 1$, we have a sharper inequality

$$l(R) \le e = n - 1.$$

The following characterization of equidimensionality is well known to experts.

Proposition 2.3 (Equidimensionality and images of cones). *Let $\varphi_p\colon X_\Sigma \to Y_\Delta$ be a surjective toric morphism and let $p\colon N \to N_Y$ be the lattice homomorphism defining φ_p. Then φ_p is equidimensional if and only if $p_{\mathbb{R}}(\sigma)$ is a cone of Δ for every $\sigma \in \Sigma$.*

Proof. This is the toric case of Abramovich–Karu's characterization of equidimensional toroidal morphisms; see [AK00, Lemma 4.1]. □

3. The case of a one-dimensional affine base

The following theorem can be regarded as an affine-line analogue of [FS20a, Theorem 3.2.9]. It is the main ingredient in the proof of the affine local criterion established in Section 4, but also seems to be of independent interest.

Theorem 3.1 (The affine-line version of [FS20a, Theorem 3.2.9]). *Let $f\colon X \to Y$ be a projective surjective toric morphism with connected fibers. Assume that $Y \simeq \mathbb{A}^1$, $\dim X = n \ge 2$, and that X is $\mathbb{Q}$-Gorenstein. Suppose moreover that*

$$-K_X \cdot C > n - 1$$

for every complete integral curve $C \subset X$.

Then there exists an equivariant isomorphism over Y

$$X \simeq \mathbb{P}^{n-1} \times Y$$

under which f is identified with the second projection.

Proof. We divide the proof into four steps.

Step 1. Choosing an extremal ray. We first choose a suitable extremal ray and show that its contraction is of fiber type.

Since $n \ge 2$, a fiber of f contains a complete curve. By assumption,

$$K_X \cdot C < 0$$

for every complete curve $C \subset X$. In particular, K_X is not f-nef.

Since f is a projective toric morphism, the relative Kleiman–Mori cone $\mathrm{NE}(X/Y)$ is a rational polyhedral cone. Hence it contains a K_X-negative extremal ray R. Let $\varphi_R\colon X \to W$ be the contraction of R. Since φ_R is a contraction over Y, there is a projective toric morphism $h\colon W \to Y$ such that $f = h \circ \varphi_R$.

Every curve whose numerical class belongs to R is complete. Therefore the hypothesis gives

$$l(R) > n - 1. \tag{3.1}$$

We claim that φ_R is a Fano contraction, that is, $\dim W < \dim X$. Suppose that φ_R is birational, and put

$$e := \max_{w\in W} \dim \varphi_R^{-1}(w).$$

By Lemma 2.2, if $e \leq n-2$, then

$$l(R) < e + 1 \leq n - 1,$$

contrary to (3.1). If $e = n-1$, the sharper part of the same theorem gives

$$l(R) \leq n - 1,$$

again contradicting (3.1). Thus φ_R is not birational. Consequently, φ_R is a Fano contraction.

Step 2. Identifying the contraction. We next show that the extremal contraction obtained above coincides with the original morphism.

Let F be a general fiber of φ_R, and put

$$r := \dim F = \dim X - \dim W.$$

Since $h\colon W \to Y$ is surjective and $\dim Y = 1$, we have $\dim W \geq 1$, and hence $r \leq n-1$.

The fiber F is a projective toric r-fold. Over the big torus of W, adjunction gives

$$K_F = K_X|_F.$$

Thus, for every integral curve $C \subset F$,

$$-K_F \cdot C = -K_X \cdot C > n - 1 \geq r. \tag{3.2}$$

By Lemma 2.1, if $F \not\simeq \mathbb{P}^r$, then there exists an integral curve $C \subset F$ such that

$$-K_F \cdot C \leq r,$$

contrary to (3.2). Therefore

$$F \simeq \mathbb{P}^r.$$

For a line $L \subset F \simeq \mathbb{P}^r$, we have

$$-K_X \cdot L = -K_F \cdot L = r + 1.$$

The hypothesis therefore gives $r + 1 > n - 1$. Since $r \leq n-1$, we obtain $r = n-1$. Consequently,

$$\dim W = 1.$$

Let $q : N_W \to N_Y$ be the lattice homomorphism defining h. Since both f and φ_R have connected fibers, the corresponding lattice homomorphisms are surjective. Hence q is surjective. Since N_W and N_Y both have rank one, q is an isomorphism. Hence h is a projective birational morphism between normal curves, and therefore h is an isomorphism. After identifying W with Y, we have $f = \varphi_R$.

In particular,

$$\rho(X/Y) = 1. \tag{3.3}$$

Step 3. The $\mathbb{Q}$-factoriality of X. We now prove that X is $\mathbb{Q}$-factorial.

