ON NON-PROJECTIVE COMPLETE TORIC VARIETIES

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Abstract. For every complete toric variety, there exists a projective toric variety which is isomorphic to it in codimension one. In this paper, we show that every smooth non-projective complete toric threefold of Picard number at most five becomes projective after a finite succession of flops or anti-flips.

Contents

1. Introduction

Throughout this paper, we will work over an algebraically closed field k of arbitrary characteristic. To the best knowledge of the authors, the following statement is not stated in the standard literature. Hence we state it here for the sake of completeness.

Theorem 1.1. *Let X be a complete toric variety. Then we can always construct a projective $\mathbb{Q}$-factorial toric variety X' which is isomorphic to X in codimension one.*

Theorem 1.1 was inspired by Kollár's problem in [K].

Problem 1.2 ([K, 5.2.2. Problem])**.** Let X be a proper algebraic threefold (smooth or with mild singularities). Can one find a (possibly very singular) projective variety X^+ such that X and X^+ are isomorphic in codimension one? This means that there are subsets $B \subset X$ and $B^+ \subset X^+$ and an isomorphism $X \setminus B \simeq X^+ \setminus B^+$ such that $\dim B \leq 1$, $\dim B^+ \leq 1$.

More generally, in this paper, we prove the following relative statement. It is obvious that if we put $Y = \operatorname{Spec} k$ in Theorem 1.3 then we can recover Theorem 1.1 as a special case.

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Theorem 1.3. *Let $f\colon X \to Y$ be a proper surjective toric morphism. Then we can construct a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & Y & \end{tikzcd}$$

such that

(i) *$f'\colon X' \to Y$ is a projective surjective toric morphism,*
(ii) *X' is $\mathbb{Q}$-factorial, and*
(iii) *ϕ is a toric birational map which is an isomorphism in codimension one.*

From the toric Mori theoretic viewpoint (see [F1], [FS], [F3], and so on), Theorem 1.3 is natural and is not difficult to prove. Hence it is natural to pose the following conjecture, which was inspired by [K, 5.2. Projectivization with flip or flop].

Conjecture 1.4. *Let X be a complete $\mathbb{Q}$-factorial toric variety. Then there exists a finite sequence consisting of flips, flops, or anti-flips*

$$X =: X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_m =: X'$$

such that X' is a projective $\mathbb{Q}$-factorial toric variety.

For the precise definition of flips, flops, and anti-flips, see Definition 2.2 below. The following observation seems to be more or less natural to the experts of the minimal model program.

Let X be a complete $\mathbb{Q}$-factorial toric variety. Then, by Theorem 1.1, we can take a projective $\mathbb{Q}$-factorial toric variety X' which is isomorphic to X in codimension one. We take an ample Cartier divisor H' on X'. Let H be the strict transform of H' on X. Then H is movable since X is isomorphic to X' in codimension one. Suppose that we can run the H-minimal model program. Then we have a finite sequence of flips

$$X =: X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_m =: X^\dagger$$

such that $H^\dagger$, which is the pushforward of H on $X^\dagger$, is nef and big. Note that every nef $\mathbb{Q}$-Cartier divisor on a complete toric variety is semiample. Hence we have a birational morphism

$$\Phi_{|lH^\dagger|}\colon X^\dagger \to X' \simeq \operatorname{Proj} \bigoplus_{m=0}^{\infty} H^0(X', \mathcal{O}_{X'}(mH'))$$

for some sufficiently large and divisible positive integer l. Since X' is $\mathbb{Q}$-factorial and $X^\dagger$ is isomorphic to X' in codimension one, $\Phi_{|lH^\dagger|}\colon X^\dagger \to X'$ is an isomorphism. Then $X^\dagger$ is projective. In particular, Conjecture 1.4 holds. Unfortunately, however, we do not know whether we can run the minimal model program for non-projective complete toric varieties or not. Therefore, Conjecture 1.4 is open. Now we have no good ideas to formulate the toric Mori theory for non-projective complete toric varieties.

One of the main results of this paper is as follows.

Theorem 1.5 (see Theorem 4.8). *Conjecture 1.4 holds true for any smooth complete toric threefold of Picard number at most five.*

Since every complete toric surface is projective, Theorem 1.5 is a first non-trivial case of Conjecture 1.4 (see Remark 1.6 below).

Remark 1.6. It is well known that every smooth complete toric variety with Picard number at most three is always projective in any dimension (see [KS]). Similarly, every $\mathbb{Q}$-factorial complete toric variety with Picard number at most two is projective (see [RT]). Note that the fan Σ in [FP, Example 1] defines a $\mathbb{Q}$-factorial non-projective complete toric threefold with Picard number three and no nontrivial nef line bundles. We also note that there is a smooth non-projective complete toric threefold with Picard number four (see Example 4.1 below).

In Section 4, we will prove Theorem 1.5 by using Oda's classification table of smooth complete toric threefolds with Picard number at most five in [O1] and [O2]. We think that the most interesting non-projective complete toric threefold with Picard number five is the one labeled as [8-13″] in [O1]. As a byproduct of the proof of Theorem 1.5, we have the following result.

Theorem 1.7 (see Corollary 5.4)**.** *There exists a smooth non-projective complete toric threefold X with Picard number five satisfying the following property. If $X \dashrightarrow X'$ is a toric birational map which is an isomorphism in codimension one to a projective toric variety X', then X' must be singular.*

Theorem 1.7 says that we can not make X' smooth even when X is smooth in Theorem 1.3. We know nothing about Conjecture 1.4 in dimension at least four.

The smooth complete toric varieties with Picard number at most five, which were classified by Tadao Oda (see [O1] and [O2]), are very special among all toric varieties. Therefore, it is not clear whether Theorem 1.5 supports Conjecture 1.4 or not. On the other hand, they give some interesting examples. It is well known that the toric variety labeled as [7-5] in [O1] is the simplest smooth non-projective complete toric variety. The one labeled as [8-12] in [O1] is the first example of smooth complete toric threefolds with Picard number five and no non-trivial nef line bundles (see [FP, Example 1]). In this paper, we show that the toric variety labeled as [8-13″] in [O1] gives an example that satisfies the property in Theorem 1.7. We hope that the description in Section 4 will be useful for constructing some interesting examples or checking some conjectures.

We look at the organization of this paper. In Section 2, we collect some definitions necessary for this paper. In Section 3, we give a detailed proof of Theorem 1.3 based on the toric Mori theory for the sake of completeness. We do not need any combinatorial arguments in this section. The main part of this paper is Section 4, where we prove Theorem 1.5. More precisely, we check Theorem 1.5 by using Oda's classification table of smooth complete toric threefolds with Picard number at most five. In the final section: Section 5, we prove Theorem 1.7.

The reader can find other interesting non-projective complete toric varieties (resp. non-projective complete surfaces) in [Bo] and [F2] (resp. [F5, Section 12]).

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2. Preliminaries

In this section, we collect some definitions and properties of toric varieties necessary for this paper. For the basic definitions and results of toric geometry, see [CLS], [Fl], [O1],

[O2], and so on. For the basic definitions of the minimal model theory, see [F4]. The following lemma is very well known.

Lemma 2.1 ($\mathbb{Q}$-factoriality). *Let $X = X_\Sigma$ be a toric variety with its fan Σ. Then X is* $\mathbb{Q}$-factorial *if and only if each cone $\sigma \in \Sigma$ is simplicial.*

We need flips, flops, and anti-flips.

Definition 2.2 (Flips, flops, and anti-flips). Let $\varphi\colon X \to W$ be a small projective toric birational morphism from a $\mathbb{Q}$-factorial toric variety X such that $\rho(X/W) = 1$. Let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. If $-D$ is φ-ample, then $\varphi\colon X \to W$ is called a *D-flipping contraction.* In this setting, we can always construct the following commutative diagram:

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr, "\varphi"'] & & X^+ \arrow[dl, "\varphi^+"] \\ & W & \end{tikzcd}$$

such that

(i) $\varphi^+\colon X^+ \to W$ is a small projective toric birational morphism,
(ii) X^+ is $\mathbb{Q}$-factorial,
(iii) $\rho(X^+/W) = 1$, and
(iv) D^+ is φ^+-ample, where D^+ is the strict transform of D on X^+.

We usually say that $\phi\colon X \dashrightarrow X^+$ is a *D-flip.* Note that we sometimes simply say that $\phi\colon X \dashrightarrow X^+$ is a *flip* if there is no danger of confusion. Let r be a positive integer such that rD is Cartier. Then it is well known that $\varphi^+\colon X^+ \to W$ is the natural projection

$$\pi\colon \mathrm{Proj}_W \bigoplus_{m=0}^{\infty} \varphi_*\mathcal{O}_X(mrD) \to W.$$

If $-K_X$ (resp. K_X) is φ-ample, then $\phi\colon X \dashrightarrow X^+$ is simply called a *flip* (resp. an *anti-flip*). If K_X is φ-numerically trivial, then $\phi\colon X \dashrightarrow X^+$ is called a *flop.*

For the combinatorial description of $\varphi\colon X \to W$ and $\varphi^+\colon X^+ \to W$, see [Re], [M], [Sa2], and so on.

Remark 2.3. For simplicity, we assume that X is a $\mathbb{Q}$-factorial projective toric variety. In Definition 2.2, the relative Kleiman–Mori cone $\mathrm{NE}(X/W)$ is obviously a half line. However, $\mathrm{NE}(X/W)$ is not necessarily an extremal ray of $\mathrm{NE}(X)$. Note that $\mathrm{NE}(X/W)$ is an extremal ray of $\mathrm{NE}(X)$ if and only if W is projective.

We make a small remark about smooth toric threefolds. Note that we are mainly interested in threefolds in this paper.

Remark 2.4. If X is a smooth toric threefold, then X has no flipping contractions (see, for example, [FSTU]). If X is a smooth threefold and $\phi\colon X \dashrightarrow X^+$ is a flop in Definition 2.2, then X^+ is also a smooth threefold.

The following notion of primitive collections and relations introduced by Batyrev [Ba1] is convenient to describe the combinatorial information of a simplicial complete fan. For more informations, please see [Ba1], [Ba2] and [Sa1].

Definition 2.5 (Primitive collections). Let $X = X_\Sigma$ be a $\mathbb{Q}$-factorial complete toric variety with its fan Σ. Let $\mathrm{G}(\Sigma)$ be the set of all primitive generators for one-dimensional cones in Σ. Then, we call a non-empty subset $P \subset \mathrm{G}(\Sigma)$ a *primitive collection* if P does not generate a cone in Σ, while any proper subset of P generates a cone in Σ.