Suppose that X is not $\mathbb{Q}$-factorial. Let $\pi\colon \widetilde{X} \to X$ be a small projective toric $\mathbb{Q}$-factorialization. Since π is small and K_X is $\mathbb{Q}$-Cartier,

$$K_{\widetilde{X}} = \pi^* K_X. \tag{3.4}$$

Put

$$\widetilde{f} := f \circ \pi \colon \widetilde{X} \to Y.$$

Since π is nontrivial and projective, $\rho(\widetilde{X}/X) > 0$, and hence

$$\rho(\widetilde{X}/Y) > \rho(X/Y) = 1. \tag{3.5}$$

The divisor $K_{\widetilde{X}}$ is not $\widetilde{f}$-nef, so we can choose a $K_{\widetilde{X}}$-negative extremal ray $\widetilde{R} \subset \mathrm{NE}(\widetilde{X}/Y)$. A curve whose class belongs to $\widetilde{R}$ is not π-exceptional, because by (3.4) the divisor $K_{\widetilde{X}}$ has degree zero on every π-exceptional curve.

Let $\widetilde{C}$ be an integral curve with $[\widetilde{C}] \in \widetilde{R}$, and let $C := \pi(\widetilde{C})$ with the reduced structure. If the mapping degree of $\widetilde{C} \to C$ is $m > 0$, then

$$-K_{\widetilde{X}} \cdot \widetilde{C} = -\pi^* K_X \cdot \widetilde{C} = m(-K_X \cdot C) > n - 1.$$

Thus

$$l(\widetilde{R}) > n - 1. \tag{3.6}$$

Repeating the arguments of Steps 1 and 2 for the fixed extremal ray $\widetilde{R}$, using (3.6), we see that this contraction is a Fano contraction and that its base is isomorphic to Y. Hence it is precisely the morphism $\widetilde{f}\colon \widetilde{X} \to Y$. Since it is the contraction of a single extremal ray, this gives $\rho(\widetilde{X}/Y) = 1$, contrary to (3.5). Therefore X is $\mathbb{Q}$-factorial.

Step 4. Description of the fan. Finally, we determine the fan explicitly and conclude the proof.

Let

$$p\colon N \longrightarrow N_Y \simeq \mathbb{Z}$$

be the lattice homomorphism defining f. Since f has connected fibers, p is surjective. Put

$$N_F := \ker p.$$

The general fiber is $\mathbb{P}^{n-1}$. Therefore its fan in $(N_F)_{\mathbb{R}}$ has primitive ray generators

$$v_1, \ldots, v_n$$

which, after choosing a basis of N_F, may be written as

$$v_1 = e_1, \quad \ldots, \quad v_{n-1} = e_{n-1}, \qquad v_n = -e_1 - \cdots - e_{n-1}. \tag{3.7}$$

Since X is $\mathbb{Q}$-factorial and $\rho(X/Y) = 1$, the standard toric divisor sequence shows that there is exactly one nonvertical ray of the fan of X. Let v_+ be its primitive generator. Choose $e_n \in N$ such that

$$p(e_n) = 1.$$

Then

$$N = N_F \oplus \mathbb{Z} e_n,$$

and we may write

$$v_+ = b_1 e_1 + \cdots + b_{n-1} e_{n-1} + a e_n, \qquad a > 0. \tag{3.8}$$

By replacing e_n with

$$e_n + c_1 e_1 + \cdots + c_{n-1} e_{n-1}$$

for suitable integers c_i, we may assume

$$0 \le b_i < a \qquad (1 \le i \le n-1). \tag{3.9}$$

Properness of f and the description of the general fiber imply that the maximal cones of the fan of X are

$$\mathrm{Cone}(v_+, v_1, \dots, \widehat{v_i}, \dots, v_n), \qquad 1 \le i \le n. \tag{3.10}$$

We claim that $b_i = 0$ for every i. Suppose otherwise. After renumbering the v_i's, we may assume that

$$0 < b_1 < a.$$

Let

$$\mu := \mathrm{Cone}(v_2, \dots, v_{n-1}, v_+),$$

where the list $v_2, \dots, v_{n-1}$ is empty when $n = 2$, and put

$$C := V(\mu).$$

The curve C is contained in the fiber over the origin of $\mathbb{A}^1$, and hence is complete.

Let $D_i := V(v_i)$ and $D_+ := V(v_+)$. The principal divisor relations give

$$D_i - D_n + b_i D_+ \sim 0 \qquad (1 \le i \le n-1)$$

and

$$aD_+ \sim 0.$$

Consequently,

$$D_+ \cdot C = 0 \quad \text{and} \quad D_i \cdot C = D_n \cdot C \qquad (1 \le i \le n-1). \tag{3.11}$$

By the standard toric intersection formula,

$$D_1 \cdot C = \frac{\mathrm{mult}(\mu)}{\mathrm{mult}(\mathrm{Cone}(v_1, \dots, v_{n-1}, v_+))} = \frac{\gcd(a, b_1)}{a}. \tag{3.12}$$

Using

$$-K_X = D_1 + \cdots + D_n + D_+$$

and (3.11), we obtain

$$-K_X \cdot C = n\frac{\gcd(a, b_1)}{a}. \tag{3.13}$$

Since $0 < b_1 < a$, the integer $\gcd(a, b_1)$ is a proper divisor of a, and hence

$$\frac{\gcd(a, b_1)}{a} \le \frac{1}{2}.$$

Therefore

$$-K_X \cdot C \le \frac{n}{2} \le n - 1,$$

contrary to the hypothesis.

Thus $b_1 = \cdots = b_{n-1} = 0$. It follows from (3.8) that $v_+ = ae_n$. Since v_+ is primitive, we must have $a = 1$. Therefore $v_+ = e_n$.

By (3.7), (3.10), and $v_+ = e_n$, the fan of X is the product of the standard fan of $\mathbb{P}^{n-1}$ and the fan of $\mathbb{A}^1$. Hence there is an equivariant isomorphism over Y

$$X \simeq \mathbb{P}^{n-1} \times \mathbb{A}^1.$$

Under this isomorphism, f is the second projection. □

Although Theorem 3.1 is sufficient for our purposes, the proof suggests that its numerical assumption may not be optimal. Since this theorem is the key local ingredient of Theorem 4.1, a positive answer to the following question would immediately strengthen both the affine criterion and its global application to long extremal toric contractions (cf. [FS20b]).

Question 3.2. Is it possible to weaken the assumption $-K_X \cdot C > n-1$ in Theorem 3.1 to require only that $-K_X \cdot C > \frac{n}{2}$ for every complete integral curve, and $-K_X \cdot C > n-1$ for every curve C contained in a general fiber?

Note that the condition $-K_X \cdot C > n-1$ for every curve C contained in a general fiber is equivalent to the general fiber being isomorphic to $\mathbb{P}^{n-1}$. If Theorem 3.1 holds under these weaker assumptions, then in Theorem 4.1 one may replace (4.1) by the condition

$$-K_{X/Y} \cdot C > \frac{d+1}{2}$$

for every complete integral curve $C \subset X$, together with

$$-K_{X/Y} \cdot C > d$$

for every curve C contained in a general fiber.

4. Main Theorem

We first explain the relative canonical divisor used throughout this section. We fix the standard torus-invariant canonical divisors

$$K_X = -\sum_{\rho \in \Sigma(1)} D_\rho, \qquad K_Y = -\sum_{\alpha \in \Delta(1)} D_\alpha.$$

Let

$$\varphi = \varphi_p \colon X_\Sigma \longrightarrow Y_\Delta$$

be a proper surjective equidimensional toric morphism with connected fibers. Note that $p\colon N \to N_Y$ is the lattice homomorphism defining φ. For $\rho \in \Sigma(1)$ and $\alpha \in \Delta(1)$, we denote by $u_\rho \in N$ and $w_\alpha \in N_Y$ the primitive lattice generators of ρ and α, respectively. Since Y is normal, the complement of its smooth locus Y_{sm} has codimension at least two. Equidimensionality gives

$$\operatorname{codim}_X \varphi^{-1}(Y \setminus Y_{\mathrm{sm}}) \geq 2.$$

Hence, for a torus-invariant Weil divisor D on Y, the Cartier pullback of $D|_{Y_{\mathrm{sm}}}$ on $\varphi^{-1}(Y_{\mathrm{sm}})$ has a unique closure as a Weil divisor on X. We denote this closure by

$$\varphi^{[*]}D.$$

If D is $\mathbb{Q}$-Cartier, then $\varphi^{[*]}D$ agrees with the usual pullback φ^*D.

More explicitly, write

$$D = \sum_{\alpha \in \Delta(1)} a_\alpha D_\alpha.$$

By Proposition 2.3, every ray of Σ is mapped either to the origin or to a ray of Δ. If $p(u_\rho) = m_\rho w_\alpha$ with $m_\rho > 0$, then the coefficient of D_ρ in $\varphi^{[*]}D$ is $m_\rho a_\alpha$. A vertical divisor, that is, a divisor corresponding to a ray contained in $\ker(p)_{\mathbb{R}}$, has coefficient zero in $\varphi^{[*]}D$.

We define the toric relative canonical divisor by

$$K_{X/Y} := K_X - \varphi^{[*]}K_Y.$$

When K_Y is $\mathbb{Q}$-Cartier, this is the usual relative canonical divisor $K_X - \varphi^*K_Y$.