From all the primitive collections of a fan Σ, we can know the combinatorial information of Σ. Moreover, when X is smooth, we can recover Σ from the data of the following primitive relations.

Definition 2.6 (Primitive relations)**.** Let $X = X_\Sigma$ be a *smooth* complete toric variety. For a primitive collection $P = \{v_1, \ldots, v_r\} \subset \mathrm{G}(\Sigma)$, there exists the unique cone $\sigma \in \Sigma$ which contains $v_1 + \cdots + v_r$ in its relative interior. Let $\sigma \cap \mathrm{G}(\Sigma) = \{w_1, \ldots, w_s\}$. Thus, we have the equality

$$v_1 + \cdots + v_r = a_1 w_1 + \cdots + a_s w_s \quad (a_1, \ldots, a_s \in \mathbb{Z}_{>0}),$$

which we call the *primitive relation* for P.

The following lemma is also well known.

Lemma 2.7 ([O2, Corollary 2.14])**.** *Let $X = X_\Sigma$ be a complete toric variety with its fan Σ. Then X is projective if and only if there exists a strictly convex Σ-linear support function.*

For the details of Lemma 2.7, see [CLS], [Fl], [O1], [O2], and so on. In our setting, we have:

Lemma 2.8. *Let $X = X_\Sigma$ be a $\mathbb{Q}$-factorial complete toric variety and let $G(\Sigma) = \{v_1, \ldots, v_l\}$ be the set of all primitive generators for one-dimensional cones in Σ. Let $D = \sum_i d_i D_i$ be an effective Cartier divisor on X, where D_i is the torus-invariant prime divisor corresponding to v_i for every i. Let Ψ_D be the piecewise linear function associated to D, that is, Ψ_D is the piecewise linear function defined by $\Psi_D(v_i) = d_i$ for every i. Then D is ample if and only if $-\Psi_D$ is strictly convex.*

We will repeatedly use the following lemma to prove the non-projectivity of a given toric variety in Section 4. It is obvious by Lemma 2.8.

Lemma 2.9. *Let $X = X_\Sigma$ be a smooth complete toric variety, $\mathrm{G}(\Sigma) = \{v_1, \ldots, v_l\}$ and $D_1, \ldots, D_l$ the torus-invariant prime divisors on X corresponding to $v_1, \ldots, v_l$, respectively. If an effective divisor $D = \sum_i d_i D_i$ $(d_i \geq 0)$ is ample, then for every primitive relation*

$$v_{i_1} + \cdots + v_{i_r} = a_1 v_{j_1} + \cdots + a_s v_{j_s} \quad (a_1, \ldots, a_s \in \mathbb{Z}_{>0}),$$

the inequality

$$d_{i_1} + \cdots + d_{i_r} - (a_1 d_{j_1} + \cdots + a_s d_{j_s}) > 0$$

holds.

3. Proof of Theorem 1.3

In this section, we give a detailed proof of Theorem 1.3 based on the toric Mori theory for the reader's convenience (see, for example, [F1], [FS], [F3], and so on). We start with an easy lemma. We prove it here for the sake of completeness.

Lemma 3.1. *Let X be a toric variety. Then there exists a projective birational toric morphism $X' \to X$ from a smooth quasi-projective toric variety X'.*

Proof of Lemma 3.1. By Sumihiro's equivariant completion (see [Su, Theorem 3] and [O2, p.17]), we may assume that X is complete. By Chow's lemma for toric varieties (see [O2, Proposition 2.17]), we can construct a projective birational toric morphism $X'' \to X$ from a projective toric variety X''. By the toric desingularization theorem (see, for example, [CLS, Theorem 11.19]), we can construct a projective birational toric morphism $X' \to X''$ from a smooth toric variety X'. Then the induced map $X' \to X$ is a desired morphism. We finish the proof. $\square$

Let us prove Theorem 1.3.

Proof of Theorem 1.3. By taking a small projective $\mathbb{Q}$-factorialization (see [F1, Corollary 5.9]), we may assume that X is $\mathbb{Q}$-factorial. If X is projective over Y, then there is nothing to prove. Hence, from now, we may assume that X is not projective over Y. We take a projective birational toric morphism $p\colon V \to X$ from a smooth quasi-projective toric variety V (see Lemma 3.1). Since X is $\mathbb{Q}$-factorial, we can take an ample Cartier divisor A on V such that $H := p_*A$ is Cartier. We put $D := p^*H + E$, where E is an effective Cartier divisor on V with $\mathrm{Supp} E = \mathrm{Exc}(p)$. We note that

$$\bigoplus_{m=0}^{\infty} (f \circ p)_* \mathcal{O}_V(mD)$$

is a finitely generated $\mathcal{O}_Y$-algebra (see [FS, Corollary 5.8]) and that

$$(f \circ p)_* \mathcal{O}_V(mD) \simeq f_* \mathcal{O}_X(mH)$$

holds for every non-negative integer m. We consider

$$Z := \mathrm{Proj}_Y \bigoplus_{m=0}^{\infty} (f \circ p)_* \mathcal{O}_V(mD) \simeq \mathrm{Proj}_Y \bigoplus_{m=0}^{\infty} f_* \mathcal{O}_X(mH).$$

Then Z is a toric variety which is projective over Y. By running the D-minimal model program over Y (see [F1, 5.3] and [FS, 3.1]), we have a sequence of flips and divisorial contractions with respect to D over Y starting from V:

$$V =: V_0 \dashrightarrow V_1 \dashrightarrow \cdots \dashrightarrow V_l =: V'$$

such that D', which is the pushforward of D on V', is nef and big over Y. Note that V_i is $\mathbb{Q}$-factorial for every i because V is. By the definition of Z, we have a birational contraction morphism $V' \to Z$ over Y. Let

$$\begin{tikzcd} & W \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ V \arrow[rr, dashed] & & V' \end{tikzcd}$$

be a common resolution. Then by applying the negativity lemma (see, for example, [FS, Lemma 4.10]) to each step of the above minimal model program, we can write

$$\alpha^* D = \beta^* D' + G$$

where G is an effective β-exceptional $\mathbb{Q}$-divisor on W. Let m be a sufficiently large and divisible positive integer. Then

$$mG = \mathrm{Bs}|m(\beta^* D' + G)| = \mathrm{Bs}|m\alpha^* D| = \mathrm{Bs}|m\alpha^*(p^*H + E)| \supset m\alpha^* E$$

holds. Hence $\alpha^* E$ is β-exceptional. This implies that every irreducible component of E is contracted by $V \dashrightarrow V'$. Therefore, the induced birational map $\psi\colon X \dashrightarrow Z$ over

Y is a contraction, that is, ψ^{-1} contracts no divisors. We take a small projective $\mathbb{Q}$-factorialization $Z' \to Z$ (see [F1, Corollary 5.9]). Then the induced map $\psi' \colon X \dashrightarrow Z'$ over Y is also a contraction. If ψ' contracts some divisors, then we take a composite of blow-ups $X' \to Z'$ extracting those divisors (cf. [F1, Theorem 5.5]). Thus the induced toric birational map $X \dashrightarrow X'$ over Y is an isomorphism in codimension one and X' is a $\mathbb{Q}$-factorial toric variety which is projective over Y by construction. This is what we wanted. □

Now Theorem 1.1 is obvious.

Proof of Theorem 1.1. We put $Y = \operatorname{Spec} k$. Then Theorem 1.1 is an obvious consequence of Theorem 1.3. □

4. Smooth complete toric threefolds with $\rho \leq 5$

In this section, we confirm Conjecture 1.4 for smooth complete toric threefolds of Picard number at most five. If the Picard number of a smooth complete toric variety is less than four, then the variety is always projective (see Remark 1.6). So, first, we quickly review the famous smooth non-projective complete toric threefold of Picard number four described in [O1, Proposition 9.4].

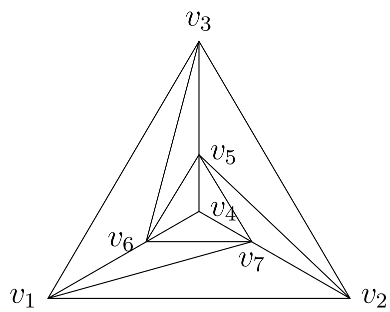
Example 4.1. Let $W := X_\Sigma$ be the smooth complete toric threefold associated to the fan Σ in $\mathbb{R}^3$ whose one-dimensional cones are generated by

$$v_1 = (1,0,0), v_2 = (0,1,0), v_3 = (0,0,1), v_4 = (-1,-1,-1),$$
$$v_5 = (-1,-1,0), v_6 = (0,-1,-1), v_7 = (-1,0,-1),$$

and the three-dimensional cones of Σ are

$$\langle v_1,v_2,v_3\rangle, \langle v_1,v_2,v_7\rangle, \langle v_1,v_3,v_6\rangle, \langle v_1,v_6,v_7\rangle, \langle v_2,v_3,v_5\rangle,$$
$$\langle v_2,v_5,v_7\rangle, \langle v_3,v_5,v_6\rangle, \langle v_4,v_5,v_6\rangle, \langle v_4,v_5,v_7\rangle, \langle v_4,v_6,v_7\rangle.$$

The configuration of cones of Σ are as the following diagram:



We should remark that there is the extra three-dimensional cone $\langle v_1,v_2,v_3\rangle$. The variety W is the smooth complete toric threefold of type [7-5] in [O1].

One can easily see there exist exactly three flopping curves

$$V(\langle v_1,v_7\rangle),\ V(\langle v_2,v_5\rangle) \text{ and } V(\langle v_3,v_6\rangle),$$

that is, there are primitive relations

$$v_2+v_6 = v_1+v_7,\ v_3+v_7 = v_2+v_5 \text{ and } v_1+v_5 = v_3+v_6,$$

correspondingly. This configuration of cones associated to the three flopping curves causes the non-projectivity of W. More precisely, the above primitive relations and Lemma 2.9 imply that there is no effective ample Cartier divisor on W. Hence W is non-projective. As

Remark in [O1, p.71], after one of these three flops, W becomes projective. In particular, Conjecture 1.4 is true for W. We remark that among smooth complete toric threefolds of Picard number four, only W is non-projective by the classification in [O1].

Next, we consider the case of Picard number five. By the classification in [O1], it is well known that there exist exactly eleven types

[8-2], [8-5′], [8-5″], [8-8], [8-10], [8-11], [8-12], [8-13′], [8-13″], [8-14′] and [8-14″]

of smooth complete toric threefolds of Picard number five which cannot be blown-down to smooth threefolds. Among them, [8-2], [8-10] and [8-11] are projective. Let us start with the proof of the projectivity of these varieties for reader's convenience. We emphasize that the projectivity of the smooth complete toric threefolds of type [8-11] was not announced in [O1]. In the following, we use the notation in [FP] which is slightly different from the one in [O1].