Theorem 4.1 (Affine local projective-space-bundle criterion)**.** *Let*

$$\varphi\colon X \longrightarrow Y$$

be a proper surjective toric morphism with connected fibers. Assume that Y is an affine toric variety and that φ is equidimensional of relative dimension

$$d := \dim X - \dim Y > 0.$$

Assume moreover that the toric relative canonical divisor $K_{X/Y}$ is $\mathbb{Q}$-Cartier and that

$$-K_{X/Y} \cdot C > d \tag{4.1}$$

for every complete integral curve $C \subset X$. Then there is an equivariant isomorphism over Y

$$X \simeq \mathbb{P}^d \times Y,$$

under which φ is the second projection.

Remark 4.2. The proof of Theorem 4.1 also gives the following slightly more general statement. One may replace $K_{X/Y}$ by

$$K_X - \varphi^{[*]} D$$

for any torus-invariant Weil divisor D on Y, provided that this divisor is $\mathbb{Q}$-Cartier and the corresponding curve inequality holds. Taking $D = 0$ gives the K_X-version, while taking $D = K_Y$ gives the relative canonical formulation stated above.

If $\varphi\colon X \to Y$ is a projective space bundle, then $K_{X/Y}$ is always Cartier. On the other hand, K_X need not be $\mathbb{Q}$-Cartier when K_Y is not $\mathbb{Q}$-Cartier. Therefore, it seems natural to use $K_{X/Y}$ to formulate Theorem 4.1.

Remark 4.3. In Theorem 4.1, since Y is affine, every complete curve in X is contracted by φ. Hence (4.1) and the toric relative ampleness criterion imply that $-K_{X/Y}$ is φ-ample. Therefore the proper morphism φ is projective.

4.1. **The general fiber.**

Lemma 4.4. *Under the assumptions of Theorem 4.1, the general fiber of φ is isomorphic to $\mathbb{P}^d$.*

Proof. Let $T_Y \subset Y$ be the dense torus and let F be a fiber over a point of T_Y. After choosing a splitting of the exact sequence of lattices over the dense torus, the restriction of φ over T_Y is a product with fiber F. Since the standard torus-invariant canonical divisor K_Y restricts to zero on T_Y, we obtain

$$K_{X/Y}|_F = K_F.$$

Every curve $C \subset F$ is complete in X, and hence

$$-K_F \cdot C = -K_{X/Y} \cdot C > d.$$

Lemma 2.1 now implies that $F \simeq \mathbb{P}^d$. □

Since φ has connected fibers, $p\colon N \to N_Y$ is surjective. Put $N_F := \ker(p)$. By Lemma 4.4, the fan in $(N_F)_{\mathbb{R}}$ is the standard fan of $\mathbb{P}^d$. We choose its primitive ray generators

$$v_0, v_1, \ldots, v_d \in N_F$$

so that $v_1, \ldots, v_d$ is a basis of N_F and

$$v_0 + v_1 + \cdots + v_d = 0. \tag{4.2}$$

For $0 \le i \le d$, put

$$\gamma_i := \mathrm{Cone}(v_0, \ldots, \widehat{v_i}, \ldots, v_d).$$

4.2. **Unique primitive lifts of the rays of the base.** We now return to Theorem 4.1. By separating a torus factor, we may assume that

$$Y = U_\tau$$

for a full-dimensional strongly convex cone $\tau \subset (N_Y)_{\mathbb{R}}$.

Lemma 4.5 (Unique primitive lift)**.** *For every ray $\alpha \in \Delta(1)$, there is exactly one ray $\rho_\alpha \in \Sigma(1)$ whose image is contained in α and is nonzero. If w_α and u_{ρ_α} are the primitive generators, then*

$$p(u_{\rho_\alpha}) = w_\alpha.$$

The rays contained in $\ker(p)_{\mathbb{R}}$ are exactly $\mathbb{R}_{\geq 0}v_0, \dots, \mathbb{R}_{\geq 0}v_d$.

Proof. Fix a ray $\alpha \in \Delta(1)$, with primitive generator w_α. The toric open subset $U_\alpha \subset Y$ is noncanonically isomorphic to

$$\mathbb{A}^1 \times (\mathbb{G}_m)^{\dim Y - 1}.$$

Put

$$N_\alpha := p^{-1}(\mathbb{Z}w_\alpha).$$

The part of the fan of X lying over U_α is contained in $(N_\alpha)_{\mathbb{R}}$. After choosing a complementary lattice, we obtain a product decomposition

$$\varphi^{-1}(U_\alpha) \simeq V_\alpha \times (\mathbb{G}_m)^{\dim Y - 1},$$

where

$$f_\alpha : V_\alpha \longrightarrow \mathbb{A}^1$$

is a projective surjective equidimensional toric morphism with connected fibers and relative dimension d.