[8-2] Let $a \in \mathbb{Z}$ and $Z_2(a) := X_\Sigma$ the smooth complete toric threefold associated to the fan Σ in $\mathbb{R}^3$ whose one-dimensional cones are generated by

$$v_1 = (1,0,0), v_2 = (0,-2,-1), v_3 = (0,-1,0), v_4 = (0,0,1),$$

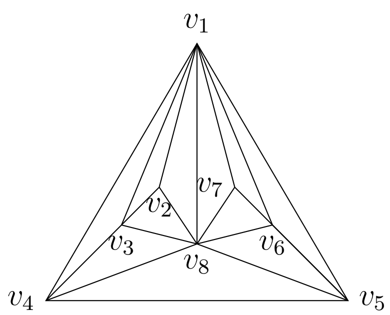
$$v_5 = (0,1,a), v_6 = (0,0,-1), v_7 = (0,-1,-1), v_8 = (-1,-3,-2),$$

and the three-dimensional cones of Σ are

$$\langle v_1,v_2,v_3\rangle, \langle v_1,v_2,v_8\rangle, \langle v_1,v_3,v_4\rangle, \langle v_1,v_4,v_5\rangle, \langle v_1,v_5,v_6\rangle, \langle v_1,v_6,v_7\rangle,$$

$$\langle v_1,v_7,v_8\rangle, \langle v_2,v_3,v_8\rangle, \langle v_3,v_4,v_8\rangle, \langle v_4,v_5,v_8\rangle, \langle v_5,v_6,v_8\rangle, \langle v_6,v_7,v_8\rangle.$$

The configuration of cones of Σ are as the following diagram:

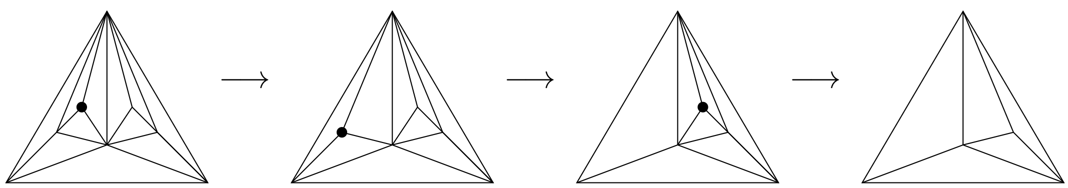


There is the extra three-dimensional cone $\langle v_1, v_4, v_5\rangle$.

We obtain the sequence

$$Z_2(a) \xrightarrow{\varphi_1} Y_1 \xrightarrow{\varphi_2} Y_2 \xrightarrow{\varphi_3} Y_3$$

of birational morphisms associated to the diagrams



of fans in $\mathbb{R}^3$. The morphisms φ_1, φ_2 and φ_3 are associated to the star subdivisions along

$$v_2 = \frac{1}{2}(v_1 + v_3 + v_8),\ v_3 = \frac{1}{3}(v_1 + 2v_4 + v_8) \text{ and } v_7 = \frac{1}{3}(v_1 + v_6 + v_8)$$

of fans, respectively. Namely, φ_1, φ_2 and φ_3 are divisorial contractions which contract $V(\langle v_2\rangle) \subset Z_2(a)$, $V(\langle v_3\rangle) \subset Y_1$ and $V(\langle v_7\rangle) \subset Y_2$, respectively. Since Y_3 is $\mathbb{Q}$-factorial and its Picard number is two, Y_3, Y_2, Y_1 and $Z_2(a)$ are projective (see Remark 1.6).

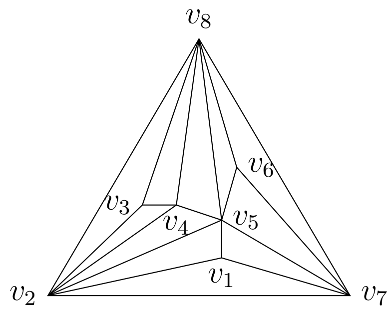
[8-10] Let $Z_{10} := X_\Sigma$ be the smooth complete toric threefold associated to the fan Σ in $\mathbb{R}^3$ whose one-dimensional cones are generated by

$$v_1 = (0,0,1), v_2 = (-1,-1,-1), v_3 = (0,-1,-2), v_4 = (0,0,-1),$$
$$v_5 = (0,1,0), v_6 = (1,0,-1), v_7 = (1,0,0), v_8 = (1,-1,-2),$$

and the three-dimensional cones of Σ are

$$\langle v_1,v_2,v_5\rangle, \langle v_1,v_2,v_7\rangle, \langle v_1,v_5,v_7\rangle, \langle v_2,v_3,v_4\rangle, \langle v_2,v_3,v_8\rangle, \langle v_2,v_4,v_5\rangle,$$
$$\langle v_2,v_7,v_8\rangle, \langle v_3,v_4,v_8\rangle, \langle v_4,v_5,v_8\rangle, \langle v_5,v_6,v_7\rangle, \langle v_5,v_6,v_8\rangle, \langle v_6,v_7,v_8\rangle.$$

The configuration of cones of Σ are as the following diagram:

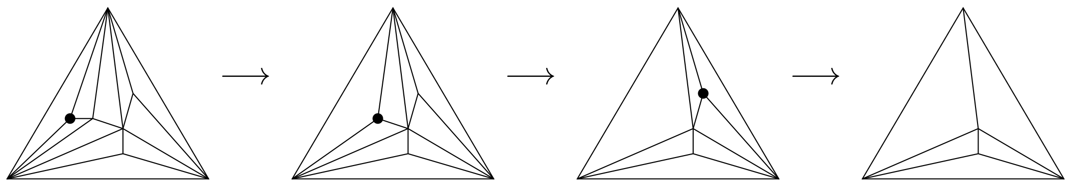


There is the extra three-dimensional cone $\langle v_2,v_7,v_8\rangle$.

We obtain the sequence

$$Z_{10} \xrightarrow{\varphi_1} Y_1 \xrightarrow{\varphi_2} Y_2 \xrightarrow{\varphi_3} Y_3$$

of birational morphisms associated to the diagrams



of fans in $\mathbb{R}^3$. The morphisms φ_1, φ_2 and φ_3 are associated to the star subdivisions along

$$v_3 = \frac{1}{2}(v_2+v_4+v_8), \ v_4 = \frac{1}{3}(v_2+2v_5+v_8) \ \text{ and } v_6 = \frac{1}{2}(v_5+v_7+v_8)$$

of fans, respectively. Namely, φ_1, φ_2 and φ_3 are divisorial contractions which contract $V(\langle v_3\rangle) \subset Z_{10}$, $V(\langle v_4\rangle) \subset Y_1$ and $V(\langle v_6\rangle) \subset Y_2$, respectively. Since Y_3 is $\mathbb{Q}$-factorial and its Picard number is two, Y_3, Y_2, Y_1 and Z_{10} are projective (see Remark 1.6).

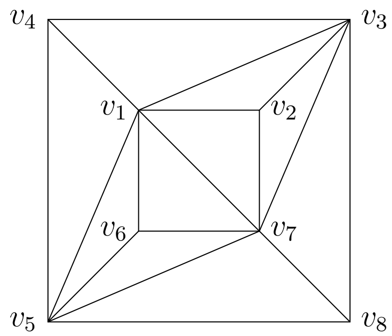
[8-11] Let $a,b \in \mathbb{Z}$ and $Z_{11}(a,b) := X_\Sigma$ the smooth complete toric threefold associated to the fan Σ in $\mathbb{R}^3$ whose one-dimensional cones are generated by

$$v_1 = (1,0,0), v_2 = (0,-1,0), v_3 = (0,0,1), v_4 = (1,1,a),$$
$$v_5 = (0,0,-1), v_6 = (0,-1,-1), v_7 = (-1,-2,-1), v_8 = (0,1,b),$$

and the three-dimensional cones of Σ are

$$\langle v_1,v_2,v_3\rangle, \langle v_1,v_3,v_4\rangle, \langle v_1,v_4,v_5\rangle, \langle v_1,v_5,v_6\rangle, \langle v_1,v_6,v_7\rangle, \langle v_1,v_2,v_7\rangle,$$
$$\langle v_2,v_3,v_7\rangle, \langle v_5,v_6,v_7\rangle, \langle v_3,v_4,v_8\rangle, \langle v_4,v_5,v_8\rangle, \langle v_5,v_7,v_8\rangle, \langle v_3,v_7,v_8\rangle.$$

The configuration of cones of Σ are as the following diagram:

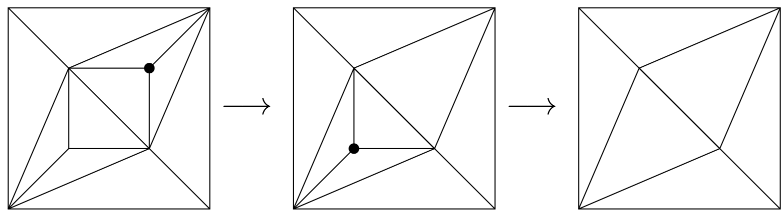


In this case, there are two extra three-dimensional cones $\langle v_4, v_5, v_8\rangle$ and $\langle v_3, v_4, v_8\rangle$.

We obtain the sequence

$$Z_{11}(a,b) \xrightarrow{\varphi_1} Y_1 \xrightarrow{\varphi_2} Y_2$$

of birational morphisms associated to the diagrams



of fans in $\mathbb{R}^3$. The morphisms φ_1 and φ_2 are associated to the star subdivisions along

$$v_2 = \frac{1}{2}(v_1 + v_3 + v_7) \text{ and } v_6 = \frac{1}{2}(v_1 + v_5 + v_7)$$

of fans, respectively. Namely, φ_1 and φ_2 are divisorial contractions which contract $V(\langle v_2\rangle) \subset Z_{11}(a,b)$ and $V(\langle v_6\rangle) \subset Y_1$, respectively. Since $v_3 + v_5 = 0$, there exists a surjective toric morphism $\phi : Y_2 \to S$ whose every fiber is isomorphic to $\mathbb{P}^1$ set-theoretically, where S is a complete toric surface. Therefore, ϕ is a projective morphism since we can easily find a ϕ-ample Cartier divisor on Y. Thus, Y_2, Y_1 and $Z_{11}(a,b)$ are projective.

From now, we confirm our Conjecture 1.4 for remaining eight types of smooth complete toric threefolds of Picard number five. The varieties of these types are non-projective except for few cases. In fact, for few values of parameters $a, b, c, d \in \mathbb{Z}$ below, the varieties become projective.

[8-5′] Let $a \in \mathbb{Z}$ and $Z'_5(a) := X_\Sigma$ the smooth complete toric threefold associated to the fan Σ in $\mathbb{R}^3$ whose one-dimensional cones are generated by

$$v_1 = (1,0,0), v_2 = (0,1,0), v_3 = (0,0,1), v_4 = (0,-1,-a),$$

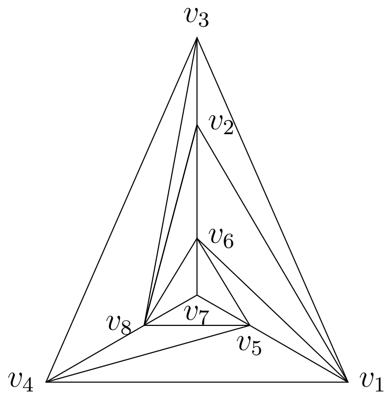
$$v_5 = (0,0,-1), v_6 = (-1,1,-1), v_7 = (-1,0,-1), v_8 = (-1,-1,0),$$

and the three-dimensional cones of Σ are

$$\langle v_1, v_2, v_3\rangle, \langle v_1, v_3, v_4\rangle, \langle v_1, v_4, v_5\rangle, \langle v_1, v_5, v_6\rangle, \langle v_1, v_2, v_6\rangle, \langle v_2, v_3, v_8\rangle,$$

$$\langle v_3, v_4, v_8\rangle, \langle v_4, v_5, v_8\rangle, \langle v_5, v_6, v_7\rangle, \langle v_5, v_7, v_8\rangle, \langle v_6, v_7, v_8\rangle, \langle v_2, v_6, v_8\rangle.$$

The configuration of cones of Σ are as the following diagram:



There is the extra three-dimensional cone $\langle v_1, v_3, v_4\rangle$.

We have already proved that $Z_5'(a)$ is non-projective for $a \neq 0, -1$ in [FP, Example 2] (see Remark 4.2 below). If $a = -1$, then we have primitive relations

$$v_1 + v_8 = v_4 + v_5, \quad v_4 + v_6 = v_2 + v_8, \quad \text{and} \quad v_2 + v_5 = v_1 + v_6.$$

Thus $Z_5'(-1)$ is non-projective by the above primitive relations and Lemma 2.9. If $a = 0$, then we have a primitive relation

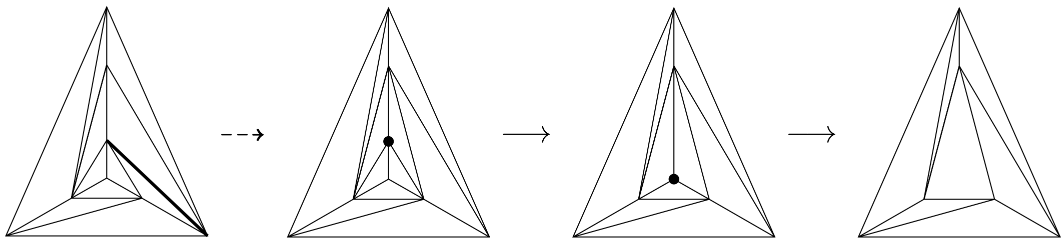
$$v_1 + v_8 = v_4.$$

Then we have a contraction $Z_5'(0) \to V$ by removing v_4. By construction, V is a smooth complete toric threefold with Picard number four. Since $v_3 + v_5 = 0$, V is not isomorphic to W in Example 4.1. This means that V and $Z_5'(0)$ are projective.

We obtain the sequence

$$Z_5'(a) \overset{\psi}{\dashrightarrow} X' \xrightarrow{\varphi_1} Y_1 \xrightarrow{\varphi_2} Y_2$$

of birational maps associated to the diagrams



of fans in $\mathbb{R}^3$. The rational map ψ is the flop associated to the primitive relation

$$v_2 + v_5 = v_1 + v_6,$$

that is, the fan corresponding to X' is obtained from Σ by removing $\langle v_1, v_6\rangle$ and by adding $\langle v_2, v_5\rangle$. The morphisms φ_1 and φ_2 are associated to the star subdivisions along

$$v_6 = v_2 + v_7 \text{ and } v_7 = v_2 + v_5 + v_8$$

of fans, respectively. Namely, ψ is the flop along $V(\langle v_1, v_6\rangle)$, φ_1 is the blow-up of Y_1 along the curve $V(\langle v_2, v_7\rangle)$ and φ_2 is the blow-up of Y_2 at the point $V(\langle v_2, v_5, v_8\rangle)$. Since Y_2 is smooth and its Picard number is three, Y_2, Y_1 and X' are projective (see Remark 1.6). Conjecture 1.4 is true for this case.

Remark 4.2. The variety $Z_5'(a)$ is studied in [FP, Example 2], where we prove that $\mathrm{NE}(Z_5'(a)) = \mathbb{R}^5$ holds when $a \neq 0, -1$. In particular, $Z_5'(a)$ is non-projective when $a \neq 0, -1$. We put $n = (1, 0, 0)$, $n' = (0, 1, 0)$, and $n'' = (0, 0, 1)$. Then the above description of $Z_5'(a)$ coincides with [8-5$'$] in [O1, p.78].

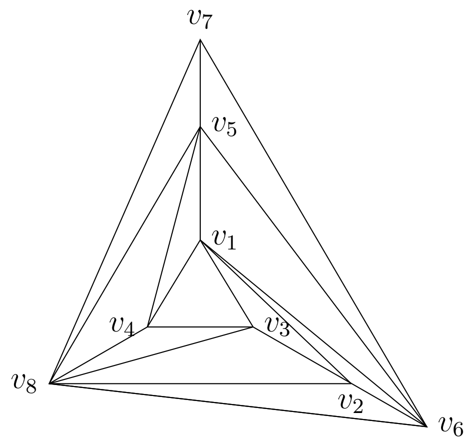
[8-5″] Let $Z_5'' := X_\Sigma$ be the smooth complete toric threefold associated to the fan Σ in $\mathbb{R}^3$ whose one-dimensional cones are generated by

$$v_1 = (0,1,0), v_2 = (0,-1,-1), v_3 = (1,0,0), v_4 = (0,0,1),$$
$$v_5 = (-1,0,-1), v_6 = (-1,-2,-2), v_7 = (-1,-1,-1), v_8 = (-1,-1,0),$$

and the three-dimensional cones of Σ are

$$\langle v_1, v_2, v_3\rangle, \langle v_1, v_3, v_4\rangle, \langle v_1, v_4, v_5\rangle, \langle v_1, v_5, v_6\rangle, \langle v_1, v_2, v_6\rangle, \langle v_2, v_3, v_8\rangle,$$
$$\langle v_3, v_4, v_8\rangle, \langle v_4, v_5, v_8\rangle, \langle v_5, v_6, v_7\rangle, \langle v_5, v_7, v_8\rangle, \langle v_6, v_7, v_8\rangle, \langle v_2, v_6, v_8\rangle.$$

The configuration of cones of Σ are as the following diagram:



There is the extra three-dimensional cone $\langle v_6, v_7, v_8\rangle$.

We have primitive relations

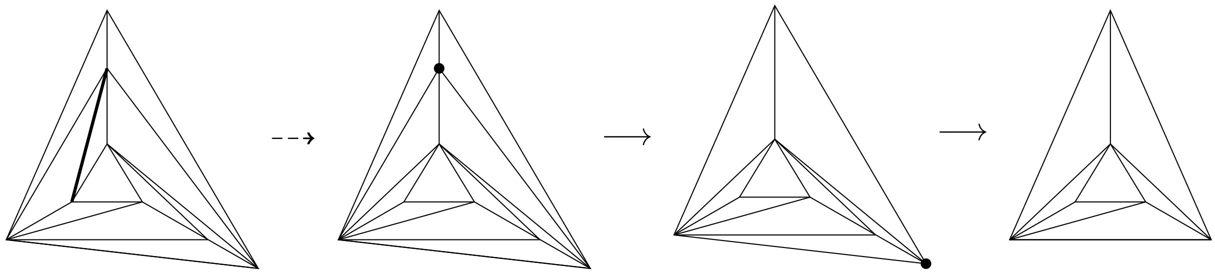
$$v_1 + v_8 = v_4 + v_5, \quad v_2 + v_4 = v_3 + v_8, \quad \text{and} \quad v_3 + v_5 = v_1 + v_2.$$

Thus we obtain that Z_5'' is non-projective by the above primitive relations and Lemma 2.9.

We obtain the sequence

$$Z_5'' \overset{\psi}{\dashrightarrow} X' \xrightarrow{\varphi_1} Y_1 \xrightarrow{\varphi_2} Y_2$$

of birational maps associated to the diagrams



of fans in $\mathbb{R}^3$. The rational map ψ is the flop associated to the primitive relation

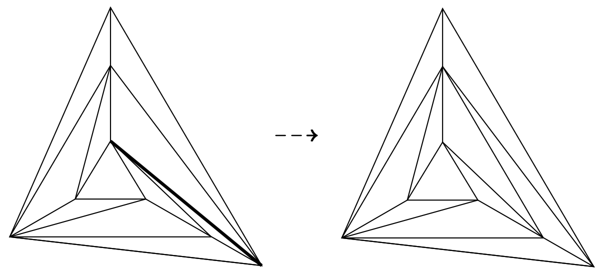
$$v_1 + v_8 = v_4 + v_5,$$

that is, the fan corresponding to X' is obtained from Σ by removing $\langle v_4, v_5\rangle$ and by adding $\langle v_1, v_8\rangle$. The morphisms φ_1 and φ_2 are associated to the star subdivisions along

$$v_5 = v_1 + v_7 \text{ and } v_6 = v_2 + v_7$$

of fans, respectively. Namely, ψ is the flop along $V(\langle v_4, v_5\rangle)$, φ_1 is the blow-up of Y_1 along the curve $V(\langle v_1, v_7\rangle)$ and φ_2 is the blow-up of Y_2 along the curve $V(\langle v_2, v_7\rangle)$. Since Y_2 is smooth and its Picard number is three, Y_2, Y_1 and X' are projective (see Remark 1.6). Conjecture 1.4 is true for this case.