The variety U_α is smooth. Hence $K_Y|_{U_\alpha}$ is Cartier, and on $\varphi^{-1}(U_\alpha)$ we have

$$K_X = K_{X/Y} + \varphi^*(K_Y|_{U_\alpha}).$$

It follows that K_X is $\mathbb{Q}$-Cartier on this open set, and therefore V_α is $\mathbb{Q}$-Gorenstein. If $C \subset V_\alpha$ is a complete curve, then $C \times \{1\}$ is a complete curve in X. Since it is vertical over U_α, the Cartier divisor $\varphi^*(K_Y|_{U_\alpha})$ has degree zero on it. Consequently,

$$-K_{V_\alpha} \cdot C = -K_X \cdot (C \times \{1\}) = -K_{X/Y} \cdot (C \times \{1\}) > d.$$

Theorem 3.1 gives

$$V_\alpha \simeq \mathbb{P}^d \times \mathbb{A}^1.$$

The fan of this product contains exactly the $d+1$ vertical rays and one horizontal ray, and the primitive generator of the horizontal ray maps to w_α. This proves the claim for every α. The assertion about the vertical rays follows from the fan of the general fiber. $\square$

4.3. **The weak splitting of the fan.** The next lemma shows that, once the rays have been controlled, the real cone structure is forced by equidimensionality and properness.

Lemma 4.6 (Weak splitting)**.** *For every face $\sigma \preceq \tau$, set*

$$\widehat{\sigma} := \operatorname{Cone}\big(u_{\rho_\alpha} \mid \alpha \in \sigma(1)\big).$$

Then $\widehat{\sigma}$ is a cone of Σ, and

$$p_{\mathbb{R}}|_{\widehat{\sigma}} : \widehat{\sigma} \xrightarrow{\sim} \sigma$$

is an isomorphism of real cones. The cones $\widehat{\sigma}$ form a subfan $\widehat{\Delta} \subset \Sigma$.

Furthermore,

$$\Sigma = \{\widehat{\sigma} + \gamma \mid \sigma \preceq \tau,\ \gamma \in \Sigma_F\}, \tag{4.3}$$

where Σ_F is the standard fan of $\mathbb{P}^d$.

Proof. Fix a face $\sigma \preceq \tau$. By Proposition 2.3, the image of every cone of Σ is a cone of Δ.

Consider the fiber over a point of the orbit $O(\sigma)$. If a cone $\kappa \in \Sigma$ maps onto σ, then the corresponding orbit has relative dimension

$$(\dim X - \dim \kappa) - (\dim Y - \dim \sigma) = d + \dim \sigma - \dim \kappa. \tag{4.4}$$

Equidimensionality implies that this number is at most d, so $\dim \kappa \geq \dim \sigma$. Since the fiber has dimension exactly d, some stratum of a d-dimensional irreducible component gives a cone κ for which equality holds. Thus

$$p_{\mathbb{R}}(\kappa) = \sigma, \qquad \dim \kappa = \dim \sigma.$$

The restriction of $p_{\mathbb{R}}$ to the span of κ is therefore an isomorphism. Hence the extremal rays of κ map bijectively to the extremal rays of σ. By Lemma 4.5, each ray of σ has only one possible lift. Consequently,

$$\kappa = \widehat{\sigma}.$$

This proves that $\widehat{\sigma} \in \Sigma$ and that it maps isomorphically to σ. The same cone is forced for every face, so

$$\sigma' \preceq \sigma \quad \Longrightarrow \quad \widehat{\sigma'} \preceq \widehat{\sigma}.$$

Thus the $\widehat{\sigma}$ form a subfan.

We now consider the unique maximal cone τ of the affine base. Put $H := \operatorname{Span}_{\mathbb{R}}(\widehat{\tau})$. Since $p_{\mathbb{R}}|_H$ is an isomorphism, we have a direct-sum decomposition of real vector spaces

$$N_{\mathbb{R}} = H \oplus (N_F)_{\mathbb{R}}.$$

The orbit closure $V(\widehat{\tau})$ lies in the proper fiber over the closed orbit of Y, so its fan

$$\operatorname{Star}_{\Sigma}(\widehat{\tau})$$

in $N_{\mathbb{R}}/H$ is complete. Every ray of this star is represented by a ray of Σ which is not contained in $\widehat{\tau}$. By Lemma 4.5, all nonvertical rays over τ already belong to $\widehat{\tau}$. Since the star is a complete d-dimensional fan, all of the $d+1$ vertical rays must occur. Hence the rays of the star are precisely the vertical rays

$$\mathbb{R}_{\geq 0} v_0, \ldots, \mathbb{R}_{\geq 0} v_d.$$

There is only one complete fan with these $d+1$ rays, namely the standard fan of $\mathbb{P}^d$. Therefore

$$\operatorname{Star}_{\Sigma}(\widehat{\tau}) = \Sigma_F.$$

It follows that the maximal cones containing $\widehat{\tau}$ are exactly

$$\widehat{\tau} + \gamma_i, \qquad 0 \leq i \leq d.$$

Their union is

$$\widehat{\tau} + |(\Sigma_F)| = \widehat{\tau} + (N_F)_{\mathbb{R}} = p_{\mathbb{R}}^{-1}(\tau),$$

which is the support of the fan over Y by properness.