Remark 4.3. Let $Z_5'' \dashrightarrow Z'$ be the flop along $V(\langle v_1, v_6\rangle)$ associated to the diagrams:



Namely, this flop is associated to the primitive relation

$$v_2 + v_5 = v_1 + v_6.$$

Let $Z' \to W'$ be the divisorial contraction which contracts $V(\langle v_6\rangle)$, that is, Z' is obtained from W' by the star subdivision along

$$v_6 = v_2 + v_7.$$

Then W' is isomorphic to W in Example 4.1. This can be confirmed by the automorphism

$$(x, y, z) \mapsto (z, y, x)$$

of $\mathbb{R}^3$, and by the changing

$$1 \mapsto 2, 2 \mapsto 5, 4 \mapsto 1, 5 \mapsto 7, 7 \mapsto 4, 8 \mapsto 6$$

of indices of v_i's. In particular, Z' is non-projective (see Remark 4.7).

[8-8] Let $Z_8 := X_\Sigma$ be the smooth complete toric threefold associated to the fan Σ in $\mathbb{R}^3$ whose one-dimensional cones are generated by

$$v_1 = (0, 0, 1), v_2 = (1, 0, 0), v_3 = (0, -1, -1), v_4 = (-1, -2, -1),$$

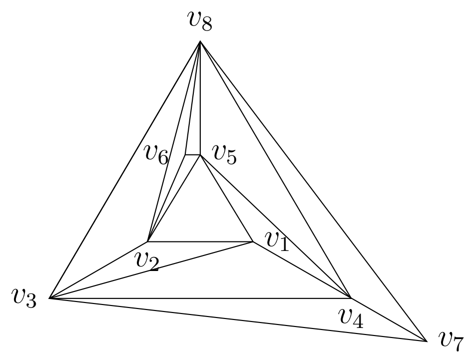
$$v_5 = (0, 1, 0), v_6 = (0, 0, -1), v_7 = (-1, -2, -2), v_8 = (-1, -1, -2),$$

and the three-dimensional cones of Σ are

$$\langle v_1, v_2, v_3\rangle, \langle v_1, v_3, v_4\rangle, \langle v_1, v_4, v_5\rangle, \langle v_1, v_2, v_5\rangle, \langle v_2, v_5, v_6\rangle, \langle v_3, v_4, v_7\rangle,$$

$$\langle v_2, v_3, v_8\rangle, \langle v_3, v_7, v_8\rangle, \langle v_4, v_7, v_8\rangle, \langle v_4, v_5, v_8\rangle, \langle v_5, v_6, v_8\rangle, \langle v_2, v_6, v_8\rangle.$$

The configuration of cones of Σ are as the following diagram:



There is the extra three-dimensional cone $\langle v_3, v_7, v_8\rangle$.

We have primitive relations

$$v_1 + v_8 = v_4 + v_5, \quad v_2 + v_4 = v_1 + 2v_3, \quad v_3 + v_5 = v_6, \quad \text{and} \quad v_3 + v_6 = v_2 + v_8.$$

Hence Z_8 is not projective by the above primitive relations and Lemma 2.9. Here, we should remark that the relation

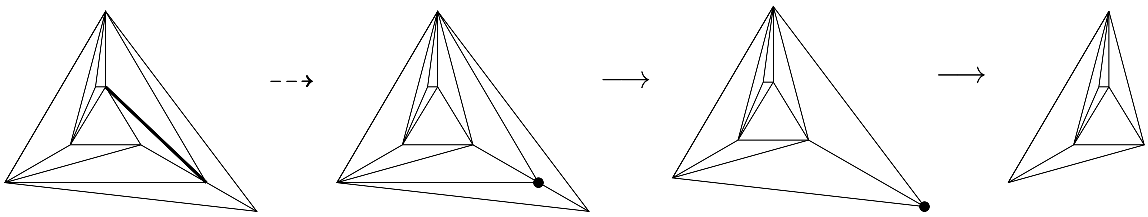
$$2v_3 + v_5 = v_2 + v_8$$

obtained simply from the third and fourth primitive relations above makes it easier to understand the non-projectiveness for Z_8, though it is not a primitive relation.

We obtain the sequence

$$Z_8 \overset{\psi}{\dashrightarrow} X' \xrightarrow{\varphi_1} Y_1 \xrightarrow{\varphi_2} Y_2$$

of birational maps associated to the diagrams



of fans in $\mathbb{R}^3$. The rational map ψ is the flop associated to the primitive relation

$$v_1 + v_8 = v_4 + v_5,$$

that is, the fan corresponding to X' is obtained from Σ by removing $\langle v_4, v_5\rangle$ and by adding $\langle v_1, v_8\rangle$. The morphisms φ_1 and φ_2 are associated to the star subdivisions along

$$v_4 = v_1 + v_7 \text{ and } v_7 = v_1 + v_3 + v_8$$

of fans, respectively. Namely, ψ is the flop along $V(\langle v_4, v_5\rangle)$, φ_1 is the blow-up of Y_1 along the curve $V(\langle v_1, v_7\rangle)$ and φ_2 is the blow-up of Y_2 at the point $V(\langle v_1, v_3, v_8\rangle)$. Since Y_2 is smooth and its Picard number is three, Y_2, Y_1 and X' are projective (see Remark 1.6). Conjecture 1.4 is true for this case.

[8-12] Let $Z_{12} := X_\Sigma$ be the smooth complete toric threefold associated to the fan Σ in $\mathbb{R}^3$ whose one-dimensional cones are generated by

$$v_1 = (1, 0, 0), v_2 = (0, 1, 0), v_3 = (0, 0, 1), v_4 = (0, -1, -1),$$

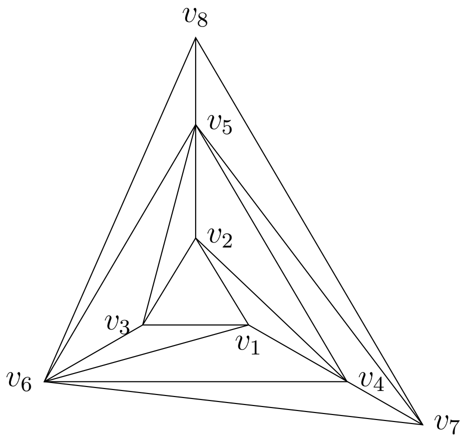
$$v_5 = (-1, 0, -1), v_6 = (-2, -1, 0), v_7 = (-1, -1, -1), v_8 = (-2, -1, -1),$$

and the three-dimensional cones of Σ are

$$\langle v_1, v_2, v_3\rangle, \langle v_1, v_2, v_4\rangle, \langle v_1, v_3, v_6\rangle, \langle v_1, v_4, v_6\rangle, \langle v_2, v_3, v_5\rangle, \langle v_2, v_4, v_5\rangle,$$

$$\langle v_3, v_5, v_6\rangle, \langle v_4, v_5, v_7\rangle, \langle v_4, v_6, v_7\rangle, \langle v_5, v_6, v_8\rangle, \langle v_5, v_7, v_8\rangle, \langle v_6, v_7, v_8\rangle.$$

The configuration of cones of Σ are as the following diagram:

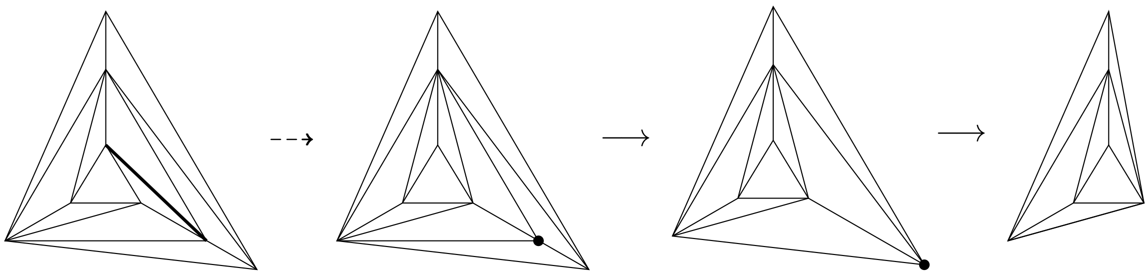


There is the extra three-dimensional cone $\langle v_6, v_7, v_8\rangle$.

We obtain the sequence

$$Z_{12} \overset{\psi}{\dashrightarrow} X' \xrightarrow{\varphi_1} Y_1 \xrightarrow{\varphi_2} Y_2$$

of birational maps associated to the diagrams



of fans in $\mathbb{R}^3$. The rational map ψ is the flop associated to the primitive relation

$$v_1 + v_5 = v_2 + v_4,$$

that is, the fan corresponding to X' is obtained from Σ by removing $\langle v_2, v_4\rangle$ and by adding $\langle v_1, v_5\rangle$. The morphisms φ_1 and φ_2 are associated to the star subdivisions along

$$v_4 = v_1 + v_7 \text{ and } v_7 = v_1 + v_8$$

of fans, respectively. Namely, ψ is the flop along $V(\langle v_2, v_4\rangle)$, φ_1 is the blow-up of Y_1 along the curve $V(\langle v_1, v_7\rangle)$ and φ_2 is the blow-up of Y_2 along the curve $V(\langle v_1, v_8\rangle)$. Since Y_2 is smooth and its Picard number is three, Y_2, Y_1 and X' are projective (see Remark 1.6). Conjecture 1.4 is true for this case.

Remark 4.4. The variety Z_{12} is studied in [FP, Example 1], which is the first example of smooth complete toric threefolds with Picard number five and no non-trivial nef line bundles. In particular, it is non-projective. If we put $n = (-1, -1, -1)$, $n' = (1, 0, 0)$, and $n'' = (0, 0, 1)$, then Z_{12} coincides with [8-12] in [O1, p.79]. Also, there is another description for Z_{12} in [F4, Example 2.2.8].

[8-13′] Let $a, b \in \mathbb{Z}$ and $Z'_{13}(a, b) := X_\Sigma$ the smooth complete toric threefold associated to the fan Σ in $\mathbb{R}^3$ whose one-dimensional cones are generated by

$$v_1 = (-1, b, 0), v_2 = (0, -1, 0), v_3 = (1, -1, 0), v_4 = (-1, 0, -1),$$

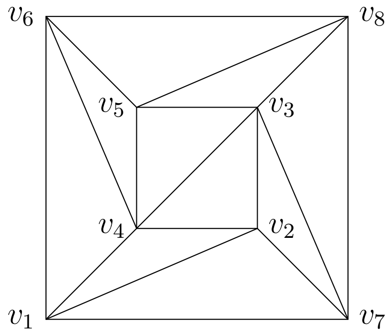
$$v_5 = (0, 0, -1), v_6 = (0, 1, 0), v_7 = (0, 0, 1), v_8 = (1, 0, a),$$

and the three-dimensional cones of Σ are

$$\langle v_1, v_2, v_4\rangle, \langle v_2, v_3, v_4\rangle, \langle v_3, v_4, v_5\rangle, \langle v_4, v_5, v_6\rangle, \langle v_1, v_4, v_6\rangle, \langle v_1, v_6, v_7\rangle,$$

$$\langle v_1, v_2, v_7\rangle, \langle v_2, v_3, v_7\rangle, \langle v_3, v_5, v_8\rangle, \langle v_5, v_6, v_8\rangle, \langle v_3, v_7, v_8\rangle, \langle v_6, v_7, v_8\rangle.$$

The configuration of cones of Σ are as the following diagram:



In this case, there are two extra three-dimensional cones $\langle v_1, v_6, v_7\rangle$ and $\langle v_6, v_7, v_8\rangle$.

We have primitive relations

$$v_2 + v_8 = v_3 + av_7 \quad \text{and} \quad v_3 + v_6 = av_5 + v_8$$

when $a > 0$, and

$$v_2 + v_8 = v_3 + (-a)v_5 \quad \text{and} \quad v_3 + v_6 = (-a)v_7 + v_8$$

when $a < 0$. Moreover, we have primitive relations

$$v_1 + v_5 = v_4 + bv_6 \quad \text{and} \quad v_4 + v_7 = v_1 + bv_2$$

when $b > 0$, and

$$v_1 + v_5 = (-b)v_2 + v_4 \quad \text{and} \quad v_4 + v_7 = v_1 + (-b)v_6$$

when $b < 0$. By the above primitive relations and Lemma 2.9, we can prove that there is no effective ample Cartier divisor on $Z'_{13}(a, b)$ when $ab \neq 0$. Therefore, $Z'_{13}(a, b)$ is not projective when $ab \neq 0$. When $a = 0$, we have $v_3 + v_6 = v_8$. Therefore, we have a contraction $Z'_{13}(0, b) \to V_1$ by removing v_8. Similarly, when $b = 0$, we have $v_4 + v_7 = v_1$. Then we have a contraction $Z'_{13}(a, 0) \to V_2$ by removing v_1. Note that V_1 and V_2 are smooth complete toric threefolds with Picard number four. They are not isomorphic to W in Example 4.1 since $v_2 + v_6 = v_5 + v_7 = 0$. Therefore, V_1 and V_2 are both projective. This implies that $Z'_{13}(a, b)$ is projective when $ab = 0$.

Let $Z'_{13}(a, b) \dashrightarrow X'$ be the flop along $V(\langle v_3, v_4\rangle)$, that is, there is a primitive relation

$$v_2 + v_5 = v_3 + v_4.$$

We can see that

$$X' = Z''_{13}(-1, a, b - 1, -1)$$

in [8-13″] below by the automorphism

$$(x, y, z) \mapsto (x, x + y, z)$$

of $\mathbb{R}^3$, and by the changing

$$1 \mapsto 8, 2 \mapsto 3, 3 \mapsto 2, 4 \mapsto 5, 5 \mapsto 4, 8 \mapsto 1$$

of indices of v_i's. Since Conjecture 1.4 holds for X' by the argument below, it holds for $Z'_{13}(a, b)$, too.

Remark 4.5. The variety $Z'_{13}(a, b)$ is studied in [FP, Example 3]. Our description here coincides with [8-13′] in [O1, p.79] if we put $n = (1, -1, 0)$, $n' = (0, 1, 0)$, and $n'' = (0, 0, 1)$.

Remark 4.6. The reader can find some related topics on [8-13′] in [Bo, 4.1.2], where the toric variety $Z'_{13}(a \pm 1, b \pm 1)$ can be obtained from $Z'_{13}(a,b)$ by an elementary transformation (see [Bo, Proposition 4]).

[8-13″] Let $a,b,c,d \in \mathbb{Z}$ and $Z''_{13}(a,b,c,d) := X_\Sigma$ the smooth complete toric threefold associated to the fan Σ in $\mathbb{R}^3$ whose one-dimensional cones are generated by

$$v_1 = (1,1,b), v_2 = (1,0,0), v_3 = (0,-1,0), v_4 = (0,0,-1),$$

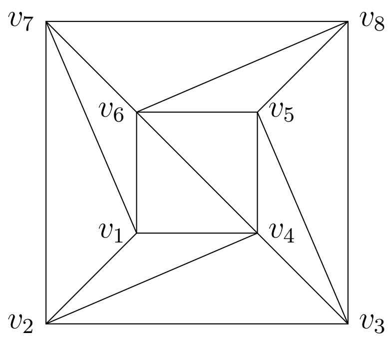
$$v_5 = (-1,a,d), v_6 = (0,1,0), v_7 = (0,0,1), v_8 = (-1,c,d+1),$$

and the three-dimensional cones of Σ are

$$\langle v_1,v_2,v_4\rangle, \langle v_2,v_3,v_4\rangle, \langle v_3,v_4,v_5\rangle, \langle v_4,v_5,v_6\rangle, \langle v_1,v_4,v_6\rangle, \langle v_1,v_6,v_7\rangle,$$

$$\langle v_1,v_2,v_7\rangle, \langle v_2,v_3,v_7\rangle, \langle v_3,v_5,v_8\rangle, \langle v_5,v_6,v_8\rangle, \langle v_3,v_7,v_8\rangle, \langle v_6,v_7,v_8\rangle.$$

The configuration of cones of Σ are as the following diagram:



In this case, there are two extra three-dimensional cones $\langle v_2,v_3,v_7\rangle$ and $\langle v_3,v_7,v_8\rangle$.

We have primitive relations

$$v_1 + v_3 = v_2 + bv_7 \quad \text{and} \quad v_2 + v_6 = v_1 + bv_4$$

when $b > 0$, and

$$v_1 + v_3 = v_2 + (-b)v_4 \quad \text{and} \quad v_2 + v_6 = v_1 + (-b)v_7$$

when $b < 0$. Moreover, we have primitive relations

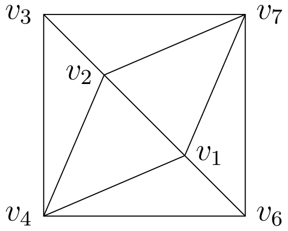
$$v_4 + v_8 = v_5 + (a-c)v_3 \quad \text{and} \quad v_5 + v_7 = v_8 + (a-c)v_6$$

when $a > c$, and

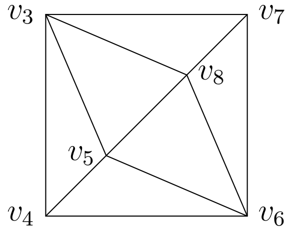
$$v_4 + v_8 = v_5 + (c-a)v_6 \quad \text{and} \quad v_5 + v_7 = v_8 + (c-a)v_3$$

when $a < c$. By the above primitive relations and Lemma 2.9, we can prove that there exists no effective ample Cartier divisor on $Z''_{13}(a,b,c,d)$ if $a \neq c$ and $b \neq 0$. This means that $Z''_{13}(a,b,c,d)$ is not projective when $a \neq c$ and $b \neq 0$.

Since $v_3 + v_6 = v_4 + v_7 = 0$ holds, there exists a morphism $Z''_{13}(a,b,c,d) \to \mathbb{P}^1$ with the general fiber $\mathbb{P}^1 \times \mathbb{P}^1$ associated to the first projection $\mathbb{R}^3 \ni (x,y,z) \mapsto x \in \mathbb{R}$. The followings are the pictures for two sub-fans $\Sigma_+ \subset \Sigma$ in $\{(x,y,z) \,|\, x \geq 0\} \subset \mathbb{R}^3$ and $\Sigma_- \subset \Sigma$ in $\{(x,y,z) \,|\, x \leq 0\} \subset \mathbb{R}^3$:



Σ_+



Σ_-

For Σ_+, we have the primitive relation:

$$v_2 + v_6 = \begin{cases} v_1 & \cdots \quad b = 0 \\ v_1 + bv_4 & \cdots \quad b > 0 \\ v_1 + (-b)v_7 & \cdots \quad b < 0 \end{cases}$$

We remark that $Z''_{13}(a, 0, c, d)$ can be blown-down to a smooth threefold V. It is not difficult to see that V is not isomorphic to W in Example 4.1 since $v_3 + v_6 = v_4 + v_7 = 0$. Hence V is projective. Therefore, $Z''_{13}(a, 0, c, d)$ itself is projective. So, let $b > 0$. Then, after the anti-flip (flop if $b = 1$) $Z''_{13}(a, b, c, d) \dashrightarrow X'$ along $V(\langle v_1, v_4\rangle)$, there exists the relation $v_2+v_6+bv_7 = v_1$. Thus, we obtain the divisorial contraction $X' \to Y$ by removing the one-dimensional cone generated by v_1. Note that Y is smooth. The case where $b < 0$ is completely similar.

On the other hand, for Σ_-, we have the primitive relation:

$$v_5 + v_7 = \begin{cases} v_8 & \cdots \quad a = c \\ v_8 + (a - c)v_6 & \cdots \quad a > c \\ v_8 + (c - a)v_3 & \cdots \quad a < c \end{cases}$$

So, we can do the same operation as Σ_+ for the other side Σ_- independently. It can be summarized as follows. If $a - c = 0$, then we can remove v_8, which is a divisorial contraction to a smooth threefold. Then we can check that $Z''_{13}(a, b, a, d)$ is projective as in the case where $b = 0$. If $a-c = \pm 1$, then we can remove v_8 after a flop. If $a-c \neq 0, \pm 1$, then we can remove v_8 after an anti-flip.

Thus, if $b \neq 0$ and $a - c \neq 0$, then Z''_{13} is not projective and we have the sequence

$$Z''_{13} \overset{\psi'}{\dashrightarrow} X' \overset{\psi''}{\dashrightarrow} X'' \xrightarrow{\varphi_1} Y_1 \xrightarrow{\varphi_2} Y_2$$

of complete toric threefolds, where ψ', ψ'' are anti-flips (or flops) and φ_1, φ_2 are divisorial contractions which contract a divisor to a smooth point. Since Y_2 is a smooth complete toric threefold of Picard number three, Y_2, Y_1 and X'' are projective. So, Conjecture 1.4 holds for Z''_{13}.