Finally, every cone of Σ is contained in one of these maximal product cones. Indeed, the product cones themselves are cones of Σ and cover the support; the relative interior of a cone of Σ cannot be covered by finitely many proper faces. Thus every cone is a face of some $\widehat{\tau} + \gamma_i$. Because the sum is direct in real vector spaces, every such face is of the form $\widehat{\sigma} + \gamma$, with $\sigma \preceq \tau$ and $\gamma \in \Sigma_F$. This proves (4.3). $\square$

4.4. **Eliminating the lattice defect.** Although the fan is now a product at the level of real cones, the horizontal lattice may still have finite index in the lattice of the base. The relative anticanonical inequality eliminates this last possibility.

Lemma 4.7 (Saturation of the horizontal lattice)**.** *For every face $\sigma \preceq \tau$, one has*

$$p(\widehat{\sigma} \cap N) = \sigma \cap N_Y. \tag{4.5}$$

In particular, Σ is split by $\widehat{\Delta}$ and Σ_F in the sense of [CLS11, Definition 3.3.18]*.*

Proof. It is enough to prove the assertion for the maximal cone τ; the statement for its faces then follows by restriction.

Set

$$H := \mathrm{Span}_{\mathbb{R}}(\widehat{\tau}), \qquad L := N \cap H.$$

The map $p|_L$ is injective, and $p(L)$ is a finite-index sublattice of N_Y. Suppose, for a contradiction, that

$$p(L) \subsetneq N_Y.$$

Since H is rational, L is saturated in N, and hence

$$\overline{N} := N/L$$

is a lattice of rank d. Moreover, $N_F \cap L = 0$. Surjectivity of p gives an exact sequence

$$0 \longrightarrow N_F \longrightarrow \overline{N} \longrightarrow N_Y/p(L) \longrightarrow 0. \tag{4.6}$$

Thus N_F is a proper finite-index sublattice of $\overline{N}$.

For $1 \leq i \leq d$, put

$$\delta_i := \mathrm{Cone}(v_1, \dots, \widehat{v}_i, \dots, v_d)$$

and consider the rank-one lattice

$$\overline{N}_i := \overline{N} \Big/ \big(\overline{N} \cap \mathrm{Span}_{\mathbb{R}}(\delta_i)\big).$$

Let e_i be its primitive generator in the direction of v_i, and write

$$\overline{v}_i = q_i e_i, \qquad q_i \geq 1.$$

By (4.2), the image of v_0 is $-q_i e_i$.

We claim that $q_i \geq 2$ for at least one i. Let $\varepsilon_1, \dots, \varepsilon_d$ be the basis of $\mathrm{Hom}(N_F, \mathbb{Z})$ dual to $v_1, \dots, v_d$. The equality $q_i = 1$ is equivalent to ε_i taking integral values on $\overline{N}$. If $q_i = 1$ for every i, then every $x \in \overline{N}$ has an expression

$$x = \sum_{i=1}^{d} \varepsilon_i(x) v_i$$

with integral coefficients. Hence $x \in N_F$, which contradicts $N_F \subsetneq \overline{N}$. The claim follows.

Fix i with $q_i \geq 2$ and put

$$\eta_i := \widehat{\tau} + \delta_i.$$

By Lemma 4.6, this is a cone of Σ of codimension one. Thus

$$C_i := V(\eta_i)$$

is a complete torus-invariant curve in the fiber over the closed orbit of Y. The two maximal cones adjacent to η_i are

$$\widehat{\tau} + \gamma_0 = \eta_i + \mathbb{R}_{\geq 0} v_i$$

and

$$\widehat{\tau} + \gamma_i = \eta_i + \mathbb{R}_{\geq 0} v_0.$$

Let m_+ and m_- be the local rational linear functions defining $-K_{X/Y}$ on these two cones. We use the convention that the local linear function associated to a torus-invariant divisor takes the coefficient of the corresponding invariant prime divisor on each primitive ray generator. Since $K_{X/Y}$ is $\mathbb{Q}$-Cartier, such functions exist and agree on η_i. The divisor $\varphi^{[*]}K_Y$ has coefficient zero along every vertical divisor. Hence $-K_{X/Y}$ has coefficient 1 along every vertical ray, just as $-K_X$ does. Therefore

$$m_+(v_i) = 1.$$

The second cone contains $v_0, v_1, \ldots, \widehat{v}_i, \ldots, v_d$, and m_- takes the value 1 on all of these rays. From (4.2),

$$v_i = -\left(v_0 + \sum_{j\neq i} v_j\right),$$

so

$$m_-(v_i) = -d.$$

Hence

$$(m_+ - m_-)(v_i) = d + 1.$$

The quotient lattice attached to the invariant curve C_i is naturally $\overline{N}_i$, and the image of v_i is q_ie_i. The standard toric slope formula for a $\mathbb{Q}$-Cartier divisor on an invariant curve gives

$$-K_{X/Y} \cdot C_i = (m_+ - m_-)(e_i) = \frac{d+1}{q_i} \leq \frac{d+1}{2} \leq d.$$