[8-14′] Let $a \in \mathbb{Z}$ and $Z'_{14}(a) := X_\Sigma$ the smooth complete toric threefold associated to the fan Σ in $\mathbb{R}^3$ whose one-dimensional cones are generated by

$$v_1 = (0, 0, -1), v_2 = (1, 0, 0), v_3 = (0, 1, 0), v_4 = (-1, -1, a),$$

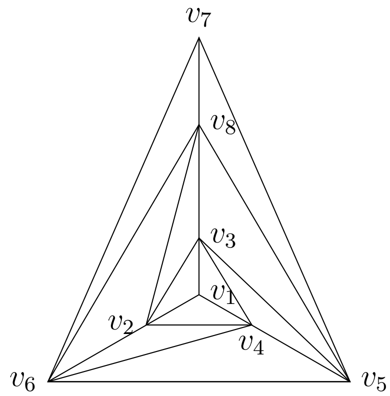
$$v_5 = (-1, -1, a + 1), v_6 = (1, 0, 1), v_7 = (0, 0, 1), v_8 = (0, 1, 1),$$

and the three-dimensional cones of Σ are

$$\langle v_1, v_2, v_3\rangle, \langle v_1, v_3, v_4\rangle, \langle v_1, v_2, v_4\rangle, \langle v_3, v_4, v_5\rangle, \langle v_4, v_5, v_6\rangle, \langle v_2, v_4, v_6\rangle,$$

$$\langle v_5, v_6, v_7\rangle, \langle v_2, v_3, v_8\rangle, \langle v_3, v_5, v_8\rangle, \langle v_5, v_7, v_8\rangle, \langle v_6, v_7, v_8\rangle, \langle v_2, v_6, v_8\rangle.$$

The configuration of cones of Σ are as the following diagram:



There is the extra three-dimensional cone $\langle v_5, v_6, v_7\rangle$.

We have primitive relations

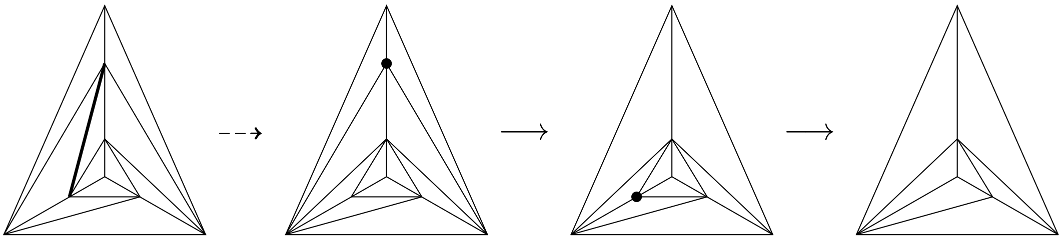
$$v_2 + v_5 = v_4 + v_6, \quad v_3 + v_6 = v_2 + v_8, \quad \text{and} \quad v_4 + v_8 = v_3 + v_5.$$

This implies that $Z'_{14}(a)$ is non-projective by the above primitive relations and Lemma 2.9.

We obtain the sequence

$$Z'_{14}(a) \overset{\psi}{\dashrightarrow} X' \xrightarrow{\varphi_1} Y_1 \xrightarrow{\varphi_2} Y_2$$

of birational maps associated to the diagrams



of fans in $\mathbb{R}^3$. The rational map ψ is the flop associated to the primitive relation

$$v_3 + v_6 = v_2 + v_8,$$

that is, the fan corresponding to X' is obtained from Σ by removing $\langle v_2, v_8\rangle$ and by adding $\langle v_3, v_6\rangle$. The morphisms φ_1 and φ_2 are associated to the star subdivisions along

$$v_8 = v_3 + v_7 \text{ and } v_2 = v_1 + v_6$$

of fans, respectively. Namely, ψ is the flop along $V(\langle v_2, v_8\rangle)$, φ_1 is the blow-up of Y_1 along the curve $V(\langle v_3, v_7\rangle)$ and φ_2 is the blow-up of Y_2 along the curve $V(\langle v_1, v_6\rangle)$. Since Y_2 is smooth and its Picard number is three, Y_2, Y_1 and X' are projective (see Remark 1.6). Conjecture 1.4 is true for this case.

[8-14″] Let $a, b \in \mathbb{Z}$ and $Z''_{14}(a,b) := X_\Sigma$ the smooth complete toric threefold associated to the fan Σ in $\mathbb{R}^3$ whose one-dimensional cones are generated by

$$v_1 = (-1, a, b), v_2 = (0, 1, 0), v_3 = (0, -1, -1), v_4 = (0, 0, 1),$$

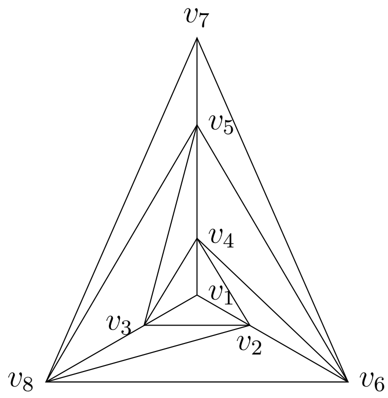
$$v_5 = (1, 0, 1), v_6 = (1, 1, 0), v_7 = (1, 0, 0), v_8 = (1, -1, -1),$$

and the three-dimensional cones of Σ are

$$\langle v_1, v_2, v_3\rangle, \langle v_1, v_3, v_4\rangle, \langle v_1, v_2, v_4\rangle, \langle v_3, v_4, v_5\rangle, \langle v_4, v_5, v_6\rangle, \langle v_2, v_4, v_6\rangle,$$

$$\langle v_5, v_6, v_7\rangle, \langle v_2, v_3, v_8\rangle, \langle v_3, v_5, v_8\rangle, \langle v_5, v_7, v_8\rangle, \langle v_6, v_7, v_8\rangle, \langle v_2, v_6, v_8\rangle.$$

The configuration of cones of Σ are as the following diagram:



There is the extra three-dimensional cone $\langle v_6, v_7, v_8\rangle$.

We have primitive relations

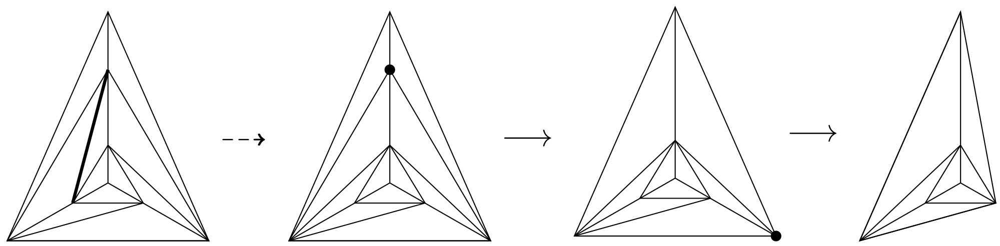
$$v_2 + v_5 = v_4 + v_6, \quad v_3 + v_6 = v_2 + v_8, \quad \text{and} \quad v_4 + v_8 = v_3 + v_5.$$

Hence $Z''_{14}(a, b)$ is not projective by the above primitive relations and Lemma 2.9.

We obtain the sequence

$$Z''_{14}(a, b) \overset{\psi}{\dashrightarrow} X' \xrightarrow{\varphi_1} Y_1 \xrightarrow{\varphi_2} Y_2$$

of birational maps associated to the diagrams



of fans in $\mathbb{R}^3$. The rational map ψ is the flop associated to the primitive relation

$$v_4 + v_8 = v_3 + v_5,$$

that is, the fan corresponding to X' is obtained from Σ by removing $\langle v_3, v_5\rangle$ and by adding $\langle v_4, v_8\rangle$. The morphisms φ_1 and φ_2 are associated to the star subdivisions along

$$v_5 = v_4 + v_7 \text{ and } v_6 = v_2 + v_7$$

of fans, respectively. Namely, ψ is the flop along $V(\langle v_3, v_5\rangle)$, φ_1 is the blow-up of Y_1 along the curve $V(\langle v_4, v_7\rangle)$ and φ_2 is the blow-up of Y_2 along the curve $V(\langle v_2, v_7\rangle)$. Since Y_2 is smooth and its Picard number is three, Y_2, Y_1 and X' are projective (see Remark 1.6). Conjecture 1.4 is true for this case.

Remark 4.7. We have to check the remaining case, that is, the blow-ups of the smooth non-projective complete toric threefold W of Picard number four in Example 4.1. However, one can easily find a flop that makes variety projective in this case. In fact, at least one of the three flopping curves on W is preserved by any blow-up.

Thus, we have the following:

Theorem 4.8. *Conjecture 1.4 is true for any smooth complete toric threefold of Picard number at most five.*

5. Proof of Theorem 1.7

In this final section, we prove Theorem 1.7. More precisely, we explicitly construct a smooth non-projective complete toric threefold satisfying the property in Theorem 1.7. In the following, we will use the notation in [8-13″]. Put $v_i = (x_i, y_i, z_i)$ for $1 \leq i \leq 8$.

Lemma 5.1. *Let Δ be a non-singular complete fan whose one-dimensional cones are $\langle v_1\rangle, \ldots, \langle v_8\rangle$ in* [8-13″]. *Suppose that the following conditions hold:*

(1) *$a, c, d \notin \{-2, -1, 0, 1\}$.*
(2) *$c > a + 1$.*
(3) *For any $2 \leq i < j \leq 8$, we have*

$$b > \left| \pm 1 - \begin{vmatrix} y_i & z_i \\ y_j & z_j \end{vmatrix} + \begin{vmatrix} x_i & z_i \\ x_j & z_j \end{vmatrix} \right|.$$

Then $\Delta = \Sigma$ holds, where Σ is the fan given in [8-13″].

Remark 5.2. The condition (3) is equivalent to the following explicit condition:

$(3)'$ $b > \max\{2, |a|, |a+2|, |c|, |c+2|, |d-1|, |d+2|, |ad+a-cd|, |ad+a-cd+2|\}$.

This can be done by simply calculating the condition (3) for any $2 \leq i < j \leq 8$.

Proof of Lemma 5.1. If $\langle v_1, v_i, v_j\rangle$ is a *non-singular* three-dimensional cone for some $2 \leq i < j \leq 8$, then we have

$$\begin{vmatrix} v_1 \\ v_i \\ v_j \end{vmatrix} = \begin{vmatrix} y_i & z_i \\ y_j & z_j \end{vmatrix} - \begin{vmatrix} x_i & z_i \\ x_j & z_j \end{vmatrix} + b \begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix} = \pm 1$$

$$\iff \quad b \begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix} = \pm 1 - \begin{vmatrix} y_i & z_i \\ y_j & z_j \end{vmatrix} + \begin{vmatrix} x_i & z_i \\ x_j & z_j \end{vmatrix}.$$

If $x_i y_j - y_i x_j \neq 0$, then the inequality

$$|b| \leq \left| \pm 1 - \begin{vmatrix} y_i & z_i \\ y_j & z_j \end{vmatrix} + \begin{vmatrix} x_i & z_i \\ x_j & z_j \end{vmatrix} \right|$$

contradicts the condition (3). Therefore, we have

$$\begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix} = 0.$$

Thus, by easy calculations, either $4 \in \{i, j\}$, $7 \in \{i, j\}$ or $(i, j) = (3, 6)$ holds by the assumptions $a, c \neq 0$ and $a \neq c$. Obviously, $\langle v_1, v_3, v_6\rangle$ is not a three-dimensional cone. Since

$$\begin{vmatrix} v_1 \\ v_4 \\ v_5 \end{vmatrix} = a + 1 \neq \pm 1, \ \begin{vmatrix} v_1 \\ v_7 \\ v_5 \end{vmatrix} = -(a+1) \neq \pm 1, \ \begin{vmatrix} v_1 \\ v_4 \\ v_8 \end{vmatrix} = c + 1 \neq \pm 1 \text{ and } \begin{vmatrix} v_1 \\ v_7 \\ v_8 \end{vmatrix} = -(c+1) \neq \pm 1$$

hold by the assumptions $a \neq 0, -2$ and $c \neq 0, -2$, the remaining possibilities for a three-dimensional cone which contains $\langle v_1\rangle$ as a face are

$$\langle v_1, v_2, v_7\rangle, \ \langle v_1, v_3, v_7\rangle, \ \langle v_1, v_6, v_7\rangle, \ \langle v_1, v_2, v_4\rangle, \ \langle v_1, v_3, v_4\rangle \text{ and } \langle v_1, v_4, v_6\rangle.$$

We remark that these six cones are contained in the half space defined by $x \geq 0$. Among them, we note that

$$\langle v_1, v_2, v_4\rangle, \ \langle v_1, v_3, v_4\rangle, \ \langle v_1, v_2, v_7\rangle \text{ and } \langle v_1, v_3, v_7\rangle$$

are contained in the half space defined by $x \geq y$, while $\langle v_1 \rangle$, $\langle v_4 \rangle$ and $\langle v_7 \rangle$ are contained in the hyperplane defined by $x = y$. Therefore, $\langle v_1, v_4, v_6 \rangle$ and $\langle v_1, v_6, v_7 \rangle$ must be in Δ. Since the relation $v_1 + v_3 + bv_4 = v_2$ tells us that $\langle v_1, v_3, v_4 \rangle$ contains $\langle v_2 \rangle$ inside by the assumption $b > 0$, the cone $\langle v_1, v_3, v_4 \rangle$ is not contained in Δ. Hence, we have $\langle v_1, v_2, v_4 \rangle \in \Delta$. Suppose that $\langle v_2, v_4, v_i \rangle \in \Delta$ is the another three-dimensional cone which contains $\langle v_2, v_4 \rangle$. Then,

$$\begin{vmatrix} v_2 \\ v_4 \\ v_i \end{vmatrix} = y_i = \pm 1$$

says that $i = 3$ by the assumptions $a \neq \pm 1$ and $c \neq \pm 1$. Thus, $\langle v_2, v_3, v_4 \rangle \in \Delta$. Moreover, suppose that $\langle v_2, v_3, v_j \rangle \in \Delta$ is the another three-dimensional cone which contains $\langle v_2, v_3 \rangle$. Then, the equality

$$\begin{vmatrix} v_2 \\ v_3 \\ v_j \end{vmatrix} = -z_j = \pm 1$$

implies that $j = 7$ by the assumptions $b \neq \pm 1$ and $d \neq -2, -1, 0, 1$. Then we obtain that $\langle v_2, v_3, v_7 \rangle$. Also, one can easily see $\langle v_1, v_2, v_7 \rangle$ must be in Δ. Hence Δ coincides with Σ in the half space defined by $x \geq 0$.

From now, we see the half space defined by $x \leq 0$. By the assumption $c > a$, the relation

$$(c - a)v_3 + v_4 + v_8 = v_5$$

says that $\langle v_3, v_4, v_8 \rangle$ contains $\langle v_5 \rangle$ inside, and $\langle v_3, v_4, v_8 \rangle \notin \Delta$. Hence we have $\langle v_3, v_4, v_5 \rangle \in \Delta$. Since $v_4 + v_7 = 0$, the another three-dimensional cone in Δ which contains $\langle v_4, v_5 \rangle$ is either $\langle v_4, v_5, v_6 \rangle$ or $\langle v_4, v_5, v_8 \rangle$. However,

$$\begin{vmatrix} v_4 \\ v_5 \\ v_8 \end{vmatrix} = c - a \neq \pm 1$$

by the assumption $c > a + 1$. This implies $\langle v_4, v_5, v_6 \rangle \in \Delta$. Finally, since also

$$\begin{vmatrix} v_5 \\ v_7 \\ v_8 \end{vmatrix} = c - a \neq \pm 1$$

holds, we have $\langle v_5, v_7, v_8 \rangle \notin \Delta$. The remaining possibilities for a three-dimensional cone which contains $\langle v_8 \rangle$ as a face are

$$\langle v_3, v_5, v_8 \rangle, \ \langle v_5, v_6, v_8 \rangle, \ \langle v_3, v_7, v_8 \rangle \text{ and } \langle v_6, v_7, v_8 \rangle,$$

because $v_3 + v_6 = 0$. Actually, Δ must contain all of them. This means that $\Delta = \Sigma$ holds. We finish the proof. □

Remark 5.3. The condition in Lemma 5.1 is only a *sufficient* condition. In fact, for example, we see $\Delta = \Sigma$ for $(a, b, c, d) = (2, 3, 5, 7)$, though the condition in Lemma 5.1 is not satisfied. Moreover, for many other cases, one can confirm that Σ is the unique non-singular complete fan whose one-dimensional cones are $\langle v_1 \rangle, \ldots, \langle v_8 \rangle$ in [8-13″] as the proof of Lemma 5.1.

For example, $(a, b, c, d) = (2, 7, 4, 2)$ satisfies the conditions in Lemma 5.1 (see Remark 5.2, too). Thus, we obtain the following.

Corollary 5.4. *Let $Z''_{13}(2,7,4,2) \dashrightarrow X'$ be a toric birational map which is an isomorphism in codimension one. Assume that X' is projective. Then X' is always singular.*

Proof. We have already proved that $Z''_{13}(2,7,4,2)$ is non-projective in Section 4. Hence this follows from Lemma 5.1. □

Note that $Z''_{13}(2,7,4,2)$ is a desired smooth non-projective complete toric threefold with Picard number five in Theorem 1.7. Corollary 5.4 says that we can not make X' smooth even if X is so in Theorem 1.3.

Remark 5.5 (see Remark 2.4)**.** Since $Z''_{13}(2,7,4,2)$ is a smooth toric threefold, it has no flipping contractions (see [FSTU]). Moreover, by Lemma 5.1, it has no flopping contractions since every three-dimensional toric flop preserves smoothness.

References

[Ba1] V. Batyrev, On the classification of smooth projective toric varieties, Tohoku Math. J. **43** (1991), no. 4, 569–585.

[Ba2] V. Batyrev, On the classification of toric Fano 4-folds, Algebraic Geometry, 9, J. Math. Sci. (New York) **94** (1999), 1021–1050.

[Bo] L. Bonavero, Sur des variétés toriques non projectives, Bull. Soc. Math. France **128** (2000), no. 3, 407–431.

[CLS] D. A. Cox, J. B. Little, H. K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, **124**. American Mathematical Society, Providence, RI, 2011.

[F1] O. Fujino, Notes on toric varieties from Mori theoretic viewpoint, Tohoku Math. J. (2) **55** (2003), no. 4, 551–564.

[F2] O. Fujino, On the Kleiman–Mori cone, Proc. Japan Acad. Ser. A Math. Sci. **81** (2005), no. 5, 80–84.

[F3] O. Fujino, Equivariant completions of toric contraction morphisms, With an appendix by Fujino and Hiroshi Sato, Tohoku Math. J. (2) **58** (2006), no. 3, 303–321.

[F4] O. Fujino, *Foundations of the minimal model program*, MSJ Memoirs, **35**. Mathematical Society of Japan, Tokyo, 2017.

[F5] O. Fujino, Minimal model theory for log surfaces in Fujiki's class $\mathcal{C}$, Nagoya Math. J. **244** (2021), 256–282.

[FP] O. Fujino, S. Payne, Smooth complete toric threefolds with no nontrivial nef line bundles, Proc. Japan Acad. Ser. A Math. Sci. **81** (2005), no. 10, 174–179.

[FS] O. Fujino, H. Sato, Introduction to the toric Mori theory, Michigan Math. J. **52** (2004), no. 3, 649–665.

[FSTU] O. Fujino, H. Sato, Y. Takano, H. Uehara, Three-dimensional terminal toric flips, Cent. Eur. J. Math. **7** (2009), no. 1, 46–53.

[Fl] W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, **131**. The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ, 1993.

[KS] P. Kleinschmidt, B. Sturmfels, Smooth toric varieties with small Picard number are projective, Topology **30** (1991), no. 2, 289–299.

[K] J. Kollár, Flips, flops, minimal models, etc, Surveys in differential geometry (Cambridge, MA, 1990), 113–199, Lehigh Univ., Bethlehem, PA, 1991.

[M] K. Matsuki, *Introduction to the Mori program*, Universitext. Springer-Verlag, New York, 2002.

[N] O. Nagaya, Classification of 3-dimensional complete non-singular torus embeddings, Master's thesis, Nagoya Univ. 1976.

[O1] T. Oda, *Torus embeddings and applications*, Based on joint work with Katsuya Miyake. Tata Institute of Fundamental Research Lectures on Mathematics and Physics, **57**. Tata Institute of Fundamental Research, Bombay; Springer-Verlag, Berlin-New York, 1978.

[O2] T. Oda, *Convex bodies and algebraic geometry. An introduction to the theory of toric varieties*, Translated from the Japanese. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], **15**. Springer-Verlag, Berlin, 1988.

[Re] M. Reid, Decomposition of toric morphisms, *Arithmetic and geometry, Vol. II*, 395–418, Progr. Math., **36**, Birkhäuser Boston, Boston, MA, 1983.

[RT] M. Rossi, L. Terracini, A $\mathbb{Q}$-factorial complete toric variety with Picard number 2 is projective, J. Pure Appl. Algebra **222** (2018), no. 9, 2648–2656.

[Sa1] H. Sato, Toward the classification of higher-dimensional toric Fano varieties, Tohoku Math. J. **52** (2000), no. 3, 383–413.

[Sa2] H. Sato, Combinatorial descriptions of toric extremal contractions, Nagoya Math. J. **180** (2005), 111–120.

[Su] H. Sumihiro, Equivariant completion, J. Math. Kyoto Univ. **14** (1974), 1–28.

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