This contradicts (4.1). Therefore $p(L) = N_Y$. Since $p_{\mathbb{R}}$ maps $\widehat{\tau}$ isomorphically onto τ, we obtain

$$p(\widehat{\tau} \cap N) = \tau \cap N_Y.$$

The assertion for every face follows by restriction. □

4.5. **Proof of the affine local theorem.**

Proof of Theorem 4.1. After removing a torus factor, we may assume that the cone defining the affine toric variety Y is full-dimensional. Lemma 4.4 identifies the fiber fan with the standard fan Σ_F of $\mathbb{P}^d$. Lemma 4.5 gives a unique primitive lift of every ray of the base. Lemma 4.6 gives a horizontal subfan $\widehat{\Delta}$ and the weak product decomposition (4.3). Lemma 4.7 gives the integral lattice equality (4.5).

By Lemmas 4.6 and 4.7, Σ is split by $\widehat{\Delta}$ and Σ_F. Hence [CLS11, Theorem 3.3.19] yields a locally trivial toric fiber bundle with fiber $\mathbb{P}^d$.

Since $Y = U_\tau$ consists of a single affine toric chart, the local trivialization furnished by [CLS11, Theorem 3.3.19] is global. Consequently,

$$X \simeq \mathbb{P}^d \times Y$$

equivariantly over Y. □

4.6. **Application to long relative extremal contractions.** We record the global consequence for which the local theorem was designed.

Corollary 4.8 (Global relative projective-space-bundle statement)**.** *Let X be a projective toric variety, and let*

$$\varphi_R : X \longrightarrow W$$

be a projective toric contraction associated to an extremal ray $R \subset \mathrm{NE}(X)$. Assume that

$$d := \dim X - \dim W > 0,$$

that φ_R is equidimensional, and that the toric relative canonical divisor

$$K_{X/W} := K_X - \varphi_R^{[*]}K_W$$

is $\mathbb{Q}$-Cartier. Define the relative length of R by

$$l_{X/W}(R) := \min_{[C]\in R}\bigl(-K_{X/W}\cdot C\bigr),$$

where C runs through complete integral curves whose numerical classes belong to R. If

$$l_{X/W}(R) > d,$$

then φ_R is a toric $\mathbb{P}^d$-bundle.

Proof. Let $U \subset W$ be a maximal affine toric chart and put $X_U := \varphi_R^{-1}(U)$. The restricted morphism

$$X_U \longrightarrow U$$

is proper, surjective, equidimensional, toric, and has connected fibers. Moreover, its toric relative canonical divisor is the restriction of $K_{X/W}$ and is therefore $\mathbb{Q}$-Cartier.

Let $C \subset X_U$ be a complete curve. Since U is affine, the image of C is a point. Thus C is contracted by φ_R, and its numerical class belongs to R. Hence

$$-K_{X_U/U}\cdot C = -K_{X/W}\cdot C \geq l_{X/W}(R) > d.$$

Therefore Theorem 4.1 applies and gives

$$X_U \simeq \mathbb{P}^d \times U$$

over U. The maximal affine toric charts cover W, so these local products show that φ_R is a Zariski locally trivial toric $\mathbb{P}^d$-bundle. □

Remark 4.9. Under the assumptions of Corollary 4.8, replace the hypotheses that $K_{X/W}$ is $\mathbb{Q}$-Cartier and that $l_{X/W}(R) > d$ by the assumptions that K_X is $\mathbb{Q}$-Cartier and that

$$l(R) := \min_{[C]\in R}(-K_X\cdot C) > d,$$

where C runs through complete integral curves. Then the same conclusion holds: φ_R is a toric $\mathbb{P}^d$-bundle.

Indeed, let $U \subset W$ be a maximal affine toric chart and put $X_U := \varphi_R^{-1}(U)$. Every complete curve $C \subset X_U$ is contracted by φ_R, and hence

$$-K_X\cdot C \geq l(R) > d.$$

Therefore the K_X-version of Theorem 4.1 given in Remark 4.2 applies to $X_U \to U$. The same argument as in the proof of Corollary 4.8 shows that φ_R is a toric $\mathbb{P}^d$-bundle.

This may be viewed as an equidimensional non-$\mathbb{Q}$-factorial counterpart of [FS20b, Corollary 3.3].

Remark 4.10. Under the assumptions of Corollary 4.8, if K_W is $\mathbb{Q}$-Cartier, then

$$K_X = K_{X/W} + \varphi_R^*K_W$$

is also $\mathbb{Q}$-Cartier. Moreover,

$$-K_{X/W}\cdot C = -K_X\cdot C$$

for every curve contracted by φ_R. Thus

$$l_{X/W}(R) = l(R),$$

and the relative and usual formulations agree.

Remark 4.11. The affine theorem does not require a relative Picard-number hypothesis. In the global corollary, extremality is used only to guarantee that every complete curve in the inverse image of an affine chart belongs to the single ray R and hence satisfies the required relative anticanonical inequality.

The following example shows that the equidimensionality assumption in Corollary 4.8 cannot be dropped.

Example 4.12. Let

$$N = \mathbb{Z}^3, \qquad p\colon N \longrightarrow N_W = \mathbb{Z}^2, \qquad p(x,y,z) = (x,y).$$

The toric variety $\mathbb{P}^1 \times \mathbb{P}^2$ has fan in N with rays generated by

$$h_1 = (1,0,0), \quad h_2 = (0,1,0), \quad h_3 = (-1,-1,0),$$
$$v_+ = (0,0,1), \quad v_- = (0,0,-1).$$

The second projection $\mathbb{P}^1 \times \mathbb{P}^2 \to W = \mathbb{P}^2$ is associated to p. The rays of the fan of W are generated by

$$\bar{h}_1 = (1,0), \qquad \bar{h}_2 = (0,1), \qquad \bar{h}_3 = (-1,-1).$$

Let

$$\mu\colon \widetilde{X} \longrightarrow \mathbb{P}^1 \times \mathbb{P}^2$$

be the blow-up at the point $V(\langle h_1, h_2, v_+\rangle)$. Thus the fan of $\widetilde{X}$ has one additional ray generated by

$$e := h_1 + h_2 + v_+.$$

Its primitive relations are

$$h_1 + h_2 + v_+ = e, \qquad h_3 + e = v_+, \qquad v_- + e = h_1 + h_2,$$

and

$$h_1 + h_2 + h_3 = 0, \qquad v_+ + v_- = 0.$$

The first three relations are extremal. Let

$$\psi\colon \widetilde{X} \longrightarrow X = X_\Sigma$$

be the contraction associated to the third relation. The contraction ψ is small, and X is a non-$\mathbb{Q}$-factorial projective toric threefold. A direct check of the canonical support function shows that X is Gorenstein and that ψ is crepant. There is a toric morphism

$$\varphi_R\colon X \longrightarrow W$$

with $\rho(X/W) = 1$, associated to an extremal ray $R \subset \mathrm{NE}(X)$. Its general fiber is $\mathbb{P}^1$, so $d = 1$. On the other hand, the divisor $V(\langle e\rangle) \subset X$ is contracted to the point $V(\langle \bar{h}_1, \bar{h}_2\rangle) \in W$. Thus φ_R is not equidimensional.

The torus-invariant curves contracted by φ_R are

$$C_1 := V(\langle h_1, e\rangle), \quad C_2 := V(\langle h_2, e\rangle), \quad C_3 := V(\langle v_+, e\rangle),$$

and

$$C_4 := V(\langle h_1, h_3\rangle), \qquad C_5 := V(\langle h_2, h_3\rangle).$$

We have

$$(-K_X \cdot C_4) = (-K_X \cdot C_5) = 2.$$

On $\widetilde{X}$, put

$$C_1' := V(\langle h_1, e\rangle), \quad C_2' := V(\langle h_2, e\rangle), \quad C_3' := V(\langle v_+, e\rangle).$$

The wall relations give

$$(-K_{\widetilde{X}} \cdot C_1') = (-K_{\widetilde{X}} \cdot C_2') = (-K_{\widetilde{X}} \cdot C_3') = 2.$$

Since ψ is small and crepant,

$$(-K_X \cdot C_1) = (-K_X \cdot C_2) = (-K_X \cdot C_3) = 2.$$

Because $W = \mathbb{P}^2$ is smooth, $K_{X/W} = K_X - \varphi_R^* K_W$ and

$$-K_{X/W} \cdot C_i = -K_X \cdot C_i = 2 \qquad (1 \le i \le 5).$$

Since every effective curve on a complete toric variety is numerically equivalent to an effective torus-invariant 1-cycle, the above computations show that

$$l_{X/W}(R) = 2 > 1 = d,$$

although $\varphi_R \colon X \to W$ is not a $\mathbb{P}^1$-bundle.

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